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A hybridized formulation for the weak Galerkin mixed finite element method

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ABSTRACT

This paper presents a hybridized formulation for the weak Galerkin mixed finite element method (WG-MFEM) which was introduced and analyzed in Wang and Ye (2014) for second order elliptic equations. The WG-MFEM method was designed by using discontinuous piecewise polynomials on finite element partitions consisting of polygonal or polyhedral elements of arbitrary shape. The key to WG-MFEM is the use of a discrete weak divergence operator which is defined and computed by solving inexpensive problems locally on each element. The hybridized formulation of this paper leads to a significantly reduced system of linear equations involving only the unknowns arising from the Lagrange multiplier in hybridization. Optimal-order error estimates are derived for the hybridized WG-MFEM approximations. Some numerical results are reported to confirm the theory and a superconvergence for the Lagrange multiplier.

1. Introduction

In this paper, we are concerned with new developments of weak Galerkin finite element methods for partial differential equations. Weak Galerkin (WG) [1–4] is a generic finite element method for partial differential equations where the differential operators (e.g., gradient, divergence, curl, Laplacian etc.) in the variational form are approximated by weak forms as generalized distributions. This process often involves the solution of inexpensive problems defined locally on each element. The solution from the local problems can be regarded as a reconstruction of the corresponding differential operators. The fundamental difference between the weak Galerkin finite element method and other existing methods is the use of weak functions and weak derivatives (i.e., locally reconstructed differential operators) in the design of numerical schemes based on existing variational forms for the underlying PDE problems. Weak Galerkin is, therefore, a natural extension of the conforming Galerkin finite element method. Due to its great structural flexibility, the weak Galerkin finite element method fits well to most partial differential equations by providing the needed stability and accuracy in approximation.

The goal of this paper is to develop a new computational method which reduces the computational complexity for the weak Galerkin mixed finite element methods [2] by using the well-known hybridization technique [5–7]. Our model

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problem seeks a vector-valued function $\mathbf{q} = \mathbf{q}(\mathbf{x})$, also known as flux function, and a scalar function $u = u(\mathbf{x})$ defined in an open bounded polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ satisfying

$$\alpha\mathbf{q} + \nabla u = 0, \qquad \nabla \cdot \mathbf{q} = f, \quad \text{in } \Omega \tag{1.1}$$

and the following Dirichlet boundary condition

$$u = g, \quad \text{on } \partial\Omega, \tag{1.2}$$

where $\alpha = (\alpha_{ij}(\mathbf{x}))_{d\times d} \in [L^\infty(\Omega)]^{d\times d}$ is a symmetric, uniformly positive definite matrix on the domain Ω. A weak formulation for (1.1)–(1.2) seeks $\mathbf{q} \in H(div, \Omega)$ and $u \in L^2(\Omega)$ such that

$$(\alpha\mathbf{q}, \mathbf{v}) - (\nabla \cdot \mathbf{v}, u) = -\langle g, \mathbf{v} \cdot \mathbf{n}\rangle_{\partial\Omega}, \quad \forall \mathbf{v} \in H(div, \Omega) \tag{1.3}$$

$$(\nabla \cdot \mathbf{q}, w) = (f, w), \quad \forall w \in L^2(\Omega). \tag{1.4}$$

Here $L^2(\Omega)$ is the standard space of square integrable functions on Ω, $\nabla \cdot \mathbf{v}$ is the divergence of vector-valued functions $\mathbf{v}$ on Ω, $H(div, \Omega)$ is the Sobolev space consisting of vector-valued functions $\mathbf{v}$ such that $\mathbf{v} \in [L^2(\Omega)]^d$ and $\nabla \cdot \mathbf{v} \in L^2(\Omega)$, $(\cdot, \cdot)$ stands for the L^2-inner product in $L^2(\Omega)$, and $\langle \cdot, \cdot \rangle_{\partial\Omega}$ is the inner product in $L^2(\partial\Omega)$.

Conforming Galerkin finite element methods based on the weak formulation (1.3)–(1.4) and finite dimensional subspaces of $H(div, \Omega) \times L^2(\Omega)$ with piecewise polynomials are known as mixed finite element methods (MFEM) [8,5]. MFEMs for (1.1)–(1.2) treat $\mathbf{q}$ and u as independent unknown functions and are capable of providing accurate approximations for both unknowns [6,9–14,8,15]. All the existing MFEMs in literature possess local mass conservation that makes MFEM a competitive numerical technique in many applications such as flow of fluid in porous media including oil reservoir and groundwater contamination simulation.

Based on the weak formulation (1.3)–(1.4), a weak Galerkin mixed finite element method (WG-MFEM) was developed in [2] which provides accurate numerical approximations for both the scalar and the flux variables. Like the existing MFEMs, the WG-MFEM scheme conserves mass locally on each element. But unlike the existing MFEMs, the WG-MFEM allows the use of finite element partitions consisting of polygons $(d = 2)$ or polyhedra $(d = 3)$ of arbitrary shape that satisfy the shape-regularity assumption specified in [2]. It should be pointed out that some similar features are shared by a number of recently developed numerical methods such as Virtual Element Methods [16,17], hybridizable discontinuous Galerkin methods [18], and mimetic finite differences [19,20].

While weak functions and weak derivatives provide a great deal of flexibility for WG methods, they also introduce more degrees of freedom than the standard finite element method. The purpose of this paper is to develop a hybridized formulation for the weak Galerkin mixed finite element method [2] that shall reduce the computational complexity significantly by solving a linear system involving a small number of unknowns arising from an auxiliary function called Lagrange multiplier. The Lagrange multiplier is defined only on the element boundaries (also called wired-basket of the finite element partition). As a result, the linear system that costs the majority of the computing time depends on the dimension of the finite element space defined on the wired-basket. Optimal-order error estimates for the hybridized WG-MFEM (HWG-MFEM) approximations are established in several discrete norms. Some numerical results are presented to demonstrate the efficiency and power of the hybridized WG-FEM.

Throughout the paper, we will follow the usual notation for Sobolev spaces and norms [21,12,11]. For any open bounded domain $D \subset \mathbb{R}^d$ with Lipschitz continuous boundary, we use $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ to denote the norm and seminorms in the Sobolev space $H^s(D)$ for any $s \geq 0$, respectively. The inner product in $H^s(D)$ is denoted by $(\cdot, \cdot)_{s,D}$. The space $H^0(D)$ coincides with $L^2(D)$, for which the norm and the inner product are denoted by $\|\cdot\|_D$ and $(\cdot, \cdot)_D$, respectively. When $D = \Omega$, we shall drop the subscript D in the norm and inner product notation.

The paper is organized as follows. In Section 2, we review the discrete weak divergence operator. In Section 3, we present a hybridized weak Galerkin mixed finite element method. Section 4 is devoted to a discussion of the relation between the WG-MFEM and its hybridized version. In Section 5, we derive an error estimate for the hybridized WG-MFEM. Finally, in Section 6, we report some numerical results that demonstrate the efficiency and accuracy of the hybridized WG-MFEM, including a superconvergence for the Lagrange multiplier.

2. Weak divergence

Let K be a polygonal or polyhedral domain. A weak vector-valued function on K refers to a vector field $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}$ with $\mathbf{v}_0 \in [L^2(K)]^d$ and $\mathbf{v}_b \in [L^2(\partial K)]^d$. The first component $\mathbf{v}_0$ carries the information of $\mathbf{v}$ in K, and $\mathbf{v}_b$ represents partial or full information of $\mathbf{v}$ on ∂K. The choice of the boundary information that $\mathbf{v}_b$ represents is problem-dependent. Note that $\mathbf{v}_b$ may not necessarily be related to the trace of $\mathbf{v}_0$ on ∂K should a trace be well-defined.

Denote by $V(K)$ the space of all weak vector-valued functions on K; i.e.,

$$V(K) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \ \mathbf{v}_0 \in [L^2(K)]^d, \ \mathbf{v}_b \in [L^2(\partial K)]^d\}. \tag{2.1}$$

A weak divergence can be taken for any vector field in $V(K)$ by following the definition introduced in [2], which we summarize as follows.

Definition 2.1 (*[2]*)**.** For any $\mathbf{v} \in V(K)$, the weak divergence of $\mathbf{v}$ is defined as a linear functional, denoted by $\nabla_w \cdot \mathbf{v}$, on $H^1(K)$ whose action on each $\varphi \in H^1(K)$ is given by

$$\langle \nabla_w \cdot \mathbf{v}, \varphi \rangle_K := -(\mathbf{v}_0, \nabla\varphi)_K + \langle \mathbf{v}_b \cdot \mathbf{n}, \varphi \rangle_{\partial K}, \tag{2.2}$$

where $(\cdot, \cdot)_K$ and $\langle \cdot, \cdot \rangle_{\partial K}$ stands for the L^2-inner product in $L^2(K)$ and $L^2(\partial K)$, respectively.

The Sobolev space $[H^1(K)]^d$ can be embedded into the space $V(K)$ by an inclusion map $i_V : [H^1(K)]^d \to V(K)$ defined as follows

$$i_V(\mathbf{q}) = \{\mathbf{q}|_K, \mathbf{q}|_{\partial K}\}.$$

With the help of the inclusion map i_V, the Sobolev space $[H^1(K)]^d$ can be viewed as a subspace of $V(K)$ by identifying each $\mathbf{q} \in [H^1(K)]^d$ with $i_V(\mathbf{q})$. Analogously, a weak function $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V(K)$ is said to be in $[H^1(K)]^d$ if it can be identified with a function $\mathbf{q} \in [H^1(K)]^d$ through the above inclusion map. It is not hard to see that $\nabla_w \cdot \mathbf{v} = \nabla \cdot \mathbf{v}$ if $\mathbf{v}$ is a smooth function in $[H^1(K)]^d$.

Next, we introduce a discrete weak divergence operator by approximating $(\nabla_w \cdot)$ in a polynomial subspace of the dual of $H^1(K)$. To this end, for any non-negative integer $r \geq 0$, denote by $P_r(K)$ the set of polynomials on K with degree r or less.

Definition 2.2. A discrete weak divergence $(\nabla_{w,r} \cdot)$ is defined as the unique polynomial $(\nabla_{w,r} \cdot \mathbf{v}) \in P_r(K)$ satisfying the following equation

$$(\nabla_{w,r} \cdot \mathbf{v}, \phi)_K = -(\mathbf{v}_0, \nabla\phi)_K + \langle \mathbf{v}_b \cdot \mathbf{n}, \phi \rangle_{\partial K}, \quad \forall \phi \in P_r(K). \tag{2.3}$$

3. Hybridized WG-MFEM

Let $\mathcal{T}_h$ be a finite element partition of the domain Ω consisting of polygons in two dimensions or polyhedra in three dimensions satisfying the shape-regularity condition specified in [2]. For $T \in \mathcal{T}_h$, denote by h_T its diameter and $h = \max_{T \in \mathcal{T}_h} h_T$ the meshsize of $\mathcal{T}_h$. The set of all edges or flat faces in $\mathcal{T}_h$ is denoted as $\mathcal{E}_h$, with the subset $\mathcal{E}_h^0 = \mathcal{E}_h \setminus \partial\Omega$ consisting of all the interior edges or flat faces.

Let $k \geq 0$ be any integer. On each element $T \in \mathcal{T}_h$, we denote by $\mathbf{n}$ the outward normal direction on the boundary ∂T, and define two local finite element spaces $W_{k+1}(T) = P_{k+1}(T)$ and

$$V_k(T) = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} : \ \mathbf{v}_0 \in [P_k(T)]^d, \ \mathbf{v}_b|_e = v_b \mathbf{n}, \ v_b \in P_k(e), \ e \subset \partial T\}. \tag{3.1}$$

On the wired-basket $\mathcal{E}_h$, we introduce a finite element space using piecewise polynomials of degree k:

$$\Lambda_h = \{\lambda : \ \lambda|_e \in P_k(e), \ \forall e \in \mathcal{E}_h\}. \tag{3.2}$$

Let $\Lambda_h^0 \subset \Lambda_h$ be the subspace consisting of functions with zero boundary value

$$\Lambda_h^0 = \{\lambda \in \Lambda_h; \ \lambda|_e = 0, \ \forall e \subset \partial\Omega\}. \tag{3.3}$$

We further introduce an element-wise stabilizer as follows

$$s_T(\mathbf{r}, \mathbf{v}) = h_T \langle (\mathbf{r}_0 - \mathbf{r}_b) \cdot \mathbf{n}, \ (\mathbf{v}_0 - \mathbf{v}_b) \cdot \mathbf{n} \rangle_{\partial T}, \quad \mathbf{r}, \mathbf{v} \in V_k(T). \tag{3.4}$$

Denote by $\nabla_{w,k+1}\cdot$ the discrete weak divergence operator in the space $V_k(T)$ computed by using (2.3) on each element T. For simplicity of notation, from now on we shall drop the subscript $k+1$ from the notation $\nabla_{w,k+1}\cdot$ for the discrete weak divergence.

Hybridized WG-MFEM Scheme 3.1. *For an approximate solution of (1.1)–(1.2), find $\mathbf{q}_h = \{\mathbf{q}_0, \mathbf{q}_b\} \in V_k(T)$, $u_h \in W_{k+1}(T)$, $\lambda_h \in \Lambda_h$ such that $\lambda_h = Q_b g$ on $\partial\Omega$ and*

$$s_T(\mathbf{q}_h, \mathbf{v}) + (\alpha \mathbf{q}_0, \mathbf{v}_0)_T - (\nabla_w \cdot \mathbf{v}, u_h)_T = -\langle \lambda_h, \mathbf{v}_b \cdot \mathbf{n} \rangle_{\partial T}, \quad \forall \mathbf{v} \in V_k(T), \tag{3.5}$$

$$(\nabla_w \cdot \mathbf{q}_h, w)_T = (f, \ w)_T, \quad \forall w \in W_{k+1}(T), \tag{3.6}$$

$$\sum_{T \in \mathcal{T}_h} \langle \mathbf{q}_b \cdot \mathbf{n}, \phi \rangle_{\partial T} = 0, \quad \forall \phi \in \Lambda_h^0. \tag{3.7}$$

Here $(Q_b g)|_e \in P_k(e)$ is the L^2 projection of the boundary value $u = g$ on the edge $e \subset \partial\Omega$.

The approximation for the primal variable u is given by λ_h on $\mathcal{E}_h$ and u_h on each element T. The flux $\mathbf{q}$ is approximated by $\mathbf{q}_0$ on each element, and $\mathbf{q}_b$ is an approximation of the normal component of $\mathbf{q}$ on the element boundary.

The hybridized WG mixed finite element scheme (3.5)–(3.7) is an analogy of the hybridized mixed finite element method [5] (see [6,12,15] for more details). Note that the approximation $\mathbf{q}_h$ and u_h are defined locally on each element $T \in \mathcal{T}_h$ so that $\mathbf{q}_b$ assumes multi-values on each interior edge. More specifically, unlike the WG mixed finite element method introduced in [2], the hybridized WG-MFEM scheme (3.5)–(3.7) does not assume the "continuity" (or single-value) of $\mathbf{q}_b$ on each interior edge $e \in \mathcal{E}_h^0$. The new variable λ_h, known as the Lagrange multiplier, was added here to provide the necessary "continuity" of $\mathbf{q}_b$ on $\mathcal{E}_h^0$ through Eq. (3.7).

In the rest of this section, we shall reformulate the problem (3.5)–(3.7) by eliminating the unknowns $\mathbf{q}_h$ and u_h. The resulting linear system is symmetric and positive definite in terms of the variable λ_h. Readers are referred to [6] for a similar result for the standard mixed finite element method.

Denote by $\mathcal{M}_h$ the collection of all the local finite element spaces:

$$\mathcal{M}_h = \bigotimes_{T\in\mathcal{T}_h} V_k(T) \times W_{k+1}(T).$$

For any given $\theta \in \Lambda_h$, let $\{\mathbf{q}_h^\theta, u_h^\theta\} \in \mathcal{M}_h$ be a solution of the following problem

$$s_T(\mathbf{q}_h^\theta, \mathbf{v}) + (\alpha\mathbf{q}_0^\theta, \mathbf{v}_0)_T - (\nabla_w\cdot\mathbf{v}, u_h^\theta)_T = -\langle\theta, \mathbf{v}_b\cdot\mathbf{n}\rangle_{\partial T}, \quad \forall\mathbf{v}\in V_k(T), \tag{3.8}$$

$$(\nabla_w\cdot\mathbf{q}_h^\theta, w)_T = 0, \quad \forall w\in W_{k+1}(T). \tag{3.9}$$

Since the WG-MFEM formulation (4.5)–(4.6) has a unique solution proved in [2], the system (3.8)–(3.9) has a unique solution.

By setting

$$\mathbb{H}(\theta) := \{\mathbf{q}_0^\theta, \mathbf{q}_b^\theta, u_h^\theta\}$$

we see that $\mathbb{H}$ defines a linear operator from Λ_h to $\mathcal{M}_h$. For convenience, we decompose the operator $\mathbb{H}$ into three components $\mathbb{H} = \{\mathbb{H}_0, \mathbb{H}_b, \mathbb{H}_u\}$ so that

$$\mathbf{q}_0^\theta = \mathbb{H}_0(\theta), \qquad \mathbf{q}_b^\theta = \mathbb{H}_b(\theta), \qquad u_h^\theta = \mathbb{H}_u(\theta).$$

The first two components of $\mathbb{H}$ can be grouped together to give the following operator

$$\mathbb{H}_\mathbf{q} = \{\mathbb{H}_0, \mathbb{H}_b\}.$$

Note that the third component $\mathbb{H}_u(\theta)$ is a piecewise polynomial of degree $k+1$ which is indeed a discrete harmonic extension of θ in the domain Ω from the wired-basket $\mathcal{E}_h$.

With the help of the linear operator $\mathbb{H}$, we can reformulate the hybridized WG-MFEM scheme (3.5)–(3.7) into one involving only the unknown variable on the wired-basket $\mathcal{E}_h$.

Theorem 3.2. *The solution $\lambda_h \in \Lambda_h$ arising from the hybridized WG-MFEM scheme (3.5)–(3.7) is a solution of the following problem: Find $\lambda_h \in \Lambda_h$ satisfying $\lambda_h = Q_b g$ on $\partial\Omega$ and the following equation*

$$\sum_{T\in\mathcal{T}_h}\left\{s_T(\mathbb{H}_\mathbf{q}(\lambda_h), \mathbb{H}_\mathbf{q}(\phi)) + (\alpha\mathbb{H}_0(\lambda_h), \mathbb{H}_0(\phi))_T\right\} = (f, \mathbb{H}_u(\phi)), \quad \forall\phi\in\Lambda_h^0. \tag{3.10}$$

Proof. Note that Eq. (3.7) has test function $\phi \in \Lambda_h^0$. Owing to the operator $\mathbb{H}$, by letting $\theta = \phi$ in (3.8), we arrive at

$$-\langle\phi, \mathbf{v}_b\cdot\mathbf{n}\rangle_{\partial T} = s_T(\mathbb{H}_\mathbf{q}(\phi), \mathbf{v}) + (\alpha\mathbb{H}_0(\phi), \mathbf{v}_0)_T - (\nabla_w\cdot\mathbf{v}, \mathbb{H}_u(\phi))_T$$

for all $\mathbf{v} \in V_k(T)$. In particular, by setting $\mathbf{v} = \mathbf{q}_h$ (i.e., the solution of (3.5)–(3.6)), we obtain

$$\begin{aligned}-\langle\phi, \mathbf{q}_b\cdot\mathbf{n}\rangle_{\partial T} &= s_T(\mathbb{H}_\mathbf{q}(\phi), \mathbf{q}_h) + (\alpha\mathbb{H}_0(\phi), \mathbf{q}_0)_T - (\nabla_w\cdot\mathbf{q}_h, \mathbb{H}_u(\phi))_T\\ &= s_T(\mathbf{q}_h, \mathbb{H}_\mathbf{q}(\phi)) + (\alpha\mathbf{q}_0, \mathbb{H}_0(\phi))_T - (f, \mathbb{H}_u(\phi))_T,\end{aligned}$$

where we have used (3.6) in the second line. By summing over all the element T we have from (3.7) that

$$\sum_{T\in\mathcal{T}_h}\left\{s_T(\mathbf{q}_h, \mathbb{H}_\mathbf{q}(\phi)) + (\alpha\mathbf{q}_0, \mathbb{H}_0(\phi))_T\right\} = (f, \mathbb{H}_u(\phi)), \quad \forall\phi\in\Lambda_h^0. \tag{3.11}$$

Next, by using (3.9) and then (3.5) with $\mathbf{v} = \mathbb{H}_\mathbf{q}(\phi)$ we obtain

$$\begin{aligned}s_T(\mathbf{q}_h, \mathbb{H}_\mathbf{q}(\phi)) + (\alpha\mathbf{q}_0, \mathbb{H}_0(\phi))_T &= s_T(\mathbf{q}_h, \mathbb{H}_\mathbf{q}(\phi)) + (\alpha\mathbf{q}_0, \mathbb{H}_0(\phi))_T - (\nabla_w\cdot\mathbb{H}_\mathbf{q}(\phi), u_h)_T\\ &= -\langle\lambda_h, \mathbb{H}_b(\phi)\cdot\mathbf{n}\rangle_{\partial T}\\ &= s_T(\mathbb{H}_\mathbf{q}(\lambda_h), \mathbb{H}_\mathbf{q}(\phi)) + (\alpha\mathbb{H}_0(\lambda_h), \mathbb{H}_0(\phi))_T - (\nabla_w\cdot\mathbb{H}_\mathbf{q}(\phi), \mathbb{H}_u(\lambda_h))_T\\ &= s_T(\mathbb{H}_\mathbf{q}(\lambda_h), \mathbb{H}_\mathbf{q}(\phi)) + (\alpha\mathbb{H}_0(\lambda_h), \mathbb{H}_0(\phi))_T.\end{aligned}$$

Substituting the above equation into (3.11) yields

$$\sum_{T\in\mathcal{T}_h}\left\{s_T(\mathbb{H}_\mathbf{q}(\lambda_h), \mathbb{H}_\mathbf{q}(\phi)) + (\alpha\mathbb{H}_0(\lambda_h), \mathbb{H}_0(\phi))_T\right\} = (f, \mathbb{H}_u(\phi)), \quad \forall\phi\in\Lambda_h^0,$$

which is precisely Eq. (3.10). □

Concerning the solution existence and uniqueness for the reduced system (3.10), we have the following result.

Theorem 3.3. *There exists one and only one $\lambda_h \in \Lambda_h$ satisfying Eq. (3.10) and the boundary condition $\lambda_h = Q_b g$ on $\partial\Omega$.*

Proof. Since the number of unknowns equals the number of equations, it is sufficient to verify the solution uniqueness. To this end, let $\lambda_h^{(i)} \in \Lambda_h$ be two solutions of (3.10) satisfying the boundary condition $\lambda_h^{(i)} = Q_b g,\ i = 1, 2$. It is clear that their difference, denoted by $\sigma = \lambda_h^{(1)} - \lambda_h^{(2)}$, is a function in Λ_h^0 and satisfies

$$\sum_{T\in\mathcal{T}_h} \left\{ s_T(\mathbb{H}_{\mathbf{q}}(\sigma), \mathbb{H}_{\mathbf{q}}(\phi)) + (\alpha\mathbb{H}_0(\sigma), \mathbb{H}_0(\phi))_T \right\} = 0, \quad \forall \phi \in \Lambda_h^0.$$

By setting $\phi = \sigma$ we arrive at

$$s_T(\mathbb{H}_{\mathbf{q}}(\sigma), \mathbb{H}_{\mathbf{q}}(\sigma)) + (\alpha\mathbb{H}_0(\sigma), \mathbb{H}_0(\sigma))_T = 0, \quad \forall\, T \in \mathcal{T}_h.$$

It follows that $\mathbb{H}_0(\sigma) = 0$ on each element T and $\mathbb{H}_b(\sigma)\cdot\mathbf{n} = \mathbb{H}_0(\sigma)\cdot\mathbf{n}$ on ∂T, which implies $\mathbb{H}_b(\sigma) = 0$ on ∂T. Now using (3.8) we have

$$(\nabla_w \cdot \mathbf{v}, \mathbb{H}_u(\sigma))_T = \langle \sigma, \mathbf{v}_b\cdot\mathbf{n}\rangle_{\partial T}, \quad \forall \mathbf{v} \in V_k(T),\ T \in \mathcal{T}_h. \tag{3.12}$$

The definition of the discrete weak divergence can be applied to the left-hand side to yield

$$-(\mathbf{v}_0, \nabla\mathbb{H}_u(\sigma))_T + \langle \mathbf{v}_b\cdot\mathbf{n}, \mathbb{H}_u(\sigma)\rangle_{\partial T} = \langle \sigma, \mathbf{v}_b\cdot\mathbf{n}\rangle_{\partial T}, \quad \forall \mathbf{v} \in V_k(T). \tag{3.13}$$

Thus, by setting $\mathbf{v}_b = 0$ and varying $\mathbf{v}_0$ we obtain

$$\nabla\mathbb{H}_u(\sigma) = 0 \Longrightarrow \mathbb{H}_u(\sigma) = const, \quad \text{in } T \in \mathcal{T}_h,$$

which, together with varying $\mathbf{v}_b$ in (3.13), leads to $\mathbb{H}_u(\sigma) = \sigma$ on ∂T. It follows that both $\mathbb{H}_u(\sigma)$ and σ are constants in the domain Ω and vanishes on the boundary. Hence, $\sigma = \mathbb{H}_u(\sigma) = 0$. This completes the proof of the theorem. □

4. Equivalence between HWG-MFEM and WG-MFEM

The goal of this section is to show that the hybridized WG-MFEM approximate solution arising from (3.5)–(3.7) coincides with the solution from the corresponding WG-MFEM scheme introduced as in [2]. To this end, consider the finite element space $\tilde{V}_h = \bigotimes_{T\in\mathcal{T}_h} V_k(T)$. For any interior edge/face $e \in \mathcal{E}_h^0$, denote by T_1 and T_2 the two elements that share e as a common side. Recall that functions $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in \tilde{V}_h$ have two-sided values on e for the component $\mathbf{v}_b$: one comes from the element T_1 and other from T_2. Define the jump of $\mathbf{v}$ along $e \in \mathcal{E}_h^0$ as follows

$$[\![\mathbf{v}]\!]_e = \mathbf{v}_b|_{\partial T_1}\cdot\mathbf{n}_1 + \mathbf{v}_b|_{\partial T_2}\cdot\mathbf{n}_2, \tag{4.1}$$

where $\mathbf{n}_i$ is the outward normal direction on e as seen from the element T_i, $i = 1, 2$.

Let $k \ge 0$ be any integer. Following [2], we introduce two weak finite element spaces

$$V_h = \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} :\ \mathbf{v} \in \tilde{V}_h,\ [\![\mathbf{v}]\!]_e = 0,\ \forall e \in \mathcal{E}_h^0\},$$
$$W_h = \bigotimes_{T\in\mathcal{T}_h} W_{k+1}(T).$$

On each element $T \in \mathcal{T}_h$, we introduce three bilinear forms

$$a_T(\mathbf{r},\ \mathbf{v}) = (\alpha\mathbf{r}_0, \mathbf{v}_0)_T, \tag{4.2}$$
$$b_T(\mathbf{v},\ w) = (\nabla_w\cdot\mathbf{v},\ w)_T, \tag{4.3}$$
$$c_T(\mathbf{v},\ \sigma) = \langle \mathbf{v}_b\cdot\mathbf{n},\ \sigma\rangle_{\partial T}, \tag{4.4}$$

for $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\}$, $\mathbf{r} = \{\mathbf{r}_0, \mathbf{r}_b\} \in V_k(T)$, $w \in W_{k+1}(T)$ and $\sigma \in \Lambda_h$. In addition, set

$$a_{s,T}(\mathbf{r},\ \mathbf{v}) = a_T(\mathbf{r},\ \mathbf{v}) + s_T(\mathbf{r},\ \mathbf{v}).$$

WG-MFEM Scheme 4.1 ([2]). *For an approximation of the solution of (1.1)–(1.2), find $\bar{\mathbf{q}}_h = \{\bar{\mathbf{q}}_0, \bar{\mathbf{q}}_b\} \in V_h$ and $\bar{u}_h \in W_h$ such that*

$$\sum_{T\in\mathcal{T}_h} a_{s,T}(\bar{\mathbf{q}}_h, \mathbf{v}) - \sum_{T\in\mathcal{T}_h} b_T(\mathbf{v}, \bar{u}_h) = -\langle g,\ \mathbf{v}_b\cdot\mathbf{n}\rangle_{\partial\Omega}, \quad \forall \mathbf{v} \in V_h \tag{4.5}$$

$$\sum_{T\in\mathcal{T}_h} b_T(\bar{\mathbf{q}}_h, w) = (f,\ w), \quad \forall w \in W_h. \tag{4.6}$$

The following theorem shows that the WG-MFEM is equivalent to the hybridized WG-MFEM.

Theorem 4.2. *Let $(\mathbf{q}_h, u_h, \lambda_h) \in \mathcal{M}_h \times \Lambda_h$ be the hybridized WG-MFEM approximation of (1.1)–(1.2) obtained from (3.5)–(3.7), and $(\bar{\mathbf{q}}_h, \bar{u}_h) \in V_h \times W_h$ be the WG-MFEM approximation of (1.1)–(1.2) arising from (4.5)–(4.6). Then we have $\mathbf{q}_h = \bar{\mathbf{q}}_h$ and $u_h = \bar{u}_h$. Moreover, the hybridized WG-MFEM scheme (3.5)–(3.7) has one and only one solution.*

Proof. Assume that $(\mathbf{q}_h, u_h, \lambda_h) \in \mathcal{M}_h \times \Lambda_h$ is a solution of (3.5)–(3.7). From Eq. (3.7) we see that $[\![\mathbf{q}_h]\!] = 0$ on each interior edge $e \in \mathcal{E}_h^0$. Hence, $\mathbf{q}_h \in V_h$. Moreover, for any $\mathbf{v} \in V_h$ so that $[\![\mathbf{v}_b]\!]|_e = 0$ with $e \in \mathcal{E}_h^0$, we have

$$\sum_{T\in\mathcal{T}_h} \langle \lambda_h, \mathbf{v}_b \cdot \mathbf{n}\rangle_{\partial T} = \langle g, \mathbf{v}_b \cdot \mathbf{n}\rangle_{\partial\Omega}. \tag{4.7}$$

Therefore, for $\mathbf{v} \in V_h$ and $w \in W_h$, Eqs. (3.5)–(3.6) leads to

$$\sum_{T\in\mathcal{T}_h} a_{s,T}(\mathbf{q}_h, \mathbf{v}) - \sum_{T\in\mathcal{T}_h} b_T(\mathbf{v}, u_h) = -\langle g, \mathbf{v}_b \cdot \mathbf{n}\rangle_{\partial\Omega},$$
$$\sum_{T\in\mathcal{T}_h} b_T(\mathbf{q}_h, w) = (f,\ w),$$

which shows that $(\mathbf{q}_h, u_h)$ is a solution for the WG-MFEM. The solution existence and uniqueness for the WG-MFEM [2] then implies $\mathbf{q}_h = \bar{\mathbf{q}}_h$ and $u_h = \bar{u}_h$.

To show that the hybridized WG-MFEM scheme (3.5)–(3.7) has a unique solution, assume that $(\mathbf{q}_h^{(i)}, u_h^{(i)}, \lambda_h^{(i)}) \in \mathcal{M}_h \times \Lambda_h,\ i = 1, 2$, are two solutions. Their difference,

$$(\mathbf{q}_h, u_h, \lambda_h) := (\mathbf{q}_h^{(1)} - \mathbf{q}_h^{(2)}, u_h^{(1)} - u_h^{(2)}, \lambda_h^{(1)} - \lambda_h^{(2)}),$$

must satisfy (3.5)–(3.7) with $f \equiv 0$ and $g \equiv 0$. From the first part of this theorem, we see that

$$\mathbf{q}_h = 0, \qquad u_h = 0.$$

Thus, from (3.5) we have

$$0 = \langle \lambda_h, \mathbf{v}_b \cdot \mathbf{n}\rangle_{\partial T}, \quad \forall \mathbf{v} \in V_k(T),\ T \in \mathcal{T}_h,$$

which implies $\lambda_h = 0$ on ∂T for any $T \in \mathcal{T}_h$. This completes the proof of the theorem. □

The hybridized WG-MFEM scheme (3.5)–(3.7) consists of two parts: a local system (3.5)–(3.6) defined on each element $T \in \mathcal{T}_h$ and a global system (3.7). This scheme has a reformulation of (3.10) which involves only the Lagrange multiplier λ_h. It should be pointed out that the reduced system (3.10) is symmetric and positive definite. Since the Lagrange multiplier λ_h is defined only on the element boundary as its unknowns, the size of the linear system for the hybridized WG-MFEM is significantly smaller than the one for the WG-MFEM scheme. Thus, the hybridized WG-MFEM is an efficient implementation of the WG-MFEM scheme proposed and analyzed in [2].

5. Error estimates

Owing to the equivalence between the hybridized WG-MFEM scheme (3.5)–(3.7) and the WG-MFEM scheme of [2], all the error estimates developed in [2] can be applied to the approximate solution $(\mathbf{q}_h, u_h)$ obtained from (3.5)–(3.7). What remains to study is the convergence and error estimate for the Lagrange multiplier λ_h.

On each element $T \in \mathcal{T}_h$, denote by Q_0 the L^2 projection from $[L^2(T)]^d$ to $[P_k(T)]^d$, by Q_b the L^2 projection from $L^2(e)$ to $P_k(e)$, and by $\mathbb{Q}_h$ the L^2 projection from $L^2(T)$ onto $P_{k+1}(T)$. With the help of these operators, we define a projection operator $Q_h : V(T) \to V_k(T)$ so that for any $\mathbf{v} = \{\mathbf{v}_0, v_b\mathbf{n}\} \in V(T)$

$$Q_h\mathbf{v} = \{Q_0\mathbf{v}_0, (Q_b v_b)\mathbf{n}\}. \tag{5.1}$$

In the finite element space W_h, we introduce the following norm:

$$\|w\|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} \|\nabla w\|_T^2 + h^{-1}\sum_{e\in\mathcal{E}_h} \|[\![w]\!]\|_e^2. \tag{5.2}$$

In the space V_h, we use

$$|\!|\!|\mathbf{v}|\!|\!|^2 = \sum_{T\in\mathcal{T}_h} a_{s,T}(\mathbf{v}, \mathbf{v}).$$

For simplicity of notation, we shall use $\lesssim$ to denote "*less than or equal to*" up to a constant independent of the mesh size, variables, or other parameters appearing in the inequalities of this section.

Theorem 5.1 ([2]). *Let $(\mathbf{q}_h, u_h, \lambda_h) \in \mathcal{M}_h \times \Lambda_h$ be the approximate solution of (1.1)–(1.2) arising from the hybridized WG-MFEM scheme (3.5)–(3.7) of order $k \geq 0$. Assume that the exact solution $(\mathbf{q}, u)$ of (1.3)–(1.4) is regular such that $u \in H^{k+2}(\Omega)$ and $\mathbf{q} \in [H^{k+1}(\Omega)]^d$. Then, we have*

$$|||\mathbf{q}_h - Q_h\mathbf{q}||| + \|u_h - \mathbb{Q}_h u\|_{1,h} \lesssim h^{k+1} \left(\|u\|_{k+2} + \|\mathbf{q}\|_{k+1}\right). \tag{5.3}$$

If, in addition, the problem (1.1) with homogeneous Dirichlet boundary condition $u = 0$ has the usual H^2-regularity, then the following optimal order error estimate in L^2 holds true:

$$\|u_h - \mathbb{Q}_h u\| \lesssim h^{k+2} \left(\|u\|_{k+2} + \|\mathbf{q}\|_{k+1}\right). \tag{5.4}$$

For any $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_k(T)$ and $w \in H^1(T)$, using the definition of $\mathbb{Q}_h$ and the integration by parts we obtain

$$\begin{aligned}
(\nabla_w \cdot \mathbf{v},\ \mathbb{Q}_h w)_T &= -(\mathbf{v}_0,\ \nabla(\mathbb{Q}_h w))_T + \langle \mathbf{v}_b \cdot \mathbf{n},\ \mathbb{Q}_h w\rangle_{\partial T} \\
&= (\nabla \cdot \mathbf{v}_0,\ \mathbb{Q}_h w)_T + \langle \mathbf{v}_b \cdot \mathbf{n} - \mathbf{v}_0 \cdot \mathbf{n},\ \mathbb{Q}_h w\rangle_{\partial T} \\
&= (\nabla \cdot \mathbf{v}_0,\ w)_T + \langle \mathbf{v}_b \cdot \mathbf{n} - \mathbf{v}_0 \cdot \mathbf{n},\ \mathbb{Q}_h w\rangle_{\partial T} \\
&= -(\mathbf{v}_0,\ \nabla w)_T + \langle \mathbf{v}_0 \cdot \mathbf{n},\ w\rangle_{\partial T} + \langle \mathbf{v}_b \cdot \mathbf{n} - \mathbf{v}_0 \cdot \mathbf{n},\ \mathbb{Q}_h w\rangle_{\partial T} \\
&= -(\mathbf{v}_0,\ \nabla w)_T + \langle \mathbf{v}_0 \cdot \mathbf{n} - \mathbf{v}_b \cdot \mathbf{n},\ w - \mathbb{Q}_h w\rangle_{\partial T} + \langle \mathbf{v}_b \cdot \mathbf{n},\ w\rangle_{\partial T}.
\end{aligned} \tag{5.5}$$

By introducing the following notation

$$\ell_T(w, \mathbf{v}) = \langle \mathbf{v}_0 \cdot \mathbf{n} - \mathbf{v}_b \cdot \mathbf{n},\ \mathbb{Q}_h w - w\rangle_{\partial T},$$

we have from (5.5) that

$$(\nabla_w \cdot \mathbf{v},\ \mathbb{Q}_h w)_T = -(\mathbf{v}_0,\ \nabla w)_T + \langle \mathbf{v}_b \cdot \mathbf{n},\ w\rangle_{\partial T} + \ell_T(w, \mathbf{v}). \tag{5.6}$$

Lemma 5.2. *Let $(\mathbf{q}_h, u_h, \lambda_h) \in \mathcal{M}_h \times \Lambda_h$ be the approximate solution of (1.1)–(1.2) arising from the hybridized WG-MFEM scheme (3.5)–(3.7) of order $k \geq 0$. On each element $T \in \mathcal{T}_h$, the following identity holds true*

$$s_T(\mathbf{e}_h, \mathbf{v}) + (\alpha \mathbf{e}_0, \mathbf{v}_0)_T - (\nabla_w \cdot \mathbf{v}, \epsilon_h)_T + \langle \delta_h, \mathbf{v}_b \cdot \mathbf{n}\rangle_{\partial T} = s_T(Q_h\mathbf{q}, \mathbf{v}) + \ell_T(u, \mathbf{v}), \quad \forall \mathbf{v} \in V_k(T), \tag{5.7}$$

where

$$\mathbf{e}_h = \{\mathbf{e}_0, \mathbf{e}_b\} = Q_h\mathbf{q} - \mathbf{q}_h, \qquad \epsilon_h = \mathbb{Q}_h u - u_h, \qquad \delta_h = Q_b u - \lambda_h,$$

stand for the error between the hybridized WG-MFEM approximate solution and the L^2 projection of the exact solution.

Proof. For any $\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \in V_k(T)$, we test the first equation of (1.1) against $\mathbf{v}_0$ to obtain

$$(\alpha\mathbf{q}, \mathbf{v}_0)_T + (\nabla u, \mathbf{v}_0)_T = 0.$$

By applying the identity (5.6) to the term $(\nabla u, \mathbf{v}_0)_T$ we arrive at

$$(\alpha\mathbf{q}, \mathbf{v}_0)_T - (\nabla_w \cdot \mathbf{v}, \mathbb{Q}_h u)_T + \langle \mathbf{v}_b \cdot \mathbf{n}, u\rangle_{\partial T} = \ell_T(u, \mathbf{v}).$$

Adding $s_T(Q_h\mathbf{q}, \mathbf{v})$ to both sides of the above equation and then using the definition of Q_h, we have

$$s_T(Q_h\mathbf{q}, \mathbf{v}) + (\alpha(Q_0\mathbf{q}), \mathbf{v}_0)_T - (\nabla_w \cdot \mathbf{v}, \mathbb{Q}_h u)_T + \langle \mathbf{v}_b \cdot \mathbf{n}, u\rangle_{\partial T} = \ell_T(u, \mathbf{v}) + s_T(Q_h\mathbf{q}, \mathbf{v}).$$

Now subtracting (3.5) from the above equation gives (5.7). This completes the proof of the lemma. □

Let $T \in \mathcal{T}_h$ be an element with e as an edge/face. For any function $\varphi \in H^1(T)$, the following trace inequality has been derived for general polyhedral partitions satisfying the shape regularity assumptions **A1**–**A4** (see [2] for details):

$$\|\varphi\|_e^2 \lesssim \left(h_T^{-1}\|\varphi\|_T^2 + h_T\|\nabla\varphi\|_T^2\right). \tag{5.8}$$

Theorem 5.3. *Under the assumptions of Theorem 5.1, we have the following error estimate:*

$$\left(\sum_{T\in\mathcal{T}_h} h_T\|Q_b u - \lambda_h\|_{\partial T}^2\right)^{\frac{1}{2}} \lesssim h^{k+2}\left(\|u\|_{k+2} + \|\mathbf{q}\|_{k+1}\right). \tag{5.9}$$

Proof. By letting $\mathbf{v} = \{0, \delta_h\mathbf{n}\}$, we have $\mathbf{v}_0 = 0$ and $(\nabla_w\mathbf{v}, \epsilon_h)_T = \langle \delta_h, \epsilon_h\rangle_{\partial T}$. It follows from (5.7) that

$$\begin{aligned}
\|\delta_h\|_{\partial T}^2 &= \langle \delta_h, \mathbf{v}_b \cdot \mathbf{n}\rangle_{\partial T} \\
&= -s_T(\mathbf{e}_h, \mathbf{v}) + (\nabla_w \cdot \mathbf{v}, \epsilon_h)_T + s_T(Q_h\mathbf{q}, \mathbf{v}) + \ell_T(u, \mathbf{v}) \\
&= h_T\langle \mathbf{e}_0 \cdot \mathbf{n} - \mathbf{e}_b \cdot \mathbf{n}, \delta_h\rangle_{\partial T} + \langle \delta_h, \epsilon_h\rangle_{\partial T} - h_T\langle Q_0\mathbf{q} \cdot \mathbf{n} - Q_b(\mathbf{q} \cdot \mathbf{n}), \delta_h\rangle_{\partial T} - \langle \delta_h, \mathbb{Q}_h u - u\rangle_{\partial T}.
\end{aligned}$$

Thus, from the usual Cauchy–Schwarz inequality, we obtain

$$\|\delta_h\|_{\partial T} \lesssim h_T\|\mathbf{e}_0\cdot\mathbf{n}-\mathbf{e}_b\cdot\mathbf{n}\|_{\partial T}+\|\epsilon_h\|_{\partial T}+h_T\|Q_0\mathbf{q}\cdot\mathbf{n}-Q_b(\mathbf{q}\cdot\mathbf{n})\|_{\partial T}+\|\mathbb{Q}_h u-u\|_{\partial T}. \tag{5.10}$$

By first applying the trace inequality (5.8) to the last three terms on the right-hand side of (5.10), and then summing over all the element $T\in\mathcal{T}_h$ we obtain

$$\begin{aligned}\sum_{T\in\mathcal{T}_h} h_T\|\delta_h\|_{\partial T}^2 &\lesssim h^2\sum_{T\in\mathcal{T}_h} h_T\|\mathbf{e}_0\cdot\mathbf{n}-\mathbf{e}_b\cdot\mathbf{n}\|_{\partial T}^2+\sum_{T\in\mathcal{T}_h}(\|\epsilon_h\|_T^2+h_T^2\|\nabla\epsilon_h\|_T^2)\\ &\quad+h^2\sum_{T\in\mathcal{T}_h}(\|Q_0\mathbf{q}-\mathbf{q}\|_T^2+h_T^2\|\nabla(Q_0\mathbf{q}-\mathbf{q})\|_T^2)+\sum_{T\in\mathcal{T}_h}(\|\mathbb{Q}_h u-u\|_T^2+h_T^2\|\nabla(\mathbb{Q}_h u-u)\|_T^2)\\ &\lesssim h^2|\!|\!|\mathbf{e}_h|\!|\!|^2+\|\epsilon_h\|^2+h^2\|\epsilon_h\|_{1,h}^2+h^{2k+4}(\|\mathbf{q}\|_{k+1}^2+\|u\|_{k+2}^2).\end{aligned} \tag{5.11}$$

Combining the above inequality with the error estimates (5.3) and (5.4) yields (5.9). This completes the proof of the theorem. □

6. Numerical experiments

In this section, we present some numerical results for the hybridized WG-MFEM scheme (3.5)–(3.7) based on the lowest order element (i.e., $k=0$) for the second order elliptic problem (1.1)–(1.2). Recall that for the lowest order hybridized WG-MFEM, the corresponding finite element spaces are given by

$$\mathcal{M}_h=\bigotimes_{T\in\mathcal{T}_h} V_0(T)\times W_1(T),$$
$$\Lambda_h=\{\mu:\mu|_e\in P_0(e), e\in\mathcal{E}_h\}.$$

For any given $\mathbf{v}=\{\mathbf{v}_0,\mathbf{v}_b\}$, the discrete weak divergence $\nabla_w\cdot\mathbf{v}\in P_1(T)$ is computed locally on each element T as follows: Find $\nabla_w\cdot\mathbf{v}\in P_1(T)$ such that

$$(\nabla_w\cdot\mathbf{v},\tau)_T=-(\mathbf{v}_0,\nabla\tau)_T+\langle\mathbf{v}_b\cdot\mathbf{n},\tau\rangle_{\partial T},\quad\forall\tau\in P_1(T).$$

Let $(\mathbf{q}_h,u_h,\lambda_h)\in\mathcal{M}_h\times\Lambda_h$ be the hybridized WG-MFEM approximate solution arising from (3.5)–(3.7) and $(\mathbf{q},u)$ be the exact solution of (1.1)–(1.2), respectively. In our numerical experiments, we compare the numerical solutions with the L^2 projection of the exact solution in various norms for their difference:

$$\mathbf{e}_h=\{\mathbf{e}_0,\mathbf{e}_b\}=Q_h\mathbf{q}-\mathbf{q}_h,\qquad \epsilon_h=\mathbb{Q}_h u-u_h,\qquad \delta_h=Q_b u-\lambda_h.$$

In particular, the following norms are used to measure the scale of the error:

$$H^1\text{-norm}:\quad \|\epsilon_h\|_{1,h}=\left(\sum_{T\in\mathcal{T}_h}\|\nabla\epsilon_h\|_T^2+h^{-1}\sum_{e\in\mathcal{E}_h}\|[\![\epsilon_h]\!]\|_e^2\right)^{\frac{1}{2}},$$

$$L^2\text{-norm}:\quad |\!|\!|\mathbf{e}_h|\!|\!|=\left(\sum_{T\in\mathcal{T}_h}\int_T|\mathbf{e}_0|^2dx+h_T\int_{\partial T}|\mathbf{e}_0\cdot\mathbf{n}-\mathbf{e}_b\cdot\mathbf{n}|^2ds\right)^{\frac{1}{2}},$$

$$L^2\text{-norm}:\quad \|\delta_h\|=\left(\sum_{T\in\mathcal{T}_h}h_T\|\lambda_h-Q_b u\|_{\partial T\setminus\partial\Omega}^2\right)^{\frac{1}{2}},$$

$$L^2\text{-norm}:\quad \|\epsilon_h\|=\left(\sum_{T\in\mathcal{T}_h}\int_T|\epsilon_h|^2dx\right)^{\frac{1}{2}}.$$

6.1. Example 1

In this test case, the domain is the unit square $\Omega=(0,1)\times(0,1)$ and the coefficient is $\alpha=\frac{1}{(1+x)(1+y)}$. The exact solution is given by $u=\sin(\pi x)\sin(\pi y)$.

This numerical experiment was performed on uniform triangular partitions for the domain. The triangular partitions are constructed as follows: (1) first uniformly partition the domain into $n\times n$ sub-rectangles; (2) then divide each rectangular element by the diagonal line with a negative slope. The mesh size is denoted by $h=1/n$. Table 6.1 shows the convergence rate for the hybridized WG-MFEM solutions measured in different norms. The results show that the hybridized WG-MFEM approximates are convergent with rate $O(h)$ in H^1 and $O(h^2)$ in L^2 norms.

Recall that the primal variable u is approximated by the Lagrange multiplier λ_h as a piecewise constant function on the wired-basket $\mathcal{E}_h$. The error function δ_h, which is the difference of λ_h and the L^2 projection of the exact solution on each edge

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Table 6.1
Example 1. Convergence rate on triangular elements.

h	$\|\mathbf{e}_h\|$	Order	$\|\delta_h\|$	Order	$\|\epsilon_h\|_{1,h}$	Order	$\|\epsilon_h\|$	Order
1/4	2.93e−1	–	3.10e−2	–	1.22	–	1.91e−1	–
1/8	1.47e−1	0.99	8.17e−3	1.92	5.07e−1	1.27	4.76e−2	2.01
1/16	7.39e−2	1.00	2.07e−3	1.98	2.39e−1	1.08	1.19e−2	2.00
1/32	3.70e−2	1.00	5.20e−4	1.99	1.18e−1	1.02	2.97e−3	2.00
1/64	1.85e−2	1.00	1.30e−4	2.00	5.87e−2	1.01	7.42e−4	2.00
1/128	9.25e−3	1.00	3.25e−5	2.00	2.93e−2	1.00	1.86e−4	2.00

Table 6.2
Example 2. Convergence rate on rectangular elements.

h	$\|\mathbf{e}_h\|$	Order	$\|\delta_h\|$	Order	$\|\epsilon_h\|_{1,h}$	Order	$\|\epsilon_h\|$	Order
1/4	9.19e−1	–	1.86e−2	–	2.08	–	8.70e−1	–
1/8	4.84e−1	0.93	4.57e−3	2.03	2.09	–	2.85e−1	1.61
1/16	2.45e−1	0.98	1.14e−3	2.01	1.33	0.65	7.90e−2	1.85
1/32	1.23e−1	1.00	2.84e−4	2.00	7.29e−1	0.87	2.06e−2	1.94
1/64	6.15e−2	1.00	7.10e−5	2.00	3.78e−1	0.95	5.25e−3	1.97
1/128	3.08e−2	1.00	1.77e−5	2.00	1.92e−1	0.98	1.31e−3	2.00

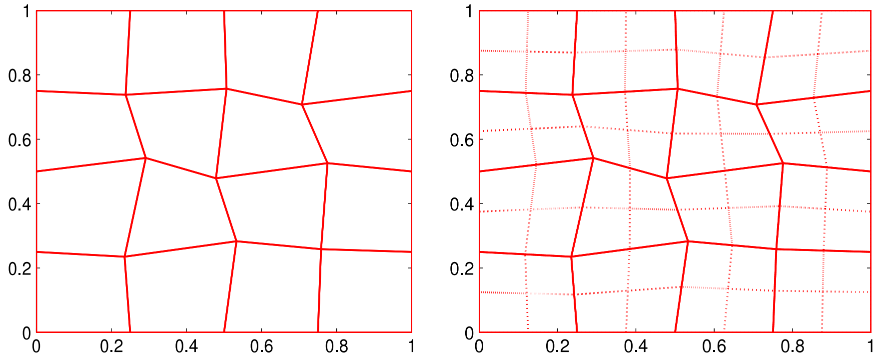

Fig. 6.1. Mesh level 1 (Left) and mesh level 2 (Right) for example 3.

$e \in \mathcal{E}_h$, is shown to be convergent at the rate of $O(h^2)$. Thus, the Lagrange multiplier is a superconvergent approximation of the exact solution at the mid-point of each edge $e \in \mathcal{E}_h$. This result is similar to the hybridized mixed finite element method.

6.2. Example 2

In our second test, the domain Ω is again the unit square. But the finite element partitions are given by the uniform rectangular meshes. The coefficient α is given by $\alpha = 1$. The data is chosen so that the exact solution is $u = \sin(\pi x)\cos(\pi y)$. The numerical results are presented in Table 6.2.

Table 6.2 shows the optimal rate of convergence for the numerical solution in H^1 and L^2 norms. Once again, we see a superconvergence for the solution on the wired-basket.

6.3. Example 3

The test problem here is the same as Example 2, but the finite element partitions consist of quadrilaterals constructed as follows. Starting with a coarse quadrilateral mesh shown as in Fig. 6.1 (Left), we successively refine each quadrilateral by connecting its barycenter with the middle points of its edges, shown as in Fig. 6.1 (Right). The numerical results are presented in Table 6.3. All the numerical results are in consistency with the theory developed in this paper.

6.4. Example 4

In this example, we shall compare the degree of freedoms(DOF) and CPU time for Scheme 4.1 (WGM method) and Scheme 3.1 (HWGM method) with linear weak Galerkin element for Example 1. Both of the algorithms are performed on the uniform triangular mesh with mesh size h. The DOF and CPU time are reported in Table 6.4. It shows that HWGM has much fewer DOF and also more efficient for solving PDEs. Mac pro Processor 2.8 GHz, Intel Core i7 and Memory 16 GB are used in the computation. Direct solver from MATLAB is used to solve the linear system.

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Table 6.3
Example 3. Convergence rate on quadrilateral elements.

h	$\|\|\|\mathbf{e}_h\|\|\|$	Order	$\|\delta_h\|$	Order	$\|\epsilon_h\|_{1,h}$	Order	$\|\epsilon_h\|$	Order
2.86e−1	8.89e−1	–	3.10e−2	–	2.03	–	8.08e−1	–
1.43e−1	4.74e−1	0.91	6.30e−3	2.30	1.95	0.56	2.71e−1	1.58
7.16e−2	2.42e−1	0.97	1.44e−3	2.13	1.27	0.62	7.66e−2	1.82
3.58e−2	1.21e−1	1.00	3.51e−4	2.04	7.08e−1	0.84	2.03e−2	1.92
1.79e−2	6.03e−2	1.01	8.71e−5	2.01	3.71e−1	0.93	5.25e−3	1.95
8.95e−3	2.95e−2	1.03	2.17e−5	2.00	1.90e−1	0.97	1.36e−3	1.95

Table 6.4
Example 4. Comparison of weak Galerkin mixed finite element methods (WGM) and hybridized weak Galerkin mixed finite element methods (HWGM) on triangular meshes.

h	WGM		HWGM	
	DOF	CPU time (s)	DOF	CPU time (s)
1/4	216	0.000689	40	0.000269
1/8	848	0.002440	176	0.000511
1/16	3 360	0.015474	736	0.001631
1/32	13 376	0.049909	3 008	0.005665
1/64	53 376	0.237016	12 160	0.029706
1/128	213 248	1.185962	48 896	0.14558

Table 6.5
Example 5. Convergence rate for lower regularity test with linear HWGM element.

h	$\|\|\|\mathbf{e}_h\|\|\|$	Order	$\|\epsilon_h\|_{1,h}$	Order	$\|\epsilon_h\|$	Order
$\gamma = 1$						
1/4	5.09e−2		1.81e−1		2.09e−2	
1/8	3.07e−2	7.30e−1	1.01e−1	8.39e−1	6.36e−3	1.71
1/16	1.72e−2	8.32e−1	5.40e−2	9.03e−1	1.82e−3	1.81
1/32	9.41e−3	8.75e−1	2.83e−2	9.33e−1	5.02e−4	1.85
1/64	5.05e−3	8.96e−1	1.47e−2	9.48e−1	1.36e−4	1.88
1/128	2.69e−3	9.10e−1	7.55e−3	9.57e−1	3.66e−5	1.90
$\gamma = 0.5$						
1/4	1.32e−1		4.70e−1		4.20e−2	
1/8	1.04e−1	3.45e−1	3.96e−1	2.47e−1	1.76e−2	1.26
1/16	7.68e−2	4.33e−1	3.04e−1	3.81e−1	6.75e−3	1.38
1/32	5.54e−2	4.70e−1	2.24e−1	4.42e−1	2.49e−3	1.44
1/64	3.96e−2	4.86e−1	1.61e−1	4.71e−1	9.00e−4	1.47
1/128	2.81e−2	4.93e−1	1.15e−1	4.86e−1	3.22e−4	1.48
$\gamma = 0.125$						
1/4	2.86e−1		8.52e−1		6.83e−2	
1/8	2.84e−1	1.01e−2	9.91e−1	–	3.80e−2	8.45e−1
1/16	2.69e−1	7.49e−2	1.01	–	1.90e−2	9.97e−1
1/32	2.51e−1	1.03e−1	9.77e−1	5.10e−2	9.11e−3	1.06
1/64	2.32e−1	1.15e−1	9.18e−1	8.87e−2	4.27e−3	1.10
1/128	2.13e−1	1.20e−1	8.53e−1	1.07e−1	1.98e−3	1.11

6.5. Example 5

In this example, we shall consider an example with a corner singularity. Here $\Omega = (0, 1) \times (0, 1)$, the permeability α is chosen as identity matrix, and the exact solution is

$$u(x, y) = x(1-x)y(1-y)r^{-2+\gamma}, \tag{6.1}$$

where $r = \sqrt{(x^2 + y^2)}$. Clearly, the exact solution admits a corner singularity at origin, and we have

$$u \in H_0^1(\Omega) \cap H^{1+\gamma-\epsilon}(\Omega), \ u \notin H^{1+\gamma}(\Omega),$$

where ϵ is any small positive number. For convenience, we set $s = \gamma - \epsilon$.

We apply the linear HWGM element for this experiment and the errors of $|||\mathbf{e}_h|||$, $\|\epsilon_h\|_{1,h}$ and $\|\epsilon_h\|$ are represented in Table 6.5. Because of the lower regularity in exact solution, it is expected that the L^2-error converges at order $O(h^{s+1})$ and the H^1-error converges at order $O(h^s)$. The error profiles and convergence rates in Table 6.5 agree very well with our theoretical expectations.

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