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Efficient numerical solution of the generalized Dirichlet–Neumann map for linear elliptic PDEs in regular polygon domains☆

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ABSTRACT

A new and novel approach for analyzing boundary value problems for linear and for integrable nonlinear PDEs was recently introduced. For linear elliptic PDEs, an important aspect of this approach is the characterization of a generalized Dirichlet–Neumann map: given the derivative of the solution along a direction of an arbitrary angle to the boundary, the derivative of the solution perpendicularly to this direction is computed *without* solving on the interior of the domain. For this computation, a collocation-type numerical method has been recently developed. Here, we study the collocation's coefficient matrix properties. We prove that, for the Laplace's equation on regular polygon domains with the same type of boundary conditions on each side, the collocation matrix is block circulant, independently of the choice of basis functions. This leads to the deployment of the FFT for the solution of the associated collocation linear system, yielding significant computational savings. Numerical experiments are included to demonstrate the efficiency of the whole computation.

1. Introduction

A new method for solving boundary value problems for linear and for integrable nonlinear PDEs in two dimensions was introduced by Fokas in [1,2]. This method involves two novel features:

(a) It yields an analytic representation of the solution, in an integral form, in the complex k-plane.
(b) It characterizes a generalized Dirichlet–Neumann map through the solution of the so-called *global relation*, an equation, valid for all complex values of k, which couples known and unknown components of the solution and its derivatives on the boundary.

For a large class of boundary value problems, the global relation can be solved analytically, and hence the generalized Dirichlet–Neumann map can be constructed in closed form. This includes linear evolution PDEs with spatial derivatives of arbitrary order on the half-line [3] and on a finite interval [4], the Laplace, the bi-harmonic and the modified Helmholtz equation in certain simple polygons [5–7], and the basic nonlinear integrable evolution PDEs on the half-line for certain

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simple boundary conditions [8,9]. However, for general boundary value problems, the global relation must be solved numerically.

In [5,10] a well conditioned and fast convergent collocation-type numerical method was developed and studied for the numerical solution of the generalized Dirichlet–Neumann map associated to the generic model problem of Laplace's equation on an arbitrary convex polygon domain. The present work, which is a natural continuation of our work in [10], aims to the efficient solution of the associated collocation linear system. For the case of regular polygon domains, with the same type of boundary conditions on each side, we prove, among other properties, that the Collocation coefficient matrix is *block circulant*, independently of the choice of basis functions. Evidently, therefore, by deploying the *Fourier Matrix*, the collocation matrix is transformed into a similar block diagonal matrix, the construction of which is implemented through the *Fast Fourier Transform* (FFT), yielding efficient implementation of direct and iterative methods for the solution of the collocation linear system.

This paper is organized as follows: Section 2 outlines some analytical results, as well as the collocation method of [10]. Section 3 presents the structure and the properties of the collocation coefficient matrix for the case of regular polygons. And, finally, Section 4 presents the implementation through the FFT as well as numerical results, for a variety of boundary conditions, to demonstrate the efficiency of the whole computation.

2. Overview

For elliptic PDEs in two dimensions it is convenient to replace the Cartesian coordinates (x, y) with the complex coordinates $(z, \bar{z}) = (x + iy, x - iy)$. In doing so and using the equations

$$\partial_z = \frac{1}{2}\left(\partial_x - i\partial_y\right), \qquad \partial_{\bar{z}} = \frac{1}{2}\left(\partial_x + i\partial_y\right),$$

Laplace's equation in the independent variable q can be written in the form

$$\frac{\partial^2 q}{\partial z \partial \bar{z}} = 0. \tag{2.1}$$

This equation is equivalent to the equation

$$\frac{\partial}{\partial \bar{z}}\left(e^{-ikz}\frac{\partial q}{\partial z}\right) = 0, \tag{2.2}$$

for an *arbitrary* complex parameter k.

The global relation.

Suppose that the real-valued function $q(z, \bar{z})$ satisfies Laplace's Eq. (2.1) in a simply connected bounded domain D with boundary ∂D. Then, Eq. (2.2) together with the complex form of Green's theorem imply the equation

$$\oint_{\partial D} e^{-ikz}\frac{\partial q}{\partial z}dz = 0, \quad k \in \mathbb{C}, \tag{2.3}$$

which is referred to (cf. [2]) as the *global relation* associated with Laplace's equation. For the case of D being a convex bounded polygon with vertices $z_1, z_2, \ldots, z_n$ (modulo n), which have indexed counter-clockwise, the global relation (2.3) becomes

$$\sum_{j=1}^{n}\int_{S_j} e^{-ikz}\frac{\partial q}{\partial z}dz = 0, \quad k \in \mathbb{C}, \tag{2.4}$$

where S_j denotes the side (z_j, z_{j+1}).

The generalized Dirichlet–Neumann map.

Using the identity

$$\frac{\partial q}{\partial z} = \frac{1}{2}e^{-i\alpha_j}\left(q_\tau^{(j)} + iq_n^{(j)}\right), \quad z \in S_j,\ \alpha_j = \arg\left(z_{j+1} - z_j\right), \tag{2.5}$$

where $q_\tau^{(j)}$ and $q_n^{(j)}$ denote the tangential and (outward) normal components of $\frac{\partial q}{\partial z}$ along the side S_j, as well as the local coordinates

$$z = m_j + sh_j, \qquad z_i < z < z_{i+1}, \qquad -\pi < s < \pi, \tag{2.6}$$

with

$$m_j = \frac{1}{2}\left(z_j + z_{j+1}\right), \qquad h_j = \frac{1}{2\pi}\left(z_{j+1} - z_j\right), \tag{2.7}$$

it was shown (e.g. [10]) that:

Proposition 2.1. *Let the real-valued function $q(z,\bar{z})$ satisfy the Laplace equation in the interior D of the convex bounded polygon with corners $\{z_i\}_{i=1}^n$ described above. Let $g^{(j)}$ denote the derivative of the solution in the direction making an angle β_j, $0 \le \beta_j \le \pi$, with the side S_j, namely*

$$\cos\left(\beta_j\right) q_\tau^{(j)} + \sin\left(\beta_j\right) q_n^{(j)} = g^{(j)}, \quad z \in S_j,\ 1 \le j \le n. \tag{2.8}$$

Let $f^{(j)}$ denote the derivative of the solution in the direction normal to the above direction, namely

$$-\sin\left(\beta_j\right) q_\tau^{(j)} + \cos\left(\beta_j\right) q_n^{(j)} = f^{(j)}, \quad z \in S_j,\ 1 \le j \le n. \tag{2.9}$$

The generalized Dirichlet–Neumann map, that is the relation between the sets $\left\{f^{(j)}\right\}_{j=1}^n$ and $\left\{g^{(j)}\right\}_{j=1}^n$, is characterized by the single equation

$$\sum_{j=1}^{n} \left|h_j\right| e^{i\left(\beta_j - km_j\right)} \int_{-\pi}^{\pi} e^{-ikh_js} \left(f^{(j)}(s) - ig^{(j)}(s)\right) ds = 0, \quad k \in \mathbb{C}. \tag{2.10}$$

Evaluating Eq. (2.10) on the following n-rays of the complex k-plane

$$k_p = -\frac{l}{h_p}, \quad l \in \mathbb{R}^+,\ p = 1, \ldots, n, \tag{2.11}$$

and multiplying the resulting equations by $\exp\left[-i(\beta_p + lm_p/h_p)\right] / \left|h_p\right|$, *Eq. (2.10) yields the following set of n equations:*

$$\sum_{j=1}^{n} \frac{\left|h_j\right|}{\left|h_p\right|} e^{i\left(\beta_j - \beta_p\right)} e^{-\frac{il}{h_p}\left(m_p - m_j\right)} \int_{-\pi}^{\pi} e^{il\frac{h_j}{h_p}s} \left(f^{(j)}(s) - ig^{(j)}(s)\right) ds = 0, \tag{2.12}$$

with $l \in \mathbb{R}^+,\ p = 1, \ldots, n$.

Suppose, now, that the function set $\left\{g^{(j)}\right\}_{j=1}^n$ is known through appropriate boundary data. Then, it becomes apparent that the generalized Dirichlet–Neumann map, in its convenient form of (2.12), may be used to determine the function set $\left\{f^{(j)}\right\}_{j=1}^n$. The end values of the unknown functions $f^{(j)}$ can be calculated by the continuity requirements $q_z^{(j)}\left(z_j\right) = q_z^{(j-1)}\left(z_j\right)$. Namely, rewriting equation (2.5) as

$$\frac{\partial q}{\partial z} = \frac{1}{2} e^{-i\left(\alpha_j - \beta_j\right)} \left(g^{(j)} + if^{(j)}\right), \quad z \in S_j, \tag{2.13}$$

assuming that $g^{(j)}$ are compatible in the corners and setting $\delta_j = \alpha_j - \beta_j$ we obtain (cf. [5])

$$f^{(j)}(\pi) = \frac{\cos\left(\delta_{j+1} - \delta_j\right) g^{(j)}(\pi) - g^{(j+1)}(\pi)}{\sin\left(\delta_{j+1} - \delta_j\right)},$$
$$f^{(j)}(-\pi) = \frac{g^{(j-1)}(\pi) - \cos\left(\delta_j - \delta_{j-1}\right) g^{(j)}(-\pi)}{\sin\left(\delta_j - \delta_{j-1}\right)}.$$

The numerical method.

The collocation-type numerical method, developed in [5,11,12,10], for the determination of the function set $\left\{f^{(j)}\right\}_{j=1}^n$, is being described in the form of the following proposition (cf. [10]):

Proposition 2.2. *Consider the generalized Dirichlet–Neumann map in Proposition 2.1. Suppose that the set $\left\{g^{(j)}\right\}_{j=1}^n$ is given. Suppose that $f^{(j)}(s)$ is approximated by*

$$f_N^{(j)}(s) = f_*^{(j)}(s) + \sum_{r=1}^{N} U_r^j \varphi_r(s), \quad j = 1, \ldots, n,\ N \text{ even integer}, \tag{2.14}$$

where $\varphi_r(s)$ are appropriate basis functions, N is even and

$$f_*^{(j)}(s) = \frac{1}{2\pi}\left[(s+\pi) f^{(j)}(\pi) - (s-\pi) f^{(j)}(-\pi)\right].$$

Then, the real coefficients U_r^j satisfy the Nn algebraic set of equations

$$\sum_{j=1}^{n} \frac{\left|h_j\right|}{\left|h_p\right|} e^{i\left(\beta_j - \beta_p\right)} e^{-i\frac{l}{h_p}\left(m_p - m_j\right)} \sum_{r=1}^{N} U_r^j F_r\left(\frac{lh_j}{h_p}\right) = G_p(l), \quad p = 1, 2, \ldots, n, \tag{2.15}$$

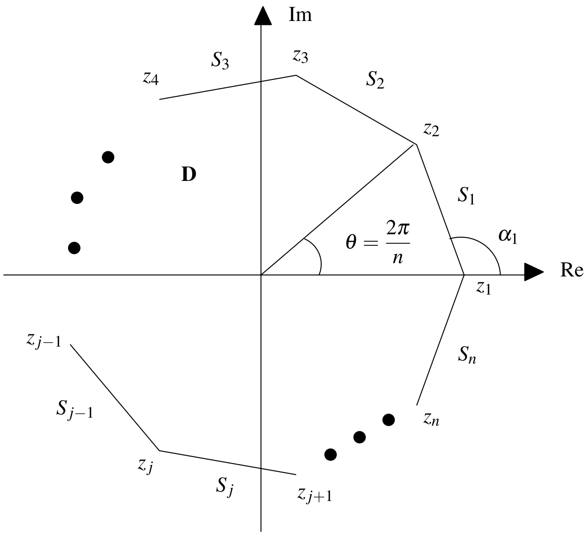

Fig. 3.1. Regular n-gon with vertices z_j, sides S_j and interior **D**.

where $G_p(l)$ denotes the known function

$$G_p(l) = i \sum_{j=1}^{n} \frac{|h_j|}{|h_p|} e^{i(\beta_j-\beta_p)} e^{-i\frac{l}{h_p}(m_p-m_j)} \int_{-\pi}^{\pi} e^{il\frac{h_j}{h_p}s} (g^{(j)}(s) + if_*^{(j)}(s))ds, \tag{2.16}$$

$F_r(l)$ denotes the integral

$$F_r(l) = \int_{-\pi}^{\pi} e^{ils} \varphi_r(s)ds, \quad r = 1, 2, \ldots, N, \tag{2.17}$$

and l is chosen as follows: For the imaginary part of Eqs. (2.15)

$$l = 1, 2, \ldots, N/2,$$

whereas for the real part of Eqs. (2.15),

$$l = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N-1}{2}.$$

It is worthwhile to point out that different choices of the basis functions $\varphi_r(s)$ affect the convergence rate and the condition number of the method, as well as the structure of the coefficient matrix (cf. [10] for a detailed treatment).

3. Matrix properties for regular polygons

Let us now consider the regular polygon, depicted in Fig. 3.1, with vertices z_j and sides S_j, $j = 1, \ldots, n$ (modulo n), indexed counter-clockwise, and interior **D**. For notational simplicity and without any loss of the generality, we assume that the polygon is centered at the origin, scaled and oriented so that one vertex (say z_1) is located at 1, that is $z_1 = 1$. Setting $\theta = \frac{2\pi}{n}$, the vertices may be written as

$$z_j = \omega^{j-1}, \quad \omega = e^{i\theta}, \ j = 1, \ldots, n, \tag{3.1}$$

and the angle α_j of the side S_j from the real axis (measured counterclockwise) is given by

$$\alpha_j = arg(z_{j+1} - z_j) = \frac{1}{2}[\pi + (2j-1)\theta] = \frac{\pi}{2} + (2j-1)\frac{\pi}{n}, \quad j = 1, \ldots, n. \tag{3.2}$$

Suppose that the real-valued function $q(z, \bar{z})$ satisfies the Laplace's equation in the interior **D** of the regular n-gon, described above, subject to the same type of *oblique Neumann* boundary conditions on all sides, that is

$$\cos(\beta)\, q_\tau^{(j)} + \sin(\beta)\, q_n^{(j)} = g^{(j)}, \quad z \in S_j, \ 1 \le j \le n. \tag{3.3}$$

Dirichlet and Neumann boundary conditions correspond to the special cases of $\beta = 0$ and $\beta = \frac{\pi}{2}$ respectively.

Recalling, now, the local coordinates of (2.6) and observing that their parametrization of (2.7) is expressed as

$$m_j = \frac{1}{2}\left(z_j + z_{j+1}\right) = |m_j| e^{i(\alpha_j - \frac{\pi}{2})} = \cos\left(\frac{\pi}{n}\right) e^{i(2j-1)\frac{\pi}{n}} = \cos\left(\frac{\pi}{n}\right) \omega^{(2j-1)/2}, \tag{3.4}$$

and

$$h_j = \frac{1}{2\pi}\left(z_{j+1} - z_j\right) = |h_j| e^{i a_j} = i\frac{1}{\pi}\sin\left(\frac{\pi}{n}\right) e^{i(2j-1)\frac{\pi}{n}} = i\frac{1}{\pi}\sin\left(\frac{\pi}{n}\right)\omega^{(2j-1)/2}, \tag{3.5}$$

it follows easily, from Proposition 2.1, that:

Corollary 3.1. *Let the real-valued function $q(z,\bar{z})$ satisfy the Laplace equation in the interior D of the regular n-gon with corners $\{z_i\}_{i=1}^n$ described above in this section. Let $g^{(j)}$, defined in (3.3), denote the derivative of the solution in the direction making an angle β, $0 \le \beta \le \pi$, with the side S_j, and let $f^{(j)}$ denote the derivative of the solution in the direction normal to the above direction. The generalized Dirichlet–Neumann map is characterized by the single equation*

$$\sum_{j=1}^{n} e^{-ikm_j} \int_{-\pi}^{\pi} e^{-ikh_j s}\left(f^{(j)}(s) - ig^{(j)}(s)\right) ds = 0, \quad k \in \mathbb{C}, \tag{3.6}$$

which, upon evaluation on the n-rays of the complex k-plane defined in (2.11), yields the following set of n equations:

$$\sum_{j=1}^{n} e^{l\phi\omega_{jp}} \int_{-\pi}^{\pi} e^{il\omega_{jp}s}\left(f^{(j)}(s) - ig^{(j)}(s)\right) ds = 0, \quad l \in \mathbb{R}^+,\ p = 1, \ldots, n \tag{3.7}$$

where

$$\phi = \pi \cot\frac{\pi}{n}, \qquad \omega_{jp} = \omega^{j-p} \tag{3.8}$$

and $\omega = e^{i\theta}$ being as in (3.1).

Proof. The relations

$$\frac{m_p - m_j}{h_p} = i\phi(\omega_{jp} - 1) \quad \text{and} \quad \frac{h_j}{h_p} = \omega_{jp}, \tag{3.9}$$

together with the appropriate simplifications of constant quantities, relax Eq. (2.12) to (3.7) and the proof follows immediately. □

As an immediate consequence of Corollary 3.1 and its combination with Proposition 2.2, one may readily obtain that:

Corollary 3.2. *Consider the generalized Dirichlet–Neumann map in Corollary 3.1. Suppose that the set $\left\{g^{(j)}\right\}_{j=1}^n$ is given through (3.3) and that the set $\left\{f^{(j)}\right\}_{j=1}^n$ is approximated by $\left\{f_N^{(j)}\right\}_{j=1}^n$ defined in (2.14). Then, the real coefficients U_r^j satisfy the Nn algebraic set of equations*

$$\sum_{j=1}^{n} e^{l\phi\omega_{jp}} \sum_{r=1}^{N} U_r^j F_r\left(l\omega_{jp}\right) = G_p(l), \quad p = 1, 2, \ldots, n, \tag{3.10}$$

where $G_p(l)$ denotes the known function

$$G_p(l) = i\sum_{j=1}^{n} e^{l\phi\omega_{jp}} \int_{-\pi}^{\pi} e^{il\omega_{jp}s}(g^{(j)}(s) + if_*^{(j)}(s))ds, \tag{3.11}$$

$F_r(l)$ denotes the integral

$$F_r(l) = \int_{-\pi}^{\pi} e^{ils}\varphi_r(s)ds, \quad r = 1, 2, \ldots, N, \tag{3.12}$$

ϕ and ω_{jp} are as defined in (3.8), and l is chosen as in Proposition 2.2, namely: For the imaginary part of Eqs. (3.10) $l = 1, 2, \ldots, N/2$, whereas for the real part of Eqs. (3.10) $l = \frac{1}{2}, \frac{3}{2}, \ldots, \frac{N-1}{2}$.

System's structure.

Following Corollary 3.2, let $C(p,j,l,r)$ be defined by

$$C(p,j,l,r) = e^{l\phi\omega_{jp}} F_r\left(l\omega_{jp}\right) = e^{l\phi\omega_{jp}} \int_{-\pi}^{\pi} e^{il\omega_{jp}s} \varphi_r(s) ds, \tag{3.13}$$

with $C_R(p,j,l,r)$ and $C_I(p,j,l,r)$ to denote its real and imaginary parts respectively, that is

$$C_R(p,j,l,r) = \mathrm{Re}[C(p,j,l,r)] \quad \text{and} \quad C_I(p,j,l,r) = \mathrm{Im}[C(p,j,l,r)]. \tag{3.14}$$

Let, also, $C_{p,j} \in \mathbb{R}^{N,N}$ denote the real $N \times N$ matrix with elements

$$C_{p,j} = \{c^{p,j}_{2l,r}\}, \quad c^{p,j}_{2l,r} = \begin{cases} C_R(p,j,l,r), & l = \frac{1}{2}, \frac{3}{2}, \dots, \frac{N-1}{2} \\ C_I(p,j,l,r), & l = 1, 2, \dots, N/2 \end{cases}, \quad r = 1, 2, \dots, N. \tag{3.15}$$

If, now, $\mathbf{U}_j \in \mathbb{R}^{N,1}$ and $\mathbf{G}_p \in \mathbb{R}^{N,1}$ denote the real vectors

$$\mathbf{U}_j = \{U^j_r\}_{r=1}^N = [U^j_1 \quad U^j_2 \quad \cdots \quad U^j_N]^T, \tag{3.16}$$

and

$$\mathbf{G}_j = \{G^p_\ell\}_{\ell=1}^N = [G^p_1 \quad G^p_2 \quad \cdots \quad G^p_N]^T, \tag{3.17}$$

where

$$G^p_\ell = \begin{cases} G^p_R(l), & l = \frac{1}{2}, \frac{3}{2}, \dots, \frac{N-1}{2}, \\ G^p_I(l), & l = 1, 2, \dots, N/2, \end{cases}, \quad \ell = 2l, \tag{3.18}$$

with $G^p_R(l)$ and $G^p_I(l)$ to denote respectively the real and imaginary parts of $G_p(l)$ in (3.11), then it can be easily seen that:

Proposition 3.1. *The linear system, described by Eqs. (3.10)–(3.12) in Corollary 3.2, is given by*

$$C\mathbf{U} = \mathbf{G}, \quad C \in \mathbb{R}^{nN,nN}, \ \mathbf{U}, \mathbf{G} \in \mathbb{R}^{nN}, \tag{3.19}$$

where

$$C = \begin{pmatrix} C_{1,1} & \cdots & C_{1,j} & \cdots & C_{1,n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ C_{p,1} & \cdots & C_{p,j} & \cdots & C_{p,n} \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ C_{n,1} & \cdots & C_{n,j} & \cdots & C_{n,n} \end{pmatrix}, \qquad \mathbf{U} = \begin{pmatrix} \mathbf{U}_1 \\ \vdots \\ \mathbf{U}_j \\ \vdots \\ \mathbf{U}_n \end{pmatrix}, \qquad \mathbf{G} = \begin{pmatrix} \mathbf{G}_1 \\ \vdots \\ \mathbf{G}_p \\ \vdots \\ \mathbf{G}_n \end{pmatrix}, \tag{3.20}$$

with $C_{p,j}$, $\mathbf{U}_j$ and $\mathbf{G}_p$ are as defined in (3.15)–(3.17).

Proof. Recall (3.13)–(3.14) and observe that the set of equations in (3.10) is written as

$$\sum_{j=1}^{n} \sum_{r=1}^{N} C(p,j,l,r) U^j_r = G_p(l), \quad p = 1, 2, \dots, n \tag{3.21}$$

or, equivalently, as

$$\sum_{j=1}^{n} \sum_{r=1}^{N} C_R(p,j,l,r) U^j_r = G^p_R(l) \quad \text{and} \quad \sum_{j=1}^{n} \sum_{r=1}^{N} C_I(p,j,l,r) U^j_r = G^p_I(l). \tag{3.22}$$

The above set of equations, by using (3.15) and (3.18), is expressed as

$$\sum_{j=1}^{n} \sum_{r=1}^{N} c^{p,j}_{2l,r} U^j_r = G^p_{2l}, \quad p = 1, 2, \dots, n \tag{3.23}$$

and, by letting $l = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots, \frac{N-1}{2}, \frac{N}{2}$, as

$$\sum_{j=1}^{n} C_{p,j} \mathbf{U}_j = \mathbf{G}_p, \quad p = 1, 2, \dots, n, \tag{3.24}$$

which completes the proof. □

Coefficient matrix properties.

To reveal, now, the properties of the coefficient matrix C of (3.20) we first prove that:

Proposition 3.2. *The coefficient matrix C, defined in relations (3.19)–(3.20) of Proposition 3.3, is block circulant. Namely,*

$$C = \mathrm{bcirc}\{C_1, C_2, \dots, C_n\} = \begin{pmatrix} C_1 & C_2 & C_3 & \cdots & C_{n-1} & C_n \\ C_n & C_1 & C_2 & \cdots & C_{n-2} & C_{n-1} \\ C_{n-1} & C_n & C_1 & \cdots & C_{n-3} & C_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ C_3 & C_4 & C_5 & \cdots & C_1 & C_2 \\ C_2 & C_3 & C_4 & \cdots & C_n & C_1 \end{pmatrix}, \tag{3.25}$$

where

$$C_j = C_{1,j}, \quad j = 1, \dots, n \tag{3.26}$$

and $C_{1,j}$ are as defined in (3.15).

Proof. It suffices to prove that, for any $\mu = 1, \dots, n-1$, the matrices $C_{p,j}$ $(p, j = 1, \dots, n)$ of (3.15) satisfy

$$C_{p,j} = C_{p',j'}, \tag{3.27}$$

where

$$p' = \begin{cases} p + \mu, & 1 \le \mu \le n - p \\ p + \mu - n, & n - p + 1 \le \mu \le n - 1 \end{cases} \tag{3.28}$$

and

$$j' = \begin{cases} j + \mu, & 1 \le \mu \le n - j \\ j + \mu - n, & n - j + 1 \le \mu \le n - 1 \end{cases}. \tag{3.29}$$

For this, recall $C(p, j, r, l)$ from (3.13) and observe that, for fixed r and l, there holds

$$C(p', j', r, l) = C(p, j, r, l) \tag{3.30}$$

since, from (3.1) and (3.8), $\omega^{\pm n} = 1$ and

$$\omega_{j'p'} = \omega^{j'-p'} = \omega^{j-p} = \omega_{jp}. \tag{3.31}$$

Evidently, therefore, for fixed r and l, we obtain, from (3.15), that

$$c_{2l,r}^{p,j} = c_{2l,r}^{p',j'}, \tag{3.32}$$

and the proof follows. □

We know (e.g. [11,13]) that the block circulant property of the coefficient matrix C allows its block diagonal factorization. To be specific, let $F_n \in \mathbb{C}^{n,n}$ denote the *Fourier Matrix*, that is

$$F_n = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)(n-1)} \end{pmatrix}, \tag{3.33}$$

where $\omega = e^{i2\pi/n}$ is as in (3.1). Let also $P_n \in \mathbb{R}^{n,n}$ denote the cyclic permutation matrix

$$P_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} O_{n-1,1} & I_{n-1} \\ 1 & O_{1,n-1} \end{pmatrix}, \tag{3.34}$$

where I_q denotes the $q \times q$ identity matrix and $O_{p,q}$ the $p \times q$ null matrix. It is, also, well known that, among other celebrated properties,

$$F_n^{-1} = \frac{1}{n} F_n^*, \tag{3.35}$$

where F_n^* denotes the *conjugate transpose*, while

$$P_n^n = I_n \quad \text{and} \quad P_n^\ell = \begin{pmatrix} O_{n-\ell,\ell} & I_{n-\ell} \\ I_\ell & O_{\ell,n-\ell} \end{pmatrix} \quad \text{for } \ell = 1, \ldots, n-1, \tag{3.36}$$

and, furthermore,

$$P_n = F_n \Omega_n F_n^{-1}, \quad \Omega_n = \mathrm{diag}(1, \omega, \omega^2, \ldots, \omega^{n-1}). \tag{3.37}$$

If we now let $\otimes$ and $\oplus$ denote the Kronecker product and direct sum of matrices, respectively, then (see Thm. 5.6.4 of [13]):

Proposition 3.3. *The block circulant matrix C, defined in (3.25) of Proposition 3.2, is expressed as*

$$C = (F_n \otimes I_N) \left(\bigoplus_{\ell=1}^{n} \left(\sum_{j=1}^{n} \omega^{(\ell-1)(j-1)} C_j \right) \right) (F_n^{-1} \otimes I_N). \tag{3.38}$$

Proof. Observing that

$$\begin{aligned} C &= \sum_{j=1}^{n} \left(P_n^{j-1} \otimes I_N\right) \left(\bigoplus_{\ell=1}^{n} C_j \right) \\ &= \sum_{j=1}^{n} \left(\left(F_n \Omega_n^{j-1} F_n^{-1}\right) \otimes I_N\right) \left(\bigoplus_{\ell=1}^{n} C_j \right) \\ &= \sum_{j=1}^{n} (F_n \otimes I_N) \left(\Omega_n^{j-1} \otimes I_N\right) \left(F_n^{-1} \otimes I_N\right) \left(\bigoplus_{\ell=1}^{n} C_j \right) \\ &= (F_n \otimes I_N) \left(\sum_{j=1}^{n} \left(\Omega_n^{j-1} \otimes I_N\right) \left(\bigoplus_{\ell=1}^{n} C_j \right) \right) (F_n^{-1} \otimes I_N) \\ &= (F_n \otimes I_N) \left(\sum_{j=1}^{n} \left(\bigoplus_{\ell=1}^{n} \omega^{(\ell-1)(j-1)} C_j \right) \right) (F_n^{-1} \otimes I_N) \end{aligned}$$

the proof follows. □

At this point we would like to conclude this section with the following remarks:

Remark 3.1. For the case of regular polygons with the same type boundary conditions on all sides, the coefficient matrix C of (3.25) is *independent* of the angle $\beta_j = \beta$ the derivative of the solution is making with the side S_j. Namely, C is *independent* of the type (i.e. Dirichlet, Neumann or Mixed) of *Boundary Conditions*.

Remark 3.2. The coefficient matrix properties revealed in this section are *independent* of the choice of *basis functions* and the number of *polygon sides*. However, certain choices of basis functions may enrich the properties of the coefficient matrix. For example, for the natural choice of *sine* basis functions, i.e.

$$\varphi_r(s) = \sin\left[r \left(\frac{\pi + s}{2} \right) \right], \tag{3.39}$$

studied in [10], and in addition to the property that the block diagonal submatrices of C are point diagonal, that is (cf. [10])

$$C_1 = \pi \ \mathrm{diag}(1, -1, -1, 1, \ldots, (-1)^{N-1}, (-1)^N), \tag{3.40}$$

it takes only a few algebraic manipulations to verify that (cf. [12]), for regular polygons with the same type of boundary conditions on all sides, there also holds

$$C_{n-j+2} = DC_jD, \quad D = \mathrm{diag}(1, -1, \ldots, 1, -1), \ j = 2, \ldots, \hat{n}, \tag{3.41}$$

where $\hat{n} = n/2$ if $n =$ even while $\hat{n} = (n+1)/2$ if $n =$ odd. Similarly, if one considers the case of a *square* domain, with the same type of boundary conditions on all sides, then, independently of the choice of basis functions, one may easily verify (cf. [14]) that

$$C_2 = C_4 = O \quad \text{and} \quad C_3 = \hat{D}C_1, \quad \hat{D} = \mathrm{diag}(d_1, \ldots, d_N), \tag{3.42}$$

with $d_r = (-1)^{r-1} e^{-r\pi}, \ r = 1, \ldots, N$.

4. Implementation & numerical verification

Having in mind that the cost for solving the linear system in (3.19), by the classical LU-factorization, is $\mathcal{O}\left(n^3N^3\right)$, in this section we are aiming at the efficient implementation of the results in Propositions 3.2 and 3.3 and its numerical verification. For this purpose, recall the relation (3.38) and observe that, upon substitution in (3.19), the linear system at hand may be written as

$$A\mathbf{u} = \mathbf{g}, \tag{4.1}$$

where

$$A = b\mathrm{diag}(A_1, \ldots, A_n) = \bigoplus_{\ell=1}^{n} A_\ell, \quad A_\ell = \left(\sum_{j=1}^{n} \omega^{(\ell-1)(j-1)} C_j\right), \tag{4.2}$$

$$\mathbf{u} = \left(F_n^{-1} \otimes I_N\right) \mathbf{U}, \tag{4.3}$$

and

$$\mathbf{g} = \left(F_n^{-1} \otimes I_N\right) \mathbf{G}, \tag{4.4}$$

with C_j, $\mathbf{U}$ and $\mathbf{G}$ and F_n to be as defined in (3.26), (3.20) and (3.33), respectively. The cost of the solution, by the classical LU, of the equivalent block diagonal system in (4.1) has already been reduced to $\mathcal{O}\left(nN^3\right)$, implying substantial savings especially for regular polygons with a large number of sides. Of course, in addition to this cost, one has to consider also the cost to convert the system from its form in (3.19) into the form of (4.1). And in order to enjoy the said cost savings, this conversion has to be efficient. In this direction, aiming at the efficient deployment of the FFT, let us first observe that

$$\begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix} = (F_n \otimes I_N) \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}, \tag{4.5}$$

and prove that:

Lemma 4.1. *Given the Fourier matrix in (3.33) there holds*

$$E(F_n \otimes I_N)E^T = \bigoplus_{r=1}^{N} F_n, \tag{4.6}$$

where $E \in \mathbb{R}^{nN,nN}$ denotes the permutation matrix

$$E = \begin{pmatrix} E_1 \\ \vdots \\ E_r \\ \vdots \\ E_N \end{pmatrix}, \qquad E_r = I_n \otimes \mathbf{e}_r^T \in \mathbb{R}^{n,nN}, \quad r = 1, \ldots, N,\ j = 1, \ldots, n, \tag{4.7}$$

with $\mathbf{e}_r$ to denote the $r-th$ unit vector of $\mathbb{R}^{N,1}$.

Proof. Simply observe that:

$$\begin{aligned} E(F_n \otimes I_N)E^T &= \begin{pmatrix} I_n \otimes \mathbf{e}_1^T \\ \vdots \\ I_n \otimes \mathbf{e}_N^T \end{pmatrix} (F_n \otimes I_N) \begin{pmatrix} I_n \otimes \mathbf{e}_1 & \cdots & I_n \otimes \mathbf{e}_N \end{pmatrix} \\ &= \begin{pmatrix} I_n \otimes \mathbf{e}_1^T \\ \vdots \\ I_n \otimes \mathbf{e}_N^T \end{pmatrix} \begin{pmatrix} F_n \otimes \mathbf{e}_1 & \cdots & F_n \otimes \mathbf{e}_N \end{pmatrix} \\ &= \begin{pmatrix} F_n & O & \cdots & O \\ O & F_n & \cdots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & F_n \end{pmatrix}, \end{aligned}$$

and the proof follows. □

Table 4.1
Regular 8-gon ($n = 8$).

Method	$N = 16$				$N = 128$			
	Regular		FFT		Regular		FFT	
	E_∞	Time	E_∞	Time	E_∞	Time	E_∞	Time
LU	4.93E−04	0.023	4.93E−04	0.002	8.22E−06	25	8.22E−06	0.345
BiCGSTAB	4.93E−04	0.004	4.93E−04	0.006	8.22E−06	0.226	8.15E−06	0.233
GMRES	4.92E−04	0.005	4.93E−04	0.006	8.74E−06	0.227	8.09E−06	0.269

Table 4.2
Regular 16-gon ($n = 16$).

Method	$N = 16$				$N = 128$			
	Regular		FFT		Regular		FFT	
	E_∞	Time	E_∞	Time	E_∞	Time	E_∞	Time
LU	2.71E−04	0.184	2.71E−04	0.004	4.46E−06	470	4.46E−06	0.742
BiCGSTAB	2.70E−04	0.013	2.71E−04	0.011	5.46E−06	0.744	3.91E−06	0.576
GMRES	2.70E−04	0.014	2.71E−04	0.014	5.82E−06	0.763	4.62E−06	0.837

Table 4.3
Regular 24-gon ($n = 24$).

Method	$N = 16$				$N = 128$			
	Regular		FFT		Regular		FFT	
	E_∞	Time	E_∞	Time	E_∞	Time	E_∞	Time
LU	1.40E−04	0.622	1.40E−04	0.006	2.23E−06	1540	2.23E−06	1.150
BiCGSTAB	1.41E−04	0.024	1.40E−04	0.018	5.72E−06	1.300	8.86E−06	0.970
GMRES	1.41E−04	0.032	1.40E−04	0.028	3.57E−06	1.510	2.23E−06	2.370

Table 4.4
Regular 32-gon ($n = 32$).

Method	$N = 16$				$N = 128$			
	Regular		FFT		Regular		FFT	
	E_∞	Time	E_∞	Time	E_∞	Time	E_∞	Time
LU	8.68E−05	1.490	8.68E−05	0.010	1.34E−06	4390	1.34E−06	1.570
BiCGSTAB	8.67E−05	0.060	8.69E−05	0.020	1.30E−06	3.160	2.08E−06	1.290
GMRES	8.74E−05	0.110	8.67E−05	0.080	1.88E−06	6.880	1.50E−06	4.980

Table 4.5
Regular 40-gon ($n = 40$).

Method	$N = 16$				$N = 128$			
	Regular		FFT		Regular		FFT	
	E_∞	Time	E_∞	Time	E_∞	Time	E_∞	Time
LU	6.05E−05	3.090	6.05E−05	0.010	8.73E−07	7560	8.73E−07	1.990
BiCGSTAB	6.02E−05	0.110	6.08E−05	0.030	8.62E−07	7.130	8.92E−07	1.860
GMRES	3.27E−04	0.310	6.08E−05	0.140	2.35E−03	12.50	1.97E−03	7.660

Table 4.6
Regular 48-gon ($n = 48$).

Method	$N = 16$				$N = 128$			
	Regular		FFT		Regular		FFT	
	E_∞	Time	E_∞	Time	E_∞	Time	E_∞	Time
LU	5.09E−05	9.180	5.09E−05	0.010	6.69E−07	13 900	6.69E−07	2.610
BiCGSTAB	4.32E−05	0.190	5.09E−05	0.040	7.73E−07	12.80	7.34E−07	2.500
GMRES	1.06E−02	0.380	2.50E−03	0.220	1.27E−02	15.50	1.32E−02	10.70

Evidently, therefore, we can write

$$\begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix} = E^T \left(\bigoplus_{r=1}^{N} F_n \right) E \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix}, \tag{4.8}$$

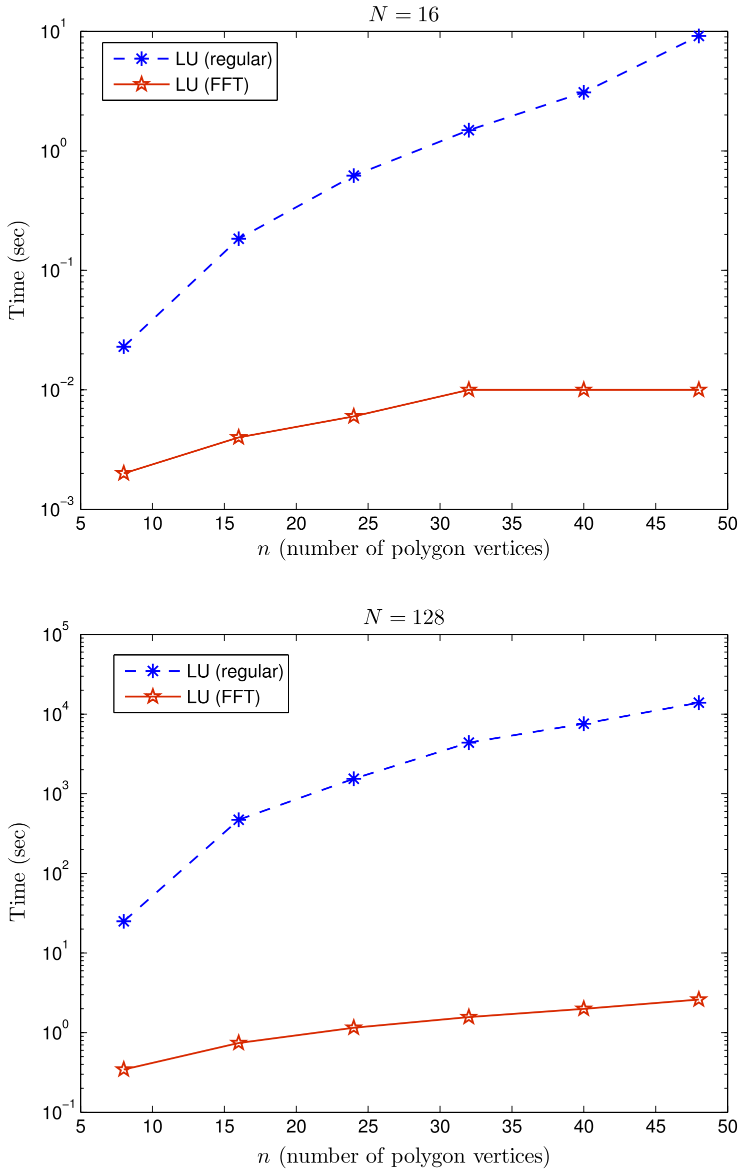

Fig. 4.1. Performance of the LU-factorization with (FFT) and without (regular) applying the FFT formulation (cases of $N = 16$ and $N = 128$).

$$\mathbf{g} = E^T \left(\bigoplus_{r=1}^{N} F_n^{-1} \right) E \, \mathbf{G}, \tag{4.9}$$

and

$$\mathbf{U} = E^T \left(\bigoplus_{r=1}^{N} F_n \right) E \, \mathbf{u}. \tag{4.10}$$

Hence, the cost of computing the vectors $\mathbf{g}$ or $\mathbf{U}$ is equivalent to the cost of applying N independent (parallel) IFFT/FFTs of order n, that is $\mathcal{O}(Nn \log n)$. Similarly, the cost of computing the matrices A_ℓ, $\ell = 1, \ldots, n$ is equivalent to the cost of performing N^2 independent (parallel) FFTs of order n, that is $\mathcal{O}(N^2 n \log n)$. Thus, by following relations (4.8)–(4.10), the total cost of constructing and solving the system in (4.1) is $\mathcal{O}(N^3 n + N^2 n \log n)$.

To demonstrate the efficiency of our implementation we consider the solution of the model Laplace's equation, with exact solution (cf. [5,11,12,10])

$$q(x, y) = \sinh(3x) \sin(3y), \tag{4.11}$$

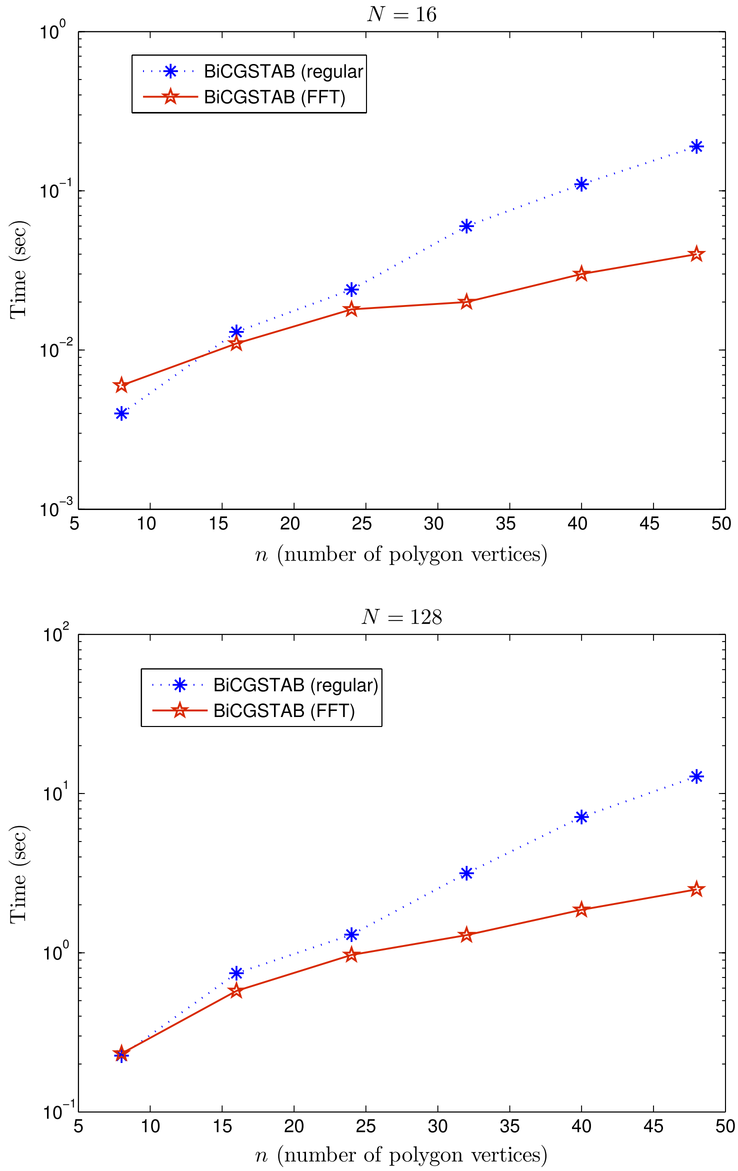

Fig. 4.2. Performance of the *BiCGSTAB* method with (FFT) and without (regular) applying the FFT formulation (cases of $N = 16$ and $N = 128$).

by the direct LU-factorization method and, two main representatives of the Krylov subspace iterative methods, the BiCGSTAB and the GMRES(10) methods. The basis functions used are the sine functions defined in (3.39), and the relative error E_∞, used to demonstrate the convergence behavior of the methods considered, is given by

$$E_\infty = \frac{\|f - f_N\|_\infty}{\|f\|_\infty}, \tag{4.12}$$

where

$$\|f\|_\infty = \max_{1\le j\le n}\left\{\max_{-\pi\le s\le\pi} |f^{(j)}(s)|\right\} \tag{4.13}$$

and

$$\|f - f_N\|_\infty = \max_{1\le j\le n}\left\{\max_{-\pi\le s\le\pi} |f^{(j)}(s) - f_N^{(j)}(s)|\right\}, \tag{4.14}$$

with $f_N^{(j)}$ as in (2.14), and the max over s is taken over a dense discretization of the interval $[-\pi, \pi]$. For the direct solution of the linear systems we have used the standard LAPACK routines, while for the computation of the right hand side vector we

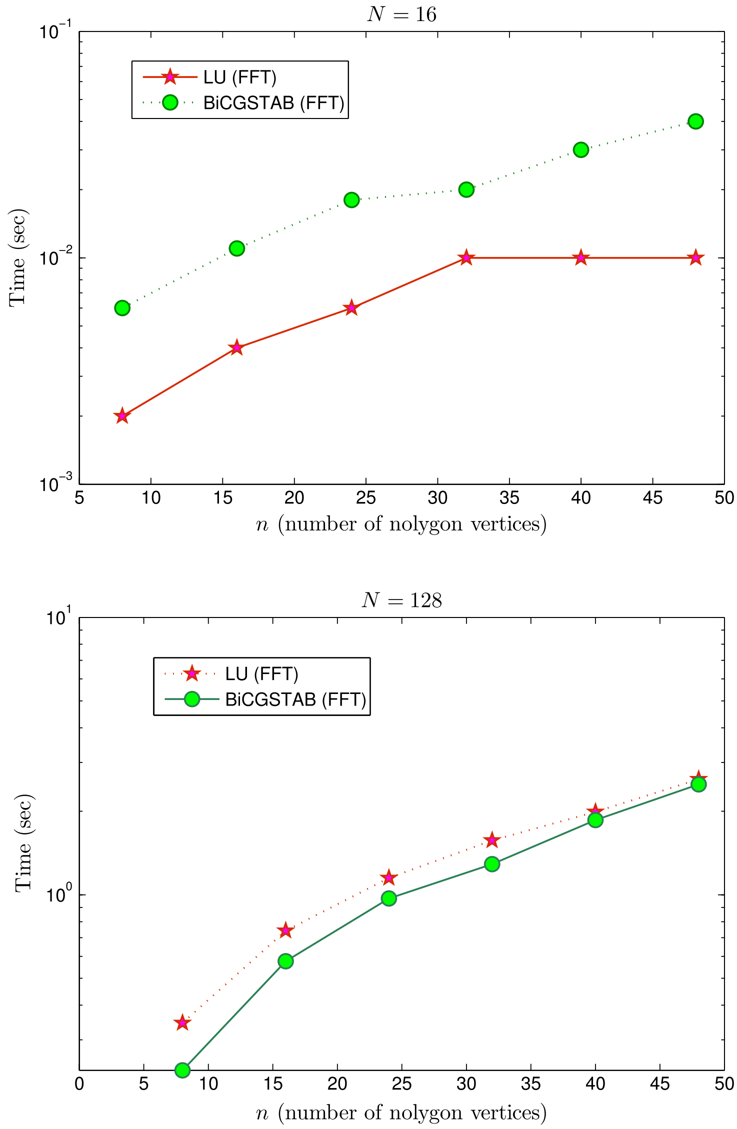

Fig. 4.3. Comparison of LU and BiCGSTAB methods, both using the FFT formulation, for the cases of $N = 16$ and $N = 128$.

have used a routine (*dqawo*) from QUADPACK implementing the modified Clenshaw–Curtis technique. We have considered the un-preconditioned forms of the both BiCGSTAB and GMRES iterative methods. However, we point out that, as we are using the sine basis functions, the diagonal blocks C_1, defined in (3.26), are point diagonal matrices (cf. [10]), hence the coefficient matrix C of (3.25) may be considered as *block Jacobi preconditioned*. The maximum number of iterations, allowed for all iterative methods to perform, is set to 200 and the zero iterate $U^{(0)}$ is set to be equal to the right hand side vector. The results we have included refer to the representative cases of regular polygons with 8, 16, 24, 32, 40 and 48 vertices. All polygons are constructed as in [5]. All experiments were conducted on a multiuser SUN V240 system using the Fortran-90 compiler.

Inspecting Tables 4.1–4.6 and Figs. 4.1–4.3 it can be readily verified that:

- Applying the FFT formulation, all direct and iterative methods perform significantly faster, especially the direct solution method (LU-factorization, Fig. 4.1).
- The un-preconditioned BiCGSTAB iterative method outperforms the un-preconditioned GMRES(10), while GMRES fails to converge within 200 iterations for polygons with medium to large number of edges.
- Iterative methods compete with direct only for large number of basis functions (Fig. 4.3).

5. Conclusions and remarks

We have studied the coefficient matrix properties of the collocation linear system associated with the Dirichlet–Neumann map for linear PDEs in regular polygon domains with the same type of boundary conditions. It was shown (Section 3) that, independently of the choice of basis functions, the matrix is *block circulant* and *independent of the boundary conditions*. These properties allowed the use of the FFT (Section 4), for the efficient solution of the collocation linear system, with significant improvement of the performance for both direct and iterative methods considered. The set of matrix properties may be expanded if one considers certain choices of basis functions (see Remark 3.2).

We note that an investigation for the matrix properties for other cases of regular and irregular polygons has already been undertaken. Preliminary results for these cases may be also found in [12] where:

- For the case of *regular polygons with different type of boundary conditions* on each side of the polygon it has been shown (cf. [12]) that the collocation matrix is *dependent* from the boundary conditions and, although not circulant, has the form

  $$C = D_1 C_1 + D_2 C_2$$

  where the matrices C_j $(j = 1, 2)$ are block circulant while the matrices D_j $(j = 1, 2)$ are diagonal. This property allows for the deployment of FFT whenever iterative methods are used for the solution of the collocation system.
- For the case of *irregular polygons* the collocation matrix is just a general matrix. A numerical treatment of this case has been included in [12].

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