ARTICLE IN PRESS

Journal of Computational and Applied Mathematics ▮ (▮▮▮▮) ▮▮▮–▮▮▮

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On inverses of infinite Hessenberg matrices

J. Abderramán Marrero [a], V. Tomeo [b,*], E. Torrano [c]

[a] Department of Mathematics Applied to Information Technology, School of Telecommunication Engineering, UPM - Technical University of Madrid, Avda Complutense s/n. Ciudad Universitaria, 28040 Madrid, Spain

[b] Department of Algebra, Faculty of Statistics, UCM - Complutense University, Avda de Puerta de Hierro s/n. Ciudad Universitaria, 28040 Madrid, Spain

[c] Department of Applied Mathematics, Faculty of Computing, UPM - Technical University of Madrid, Campus de Montegancedo, Boadilla, 28660 Madrid, Spain

ARTICLE INFO

Article history:
Received 9 July 2014
Received in revised form 16 July 2014

MSC:
15A09
15A99
47B36

Keywords:
Hessenberg matrix
Infinite matrix
Inverse matrix
Tridiagonal matrix

ABSTRACT

Here a known result on the structure of finite Hessenberg matrices is extended to infinite Hessenberg matrices. Its consequences for the example of infinite Hessenberg–Toeplitz matrices are described. The results are applied also to the inversion of infinite tridiagonal matrices via recurrence relations. Moreover, since there are available free parameters, different inverses can be associated with a given invertible tridiagonal matrix.

1. Introduction

A characterization of the finite nonsingular unreduced Hessenberg matrices is related with the distinguished rank structure of their inverse matrices [1–3]. Without loss of generality, we consider upper Hessenberg matrices. The inverses of such matrices are rank one perturbations, $\mathbf{UV} + \mathbf{T}$, of triangular matrices. The matrix $\mathbf{T}$ is (strictly) upper triangular and $\mathbf{U}$ and $\mathbf{V}$ are a column and a row vector, respectively. Their relevance to the case of finite tridiagonal matrices is immediate. We ask whether the inverses of real or complex infinite Hessenberg matrices have a similar structure, and we shall show that this is indeed the case. We propose a method for inverting infinite Hessenberg matrices that is based on these structural properties. In particular, classical inverses of general tridiagonal matrices can be generated through recurrence relations.

Some background about inversion of infinite matrices and their applications can be found in the literature; see e.g. [4–7] and the references given there. The classical role of the infinite unreduced Hessenberg matrices in orthogonal polynomials as matrix representation of the multiplication by z operator is well known [8]. In addition, infinite (transition or adjacency) Hessenberg matrices appear in signal processing, time series, and birth–death processes. Here the infinite Hessenberg matrices are regarded simply as matrices over $\mathbb{C}$ or $\mathbb{R}$. Nevertheless, the inversion of infinite Hessenberg matrices regarded as matrices representing bounded operators on the linear space ℓ^2 will be discussed briefly.

The outline is as follows. In Section 2, we recall some basic results about inverses of finite Hessenberg and tridiagonal matrices. In Section 3, we study the problem of the consistency of regarding Hessenberg matrices as the inverses of matrices

* Corresponding author. Tel.: +34 913944055.
E-mail addresses: jc.abderraman@upm.es (J. Abderramán Marrero), tomeo@estad.ucm.es (V. Tomeo), emilio@fi.upm.es (E. Torrano).

http://dx.doi.org/10.1016/j.cam.2014.07.010

of the form $\mathbf{B} = \mathbf{UV} + \mathbf{T}$. In Section 4, we propose a method for inverting infinite unreduced Hessenberg matrices, in particular Hessenberg–Toeplitz matrices and tridiagonal matrices. Its application to infinite reduced Hessenberg matrices with finitely many zeros on their subdiagonals is straightforward. In addition, recurrence relations for evaluating classical inverses of tridiagonal matrices are introduced. Finally, Section 5 contains a short remark regarding the inversion of bounded linear operators. Throughout the text the results are illustrated with appropriate examples.

2. Inverses of finite Hessenberg matrices

2.1. Unreduced Hessenberg matrices of finite order

A matrix $\mathbf{H}$ is upper Hessenberg if its elements h_{ij} satisfy $h_{ij} = 0$ for $i \geq j+2$. Here we extend and adapt a well-known lemma [1–3] to upper Hessenberg matrices. We also recall that an order n Hessenberg matrix $\mathbf{H} = (h_{ij})_{i,j=1}^{n}$ is an unreduced upper Hessenberg matrix if it has nonzero subdiagonal entries, $h_{i+1,i} \neq 0,\ i = 1, 2, \ldots, n-1$.

Lemma 1. *An $n \times n$ nonsingular matrix $\mathbf{H} = (h_{ij})_{i,j=1}^{n}$ is unreduced upper Hessenberg if and only if its inverse matrix has the form $\mathbf{B} = \mathbf{UV} + \mathbf{T}$, where $\mathbf{U}$ is a column matrix with nonzero nth component and $\mathbf{V}$ is a row matrix with nonzero first component. The matrix $\mathbf{T}$ is strictly upper triangular, having zero entries on its main diagonal and nonzero entries $t_{i,i+1} = h_{i+1,i}^{-1} \neq 0$, $1 \leq i \leq n-1$, on the superdiagonal.*

See [2] for a proof. The matrix $\mathbf{B}$ has the form

$$\mathbf{B} = \begin{pmatrix} u_1 v_1 & b_{12} & b_{13} & \cdots & b_{1n} \\ u_2 v_1 & u_2 v_2 & b_{23} & \cdots & b_{2n} \\ u_3 v_1 & u_3 v_2 & u_3 v_3 & \cdots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_n v_1 & u_n v_2 & u_n v_3 & \cdots & u_n v_n \end{pmatrix},$$

where, for $j > i$, $b_{ij} = u_i v_j + t_{ij}$. The determinant $|\mathbf{B}|$ is given by

$$|\mathbf{B}| = \frac{v_1 u_n}{\prod_{i=1}^{n-1} (-h_{i+1,i})} = (-1)^{n-1} v_1 u_n \prod_{i=1}^{n-1} t_{i,i+1}.$$

Since $\mathbf{B}$ is nonsingular, the triangular matrix $\mathbf{T}$ must have nonzero entries on its superdiagonal. The components of the vectors $\mathbf{U}$ and $\mathbf{V}$ are

$$u_i = \frac{(-1)^{i-1}}{|\mathbf{H}|} |\mathbf{H}_{n-i}^{(i)}| \frac{1}{\prod_{k=i}^{n-1} h_{k+1,k}}, \qquad v_j = (-1)^{j-1} |\mathbf{H}_{j-1}| \prod_{k=j}^{n-1} h_{k+1,k}.$$

A formally equivalent lemma holds for lower Hessenberg matrices. The entry b_{ij} of $\mathbf{B}$ has the determinantal representation

$$b_{ij} = \begin{cases} (-1)^{i+j} \dfrac{|\mathbf{H}_{j-1}| \cdot |\mathbf{H}_{n-i}^{(i)}| \cdot [h_{i,i-1} \cdots h_{j+1,j}]}{|\mathbf{H}|}, & \text{if } i \geq j; \\[2ex] (-1)^{i+j} \dfrac{|\mathbf{H}_{j-1}| \cdot |\mathbf{H}_{n-1}^{(i)}|}{|\mathbf{H}| \cdot [h_{j,j-1} \cdots h_{i+1,i}]} - \dfrac{(-1)^{i+j} |\mathbf{H}_{j-i-1}^{(i)}|}{[h_{j,j-1} \cdots h_{i+1,i}]}, & \text{if } i < j, \end{cases} \tag{1}$$

where $|\mathbf{H}_{j-1}|$ is the $(j-1)$st left principal minor, and $|\mathbf{H}^{(i)}|_{n-i}$ and $|\mathbf{H}^{(i)}|_{j-i-1}$ are the right principal minors of the matrices $\mathbf{H}_n$ and $\mathbf{H}_{j-i-1}$, respectively; see Corollary 1 in [9]. It follows immediately that, for $i < j$, the entries t_{ij} of $\mathbf{T}$ have the form

$$t_{ij} = \frac{(-1)^{i+j+1} |\mathbf{H}_{j-i-1}^{(i)}|}{\prod_{k=j-1}^{i} h_{k+1,k}}. \tag{2}$$

2.2. Unreduced tridiagonal matrices of finite order

Recall that a tridiagonal matrix having nonzero entries on both the subdiagonal and the superdiagonal is called an unreduced tridiagonal matrix. The following result is also well known [1–3,10].

Please cite this article in press as: J. Abderramán Marrero, et al., On inverses of infinite Hessenberg matrices, Journal of Computational and Applied Mathematics (2014), http://dx.doi.org/10.1016/j.cam.2014.07.010

Lemma 2. *A nonsingular matrix* $\mathbf{H} = (h_{ij})_{i,j=1}^n$ *is an unreduced tridiagonal matrix if and only if its inverse matrix* $\mathbf{B} = (b_{ij})_{i,j=1}^n$ *has the form*

$$b_{ij} = \begin{cases} u_i v_j, & \text{if } i \geq j; \\ w_i x_j, & \text{if } i \leq j, \end{cases}$$

where u_1, v_n, w_n, *and* x_1 *are nonzero.*

The proof is trivial because a tridiagonal matrix is also lower and upper Hessenberg. The result is an immediate consequence of Lemma 1. Trivially, $u_k v_k = w_k x_k$. If, in addition, the matrix is symmetric, then $u_i = x_i$ and $v_j = w_j$. Specific numerical methods for inverting finite Hessenberg and tridiagonal matrices are known; see e.g. [1,3,10–12] and the references given there.

2.3. Reduced Hessenberg matrices of finite order

For the reduced case, we consider, without loss of generality, nonsingular upper Hessenberg matrices having a zero on the subdiagonal. We can decompose the matrix in blocks and using the Schur complement. For a nonsingular reduced Hessenberg matrix $\mathbf{H}$ with $h_{k+1,k} = 0$, we have

$$\mathbf{H} = \left(\begin{array}{c|c} H_{11} & H_{12} \\ \hline 0 & H_{22} \end{array}\right) \quad \text{and} \quad \mathbf{B} = \left(\begin{array}{c|c} B_{11} & B_{12} \\ \hline 0 & B_{22} \end{array}\right)$$

where $B_{11} = H_{11}^{-1}$, $B_{22} = H_{22}^{-1}$ and $B_{12} = -B_{11} H_{21} B_{22}$. The matrix $\mathbf{B}$ is the inverse of $\mathbf{H}$ and both matrices have the same block upper triangular structure. The case of nonsingular reduced tridiagonal matrices can be handled in a similar way.

3. Consistency

We consider an infinite invertible matrix $\mathbf{B}$ having the form $\mathbf{B} = \mathbf{UV} + \mathbf{T}$, i.e. a rank one perturbation of an infinite triangular matrix, subject to the conditions $v_1 \neq 0$, $u_k \neq 0$, for all $k \geq 1$. This condition has the consequence that the nth principal section of $\mathbf{B}$, denoted $[B]_n$, has the same form $U_n V_n + T_n$. Thus $|[B]_n| \neq 0$ for all $n > 0$, and block entry $[B]_n$ is nonsingular. Consequently, its inverse is also an unreduced upper Hessenberg matrix.

It is natural to ask about the consistency of this class of Hessenberg matrices. If we consider the matrix sequence $\{H_k\}$ defined by

$$H_k = ([B]_k)^{-1}, \tag{3}$$

it need not be true that $([B]_k)^{-1}$ is a principal section of $([B]_{k+1})^{-1}$ because the last column of $([B]_k)^{-1}$ need not equal the corresponding column of $([B]_{k+1})^{-1}$. However, upon deleting the last column and the last row of $([B]_k)^{-1}$ and the last two columns and the last two rows of $([B]_{k+1})^{-1}$, the resulting matrices are the same and the resulting sequence $\{H_k\}$ is consistent. Therefore, this class of infinite unreduced Hessenberg matrices can be built section by section from $\mathbf{B}$.

Theorem 1 (*Consistency*)**.** *Given an infinite invertible matrix* $\mathbf{B}$ *of the form* $\mathbf{UV} + \mathbf{T}$, *where* $v_1 \neq 0$, $u_k \neq 0$ *for all* $k \geq 1$, *and matrix* $\mathbf{T}$ *with nonzero superdiagonal entries. Let* $\{[B]_k\}$ *be the matrix sequence of the principal sections of* $\mathbf{B}$ *as given in Lemma 1. Then, all principal sections* $[B]_k$ *are nonsingular and the matrix sequence of inverses* $\{H_k\}$ *given in (3) is consistent, i.e.*

$$[H_k]_{k-1} = [H_{k+1}]_{k-1}.$$

Proof. The nonsingularity of the principal sections of $\mathbf{B}$ is immediate from Lemma 1. Now, let $[B]_{k+1}$ be the $(k+1)$st principal section of $\mathbf{B}$ and let $([B]_{k+1})^{-1}$ be its inverse. In order to show consistency in the sequence $\{H_k\}$, we assume the following block partitions for matrices

$$[B]_{k+1} = \left(\begin{array}{c|c} [[B]_{k+1}]_{k-1} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right) \quad \text{and} \quad ([B]_{k+1})^{-1} = \left(\begin{array}{c|c} [([B]_{k+1})^{-1}]_{k-1} & H_{12} \\ \hline H_{21} & H_{22} \end{array}\right).$$

Trivially, the product $[B]_{k+1}([B]_{k+1})^{-1} = \left(\begin{array}{c|c} I_{k-1} & 0 \\ \hline 0 & I_2 \end{array}\right)$. Therefore,

$$[[B]_{k+1}]_{k-1}[([B]_{k+1})^{-1}]_{k-1} + B_{12} H_{21} = I_{k-1}.$$

In a similar way, taking the block matrix partition for the kth section, $[B]_k = \left(\begin{array}{c|c} [B_k]_{k-1} & B'_{12} \\ \hline B'_{21} & B'_{22} \end{array}\right)$ and using the product $[B]_k([B]_k)^{-1}$, with an analogous partition for $([B]_k)^{-1}$, we obtain

$$[[B]_k]_{k-1}[([B]_k)^{-1}]_{k-1} + B'_{12} H'_{21} = I_{k-1}.$$

Please cite this article in press as: J. Abderramán Marrero, et al., On inverses of infinite Hessenberg matrices, Journal of Computational and Applied Mathematics (2014), http://dx.doi.org/10.1016/j.cam.2014.07.010

But

$$B_{12}H_{21}=B'_{12}H'_{21}=\begin{pmatrix}0&0&\cdots&0&(u_1v_k+t_{1,k})h_{k,k-1}\\0&0&\cdots&0&(u_2v_k+t_{2,k})h_{k,k-1}\\0&0&\cdots&0&(u_3v_k+t_{3,k})h_{k,k-1}\\\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&\cdots&0&(u_{k-1}v_k+t_{k-1,k})h_{k,k-1}\end{pmatrix}.$$

Therefore, $[[B]_{k+1}]_{k-1}[([B]_{k+1})^{-1}]_{k-1}=[[B]_k]_{k-1}[([B]_k)^{-1}]_{k-1}$. Taking into account that $[[B]_{k+1}]_{k-1}=[[B]_k]_{k-1}=[B]_{k-1}$, with matrix $[B]_{k-1}$ nonsingular, we conclude $[([B]_{k+1})^{-1}]_{k-1}=[([B]_k)^{-1}]_{k-1}$. □

4. Inverses of infinite Hessenberg matrices

Imposing some additional conditions, we extend Lemma 1 to infinite Hessenberg matrices and Lemma 2 to infinite tridiagonal matrices. We recall that if $\mathbf{A}=(a_{ij})_{i,j=1}^{\infty}$ is an infinite matrix of complex numbers, the matrix $\mathbf{B}=(b_{ij})_{i,j=1}^{\infty}$ is a *classical inverse* of $\mathbf{A}$ if $\mathbf{AB}=\mathbf{BA}=\mathbf{I}$. It is known that an infinite matrix need not have classical inverse. For example the matrix corresponding to the right shift operator,

$$S_R=\begin{pmatrix}0&0&0&0&\cdots\\1&0&0&0&\cdots\\0&1&0&0&\cdots\\0&0&1&0&\cdots\\\vdots&\vdots&\vdots&\vdots&\ddots\end{pmatrix},$$

has no classical inverse, because the product of the first row of S_R with the first column of any other matrix is always zero. It is also known that an infinite matrix can have two classical inverse matrices; see e.g. [6]. In this case it has infinitely many classical inverses, because if $\mathbf{B}'$ and $\mathbf{B}''$ are inverses of $\mathbf{A}$, then $\alpha\mathbf{B}'+(1-\alpha)\mathbf{B}''$ is also an inverse matrix of $\mathbf{A}$, for all $\alpha\in\mathbb{C}$.

4.1. Infinite unreduced Hessenberg matrices

For infinite unreduced Hessenberg matrices we need to impose, in addition to the conditions imposed in the finite case, additional conditions on the components of the vector $\mathbf{U}$ and on the limits of the ratios of the determinants of certain submatrices, as we make precise in the following theorem.

Theorem 2. *Let $\mathbf{H}=(h_{ij})_{i,j=1}^{\infty}$ be an infinite invertible matrix and assume that the limits*

$$\lim_n\frac{|[H]_{n-i}^{(i)}|}{|[H]_n|}=\xi_i,\tag{4}$$

for $i\geq 1$, are finite and nonzero. Then $\mathbf{H}$ is an unreduced upper Hessenberg matrix if and only if its classical inverse matrix $\mathbf{B}=(b_{ij})_{i,j=1}^{\infty}$ has the form $\mathbf{B}=\mathbf{UV}+\mathbf{T}$. Here $\mathbf{U}=(u_1,u_2,u_3,\ldots)^t$ is a column vector with nonzero components, $\mathbf{V}=(v_1,v_2,v_3,\ldots)$ is a row vector with nonzero first component, and $\mathbf{T}$ is strictly upper triangular, having zeros on its main diagonal, and nonzero entries $t_{i,i+1}=h_{i+1,i}^{-1}$, $i\geq 1$, on its superdiagonal.

Proof. First, we assume that $\mathbf{H}$ is an unreduced upper Hessenberg matrix. Its principal section $[H]_n$ is a finite unreduced upper Hessenberg matrix. Direct computation of its inverse matrix $([H]_n)^{-1}$ using the adjugate matrix gives, for $i\geq j$,

$$\begin{aligned}[B]_n(i,j)&=\frac{[\mathbf{Adj}(j,i)]_n}{|[H]_n|}=\frac{(-1)^{i+j}}{|[H]_n|}\begin{vmatrix}[H]_{j-1}&D&E\\0&F&G\\0&0&[H]_{n-i}^{(i)}\end{vmatrix}\\&=(-1)^{i-1}\frac{|[H]_{n-i}^{(i)}|}{|[H]_n|}|[H]_{j-1}|\prod_{k=j}^{i-1}h_{k+1,k}.\end{aligned}$$

The entries $[B]_n(i,j)$ depend on n. Taking the limit when n tends to infinity, we have for $i\geq j$,

$$\begin{aligned}b_{ij}&=\lim_n\frac{[\mathbf{Adj}(j,i)]_n}{|[H]_n|}=(-1)^{i-1}\lim_n\frac{|[H]_{n-i}^{(i)}|}{|[H]_n|}|[H]_{j-1}|\prod_{k=j}^{i-1}h_{k+1,k}\\&=(-1)^{i-1}\frac{\xi_i}{\prod_{k=i}^{m-1}h_{k+1,k}}\cdot(-1)^{j-1}|[H]_{j-1}|\prod_{k=j}^{m-1}h_{k+1,k},\end{aligned}$$

where m is an arbitrary positive integer satisfying $m \geq \max\{i, j\}$. The block entry F is upper triangular and its main diagonal have entries from the subdiagonal of $\mathbf{H}$. Therefore, defining

$$u_i = (-1)^{i-1} \frac{\xi_i}{\prod_{k=i}^{m-1} h_{k+1,k}} \quad \text{and} \quad v_j = (-1)^{j-1} |[H]_{j-1}| \prod_{k=j}^{m-1} h_{k+1,k}, \tag{5}$$

(see also Lemmas 1 and 2 from [2]) with m as before, we obtain $b_{ij} = u_i v_j$, for $i \geq j$. With the conventions $|[H]_0| = 1$ and $|[H]_0^{(i)}| = 1$, the components v_1 and u_i are nonzero. These observations about the lower half part of $\mathbf{B}$ lead us to see that the inverse of $\mathbf{H}$ has the form

$$\mathbf{B} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \end{pmatrix} \begin{pmatrix} v_1 & v_2 & v_3 & \cdots \end{pmatrix} + \begin{pmatrix} 0 & t_{12} & t_{13} & t_{14} & \cdots \\ 0 & 0 & t_{23} & t_{24} & \cdots \\ 0 & 0 & 0 & t_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \mathbf{UV} + \mathbf{T}.$$

Hence,

$$\mathbf{B} = \begin{pmatrix} u_1 v_1 & b_{12} & b_{13} & \cdots \\ u_2 v_1 & u_2 v_2 & b_{23} & \cdots \\ u_3 v_1 & u_3 v_2 & u_3 v_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{6}$$

and, for $j > i$, by Lemma 1, Eqs. (1) and (2), and Theorem 1, there holds $b_{ij} = u_i v_j + t_{ij}$, with $t_{i,i+1} \neq 0$.

Conversely, we assume that $\mathbf{H}$ is the inverse of an infinite invertible matrix $\mathbf{B} = (b_{ij})_{i,j=1}^{\infty}$ as in (6), with entries $b_{ij} = u_i v_j$ for $i \geq j$ and $b_{ij} = u_i v_j + t_{ij}$ for $i < j$, where $1 \leq i, j$. Additionally, $t_{i,i+1} = h_{i+1,i}^{-1} \neq 0$, $u_i \neq 0$, and $v_1 \neq 0$. It is not difficult to see that, for $j - i \geq 2$, the corresponding entries of the adjugate matrix are zeros. Thus the invertible matrix $\mathbf{H} = \mathbf{B}^{-1}$ is an upper Hessenberg matrix. In addition, since $\mathbf{B}$ is invertible, the entries $t_{i,i+1} = h_{i+1,i}^{-1} \neq 0$, and $\mathbf{H}$ is unreduced. □

Note that the existence of the limits gives a sufficient condition on the form of such classical inverses, different from the bounds given for tridiagonal matrices in [5]. An equivalent result can be obtained for lower unreduced Hessenberg matrices.

4.2. *Inverses of infinite Hessenberg–Toeplitz matrices*

Theorem 2 is applied to the inversion of an infinite (invertible) Hessenberg–Toeplitz matrix

$$\mathbf{H} = \begin{pmatrix} h_1 & h_2 & h_3 & h_4 & \cdots \\ h_0 & h_1 & h_2 & h_3 & \cdots \\ 0 & h_0 & h_1 & h_2 & \cdots \\ 0 & 0 & h_0 & h_1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

By expanding along its last column we obtain the large recurrence relation for the determinant of its nth principal section,

$$|[H]_n| = \sum_{k=0}^{n-1} h_{n-k} (-h_0)^{n-1-k} |[H]_k|,$$

where $|[H]_0| = 1$. Assuming that the Hessenberg–Toeplitz matrix $\mathbf{H}$ is invertible, by Theorem 2 we obtain

$$\lim_n \frac{|[H]_{n-1}^{(1)}|}{|[H]_n|} = \lim_n \frac{|[H]_{n-1}|}{|[H]_n|} = \xi.$$

Then ξ is finite. Its value can be obtained using the large recurrence relation, by taking the limit at infinite, and solving the equation,

$$1 = h_1 \xi - h_2 h_0 \xi^2 + h_3 h_0^2 \xi^3 - h_4 h_0^3 \xi^4 + \cdots + (-1)^{n-1} h_n h_0^{n-1} \xi^n + \cdots.$$

Therefore $\xi \neq 0$ and we can evaluate the inverse $\mathbf{B} = \mathbf{UV} + \mathbf{T}$ using formulas (2) and (5). It can be seen that the upper triangular matrix $\mathbf{T}$ is also Toeplitz.

Example 1. Let **H** be the infinite upper Hessenberg matrix

$$\mathbf{H} = \begin{pmatrix} \frac{1}{2} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \frac{1}{128} & \cdots \\ \frac{1}{5} & \frac{1}{2} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \frac{1}{64} & \cdots \\ 0 & \frac{1}{5} & \frac{1}{2} & \frac{1}{8} & \frac{1}{16} & \frac{1}{32} & \cdots \\ 0 & 0 & \frac{1}{5} & \frac{1}{2} & \frac{1}{8} & \frac{1}{16} & \cdots \\ 0 & 0 & 0 & \frac{1}{5} & \frac{1}{2} & \frac{1}{8} & \cdots \\ 0 & 0 & 0 & 0 & \frac{1}{5} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The large recurrence relation is

$$|[H]_n| = \frac{1}{2}|[H]_{n-1}| - \frac{1}{40}|[H]_{n-2}| + \cdots + \frac{(-1)^{n-2}}{5^{n-2}\cdot 2^n}|[H]_1| + \frac{(-1)^{n-1}}{5^{n-1}\cdot 2^{n+1}}|[H]_0|.$$

As $[H]_n$ is Toeplitz, it satisfies $|[H]_{n-i}^{(i)}| = |[H]_{n-i}|$. Thus we have

$$\lim_n \frac{|[H]_{n-1}|}{|[H]_n|} = \xi \quad \text{and} \quad \lim_n \frac{|[H]_{n-i}|}{|[H]_n|} = \xi^i.$$

If we divide the large recurrence relation by $|[H]_n|$ and take limits, there results

$$1 = \frac{1}{2}\xi - \frac{1}{40}\xi^2 + \frac{1}{400}\xi^3 - \frac{1}{4000}\xi^4 + \cdots = \frac{20\xi + \xi^2}{40 + 4\xi},$$

so that $\xi^2 + 16\xi - 40 = 0$. Thus

$$\xi = -8 \pm 2\sqrt{26}.$$

To obtain convergence we take the positive root, $\xi = 8 + 2\sqrt{26} \simeq 2.1980$. The row vector **V** is given by (5) and we have

$$v_j = (-1)^{j-1}|[H]_{j-1}| \cdot h_0^{m-j} = \frac{(-1)^{j-1}\cdot|[H]_{j-1}|}{5^{m-j}}, \qquad v_1 = \frac{1}{5^{m-1}}.$$

The column vector **U** is also given by (5),

$$u_i = (-1)^{i-1}\xi^i 5^{m-i}.$$

For $i \geq j$

$$b_{ij} = u_i v_j = \frac{(-1)^{i+j}\xi^i|[H]_{j-1}|}{5^{i-j}},$$

which does not depend on m.

Finally,

$$\mathbf{B} = \mathbf{UV} + \mathbf{T} = \begin{pmatrix} \xi & \frac{-5\xi+10}{2} & \frac{45\xi-100}{8} & \frac{-205\xi+450}{16} & \cdots \\ -\frac{1}{5}\xi^2 & \frac{1}{2}\xi^2 & \frac{-9\xi^2+40}{8} & \frac{41\xi^2-200}{16} & \cdots \\ \frac{1}{25}\xi^3 & -\frac{1}{10}\xi^3 & \frac{7}{32}\xi^3 & \frac{-41\xi^3+400}{80} & \cdots \\ -\frac{1}{125}\xi^4 & \frac{7}{100}\xi^4 & -\frac{5}{256}\xi^4 & \frac{25}{256}\xi^4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

4.3. *Infinite unreduced tridiagonal matrices*

A finite tridiagonal matrix is written $\{a_i, b_i, c_i\}$, $1 \leq i \leq n$, where a_i, b_i, and c_i are the entries of the lower subdiagonal, the main diagonal, and the upper diagonal, respectively. We use the same notation $\mathbf{H} = \{a_i, b_i, c_i\}$ to indicate the infinite tridiagonal matrix

$$\mathbf{H} = \begin{pmatrix} b_0 & c_1 & 0 & 0 & \cdots \\ a_1 & b_1 & c_2 & 0 & \cdots \\ 0 & a_2 & b_2 & c_3 & \cdots \\ 0 & 0 & a_3 & b_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Please cite this article in press as: J. Abderramán Marrero, et al., On inverses of infinite Hessenberg matrices, Journal of Computational and Applied Mathematics (2014), http://dx.doi.org/10.1016/j.cam.2014.07.010

Recall that a tridiagonal matrix having nonzero entries on both the subdiagonal and the superdiagonal is called an unreduced tridiagonal matrix. The following result is a particular case of Theorem 2 and an extension of Lemma 2 to the infinite case.

Corollary 1. *With the assumptions of Theorem 2, an infinite invertible matrix* $\mathbf{H} = (h_{ij})_{i,j=1}^{\infty}$ *is an unreduced tridiagonal matrix if and only if the entries of its inverse matrix* $\mathbf{B} = (b_{ij})_{i,j=1}^{\infty}$ *have the forms*

$$b_{ij} = \begin{cases} u_i v_j, & \text{if } i \geq j; \\ w_i x_j, & \text{if } i \leq j, \end{cases}$$

where u_1, v_i, w_i, *and* x_1 *are nonzero.*

Trivially, $u_k v_k = w_k x_k$. If, in addition, $\mathbf{H}$ is symmetric, then $u_i = x_i$, and $v_j = w_j$.

Proposition 1. *Let* $\mathbf{H}$ *be an infinite invertible tridiagonal unreduced matrix. Then, its classical inverse* $\mathbf{B} = (b_{ij})_{i,j=1}^{\infty}$ *has entries as given in Corollary 1 and the vectors* $\mathbf{U} = (u_1, u_2, \ldots)^t$, $\mathbf{V} = (v_1, v_2, \ldots)$, $\mathbf{W} = (w_1, w_2, \ldots)^t$, *and* $\mathbf{X} = (x_1, x_2, \ldots)$ *satisfy the recurrence relations*

$$\begin{cases} u_2 = \dfrac{1 - b_0 u_1 v_1}{c_1 v_1} \\ u_i = \dfrac{-a_{i-2} u_{i-2} - b_{i-2} u_{i-1}}{c_{i-1}}, \end{cases} \qquad \begin{cases} v_2 = \dfrac{-b_0 v_1}{a_1} \\ v_i = \dfrac{-c_{i-2} v_{i-2} - b_{i-2} v_{i-1}}{a_{i-1}}, \end{cases} \quad \text{or}$$

$$\begin{cases} w_2 = \dfrac{-b_0 w_1}{c_1} \\ w_i = \dfrac{-a_{i-2} w_{i-2} - b_{i-2} w_{i-1}}{c_{i-1}}, \end{cases} \qquad \begin{cases} x_2 = \dfrac{1 - b_0 w_1 x_1}{a_1 w_1} \\ x_i = \dfrac{-c_{i-2} x_{i-2} - b_{i-2} x_{i-1}}{a_{i-1}}, \end{cases}$$

for $i \geq 3$, *with* $v_1 \neq 0$, $w_1 \neq 0$, *and* $u_1 v_1 = w_1 x_1$.

Proof. Let be $\mathbf{H}$ and $\mathbf{B}$ the matrices

$$\mathbf{H} = \begin{pmatrix} b_0 & c_1 & 0 & \cdots \\ a_1 & b_1 & c_2 & \cdots \\ 0 & a_2 & b_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \mathbf{B} = \begin{pmatrix} u_1 v_1 & w_1 x_2 & w_1 x_3 & \cdots \\ u_2 v_1 & u_2 v_2 & w_2 x_3 & \cdots \\ u_3 v_1 & u_3 v_2 & u_3 v_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $\mathbf{B} = \mathbf{UV} + \mathbf{T}$, $w_{i-1} x_i = u_{i-1} v_i + t_{i-1,i}$, and $t_{i-1,i} = \frac{1}{a_{i-1}} \neq 0$.

First we consider the matrix product $\mathbf{HB}$. If we multiply the ith row of $\mathbf{H}$ by the ith column of $\mathbf{B}$, we have, from the recurrences in Proposition 1,

$$a_{i-1} w_{i-1} x_i + b_{i-1} u_i v_1 + c_i u_{i+1} v_i = 1.$$

We now multiply the ith row of $\mathbf{H}$ by the jth column of $\mathbf{B}$. When $i \neq j$ the result is 0.

We now consider the matrix product $\mathbf{BH}$. If we multiply the ith row of $\mathbf{B}$ by the ith column of $\mathbf{H}$, we obtain

$$u_i v_{i-1} c_{i-1} + u_i v_i b_{i-1} + w_i x_{i+1} a_i = u_i v_{i-1} c_{i-1} + u_i v_i b_{i-1} + (u_i v_{i+1} + t_{i,i+1}) a_i = 1.$$

We now multiply the ith row of $\mathbf{B}$ by the jth column of $\mathbf{H}$. When $i \neq j$ the result is 0.

Therefore, $\mathbf{B}$ is the classical inverse of $\mathbf{H}$ and, conversely, $\mathbf{H}$ is the classical inverse of $\mathbf{B}$.

We must prove that the condition $u_1 v_1 = w_1 x_1$ implies $u_k v_k = w_k x_k$ for $k \geq 2$. Indeed, for $k = 2$, we have

$$u_2 v_2 = \frac{-b_0(1 - b_0 u_1 v_1)}{a_1 c_1} = \frac{-b_0(1 - b_0 w_1 x_1)}{a_1 c_1} = w_2 x_2,$$

and, additionally, we have

$$a_1 b_1 u_1 v_2 + b_1 c_1 u_2 v_1 = a_1 b_1 u_1 \frac{-b_0 v_1}{a_1} + b_1 c_1 v_1 \frac{1 - b_0 u_1 v_1}{c_1 v_1} = a_1 b_1 w_1 x_2 + b_1 c_1 w_2 x_1.$$

By the induction hypothesis we can suppose $u_{k-1} v_{k-1} = w_{k-1} x_{k-1},\; u_k v_k = w_k x_k$, and

$$a_{k-1} b_{k-1} u_{k-1} v_k + b_{k-1} c_{k-1} u_k v_{k-1} = a_{k-1} b_{k-1} w_{k-1} x_k + b_{k-1} c_{k-1} w_k x_{k-1}.$$

Therefore, we obtain $u_{k+1} v_{k+1} = w_{k+1} x_{k+1}$ and the proof is complete. □

Remark 1. In establishing Proposition 1, we use the recurrence relations subject to the conditions $v_1 \neq 0$, $w_1 \neq 0$, and $u_1 v_1 = w_1 x_1$. Three of these parameters can be chosen freely. When we choose different values for them, we obtain different classical inverses of the infinite tridiagonal matrix **H**. However, such an election is crucial in order to find a suitable matrix related with a bounded linear operator.

Example 2. We illustrate with the infinite real symmetric tridiagonal matrix

$$\mathbf{H} = \begin{pmatrix} 1 & -\frac{2}{5} & 0 & \cdots \\ -\frac{2}{5} & 1 & -\frac{2}{5} & \cdots \\ 0 & -\frac{2}{5} & 1 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where $b_i = 1$ and $a_i = c_i = -2/5$. If we choose, say, $u_1 = v_1 = 1$, we have

$$\mathbf{U} = \left(1, 0, -1, -\tfrac{5}{2}, -\tfrac{21}{4}, -\tfrac{85}{8}, \ldots\right)^t, \qquad \mathbf{V} = \left(1, \tfrac{5}{2}, \tfrac{21}{4}, \tfrac{85}{8}, \tfrac{341}{16}, \tfrac{1365}{32}, \ldots\right).$$

By symmetry we obtain a classical inverse $\mathbf{B}'$ of **H**. However, if we choose, say, $u_1 = 5/4$ and $v_1 = 1$, we have

$$\mathbf{U} = \left(\tfrac{5}{4}, \tfrac{5}{8}, \tfrac{5}{16}, \tfrac{5}{32}, \tfrac{5}{64}, \tfrac{5}{128}, \ldots\right)^t, \qquad \mathbf{V} = \left(1, \tfrac{5}{2}, \tfrac{21}{4}, \tfrac{85}{8}, \tfrac{341}{16}, \tfrac{1365}{32}, \ldots\right).$$

We obtain another classical inverse $\mathbf{B}''$ of **H**. The matrices $\mathbf{B}'$ and $\mathbf{B}''$ are

$$\mathbf{B}' = \begin{pmatrix} 1 & 0 & -1 & -\frac{5}{2} & \cdots \\ 0 & 0 & -\frac{5}{2} & -\frac{25}{4} & \cdots \\ -1 & -\frac{5}{2} & -\frac{21}{4} & -\frac{105}{8} & \cdots \\ -\frac{5}{2} & -\frac{25}{4} & -\frac{105}{8} & -\frac{425}{16} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad \mathbf{B}'' = \begin{pmatrix} \frac{5}{4} & \frac{5}{8} & \frac{5}{16} & \frac{5}{32} & \cdots \\ \frac{5}{8} & \frac{25}{16} & \frac{25}{32} & \frac{25}{64} & \cdots \\ \frac{5}{16} & \frac{25}{32} & \frac{105}{64} & \frac{105}{128} & \cdots \\ \frac{5}{32} & \frac{25}{64} & \frac{105}{128} & \frac{425}{256} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

These two matrices are examples of classical inverses of **H**. Note also that if we choose $v_1 = i$, we obtain the vector $\mathbf{V} = \left(i, \frac{5i}{2}, \frac{21i}{4}, \frac{85i}{8}, \frac{341i}{16}, \ldots\right)$. Taking, for example, $u_1 = 1$, we obtain a complex classical inverse of the real matrix **H**.

Proposition 1 yields classical inverses of a tridiagonal matrix **H**. Conversely, the next proposition shows how to obtain the inverse of an infinite matrix **B** of a particular form.

Proposition 2. *Let $\mathbf{B} = (b_{ij})_{i,j=1}^{\infty}$ be an infinite invertible matrix of the form* **UV** *for $i \geq j$ and* **WX** *for $i \leq j$. The vectors* **U**, **V**, **W**, *and* **X** *are as in Proposition 1. Then its classical inverse, the infinite tridiagonal matrix $\mathbf{H} = \{a_i, b_i, c_i\}$, is unique and its entries are given by the recursive relations*

$$\begin{cases} a_1 = \dfrac{1 - b_0 x_1}{x_2} \\ a_{i-1} = \dfrac{-c_{i-2} x_{i-2} - b_{i-2} x_{i-1}}{x_i}, \end{cases} \qquad \begin{cases} c_1 = \dfrac{1 - b_0 u_1}{u_2} \\ c_{i-1} = \dfrac{-a_{i-2} u_{i-2} - b_{i-2} u_{i-1}}{u_i}, \end{cases} \quad \textit{and}$$

$$\begin{cases} b_0 = \dfrac{b_{22}}{b_{11} b_{22} - b_{12} b_{21}} \\ b_{i-2} = \dfrac{b_{i-2,1}(b_{i-1,i-1} b_{1,i} b_{i,1} - b_{ii} b_{1,i-1} b_{i-1,1})}{c_{i-2} b_{i-1,1}(b_{ii} b_{1,i-2} b_{i-2,1} - b_{i-2,i-2} b_{1,i} b_{i,1})}, \end{cases}$$

for $i \geq 3$. The order in the computation of parameters is the following: first b_0, then a_1 and c_1, then b_1, then a_2 and c_2, and so forth, sequentially.

Proof. Without loss of generality we take $v_1 = w_1 = 1$. The matrix **B** has the form

$$\mathbf{B} = \begin{pmatrix} u_1 & x_2 & x_3 & x_4 & \cdots \\ u_2 & u_2 v_2 & w_2 x_3 & w_2 x_4 & \cdots \\ u_3 & u_3 v_2 & u_3 v_3 & w_3 x_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with $x_1 = u_1$ and $u_k v_k = w_k x_k$, for $k \geq 2$. The first column of **B** is the vector **U** and the first row of **B** is the vector **X**. Thus u_i and x_i are known for $i \geq 1$. Recurrence relations for a_i and c_i are obtained from the recurrence relations for x_i and u_i in Proposition 1. We must determine b_0 and b_i. From the recurrence for v_i, with $v_1 = 1$, we obtain b_0 and b_i. The products of rows of **H** with columns of **B** were already computed in the proof of Proposition 1. In summary, **H** is the inverse matrix of **B**, and conversely. It follows from the fact that a_i, b_i, and c_i are uniquely determined that the matrix **H** is uniquely determined. $\square$

4.4. Inverses of infinite reduced Hessenberg matrices

When an invertible Hessenberg matrix $\mathbf{H}$ has a zero entry on its subdiagonal, we can calculate its classical inverse in a way similar to how the inverse of a finite Hessenberg matrix was computed in Section 2.3.

Proposition 3. *Let $\mathbf{H}$ be an infinite invertible upper Hessenberg matrix with only a zero entry on its subdiagonal. Then its classical inverse matrix can be calculated using a decomposition into block matrices. If $\mathbf{H}$ and $\mathbf{B}$ have the forms*

$$\mathbf{H} = \left(\begin{array}{c|c} H_{11} & H_{12} \\ \hline 0 & H_{22} \end{array}\right), \qquad \mathbf{B} = \left(\begin{array}{c|c} H_{11}^{-1} & -H_{11}^{-1}H_{12}H_{22}^{-1} \\ \hline 0 & H_{22}^{-1} \end{array}\right),$$

then $\mathbf{B}$ is a classical inverse matrix of $\mathbf{H}$, where H_{11} is a finite nonsingular unreduced Hessenberg matrix, H_{22} is an infinite invertible unreduced Hessenberg matrix, and we assume that the product $-H_{11}^{-1}H_{12}H_{22}^{-1}$ exists.

Proof. This is trivial because $\mathbf{HB} = \mathbf{I}$ and $\mathbf{BH} = \mathbf{I}$. Therefore, $\mathbf{B}$ is a classical inverse of matrix $\mathbf{H}$. □

Remark 2. The product $-H_{11}^{-1}H_{12}H_{22}^{-1}$, in the block decomposition of $\mathbf{B}$, satisfies the associative property because the order of these matrices are 3×3, $3 \times \infty$, and $\infty \times \infty$. It can be proven that the product of three matrices, one of which is finite, satisfies the associative property.

Example 3. Let $\mathbf{H}$ be the infinite tridiagonal matrix $H = \{1, 2, 1\}$, with a unique zero, $h_{43} = 0$, on its subdiagonal. A classical inverse will be $\mathbf{B}$, where

$$\mathbf{H} = \left(\begin{array}{ccc|ccc} 2 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 0 & 1 & 2 & 1 & 0 & \cdots \\ \hline 0 & 0 & 0 & 2 & 1 & \cdots \\ 0 & 0 & 0 & 1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right), \qquad \mathbf{B} = \left(\begin{array}{ccc|ccc} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} & & & \cdots \\ -\frac{1}{2} & 1 & -\frac{1}{2} & B_{12} & & \cdots \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} & & & \cdots \\ \hline 0 & 0 & 0 & 1 & -1 & \cdots \\ 0 & 0 & 0 & -1 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{array}\right),$$

in which $B_{12} = -H_{11}^{-1}H_{12}H_{22}^{-1} = \begin{pmatrix} -\frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & \cdots \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \cdots \\ -\frac{3}{4} & \frac{3}{4} & -\frac{3}{4} & \cdots \end{pmatrix}$ equals

$$-\begin{pmatrix} \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{4} & -\frac{1}{2} & \frac{3}{4} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & \cdots \\ -1 & 2 & -2 & \cdots \\ 1 & -2 & 3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The case of finitely many zeros on the subdiagonal follows easily from Proposition 3.

Corollary 2. *Let $\mathbf{H}$ be an infinite invertible upper Hessenberg matrix with finitely many zeros on its subdiagonal. Then its classical inverse matrices can be calculated using a block matrix decomposition as in Proposition 3, but now entry H_{11} is a finite nonsingular reduced Hessenberg matrix.*

An analogous procedure is valid if $\mathbf{H}$ is a lower reduced Hessenberg matrix.

5. Hessenberg matrices related with bounded linear operators

In some cases, Hessenberg matrices can be regarded as bounded linear operators on ℓ^2. We recall here some basic features of the matrix representation of invertible linear operators. A bounded linear operator L between Hilbert spaces, for example from ℓ^2 to itself, is *invertible* if there exists an operator L^{-1}, obviously bounded, such that $L^{-1}Lx = x$ and $LL^{-1}y = y$ for every $x, y \in \ell^2$. The operator L^{-1} is called the inverse operator of L, and it is unique. A useful method for inverting bounded linear operators is the (iterative) power series method given by the next lemma; see e.g. Theorem 8.2 from [4].

Lemma 3. *Let $\mathbf{H}$ be a matrix representation of a bounded linear operator on ℓ^2 that satisfies $\|\mathbf{A}\| < 1$, where $\mathbf{A} = \mathbf{I} - \mathbf{H}$. Then $\mathbf{H}$ is invertible, and its inverse $\mathbf{H}^{-1}$,*

$$\mathbf{H}^{-1} = \sum_{j=0}^{\infty} \mathbf{A}^j = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + \cdots,$$

is a matrix representation of its inverse operator defined on ℓ^2.

Example 4. The infinite Hessenberg matrix from Example 1 satisfies $\mathbf{H} = \mathbf{I} - \mathbf{A}$, where $\mathbf{A}$ is the matrix

$$\mathbf{A} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{8} & -\frac{1}{16} & \cdots \\ -\frac{1}{5} & \frac{1}{2} & -\frac{1}{8} & \cdots \\ 0 & -\frac{1}{5} & \frac{1}{2} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with norm $\|\mathbf{A}\| < 1$. Thus $\mathbf{H}$, as a matrix representation of a bounded linear operator in ℓ^2, is invertible. The matrix $\mathbf{B}$ from Example 1 is a matrix representation of its inverse operator.

The infinite tridiagonal matrix from Example 2 satisfies also $\mathbf{H} = \mathbf{I} - \mathbf{A}$, where $\mathbf{A}$ is the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & \frac{2}{5} & 0 & \cdots \\ \frac{2}{5} & 0 & \frac{2}{5} & \cdots \\ 0 & \frac{2}{5} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

with norm $\|\mathbf{A}\| < 1$. Hence $\mathbf{H}$, as a matrix representation of a bounded linear operator in ℓ^2, is invertible. The matrix $\mathbf{B}''$ from Example 2 is a matrix representation of its inverse operator.

Recall that a matrix $\mathbf{H}$ can have infinitely many classical inverses. Nevertheless, if $\mathbf{H}$ is a matrix representation of an invertible bounded linear operator, it has a unique inverse. Such an inverse is also a matrix representation of its inverse bounded linear operator, as those given in Example 4.

References

[1] Y. Ikebe, On inverses of Hessenberg matrices, Linear Algebra Appl. 24 (1979) 93–97.
[2] D.K. Faddeev, Properties of a matrix, inverse of a Hessenberg matrix, J. Math. Sci. 24 (1984) 118–120.
[3] R. Vandebril, M. Van Barel, N. Mastronardi, Matrix Computation and Semiseparable Matrices, Volume I: Linear Systems, Johns Hopkins University Press, Baltimore, Maryland, USA, 2008.
[4] I. Gohberg, S. Goldberg, M.A. Kaashoek, Basic Classes of Linear Operators, Birkhäuser Verlag, Basel, 2003.
[5] P.N. Shivakumar, C. Ji, Upper and lower bounds for inverse elements of finite and infinite tridiagonal matrices, Linear Algebra Appl. 247 (1996) 297–316.
[6] K.C. Shivakumar, Moore–Penrose inverse of an invertible infinite matrix, Linear Multilinear Algebra 54 (2006) 71–77.
[7] P.N. Shivakumar, K.C. Shivakumar, A review of infinite matrices and their applications, Linear Algebra Appl. 430 (2009) 976–998.
[8] B. Simon, Orthogonal Polynomials on the Unit Circle, Part. 1: Classical Theory, in: AMS Colloquium Publications, vol. 54.1, American Mathematical Society, Providence, RI, USA, 2005.
[9] J. Abderramán Marrero, V. Tomeo, On the closed representation for the inverses of Hessenberg matrices, J. Comput. Appl. Math. 236 (2012) 2962–2970.
[10] B. Bukhberger, G.A. Emel'yanenko, Methods of inverting tridiagonal matrices, Comput. Math. Math. Phys. URSS 13 (1973) 10–20.
[11] J. Abderramán Marrero, M. Rachidi, V. Tomeo, Non-symbolic algorithms for the inversion of tridiagonal matrices, J. Comput. Appl. Math. 252 (2013) 3–11.
[12] J. Abderramán Marrero, M. Rachidi, V. Tomeo, On new algorithms for inverting Hessenberg matrices, J. Comput. Appl. Math. 252 (2013) 12–20.

Please cite this article in press as: J. Abderramán Marrero, et al., On inverses of infinite Hessenberg matrices, Journal of Computational and Applied Mathematics (2014), http://dx.doi.org/10.1016/j.cam.2014.07.010