



Algorithms for $\{K, s + 1\}$ -potent matrix constructions



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ABSTRACT

In this paper, we deal with $\{K, s + 1\}$ -potent matrices. These matrices generalize all the following classes of matrices: k -potent matrices, periodic matrices, idempotent matrices, involutory matrices, centrosymmetric matrices, mirrorsymmetric matrices, circulant matrices, among others. Several applications of these classes of matrices can be found in the literature. We develop algorithms in order to compute $\{K, s + 1\}$ -potent matrices and $\{K, s + 1\}$ -potent linear combinations of $\{K, s + 1\}$ -potent matrices. In addition, some examples are presented in order to show the numerical performance of the method.

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1. Introduction

In recent years, real applications for certain classes of matrices have been developed. Specifically, the problem of multiconductor transmission lines was studied by means of mirrorsymmetric matrices in [1,2]. Circulant matrices were applied to solve problems in several areas such as numerical computation, solid-state physics, image and signal processing, coding theory, mathematical statistics, and molecular vibration [3,4]. Some applications of centrosymmetric matrices were given in [5], for example, for solving problems in pattern recognition, antenna theory, mechanical and electrical systems, and quantum physics. In this last case, symmetric and skew-symmetric eigenvectors were used [6].

Related to the aforementioned classes of matrices, another type was introduced in [7], namely the $\{K, s + 1\}$ -potent matrices. For a given involutory matrix $K \in \mathbb{C}^{n \times n}$ ($K^2 = I_n$) and $s \in \{0, 1, 2, 3, \dots\}$, we recall that a matrix $A \in \mathbb{C}^{n \times n}$ is called $\{K, s + 1\}$ -potent if it satisfies

$$KA^{s+1}K = A.$$

When $s = 0$, the matrix A is called $\{K\}$ -centrosymmetric. It can be seen that $\{K, s + 1\}$ -potent matrices generalize all the following classes: idempotent matrices, k -potent matrices, periodic matrices, involutory matrices, centrosymmetric matrices, mirrorsymmetric matrices, circulant matrices, among others.

In [7], the authors gave properties and characterizations of $\{K, s + 1\}$ -potent matrices by using spectral theory. Later, in [8] that class of matrices was linked to other kinds of matrices (such as $\{s + 1\}$ -generalized projectors, $\{K\}$ -Hermitian matrices, normal matrices, among others). In both papers, a theoretical point of view has been used. Hence, it is interesting to know how to construct members of this class in an effective form. One of the main aims of this paper is to develop numerical methods to construct them.

Throughout this paper, K stands for an involutory matrix. We will denote by Ω_k the set of all k th roots of unity with k a positive integer; that is, if we define $\omega_k = e^{2\pi i/k}$ then $\Omega_k = \{\omega_k, \omega_k^2, \dots, \omega_k^k\}$.

The following function φ will be necessary. Let $\mathbb{N}_s = \{0, 1, 2, \dots, (s + 1)^2 - 2\}$ for $s \geq 1$, and let

$$\varphi : \mathbb{N}_s \rightarrow \mathbb{N}_s$$

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be the bijective function given by $\varphi(j) = b_j$, where b_j is the smallest nonnegative integer such that $b_j \equiv j(s+1) \pmod{(s+1)^2 - 1}$ [7].

This paper is organized as follows. In Section 2, we present an algorithm to compute $\{K, s+1\}$ -potent matrices. In Section 3, we obtain $\{K, s+1\}$ -potent matrices commuting with a given $\{K, s+1\}$ -potent matrix. In Section 4, we develop an algorithm to obtain all of the $\{K, s+1\}$ -potent linear combinations of $\{K, s+1\}$ -potent matrices. Finally, in Section 5, some numerical examples are presented in order to show the numerical performance of the methods.

2. Algorithm for computing $\{K, s+1\}$ -potent matrices

We analyze two situations: $s = 0$ and $s \geq 1$.

2.1. Case $s \geq 1$

Given an involutory matrix $K \in \mathbb{C}^{n \times n}$ and $s \in \{1, 2, 3, \dots\}$, we are interested in finding a $\{K, s+1\}$ -potent matrix $A \in \mathbb{C}^{n \times n}$. Since the cases with $K = \pm I_n$ correspond to the well-known relationship $A^{s+1} = A$, we will assume throughout that $K \neq \pm I_n$.

Since K is involutory, there is a nonsingular matrix $T = [t_1 \ \dots \ t_n]$ such that

$$K = T \begin{bmatrix} -I_r & O \\ O & I_{n-r} \end{bmatrix} T^{-1}, \quad (1)$$

where the first r eigenvectors of K are associated with the eigenvalue -1 and the t_i denote column vectors. Without loss of generality, we will assume that $r \leq n-r$. Otherwise, we pick $-K$ instead of K , obtaining the same solution. It is well known [7] that the eigenvalues of A must be included in the following set:

$$\Lambda = \left\{ 0, \omega_{(s+1)^2-1}^1, \dots, \omega_{(s+1)^2-1}^{(s+1)^2-2}, 1 \right\}$$

and that A is diagonalizable; i.e.

$$A = S \operatorname{diag}(\lambda_1, \dots, \lambda_n) S^{-1},$$

with

$$S = [s_1 \ \dots \ s_n] \quad \text{and} \quad S^{-1} = \begin{bmatrix} y_1^T \\ \vdots \\ y_n^T \end{bmatrix}.$$

It is easy to see that $y_i^T s_j = \delta_{ij}$, because $S^{-1}S = I_n$, where δ_{ij} indicates the Kronecker delta. Then, using $P_i = s_i y_i^T$, we have

$$P_i P_j = \begin{cases} 0 & \text{if } i \neq j \\ P_i & \text{if } i = j, \end{cases}$$

and by using the fact $SS^{-1} = I_n$ we get $\sum_{i=1}^n P_i = I_n$. So, matrix A can be written as

$$A = \sum_{i=1}^n \lambda_i P_i. \quad (2)$$

When all the λ_i are different, expression (2) provides the spectral decomposition of A . Otherwise, in order to obtain such a decomposition, it is sufficient to multiply the corresponding eigenvalue by the sum of all its associated P_i .

Since $KP_i K = P_{\varphi(i)}$ must hold (by Theorem 2 in [7]), we can choose

$$Ks_i = s_{\varphi(i)} \quad \text{and} \quad K^T y_i = y_{\varphi(i)} \quad (3)$$

in order to satisfy the equality $A^{s+1} = KAK$.

Noticing that $K(t_i + t_{r+i}) = -t_i + t_{r+i}$ for $i = 1, \dots, r$, we are able to establish a first method for constructing $\{K, s+1\}$ -potent matrices.

Algorithm 1.

Inputs: $K \in \mathbb{C}^{n \times n}$, $s \in \{1, 2, 3, \dots\}$.

Outputs: A $\{K, s+1\}$ -potent matrix $A \in \mathbb{C}^{n \times n}$ and the projectors P_i .

Step 1 Diagonalize K as in (1).

Step 2 If $r > n-r$, replace K with $-K$ and rearrange as in Step 1.

Table 1
The most representative cases.

| | Construction of the s_i | Construction of A |
|-------|---|--|
| D_1 | $s_1 = t_1 + t_2$ $s_2 = -t_1 + t_2$ | $A = \omega P_1 + \omega^{\varphi(1)} P_2$ |
| D_2 | $s_1 = t_1 + t_2$ $s_2 = -t_1 + t_2$ $s_3 = t_3$ | $A = \omega P_1 + \omega^{\varphi(1)} P_2 + P_3$ |
| D_3 | $s_1 = t_1 + t_2$ $s_2 = -t_1 + t_2$ $s_3 = t_3$ $s_4 = t_4$ | $A = \omega P_1 + \omega^{\varphi(1)} P_2 + P_3 + P_4$ |
| D_4 | $s_1 = t_1 + t_3$ $s_2 = -t_1 + t_3$ $s_3 = t_2 + t_4$ $s_4 = -t_2 + t_4$ | $A = \omega(P_1 + P_3) + \omega^{\varphi(1)}(P_2 + P_4)$ |
| D_5 | $s_1 = t_1 + t_3$ $s_2 = -t_1 + t_3$ $s_3 = t_2 + t_4$ $s_4 = -t_2 + t_4$ $s_5 = t_5$ | $A = \omega(P_1 + P_3) + \omega^{\varphi(1)}(P_2 + P_4) + P_5$ |

Step 3 For $i = 1, \dots, r$, compute $s_{2i-1} = t_i + t_{r+i}$ and $s_{2i} = -t_i + t_{r+i}$.

Step 4 For $i = 2r + 1, \dots, n$, set $s_i = t_i$.

Step 5 Solve the linear system $Sy_i = e_i$ for $i = 1, \dots, n$.

Step 6 Compute $P_i = s_i y_i^T$ for $i = 1, \dots, n$.

Step 7 For $i = 1, \dots, r$, compute $Q_i = \omega P_{2i-1} + \omega^{\varphi(1)} P_{2i}$.

Step 8 Compute $A = \sum_{i=1}^r Q_i + \sum_{j=2r+1}^n P_j$.

End

In order to clarify this process, the most representative cases are presented in Table 1, where $\omega := \omega_{(s+1)^2-1}$. We consider the following different involutory matrices $K_i = TD_iT^{-1}$ for $i = 1, 2, 3, 4, 5$, where $D_1 = \text{diag}(-1, 1)$, $D_2 = \text{diag}(-1, 1, 1)$, $D_3 = \text{diag}(-1, 1, 1, 1)$, $D_4 = \text{diag}(-1, -1, 1, 1)$, $D_5 = \text{diag}(-1, -1, 1, 1, 1)$.

The only step in the algorithm that needs to be justified is Step 8. In fact, A is $\{K, s+1\}$ -potent because, using expression (3), we get

$$\begin{aligned}
 A^{s+1} &= \left(\sum_{i=1}^r (\omega P_{2i-1} + \omega^{\varphi(1)} P_{2i}) + \sum_{j=2r+1}^n P_j \right)^{s+1} \\
 &= \sum_{i=1}^r (\omega^{\varphi(1)} P_{2i-1} + \omega P_{2i}) + \sum_{j=2r+1}^n P_j \\
 &= \sum_{i=1}^r (\omega K P_{2i-1} K + \omega^{\varphi(1)} K P_{2i} K) + \sum_{j=2r+1}^n K P_j K \\
 &= K A K.
 \end{aligned}$$

Although we have constructed only one $\{K, s+1\}$ -potent matrix A , it is clear that this method allows us to construct more of them (e.g., by picking other values of ω in $\Omega_{(s+1)^2-1}$).

2.2. Case $s = 0$

This case corresponds to matrices $A \in \mathbb{C}^{n \times n}$ commuting with K . The spectral theorem allows us to state that a diagonalizable matrix $A \in \mathbb{C}^{n \times n}$ is $\{K\}$ -centrosymmetric if and only if $KP_i = P_i K$, where the P_i are the projectors appearing in that decomposition. We could try a similar algorithm to the previous one. However, by using a block decomposition, a straightforward method can be developed which gives an immediate result. In fact, if matrix K is diagonalized as in (1), a simple computation provides us with the required $\{K\}$ -centrosymmetric matrices:

$$A = T \begin{bmatrix} X_A & O \\ O & Y_A \end{bmatrix} T^{-1}, \quad (4)$$

where $X_A \in \mathbb{C}^{r \times r}$ and $Y_A \in \mathbb{C}^{(n-r) \times (n-r)}$ are arbitrary matrices.

3. Obtaining $\{K, s+1\}$ -potent matrices commuting with a given $\{K, s+1\}$ -potent matrix

Let $s \geq 1$. Our next objective is to find a $\{K, s+1\}$ -potent matrix $B \in \mathbb{C}^{n \times n}$ such that $AB = BA$ for the $\{K, s+1\}$ -potent matrix $A \in \mathbb{C}^{n \times n}$ obtained by means of Algorithm 1. Since B is $\{K, s+1\}$ -potent, it must be diagonalizable. Then, in order to satisfy the condition $AB = BA$, both matrices A and B have to be simultaneously diagonalizable [9]; that is, $A = S \text{diag}(\lambda_1, \dots, \lambda_n)S^{-1}$ and $B = S \text{diag}(\mu_1, \dots, \mu_n)S^{-1}$, where the μ_i , which are in Λ , remain to be determined.

Algorithm 2.

Inputs: The $\{K, s+1\}$ -potent matrix A obtained in Algorithm 1 and the projectors P_i .

Outputs: A $\{K, s+1\}$ -potent matrix $B \in \mathbb{C}^{n \times n}$ commuting with A .

Step 1 Set $\mu := \omega_{(s+1)^2-1}^p$, where $p \notin \{1, \varphi(1), \varphi(p)\}$.

Step 2 For $i = 1, \dots, r$, compute $W_i = \mu P_{2i-1} + \mu^{\frac{\varphi(p)}{p}} P_{2i}$.

Step 3 Compute $B = \sum_{i=1}^r W_i + \sum_{i=2r+1}^n P_i$.

End

Note that μ is chosen in Λ from among the unused ω values in Algorithm 1. In Step 2 it is clear that $\mu^{\frac{\varphi(p)}{p}} = \omega^{\varphi(p)}$. In Algorithm 2, the remainder of the construction is performed as in Algorithm 1.

Using Algorithm 2, we have constructed one $\{K, s+1\}$ -potent matrix B . It is clear that this method allows us to construct more of them from the same starting matrix A (e.g., by changing now the μ in $\Omega_{(s+1)^2-1}$).

When $s = 0$, and X_A and Y_A are as in (4), then

$$B = T \begin{bmatrix} X_B & O \\ O & Y_B \end{bmatrix} T^{-1}$$

give all $\{K, 1\}$ -potent matrices B that commute with an arbitrarily constructed $\{K, 1\}$ -potent matrix A provided that $X_A X_B = X_B X_A$ and $Y_A Y_B = Y_B Y_A$ hold.

When $s \geq 1$, we notice that matrix $B = \omega A$ is also $\{K, s+1\}$ -potent, where ω is a primitive s th root of unity. Similarly, when $s = 0$, $B = \alpha A$ is $\{K, 1\}$ -potent for all $\alpha \in \mathbb{C}$ as well. In order to obtain $\{K, s+1\}$ -potent matrices by another procedure, the next section examines linear combinations of A and B using Algorithm 2.

4. An algorithm for obtaining $\{K, s+1\}$ -potent linear combinations

For the matrices A and B obtained by means of Algorithms 1 and 2, we can construct the following linear relationship:

$$C = c_1 A + c_2 B, \quad (5)$$

where c_1 and c_2 are nonzero complex numbers to be determined. In this section, we find scalars c_1 and c_2 such that C is a $\{K, s+1\}$ -potent matrix. The value $s = 0$ does not give any interesting results because all c_1 and c_2 satisfy the equality, and so we will assume that $s \geq 1$.

Since

$$A = S \text{diag}(\lambda_1, \dots, \lambda_n)S^{-1} \quad \text{and} \quad B = S \text{diag}(\mu_1, \dots, \mu_n)S^{-1},$$

by using (5), a simple computation yields to solve the following linear system:

$$\begin{bmatrix} \lambda_1 & \mu_1 \\ \vdots & \vdots \\ \lambda_n & \mu_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix},$$

where $C = S \text{diag}(\gamma_1, \dots, \gamma_n)S^{-1}$ and each γ_i ranges freely over all values in $\{0\} \cup \Omega_{(s+1)^2-1}$.

Now, we can design an algorithm to compute this class of linear combinations.

Algorithm 3.

Inputs: The $\{K, s+1\}$ -potent matrices A and B obtained in Algorithms 1 and 2, respectively.

Outputs: All values of c_1 and c_2 such that $C = c_1 A + c_2 B$ is a $\{K, s+1\}$ -potent matrix.

Step 1 Set $M = \begin{bmatrix} \lambda_1 & \mu_1 \\ \vdots & \vdots \\ \lambda_n & \mu_n \end{bmatrix}$.

Step 2 Choose $\gamma_1, \dots, \gamma_n \in \{0\} \cup \Omega_{(s+1)^2-1}$ and set $\gamma = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}$.

Step 3 Solve the linear system $M \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \gamma$.

Step 4 Repeat Steps 2 and 3 for all possible choices of γ_i in $\{0\} \cup \Omega_{(s+1)^2-1}$.

End

5. Numerical examples

Our algorithms can easily be implemented on a computer. We have used the MATLAB R2010b package. In this section, we present some numerical examples in order to show the performance of our algorithms and demonstrate their applicability.

5.1. Case $s \geq 1$

While the computational cost of [Algorithm 1](#) is $O(n^3)$, [Algorithm 2](#) has computational cost of only $O(n)$.

Note that the computational cost of [Algorithm 3](#) is basically given by Step 3. In this last case, we observe that the matrix M has at most rank 2. Then, in order to solve the system

$$\begin{bmatrix} \lambda_1 & \mu_1 \\ \vdots & \vdots \\ \lambda_n & \mu_n \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \gamma,$$

we have to choose two linearly independent equations (only one when $\text{rank}(M) = 1$). That is, we only have to solve at most a 2×2 linear system.

Example 1. For $s = 2, n = 2$, and

$$K = \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix},$$

[Algorithm 1](#) gives

$$A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}.$$

Example 2. For the same K as in [Example 1](#), [Algorithm 2](#) gives

$$B = \begin{bmatrix} -\sqrt{2}i & \frac{\sqrt{2}}{2}i \\ -\sqrt{2}i & 0 \end{bmatrix}.$$

Example 3. For the matrices A and B obtained in [Examples 1](#) and [2](#), [Algorithm 3](#) gives in addition to the zero solution.

| | | | | | | | | |
|-------|---|----|---|----|-------------|--------------|-------------|--------------|
| c_1 | 0 | 0 | 1 | -1 | $i\sqrt{2}$ | $-i\sqrt{2}$ | -1 | 1 |
| c_2 | 1 | -1 | 0 | 0 | 1 | -1 | $i\sqrt{2}$ | $-i\sqrt{2}$ |

Now, we comment on the computational time in terms of n and s . For that, we have used a Intel Core 2 Duo 2 GHz processor. For $n = 10, 30, 50, \dots, 500$, [Fig. 1](#) shows the computational time required for $s = 1, 5, 100, 200$. We can observe that, as expected, the computational time grows exponentially with s and n . Moreover, the computation time increases more rapidly with n than with s .

5.2. Case $s = 0$

In this case, the only interesting examples correspond to those constructed in [Section 2.2](#). The reason is that, for every $\{K, 1\}$ -potent matrix commuting with A , all possible linear combinations are $\{K, 1\}$ -potent matrices.

Example 4. If we consider the involutory matrix

$$K = \frac{1}{9} \begin{bmatrix} 7 & -4 & 4 \\ -4 & 1 & 8 \\ 4 & 8 & 1 \end{bmatrix},$$

the only possible $\{K, 1\}$ -potent matrices are

$$A = \frac{1}{9} \begin{bmatrix} 4a - 4c - 4b + 4d + e & -4a + 4c - 2b + 2d + 2e & -2a + 2c - 4b + 4d - 2e \\ -4a - 2c + 4b + 2d + 2e & 4a + 2c + 2b + d + 4e & 2a + c + 4b + 2d - 4e \\ -2a - 4c + 2b + 4d - 2e & 2a + 4c + b + 2d - 4e & a + 2c + 2b + 4d + 4e \end{bmatrix}$$

for a, b, c, d, e arbitrary complex numbers.

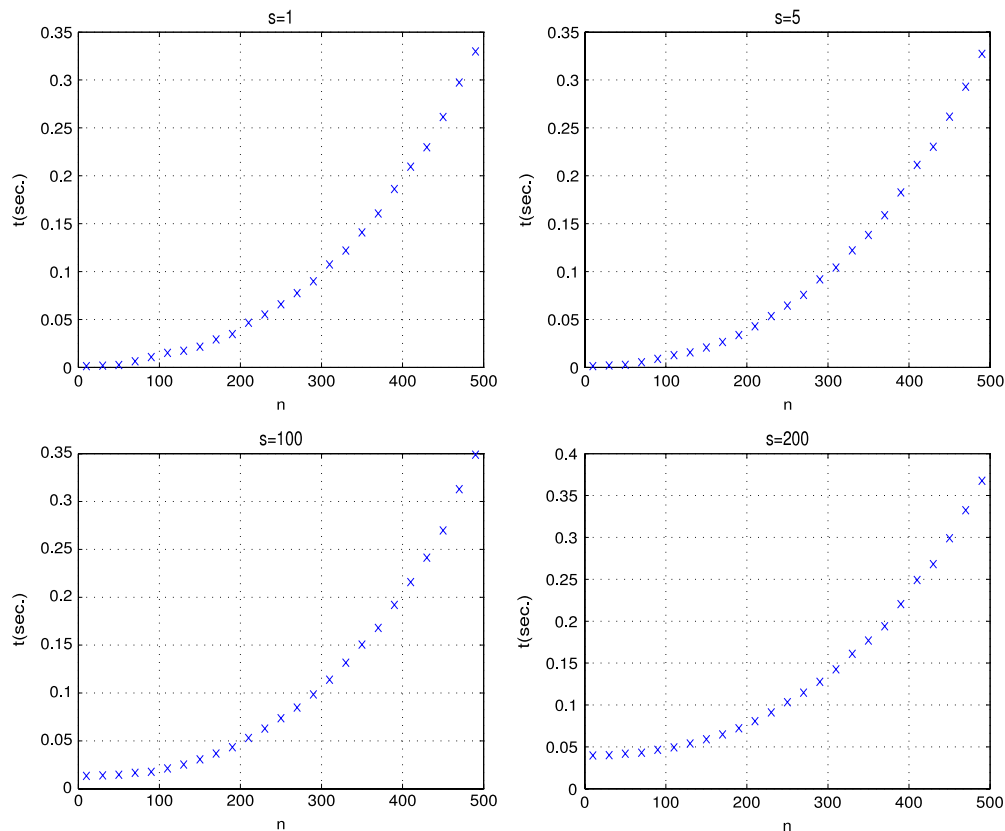


Fig. 1. Time for obtaining A with n variable and $s = 1, 5, 100, 200$.

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