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Checking strong optimality of interval linear programming with inequality constraints and nonnegative constraints

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ABSTRACT

This paper considers optimal solutions of interval linear programming problems, in a unified framework. Necessary and sufficient conditions for checking $(\mathbf{A}, \mathbf{b})$-strong optimality and $(\mathbf{A}, \mathbf{b}, \mathbf{c})$-strong optimality of interval linear programming with inequality constraints are developed. The features of the proposed methods are illustrated by some examples.

1. Introduction

The interval linear programming (IvLP) problems have been investigated by many authors, see, e.g., [1–11], among others. Nevertheless, there are only few results on the issue of optimal solutions for a general IvLP, i.e., where the objective cost vector, the coefficient matrix and the right hand vector are all interval vectors or interval matrices. In this paper, we will introduce some new concepts of optimal solutions of IvLP in a unified framework. Necessary and sufficient conditions for checking some types of optimality are developed.

Let us introduce some notation. The i-th row of a matrix $A \in \mathbb{R}^{m\times n}$ is denoted by $A_{i,\cdot}$, the j-th column by $A_{\cdot,j}$. An interval matrix is defined as

$$\mathbf{A} = [\underline{A}, \overline{A}] = \{A \in \mathbb{R}^{m\times n};\ \underline{A} \le A \le \overline{A}\},$$

where $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, and $\underline{A} \le \overline{A}$. Similarly, we define an interval vector as an one column interval matrix

$$\mathbf{b} = [\underline{b}, \overline{b}] = \{b \in \mathbb{R}^{m};\ \underline{b} \le b \le \overline{b}\},$$

where $\underline{b}, \overline{b} \in \mathbb{R}^m$, and $\underline{b} \le \overline{b}$. The set of all m-by-n interval matrices will be denoted by $\mathbb{IR}^{m\times n}$ and the set of all m-dimensional interval vectors by $\mathbb{IR}^m$.

Denote by A_c and A_Δ the center and radius matrices given by

$$A_c = \frac{1}{2}(\underline{A} + \overline{A}), \qquad A_\Delta = \frac{1}{2}(\overline{A} - \underline{A}),$$

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respectively. Then $\mathbf{A} = [A_c - A_\Delta, A_c + A_\Delta]$. Similarly, the center and radius vectors are defined as

$$b_c = \frac{1}{2}(\underline{b} + \overline{b}), \qquad b_\Delta = \frac{1}{2}(\overline{b} - \underline{b})$$

respectively. Then $\mathbf{b} = [b_c - b_\Delta, b_c + b_\Delta]$.

Let Y_m be the set of all $\{-1, 1\}$ m-dimensional vectors, i.e.

$$Y_m = \{y \in \mathbb{R}^m | \mid y \mid = e\},$$

where $e = (1, \dots, 1)^T$ is the m-dimensional vector of all 1's. For a given $y \in Y_m$, let

$$T_y = diag(y_1, \dots, y_m)$$

denote the corresponding diagonal matrix. For each $x \in \mathbb{R}^n$, we define its sign vector sign x by

$$(\text{sign } x)_i = \begin{cases} 1 & \text{if } x_i \geq 0, \\ -1 & \text{if } x_i < 0, \end{cases}$$

where $i = 1, \dots, n$. Then we have $|x| = T_z x$, where $z = \text{sign } x \in Y_n$.

For a given interval matrix $\mathbf{A} = [A_c - A_\Delta, A_c + A_\Delta]$, and for each vector $y \in Y_m$ and each vector $z \in Y_n$, we introduce the matrices

$$A_{yz} = A_c - T_y A_\Delta T_z,$$

which means

$$(A_{yz})_{ij} = (A_c)_{ij} - y_i (A_\Delta)_{ij} z_j = \begin{cases} \overline{a}_{ij} & \text{if } y_i z_j = -1, \\ \underline{a}_{ij} & \text{if } y_i z_j = 1. \end{cases}$$

where $i = 1, \dots, m, j = 1, \dots, n$. Similarly, for an interval vector $\mathbf{b} = [b_c - b_\Delta, b_c + b_\Delta]$ and for each vector $y \in Y_m$, we define vector

$$b_y = b_c + T_y b_\Delta,$$

which means

$$(b_y)_i = (b_c)_i + y_i (b_\Delta)_i = \begin{cases} \overline{b}_i & \text{if } y_i = 1, \\ \underline{b}_i & \text{if } y_i = -1. \end{cases}$$

where $i = 1, \dots, m$.

2. Unified optimal solution concepts of IvLP and some preliminaries

Consider an LP problem

$$\min\ c^T x \quad \text{subject to } x \in M(A, b), \tag{1}$$

where $M(A, b)$ is the feasible set characterized by a linear system.

Let $\mathbf{A} \in \mathbb{IR}^{m \times n}$, $\mathbf{b} \in \mathbb{IR}^m$ and $\mathbf{c} \in \mathbb{IR}^n$ be given. By an IvLP problem we mean a family of the LP (1), where $A \in \mathbf{A}, b \in \mathbf{b}, c \in \mathbf{c}$. We write it in short as

$$\min\ \mathbf{c}^T x \quad \text{subject to } x \in M(\mathbf{A}, \mathbf{b}). \tag{2}$$

By *a realization* we mean a concrete setting (1).

In the IvLP theory, one of the following canonical forms

(A) $M(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n; \mathbf{A}x = \mathbf{b}, x \geq 0\}$,
(B) $M(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n; \mathbf{A}x \leq \mathbf{b}\}$,
(C) $M(\mathbf{A}, \mathbf{b}) = \{x \in \mathbb{R}^n; \mathbf{A}x \leq \mathbf{b}, x \geq 0\}$

is usually assumed [3,7].

2.1. Optimal solution concepts of IvLP

We first review some concepts briefly. A vector $x \in \mathbb{R}^n$ is called *a weak feasible solution* of the IvLP (2) if it is a feasible solution of the LP (1) for some $A \in \mathbf{A}, b \in \mathbf{b}$. A vector $x \in \mathbb{R}^n$ is called *a strong feasible solution* of the IvLP (2) if it is a feasible solution of the LP (1) for each $A \in \mathbf{A}, b \in \mathbf{b}$ [5,7]. Recently, new concepts of the optimal solution to IvLP are proposed in a unified framework [12].

Definition 2.1 (*[12]*).

(1) A vector $x \in \mathbb{R}^n$ is called a ($\emptyset$)-strong optimal solution (or a weak optimal solution) of the IvLP (2) if it is an optimal solution of the LP (1) for some $A \in \mathbf{A}$, $b \in \mathbf{b}$ and $c \in \mathbf{c}$.
(2) A vector $x \in \mathbb{R}^n$ is called a ($\mathbf{c}$)-strong optimal solution of the IvLP (2) if for each $c \in \mathbf{c}$ it is an optimal solution of the LP (1) for some $A \in \mathbf{A}$, $b \in \mathbf{b}$.
(3) A vector $x \in \mathbb{R}^n$ is called a ($\mathbf{b}$)-strong optimal solution of the IvLP (2) if for each $b \in \mathbf{b}$ it is an optimal solution of the LP (1) for some $A \in \mathbf{A}$, $c \in \mathbf{c}$.
(4) A vector $x \in \mathbb{R}^n$ is called an ($\mathbf{A}$)-strong optimal solution of the IvLP (2) if for each $A \in \mathbf{A}$ it is an optimal solution of the LP (1) for some $b \in \mathbf{b}$, $c \in \mathbf{c}$.
(5) A vector $x \in \mathbb{R}^n$ is called an ($\mathbf{A}$, $\mathbf{b}$)-strong optimal solution of the IvLP (2) if for each $A \in \mathbf{A}$, $b \in \mathbf{b}$ it is an optimal solution of the LP (1) for some $c \in \mathbf{c}$.
(6) A vector $x \in \mathbb{R}^n$ is called an ($\mathbf{A}$, $\mathbf{c}$)-strong optimal solution of the IvLP (2) if for each $A \in \mathbf{A}$, $c \in \mathbf{c}$ it is an optimal solution of the LP (1) for some $b \in \mathbf{b}$.
(7) A vector $x \in \mathbb{R}^n$ is called a ($\mathbf{b}$, $\mathbf{c}$)-strong optimal solution of the IvLP (2) if for each $b \in \mathbf{b}$, $c \in \mathbf{c}$ it is an optimal solution of the LP (1) for some $A \in \mathbf{A}$.
(8) A vector $x \in \mathbb{R}^n$ is called an ($\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$)-strong optimal solution of the IvLP (2) if for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$ it is an optimal solution of the LP (1).

The solution concepts of IvLp are closely related with that of interval linear systems. Various solutions of interval linear equations and interval linear inequalities are investigated by different authors, see, e.g., [13–16]. Mixed interval systems are discussed in [17]. In Definition 2.1, we consider quantified solution sets where all universally quantified parameters precede all existentially quantified ones. From this point of view, the proposed solution concept can be seen as an extension of the so called *AE* solutions proposed in [18]. Some recent developments on *AE* solutions set of interval systems can be found in [19,20].

Some kinds of optimal solutions of IvLP have been studied by different authors. These optimal solutions are special cases of Definition 2.1. For example, the optimal solution discussed in [5,7,8] is a weak optimal solution (or a ($\emptyset$)-strong optimal solution). The strong optimal solution discussed in [21] is a ($\mathbf{c}$)-strong optimal solution. To discuss all kinds of optimal solutions of IvLP for all three types A, B, and C may not be a suitable task in one paper. Recently, necessary and sufficient conditions for checking ($\mathbf{A}$, $\mathbf{b}$)-strong and ($\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$)-strong optimal solutions to IvLP of type A and B are given in [12,22], respectively. In this paper, we focus on the ($\mathbf{A}$, $\mathbf{b}$)-strong optimal solutions and ($\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$)-strong optimal solutions to IvLP of type C and propose some necessary and sufficient conditions for checking them. It is worth mentioning that Hladík [7] discussed the strongly optimality which means the IvLP (2) has an optimal solution for each realization (optimal solution for each realization can be different), whereas an ($\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}$)-strong optimal solution defined above means there is a same optimal solution to each realization of the IvLP (2). It can be seen later from the theoretical conclusions and illustrative examples that the various strong optimal solutions exist not only on rare occasions.

2.2. Preliminaries

We remind some basics of tangent cones first. Let x^* be a feasible solution to a convex polyhedral set $M(A, b)$. The tangent cone to $M(A, b)$ at the point x^* is formed by all rays emanating from x^* and intersecting $M(A, b)$ in at least one point distinct from x^*. Hladík [7] presented the tangent cone to the feasible set $M(A, b)$ of some linear constraints below.

The tangent cone to $M(A, b) = \{x \in \mathbb{R}^n; Ax = b, x \geq 0\}$ at the point x^* reads

$$Ax = 0,$$
$$x_{t_j} \geq 0 \quad t_j \in F,$$

where $F = \{t_j | j = 1, \ldots, p, x^*_{t_j} = 0\}$.

The tangent cone to $M(A, b) = \{x \in \mathbb{R}^n; Ax \leq b\}$ at the point x^* reads

$$A_{r_i,\cdot}x \leq 0 \quad r_i \in G,$$

where $G = \{r_i | i = 1, \ldots, q, A_{r_i,\cdot}x^* = b_{r_i}\}$.

Finally, the tangent cone to $M(A, b) = \{x \in \mathbb{R}^n; Ax \leq b, x \geq 0\}$ at the point x^* can be transformed from tangent cone above.

Lemma 2.1. *Let $x^* \in M(A, b) = \{x \in \mathbb{R}^n; Ax \leq b, x \geq 0\}$. Denote*

$$F = \{t_j | j = 1, \ldots, p, x^*_{t_j} = 0\},$$
$$G = \{r_i | i = 1, \ldots, q, A_{r_i,\cdot}x^* = b_{r_i}\}.$$

Then the tangent cone at x^* reads

$$A_{r_i,\cdot}x \leq 0 \quad r_i \in G,$$
$$x_{t_j} \geq 0 \quad t_j \in F.$$

Proof. The linear system $Ax \leq b, x \geq 0$ is equivalent to

$$\begin{pmatrix} A \\ -I \end{pmatrix} x \leq \begin{pmatrix} b \\ 0 \end{pmatrix},$$

where I is an $n \times n$ identity matrix, and 0 is an $n \times 1$ null matrix. Let

$$A^1 = \begin{pmatrix} A \\ -I \end{pmatrix}, \qquad b^1 = \begin{pmatrix} b \\ 0 \end{pmatrix},$$

then we have $A^1 x \leq b^1$, so the tangent cone to $M(A^1, b^1) = \{x \in \mathbb{R}^n; A^1 x \leq b^1\}$ at x^* is

$$A^1{}_{s_i,\cdot} x \leq 0 \quad s_i \in G^1, \tag{3}$$

where

$$\begin{aligned} G^1 &= \{s_i | i = 1, \ldots, u, A^1{}_{s_i,\cdot} x^* = b^1{}_{s_i}\} \\ &= \{r_i | i = 1, \ldots, q, A_{r_i,\cdot} x^* = b_{r_i}\} \bigcup \{t_j | j = 1, \ldots, p, -I_{t_j,\cdot} x^* = 0\} \end{aligned}$$

Hence, the tangent cone (3) is equivalent to

$$\begin{aligned} &A_{r_i,\cdot} x \leq 0 \quad r_i \in G, \\ &x_{t_j} \geq 0 \quad t_j \in F. \end{aligned}$$

This completes the proof of Lemma 2.1. □

Without loss of generality, let a tangent cone to $M(A, b)$ at x^* be described by $Dx \leq 0$.

Lemma 2.2 ([7]). *A vector $x^* \in M(A, b)$ is an optimal solution of the LP problem*

$$\min\ c^T x \quad \textit{subject to } x \in M(A, b)$$

if and only if the linear inequality system

$$\begin{cases} Dy \leq 0, \\ c^T y \leq -1 \end{cases} \tag{4}$$

has no solution.

Thus, x^* is an optimal for some $c \in \mathbf{c}$ if and only if the system (4) is unsolvable for some $c \in \mathbf{c}$.
The following result holds for a special case of IvLP (interval objective function only).

Theorem 2.1 ([7]). *A vector $x^* \in M(A, b)$ is a weak optimal solution of the IvLP problem*

$$\min\ \mathbf{c}^T x \quad \textit{subject to } x \in M(A, b)$$

if and only if there is no solution to the linear system

$$\begin{cases} D(x^1 - x^2) \leq 0, \\ \overline{c}^T x^1 - \underline{c}^T x^2 \leq -1, \\ x^1, x^2 \geq 0. \end{cases} \tag{5}$$

Now we review some concepts for later use [5,7].

A system is feasible means that the system has a nonnegative solution, and the nonnegative solution is a feasible solution to the system. *An interval system is feasible* means that some realizations of the interval system have a nonnegative solution, and the nonnegative solution is a weak feasible solution to the interval system. *An interval system is strong feasible* means that each realization of the interval system has a nonnegative solution, and a solution which is a common feasible solution to each realization of the interval system is called a *strong feasible solution to the interval system. A strong feasible solution to an IvLP* means that it is a feasible solution to each realization of the IvLP.

Theorem 2.2 ([14]). *A system $\mathbf{A}x \leq \mathbf{b}$ is strong feasible if and only if the system*

$$\overline{A}x \leq \underline{b} \tag{6}$$

is feasible.

Corollary 2.1. *A vector $x^* \in \mathbb{R}^n$ is a strong feasible solution of $\mathbf{A}x \leq \mathbf{b}$ if and only if it is a feasible solution to the system*

$$\overline{A}x \leq \underline{b}. \tag{7}$$

Proof. "Only if": Let x^* be a strong feasible solution to $\mathbf{A}x \leq \mathbf{b}$, then for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, x^* is a feasible solution to $Ax \leq b$. Hence x^* is a feasible solution to (7).

"If": Let x^* be a feasible solution to (7). Note that $x^* \geq 0$, thus for each $A \in \mathbf{A}$, $b \in \mathbf{b}$, the inequality

$$Ax \leq \overline{A}x \leq \underline{b} \leq b$$

holds. Hence x^* is a strong feasible solution to $\mathbf{A}x \leq \mathbf{b}$. This completes the proof of the corollary. □

Theorem 2.3 (*[14]*). *A system $\mathbf{A}x \leq \mathbf{b}$ is weakly solvable if and only if the system*

$$A_{ez}x \leq \overline{b} \tag{8}$$

is solvable for some $z \in Y_n$.

From the proof in [14] we know that a weak solution of $\mathbf{A}x \leq \mathbf{b}$ is a solution of the system (8). Similarly, a solution of the system (8) is a weak solution of $\mathbf{A}x \leq \mathbf{b}$.

Let $\mathbf{A}$ be a 1-by-n interval matrix, then we have $e = 1$ in the system (8). Hence we get the corollary below and it will be used in Section 4 for clarity.

Corollary 2.2. *Let $\mathbf{c} \in \mathbb{IR}^n$, $\mathbf{b} \in \mathbb{IR}^1$, then the system $\mathbf{c}^T x \leq \mathbf{b}$ is weakly solvable if and only if the inequality*

$$(c^T)_{ez}x \leq \overline{b} \tag{9}$$

is solvable for some $z \in Y_n$, where $e = 1$.

Let $\mathbf{A} \in \mathbb{IR}^{m\times n}$, $\mathbf{b} \in \mathbb{IR}^m$ and $\mathbf{c} \in \mathbb{IR}^n$. Consider the IvLP of type C

$$\min\ \mathbf{c}^T x \quad \text{subject to } \mathbf{A}x \leq \mathbf{b},\ x \geq 0. \tag{10}$$

In next two sections, we will discuss the IvLP (10) and propose two necessary and sufficient conditions for checking $(\mathbf{A}, \mathbf{b})$-strong optimality and $(\mathbf{A}, \mathbf{b}, \mathbf{c})$-strong optimality of given vectors respectively.

3. (A, b)-strong optimal solutions

In this section, we will propose the necessary and sufficient conditions for checking $(\mathbf{A}, \mathbf{b})$-strong optimality of given vectors.

Theorem 3.1. *Let $x^* \in \mathbb{R}^n$. Denote*

$$F = \{t_j | j = 1, \ldots, p, x^*_{t_j} = 0\},$$
$$G = \{r_i | i = 1, \ldots, q, \underline{A}_{r_i,\cdot}x^* = \overline{b}_{r_i}\}.$$

Then x^* is an $(\mathbf{A}, \mathbf{b})$-strong optimal solution to (10) if and only if x^* is a strong feasible solution to (10), and the linear system

$$\begin{cases} \underline{A}_{r_i,\cdot}(x^1 - x^2) \leq 0 \quad i = 1, \ldots, q, & \text{(a)} \\ (x^1 - x^2)_{t_j} \geq 0 \quad j = 1, \ldots, p, & \text{(b)} \\ \overline{c}^T x^1 - \underline{c}^T x^2 \leq -1, & \text{(c)} \\ x^1, x^2 \geq 0 & \text{(d)} \end{cases} \tag{11}$$

has no solution.

Proof. "Only if": Let x^* be an $(\mathbf{A}, \mathbf{b})$-strong optimal solution to (10), then x^* is a strong feasible solution to (10) and x^* is a weak optimal solution to the IvLP problem

$$\min\ \mathbf{c}^T x \quad \text{subject to } \underline{A}x \leq \overline{b},\ x \geq 0. \tag{12}$$

The tangent cone to the feasible region of (12) at x^* reads

$$\begin{cases} \underline{A}_{r_i,\cdot}x \leq 0 & i = 1, \ldots, q, \\ x_{t_j} \geq 0 & j = 1, \ldots, p. \end{cases} \tag{13}$$

Let matrix $E = (e_{ij})_{n\times n}$, where

$$e_{ij} = \begin{cases} 1 & i = j \in F, \\ 0 & \text{others.} \end{cases}$$

Denote

$$D = \begin{pmatrix} \underline{A}_{r_1,\cdot} \\ \underline{A}_{r_2,\cdot} \\ \vdots \\ \underline{A}_{r_q,\cdot} \\ -E \end{pmatrix}.$$

Then the tangent cone (13) can be written as $Dx \leq 0$.

Note that x^* is a weak optimal solution to the IvLP (12), then it follows from Theorem 2.1 that the associated system (5)

$$\begin{cases} D(x^1 - x^2) \leq 0, \\ \overline{c}^T x^1 - \underline{c}^T x^2 \leq -1, \\ x^1, x^2 \geq 0 \end{cases}$$

has no solution. Hence the linear system (11)(a)–(d) has no solution.

"If": Let x^* be a strong feasible solution to (10), then x^* is a feasible solution to the IvLP problem (12). Because the linear system (11)(a)–(d) has no solution, from the proof above we have the associated system (5) has no solution. It follows from Theorem 2.1 that x^* is a weak optimal solution to the IvLP (12), that is, for some $c^0 \in \mathbf{c}$ we have $c^0 x^* = \min\{c^0 x | x \in M(\underline{A}, \overline{b})\}$.

For each $A \in \mathbf{A}$, $b \in \mathbf{b}$, x^* is a feasible solution to the IvLP problem

$$\min\ \mathbf{c}^T x \quad \text{subject to } Ax \leq b, x \geq 0. \tag{14}$$

It is obvious that

$$M(A, b) = \{x \in \mathbb{R}^n | Ax \leq b, x \geq 0\} \subseteq M(\underline{A}, \overline{b}) = \{x \in \mathbb{R}^n | \underline{A}x \leq \overline{b}, x \geq 0\}.$$

Thus, we have

$$c^0 x^* = \min\{c^0 x | x \in M(\underline{A}, \overline{b})\} \leq \min\{c^0 x | x \in M(A, b)\}.$$

Because $x^* \in M(A, b)$, we have $c^0 x^* = \min\{c^0 x | x \in M(A, b)\}$, that is, x^* is an optimal solution to the LP problem

$$\min\ (c^0)^T x \quad \text{subject to } Ax \leq b, x \geq 0.$$

Thus, x^* is a weak optimal solution to the IvLP (14) for each $A \in \mathbf{A}$, $b \in \mathbf{b}$. Hence x^* is an $(\mathbf{A}, \mathbf{b})$-strong optimal solution to (10). This completes the proof of the theorem. □

4. (A, b, c)-strong optimal solutions

In this section, we will propose the necessary and sufficient conditions for checking $(\mathbf{A}, \mathbf{b}, \mathbf{c})$-strong optimality of given vectors.

Theorem 4.1. *Let $x^* \in \mathbb{R}^n$. Denote*

$$F = \{t_j | j = 1, \dots, p, x^*_{t_j} = 0\},$$

$$G = \{r_i | i = 1, \dots, q, \underline{A}_{r_i,\cdot} x^* = \overline{b}_{r_i}\}.$$

Then x^* is an $(\mathbf{A}, \mathbf{b}, \mathbf{c})$-strong optimal solution to (10) if and only if x^* is a strong feasible solution to (10), and for each $c \in \mathbf{c}$ the linear system

$$\begin{cases} \underline{A}_{r_i,\cdot} y \leq 0 \quad i = 1, \dots, q, & \text{(a)} \\ y_{t_j} \geq 0 \quad j = 1, \dots, p, & \text{(b)} \\ c^T y \leq -1 & \text{(c)} \end{cases} \tag{15}$$

has no solution.

Proof. "Only if": Let x^* be an $(\mathbf{A}, \mathbf{b}, \mathbf{c})$-strong optimal solution to (10), then x^* is a strong feasible solution to (10) and for each $c \in \mathbf{c}$, x^* is an optimal solution to the LP problem

$$\min\ c^T x \quad \text{subject to } \underline{A}x \leq \overline{b}, x \geq 0. \tag{16}$$

The tangent cone to the feasible region of (16) at x^* reads

$$\begin{cases} \underline{A}_{r_i,\cdot} x \leq 0 \quad i = 1, \dots, q, \\ x_{t_j} \geq 0 \quad j = 1, \dots, p. \end{cases} \tag{17}$$

Let matrix $E = (e_{ij})_{n\times n}$, where

$$e_{ij} = \begin{cases} 1 & i = j \in F, \\ 0 & \text{others.} \end{cases}$$

Denote

$$D = \begin{pmatrix} \underline{A}_{r_1,\cdot} \\ \underline{A}_{r_2,\cdot} \\ \vdots \\ \underline{A}_{r_q,\cdot} \\ -E \end{pmatrix}.$$

Then the tangent cone (17) can be written as $Dx \le 0$.

Note that x^* is an optimal solution to the LP (16), then it follows from Lemma 2.2 that the associated system (4)

$$\begin{cases} Dy \le 0, \\ c^T y \le -1 \end{cases}$$

has no solution. Hence for each $c \in \mathbf{c}$ the linear system (15)(a)–(c) has no solution.

"If": Let x^* be a strong feasible solution to (10), then for each $c \in \mathbf{c}$, x^* is a feasible solution to the LP (16). Because for each $c \in \mathbf{c}$ the linear system (15)(a)–(c) has no solution, from the proof above we have the associated system (4) has no solution. It follows from Lemma 2.2 that x^* is an optimal solution to the LP (16), that is, for each $c \in \mathbf{c}$ we have $cx^* = \min\{cx | x \in M(\underline{A}, \overline{b})\}$.

For each $A \in \mathbf{A}$, $b \in \mathbf{b}$, $c \in \mathbf{c}$, x^* is a feasible solution to the LP problem

$$\min\ c^T x \quad \text{subject to } Ax \le b, x \ge 0. \tag{18}$$

It is obvious that

$$M(A, b) = \{x \in \mathbb{R}^n | Ax \le b, x \ge 0\} \subseteq M(\underline{A}, \overline{b}) = \{x \in \mathbb{R}^n | \underline{A}x \le \overline{b}, x \ge 0\}.$$

Thus, we have

$$cx^* = \min\{cx | x \in M(\underline{A}, \overline{b})\} \le \min\{cx | x \in M(A, b)\}.$$

Because $x^* \in M(A, b)$, we have $cx^* = \min\{cx | x \in M(A, b)\}$, that is, x^* is an optimal solution to the LP (18). Hence x^* is an $(\mathbf{A}, \mathbf{b}, \mathbf{c})$-strong optimal solution to (10). This completes the proof of the theorem. □

Theorem 4.1 presents the necessary and sufficient conditions for checking $(\mathbf{A}, \mathbf{b}, \mathbf{c})$-strong optimality of given vectors. However, it is not effective since it demands that the linear system (15)(a)–(c) has no solution for each $c \in \mathbf{c}$. To make the checking conditions more effective, we first prove a lemma. It is easy to see that the necessary and sufficient conditions described in Theorem 4.1 are equivalent to the conditions that x^* is a strong feasible solution to (10), and the interval linear system

$$\begin{cases} \underline{A}_{r_i,\cdot} y \le 0 & i = 1, \dots, q, \\ y_{t_j} \ge 0 & j = 1, \dots, p, \\ \mathbf{c}^T y \le -1 \end{cases} \tag{19}$$

has no weak solution.

Lemma 4.1. *Let $x^* \in \mathbb{R}^n$ be a strong feasible solution to (10). Denote*

$$F = \{t_j | j = 1, \dots, p, x^*_{t_j} = 0\},$$

$$G = \{r_i | i = 1, \dots, q, \underline{A}_{r_i,\cdot} x^* = \overline{b}_{r_i}\}.$$

Then the interval linear system (19) has no weak solution if and only if for each $h = (h_1, \dots, h_{n-p})^T \in Y_{n-p}$, the linear system

$$\begin{cases} \underline{A}_{r_i,\cdot} y \le 0 & i = 1, \dots, q, \\ y_{t_j} \ge 0 & j = 1, \dots, p, \\ \displaystyle\sum_{j=1}^{p} \underline{c}_{t_j} y_{t_j} + \sum_{j=p+1}^{n} ((c_c)_{t_j} + h_{j-p} (c_\Delta)_{t_j}) y_{t_j} \le -1 \end{cases} \tag{20}$$

has no solution.

Proof. "Only if": We prove this part by contradiction. Assume that the interval linear system (19) has no weak solution, but the linear system (20) has a solution y^* for some $h \in Y_{n-p}$. Then y^* satisfies

$$\sum_{j=1}^{p} \underline{c}_{t_j} y_{t_j} + \sum_{j=p+1}^{n} ((c_c)_{t_j} + h_{j-p}(c_\Delta)_{t_j}) y_{t_j} \le -1.$$

Let

$$-h^1_{t_j} = \begin{cases} -1 & j = 1, \dots, p, \\ h_{j-p} & j = p+1, \dots, n. \end{cases}$$

Clearly, $h^1 = (h^1_{t_1}, \dots, h^1_{t_n})^T \in Y_n$ and the system

$$\begin{aligned} (c^T)_{eh^1} y^* &= \sum_{j=1}^{n} ((c_c)_{t_j} - e(c_\Delta)_{t_j} h^1_{t_j}) y^*_{t_j} \\ &= \sum_{j=1}^{n} ((c_c)_{t_j} + (-h^1_{t_j})(c_\Delta)_{t_j}) y^*_{t_j} \\ &= \sum_{j=1}^{p} \underline{c}_{t_j} y^*_{t_j} + \sum_{j=p+1}^{n} ((c_c)_{t_j} + h_{j-p}(c_\Delta)_{t_j}) y^*_{t_j} \le -1 \end{aligned}$$

holds, where $e = 1$. From Corollary 2.2 we know that y^* is a weak solution to $\mathbf{c}^T y \le -1$, that is, y^* is a solution to $c^T y \le -1$ for some $c \in \mathbf{c}$. Hence y^* is a solution to the linear system

$$\begin{aligned} &\underline{A}_{r_i,\cdot} y \le 0 \quad i = 1, \dots, q, \\ &y_{t_j} \ge 0 \quad j = 1, \dots, p, \\ &c^T y \le -1 \end{aligned}$$

for some $c \in \mathbf{c}$, thus, y^* is a weak solution to the interval linear system (19). This is a contradiction, so the linear system (20) has no solution for each $h \in Y_{n-p}$.

"If": We prove this part by contradiction. Assume that for each $h \in Y_{n-p}$ the linear system (20) has no solution, but the interval linear system (19) has a weak solution y^*. Then y^* is a weak solution to $\mathbf{c}^T y \le -1$. From Corollary 2.2 we know that y^* is a solution to $(c^T)_{eh^1} y \le -1$ for some $h^1 = (h^1_{t_1}, \dots, h^1_{t_n})^T \in Y_n$. Let $h_{j-p} = -h^1_{t_j}, j = p+1, \dots, n$, and clearly, $h \in Y_{n-p}$. Because $y^*_{t_j} \ge 0\, j = 1, \dots, p$, the inequality

$$\begin{aligned} \sum_{j=1}^{p} \underline{c}_{t_j} y^*_{t_j} + \sum_{j=p+1}^{n} ((c_c)_{t_j} + h_{j-p}(c_\Delta)_{t_j}) y^*_{t_j} &\le \sum_{j=1}^{p} ((c_c)_{t_j} + (-h^1_{t_j})(c_\Delta)_{t_j}) y^*_{t_j} + \sum_{j=p+1}^{n} ((c_c)_{t_j} + h_{j-p}(c_\Delta)_{t_j}) y^*_{t_j} \\ &= \sum_{j=1}^{n} ((c_c)_{t_j} + (-h^1_{t_j})(c_\Delta)_{t_j}) y^*_{t_j} \\ &= \sum_{j=1}^{n} ((c_c)_{t_j} - e(c_\Delta)_{t_j} h^1_{t_j}) y^*_{t_j} \\ &= (c^T)_{eh^1} y^* \le -1 \end{aligned}$$

holds. Hence y^* satisfies the linear system (20) for the $h \in Y_{n-p}$. This is a contradiction, so the interval linear system (19) has no weak solution. This completes the proof of the lemma. □

The above discussion leads to Theorem 4.2.

Theorem 4.2. *Let $x^* \in \mathbb{R}^n$. Denote*

$$F = \{t_j | j = 1, \dots, p, x^*_{t_j} = 0\},$$

$$G = \{r_i | i = 1, \dots, q, \underline{A}_{r_i,\cdot} x^* = \overline{b}_{r_i}\}.$$

Then x^* is an $(\mathbf{A}, \mathbf{b}, \mathbf{c})$-strong optimal solution to (10) if and only if x^* is a strong feasible solution to (10), and for each $h \in Y_{n-p}$ the linear system (20) has no solution.

Obviously, Theorem 4.2 is more effective than Theorem 4.1, since Theorem 4.2 has only 2^{n-p} linear systems, whereas Theorem 3.1 has infinitely many linear systems, to be solved.

5. Illustrative examples

In this section, we will solve two examples by using the presented methods in the previous sections.

Example 1. Consider the IvLP problem

$$\begin{aligned}&\min\ [-1,0]x_1+[-2,0]x_2+[-1,2]x_3\\&\text{s.t}\ x_1+2x_2+[-1,1]x_3\le 3,\\&x_1+[-1,3]x_2+[1,2]x_3\le[4,5],\\&x_1,x_2,x_3\ge 0.\end{aligned}\tag{21}$$

Let $x^*=(1,1,0)^T$. Clearly, x^* solves $\overline{A}x\le\underline{b}$, $x\ge 0$, hence x^* is a strong feasible solution to (21).

To use the conditions given in Theorem 3.1, first, note that the sets

$$F=\{t_j|j=1,\dots,p,x^*_{t_j}=0\}=\{3\},\qquad G=\{r_i|i=1,\dots,q,\underline{A}_{r_i,\cdot}x^*=\overline{b}_{r_i}\}=\{1\}.$$

Then, the corresponding linear system (11)(a)–(d) is given by

$$\begin{aligned}&(1,2,-1)(x^1-x^2)\le 0,\\&(x^1-x^2)_3\ge 0,\\&(0,0,2)x^1-(-1,-2,-1)x^2\le -1,\\&x^1,x^2\ge 0,\end{aligned}$$

where $x^1=(x^1_1,x^1_2,x^1_3)$, $x^2=(x^2_1,x^2_2,x^2_3)$. The linear system above can be written as

$$\begin{cases}(x^1_1-x^2_1)+2(x^1_2-x^2_2)-(x^1_3+x^2_3)\le 0, & \text{(a)}\\ x^1_3-x^2_3\ge 0, & \text{(b)}\\ 2x^1_3+x^2_1+2x^2_2+x^2_3\le -1, & \text{(c)}\\ x^1_1,x^2_1,x^1_2,x^2_2,x^1_3,x^2_3\ge 0. & \text{(d)}\end{cases}\tag{22}$$

From (22)(c) and $x^2_1,x^1_2,x^1_3,x^2_3\ge 0$, we have the linear system (22)(a)–(d) has no solution. From Theorem 3.1 we know that x^* is an $(\mathbf{A},\mathbf{b})$-strong optimal solution to (21).

We describe in detail, a method by which a cost vector $c\in\mathbf{c}$ can be found such that x^* is an optimal solution to a realization of (21) for given $A\in\mathbf{A}$ and $b\in\mathbf{b}$. For a given $A\in\mathbf{A}$ and $b\in\mathbf{b}$, say for example,

$$A=\begin{pmatrix}1&2&0\\1&1&1\end{pmatrix},\qquad b=\begin{pmatrix}3\\4\end{pmatrix}.$$

Then we have the IvLP problem

$$\begin{aligned}&\min\ [-1,0]x_1+[-2,0]x_2+[-1,2]x_3\\&\text{s.t}\ x_1+2x_2\le 3,\\&x_1+x_2+x_3\le 4,\\&x_1,x_2,x_3\ge 0.\end{aligned}$$

Let us check that x^* is a weak optimal solution to the IvLP above, that is, for some $c=(c_1,c_2,c_3)\in\mathbf{c}$, x^* is an optimal solution to the LP

$$\begin{aligned}&\min\ c_1x_1+c_2x_2+c_3x_3\\&\text{s.t}\ x_1+2x_2\le 3,\\&x_1+x_2+x_3\le 4,\\&x_1,x_2,x_3\ge 0.\end{aligned}\tag{23}$$

Let the set $G^1=\{r^1_i|i=1,\dots,q^1,A_{r^1_i,\cdot}x^*=b_{r^1_i}\}=\{1\}$, hence from Lemma 2.2, we know the linear system

$$\begin{aligned}&(1,2,0)y\le 0,\\&y_3\ge 0,\\&c^Ty\le -1\end{aligned}$$

has no solution, that is, the linear system

$$\begin{aligned}&y_1+2y_2\le 0,\\&y_3\ge 0,\\&c_1y_1+c_2y_2+c_3y_3\le -1\end{aligned}$$

has no solution. Clearly, the linear system above has no solution when $c = (-1, -2, 0)^T$. Substituting $c = (-1, -2, 0)^T$ into the LP (23), we have the LP problem

$$\begin{aligned}
&\min \; -x_1 - 2x_2 \\
&\text{s.t}\, x_1 + 2x_2 \le 3, \\
&x_1 + x_2 + x_3 \le 4, \\
&x_1, x_2, x_3 \ge 0.
\end{aligned}$$

It is easy to know that x^* is an optimal solution to the LP above.

Example 2. Consider the IvLP problem

$$\begin{aligned}
&\min \; -x_1 - 2x_2 + [1, 2]x_3 \\
&\text{s.t}\, x_1 + 2x_2 + [-1, 1]x_3 \le 3, \\
&x_1 + [-1, 3]x_2 + [1, 2]x_3 \le [4, 5], \\
&x_1, x_2, x_3 \ge 0.
\end{aligned} \tag{24}$$

Let $x^* = (1, 1, 0)^T$. Clearly, x^* solves $\overline{A}x \le \underline{b}$, $x \ge 0$, hence x^* is a strong feasible solution to (24).

To use the conditions given in Theorem 4.2, first, note that the sets

$$F = \{t_j | j = 1, \dots, p, x^*_{t_j} = 0\} = \{3\}, \qquad G = \{r_i | i = 1, \dots, q, \underline{A}_{r_i,\cdot}x^* = \overline{b}_{r_i}\} = \{1\}.$$

Then, the corresponding linear system (20) is given by

$$\begin{cases}
\underline{A}_{1,\cdot}y \le 0, \\
y_3 \ge 0, \\
\underline{c}_3 y_3 + [(c_c)_1 + h_1(c_\Delta)_1]y_1 + [(c_c)_2 + h_2(c_\Delta)_2]y_2 \le -1,
\end{cases} \tag{25}$$

where $h = (h_1, h_2)^T \in Y_2$, $y = (y_1.y_2, y_3)^T$. Note that $(c_\Delta)_1 = (c_\Delta)_2 = 0$ and hence $h_1(c_\Delta)_1 = h_2(c_\Delta)_2 = 0$, so the linear system (25) becomes

$$\begin{cases}
y_1 + 2y_2 - y_3 \le 0, \\
y_3 \ge 0, \\
-y_1 - 2y_2 + y_3 \le -1.
\end{cases} \tag{26}$$

It is easy to see that the linear system (26) has no solution. From Theorem 4.2 we know that x^* is an $(\mathbf{A}, \mathbf{b}, \mathbf{c})$-strong optimal solution to (24).

6. Conclusion

In this paper, we introduce new concepts of optimal solutions of IvLP in a unified framework. Some previously discussed concepts of IvLP, e.g., weak optimal solutions and strong optimal solutions, are special cases in this framework. Efficient methods to check $(\mathbf{A}, \mathbf{b})$-strong optimality and $(\mathbf{A}, \mathbf{b}, \mathbf{c})$-strong optimality of given vectors for type C IvLP are developed. The necessary and sufficient conditions for checking $(\mathbf{A}, \mathbf{b})$-strong optimality and $(\mathbf{A}, \mathbf{b}, \mathbf{c})$-strong optimality of given vectors for type B IvLP are to appear in a forthcoming paper.

Future work includes checking methods for $(\mathbf{A})$-strong optimal solutions, $(\mathbf{b})$-strong optimal solutions, $(\mathbf{c})$-strong optimal solutions, $(\mathbf{b}, \mathbf{c})$-strong optimal solutions and $(\mathbf{A}, \mathbf{c})$-strong optimal solutions for type A, B and C type IvLP, respectively. Interval nonlinear programming problems are also studied by some authors [23–28]. The methodology of this paper can be applicable to make a generalization of the solution concepts for these interval programs.

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