Accepted Manuscript

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PII: S0377-0427(16)30288-6
DOI: http://dx.doi.org/10.1016/j.cam.2016.06.017
Reference: CAM 10684

To appear in: *Journal of Computational and Applied Mathematics*

Received date: 3 November 2015
Revised date: 5 March 2016

Weighted Lupaş q-Bézier Curves

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Abstract

This paper is concerned with a new generalization of rational Bernstein-Bézier curves involving q-integers as shape parameters. A one parameter family of rational Bernstein-Bézier curves, weighted Lupaş q-Bézier curves, is constructed based on a set of Lupaş q-analogue of Bernstein functions which is proved to be a normalized totally positive basis. The generalized rational Bézier curve is investigated from a geometric point of view. The investigation provides the geometric meaning of the weights and the representation for conic sections. We also obtain degree evaluation and de Casteljau algorithms by means of homogeneous coordinates. Numerical examples show that weighted Lupaş q-Bézier curves have more modeling flexibility than classical rational Bernstein-Bézier curves and Lupaş q-Bézier curves, and meanwhile they provide better approximations to the control polygon than rational Phillips q-Bézier curves.

Keywords: Lupaş q-analogue of Bernstein operator, Weighted Lupaş q-Bernstein basis, Normalized totally positive basis, Rational Bézier curve, Conic sections, Degree elevation, de Casteljau algorithm, Shape parameter

1. Introduction

Bernstein polynomials and classical Bézier methods are of fundamental importance for parametric curves and surfaces modeling in Computer Aided

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Geometric Design($CAGD$). When representing a parametric curve or surface, it is important which basis functions are used if we wish to preserve the shape of the curve or surface. Totally positive bases present good shape preserving properties due to the variation diminishing properties of totally positive matrices. The shape of a parametrically defined curve mimics the shape of its control polygon when the corresponding blending functions form a totally positive system [1, 2]. Classical Bernstein polynomials of degree n form a normalized totally positivity basis of the polynomial space of degree n [3]. Classical rational Bézier curves and NURBS curves also have totally positive bases and geometric convexity preserving property [4]. Now there are many studies involving total positivity and the shape of curves [5, 6] and the Bernstein-Bézier form. In this paper, we are going to construct a kind of generalized rational Bernstein-Bézier curves with totally positive bases and including q-integers as shape parameters.

Recently, the rapid development of q-calculus has led to the discovery of new generalizations of Bernstein polynomials involving q-integers [7, 8]. The applications of q-calculus in the area of approximation theory were initiated by Lupaş [9], who first introduced the q-analogue of Bernstein polynomials, Lupaş q-analogue of Bernstein operators. Ten years later Phillips [10] introduced another generalization on Bernstein polynomials based on q-integers, q-Bernstein polynomials, which became popular not only in the area of approximation theory [11, 12, 13, 14] but also in CAGD. In 2003, Oruç and Phillips [15] constructed a generalized Bézier curve, Phillips q-Bézier curve, using q-Bernstein polynomials as basis and applied the concept of total positivity to investigate the shape properties of the curve. In 2007, Dişibüyük and Oruç generalized Phillips q-Bézier curves to their rational counterparts [16] based on a de Casteljau type algorithm and then in 2008 they defined tensor product q-Bernstein Bézier patches [17]. In 2012, Simeonova, Zafirisa and Goldman [18] established q-blossoming, recursive evaluation algorithms and recursive subdivision algorithms for Phillips q-Bézier curve. More intensively, Phillips q-Bézier curve are generalized to the q-analogues of classical B-splines, quantum B-splines [19], and several fundamental formulas for classical B-splines are extended to quantum B-splines [20]. Quantum splines are piecewise polynomials whose quantum derivatives up to some order agree at the joins. By putting a tolerance on the value of q, quantum splines allow us to model tolerances, jumps, and even quantum leaps in the derivatives at the joins.

Although Lupaş q-analogue of Bernstein operators are the first q-analogue of Bernstein operators, which reduce to the classical Bernstein polynomial when $q = 1$, they are less known [21, 22, 23, 24] than q-Bernstein polynomials. However, Lupaş q-analogue of Bernstein operators have an advantage of generating positive linear operators for all $q > 0$, whereas q-Bernstein polynomials generate positive linear operators only if $q \in (0, 1)$. In 2010, Phillips [25] indicated that he was not aware of any work on the practical application of Lupaş q-analogue of Bernstein operator in CAGD and hoped that someone will pursue that. In 2014, Han, Chu and Qiu [26] introduced Lupaş q-Bézier curves and obtained their degree evaluation and de Casteljau algorithms. Lupaş q-Bézier curves are generalized Bézier curves based on Lupaş q-analogue of Bernstein operators and share many properties with Bézier curves, such as end-point property of interpolation and derivative, variation diminishing property, etc. In this paper, in order to represent conic sections exactly, we construct a one parameter family of rational Bernstein-Bézier curves by adding weights to Lupaş q-analogue of Bernstein operators. We call them *weighted* Lupaş q-Bézier curves rather than *rational* Lupaş q-Bézier curves, because Lupaş q-analogue of Bernstein operators are rational functions rather than polynomials as q-Bernstein polynomials are when $q \neq 1$. The shape-preserving properties of the new curves are gained through the view of normalized totally positive bases as well.

Let us now recall some preliminaries. Given a real number $q > 0$ and any non-negative integer k, we define $[k]$ as

$$[k] = \begin{cases} (1 - q^k)/(1 - q), & q \neq 1, \\ k, & q = 1, \end{cases}$$

and call $[k]$ a *q-integer*. Note that $[k]$ is a continuous function of q. We next define $[k]!$ as

$$[k]! = \begin{cases} [k][k-1]\cdots[1], & k \geq 1, \\ 1, & k = 0, \end{cases}$$

and call $[k]!$ a *q-factorial*. For integers $0 \leq k \leq n$, the *q-binomial coefficient* $\begin{bmatrix} n \\ k \end{bmatrix}$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n][n-1]\cdots[n-k+1]}{[k]!} = \frac{[n]!}{[k]![n-k]!},$$

and has the value 1 when k=0 and value 0 otherwise. It satisfies the Pascal-type relations:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix},$$

and

$$\begin{bmatrix} n \\ k \end{bmatrix} = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

In 1987, Lupaş introduced a wonderful generalization of the Bernstein polynomials involving q-integers but using rational functions rather than polynomials. Let $f \in C[0,1]$, linear operator $L_{n,q}(f;t) : C[0,1] \to C[0,1]$ is defined by

$$L_{n,q}(f;t) = \sum_{i=0}^{n} b_{n,i}(t;q) f_i, \tag{1}$$

where

$$f_i = f\Big(\frac{[i]}{[n]}\Big), b_{n,i}(t;q) = \frac{a_{n,i}(t;q)}{w_n(t;q)},$$

$$a_{n,i}(t;q) = \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-1)/2} t^i (1-t)^{n-i},$$

$$w_n(t;q) = \sum_{i=0}^{n} a_{n,i}(t;q) = \prod_{j=1}^{n} (1 - t + q^{j-1} t).$$

and $L_{n,q}(f;t)$ is called *Lupaş q-analogue of Bernstein operator*, $b_{n,i}(t;q)$ is Lupaş q-analogue of Bernstein functions of degree n.

The layout of this paper is as follows : In Section 2, we introduce weighted Lupaş q-analogue of Bernstein functions and prove they form a normalized totally positive (NTP) basis of a rational space with common denominator. By means of the NTP basis, we construct weighted Lupaş q-Bézier curves in Section 3 and obtain their basic properties, degree elevation formula and de Casteljau algorithms. In Section 4, we discuss the geometric meaning of weights and represent conic sections using weighted Lupaş q-Bézier curves. The effects on the shape of the curves by weights and q-integers are shown in Section 5. We close in Section 6 with a short summary of our work, along with a brief discussion of some promising problems for future research.

2. Weighted Lupaş q-Bernstein basis: a normalized totally positive basis

2.1. Weighted Lupaş q-analogue of Bernstein functions

By adding positive weights, we obtain weighted Lupaş q-analogue of Bernstein functions from Lupaş q-analogue of Bernstein functions.

Definition 2.1. Given a real number $q > 0$, $t \in [0, 1]$, and any positive real numbers $\omega_0, \omega_1, \cdots, \omega_n$, we define *weighted Lupaş q-analogue of Bernstein functions of degree n* as

$$r_{n,i}(t;q) = \frac{\omega_i a_{n,i}(t;q)}{\sum_{i=0}^{n} \omega_i a_{n,i}(t;q)}, i = 0, 1, \cdots, n. \quad (2)$$

where

$$a_{n,i}(t;q) = \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-1)/2} t^i (1-t)^{n-i}, i = 0, 1, \cdots, n. \quad (3)$$

From the properties of Lupaş q-analogue of Bernstein functions [26], we can obtain the properties of weighted Lupaş q-analogue of Bernstein function immediately as follows:

- ***Non-negative*:** $r_{n,i}(t;q) \geq 0, \quad i = 0, 1, \cdots, n, \quad t \in [0, 1]$.
- ***Partition of unity*:** $\sum_{i=0}^{n} r_{n,i}(t;q) = 1, \quad t \in [0, 1]$.
- ***End-point property*:**
  $r_{n,i}(0;q) = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0; \end{cases}$ and $r_{n,i}(1;q) = \begin{cases} 1, & i = n, \\ 0, & i \neq n. \end{cases}$
- ***q-inverse symmetry*:** When $\omega_i = \omega_{n-i}$, for $i = 0, 1, \cdots, n$, we have $r_{n,n-i}(t;q) = r_{n,i}(1-t; 1/q)$.
- ***Reducibility*:** When all of the weights $\omega_i = \omega \neq 0$ $(i = 0, 1, \cdots, n)$, formula (2) reduces to Lupaş q-analogue of Bernstein function; when $q = 1$, formula (2) reduces to the classical rational Bernstein functions.

Figure 1 shows the quadratic weighted Lupaş q-analogue of Bernstein functions with $q = 0.2$ and $q = 5$, respectively: $\omega_0 = \omega_1 = \omega_2 = 1$ for blue lines and $\omega_0 = \omega_2 = 1, \omega_1 = 2$ for red dashed lines.

Let $\Omega_n(t;q)$ denote the denominator of weighted Lupaş q-analogue of Bernstein function of degree n, and $B_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}$ be the classical Bernstein polynomials of degree n, we have

$$\omega_i a_{n,i}(t;q) = \lambda_{n,i} B_{n,i}(t), \quad (4)$$

where

$$\lambda_{n,i} = \frac{\omega_i}{\binom{n}{i}} \Big(\sum_{\substack{K \cup L = 1,2,\cdots,n \\ |K|=(n-i), |L|=i}} \prod_{k \in K} 1 \prod_{l \in L} q^{l-1} \Big), i = 0, 1, \cdots, n.$$

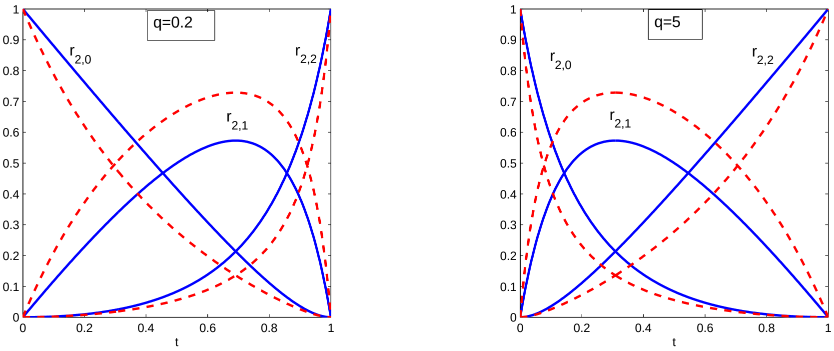

Figure 1: Quadratic weighted Lupaş q-analogue of Bernstein functions with $q = 0.2$ and $q = 5$.

Then weighted Lupaş q-analogue of Bernstein function of degree n can be written as

$$r_{n,i}(t;q) = \frac{\lambda_{n,i} B_{n,i}(t)}{\Omega_n(t;q)}, i = 0, 1, \cdots, n. \tag{5}$$

Thus, we can use weighted Lupaş q-analogue of Bernstein functions to span the space of rational functions of degree n with the same denominator $\Omega_n(t;q)$,

$$\mathbb{R}_n = span\Big\{r_{n,i}(t;q)|i = 0, 1, \cdots, n\Big\} = \Big\{P(t)/\Omega_n(t;q)|P(t) \in \mathbb{P}_n\Big\}, \tag{6}$$

where $\mathbb{P}_n$ is the space of polynomials of degree n. Since weighted Lupaş q-analogue of Bernstein functions of degree n are linearly independent, they are the basis of the space $\mathbb{R}_n$.

2.2. *Total positivity of weighted Lupaş q-analogue of Bernstein basis functions*

In order to discuss the total positivity of weighted Lupaş q-analogue of Bernstein functions, we first recall some definitions [27] about totally positive matrix, totally positive function sequences, and totally positive basis.

Definition 2.2. A real matrix A is called *totally positive* (respectively, *strictly totally positive*) if all its minors are nonnegative (respectively, positive), that is,

$$A\begin{pmatrix} i_1, & i_2, & \cdots, & i_k \\ j_1, & j_2, & \cdots, & j_k \end{pmatrix} = det\begin{bmatrix} a_{i_1,j_1} & \cdots & a_{i_1,j_k} \\ \cdots & \cdots & \cdots \\ a_{i_k,j_1} & \cdots & a_{i_k,j_k} \end{bmatrix} \geq 0\,(resp., > 0), \tag{7}$$

for all $i_1 < i_2 < \cdots < i_k$ and $j_1 < j_2 < \cdots < j_k$.

Karlin [28] pointed out that whether a matrix A is strictly totally positive by testing the positivity of only those minors that are formed from consecutive rows and columns, rather than having to examine all minors. That is, a real matrix A is strictly totally positive if

$$A\begin{pmatrix} i, & i+1, & i+2, & \cdots, & i+k \\ j, & j+1, & j+2, & \cdots, & j+k \end{pmatrix} > 0,$$

for all i, j and k. A better criterion was given in [29] and a criterion for nonsingular totally positive matrices can be seen in [30].

Definition 2.3. We call that a sequence $\Big(U_0(t), U_1(t)\cdots, U_n(t)\Big)$ of real-valued functions on an interval I is *totally positive* if, for any points $0 < t_0 < t_1 < \cdots < t_n$, the collocation matrix $\Big(U_j(t_i)\Big)_{i,j=0}^{n}$ is totally positive. When the totally positive functions $\Big(U_0(t), U_1(t)\cdots, U_n(t)\Big)$ are linearly independent, we refer to them as a *totally positive basis* (TP basis); if the totally positive basis satisfy $\sum_{i=0}^{n} U_i(t) = 1$, we refer to it as a *normalized totally positive basis* (NTP basis).

Then we obtain the total positivity of weighted Lupaş q-analogue of Bernstein basis functions directly from the definition of NTP basis.

Theorem 2.1. Given a real number $q > 0$, $t \in [0, 1]$, weighted Lupaş q-analogue of Bernstein basis functions $r_{n,0}(t; q), r_{n,1}(t; q), \cdots, r_{n,n}(t; q)$ form a normalized totally positive basis of the rational function space $\mathbb{R}^n$.

Proof. For $0 \le t_0 < t_1 < \cdots < t_n \le 1$, let A_n denote the collocation matrix $\Big\{r_{n,j}(t_i; q)\Big\}_{i,j=0}^{n}$, that is

$$A_n = \begin{pmatrix} r_{n,0}(t_0; q) & r_{n,1}(t_0; q) & \cdots & r_{n,n}(t_0; q) \\ r_{n,0}(t_1; q) & r_{n,1}(t_1; q) & \cdots & r_{n,n}(t_1; q) \\ \vdots & \vdots & \vdots & \vdots \\ r_{n,0}(t_n; q) & r_{n,1}(t_n; q) & \cdots & r_{n,n}(t_n; q) \end{pmatrix}. \tag{8}$$

In order to get totally positive basis, we need to prove A_n is a totally positive matrix for any n. We use induction on n. The result holds for $n = 1$, since all its elements are nonnegative and

$$det(A_1) = \frac{\omega_0}{\omega_0(1-t_0)+\omega_1 t_0} \cdot \frac{\omega_1}{\omega_0(1-t_1)+\omega_1 t_1} \cdot (t_1 - t_0) \geq 0.$$

Let us assume that the matrix A_n is totally positive for some $n \geq 2$, that is to say, all its elements and its minors are nonnegative.

For A_{n+1}, all its elements and the minors of order $k(2 \leq k \leq n)$ are nonnegative. Further,

$$det(A_{n+1}) = \rho_1 \cdot \rho_2 \cdot \rho_3 \cdot det(D), \tag{9}$$

where

$$\rho_1 = \prod_{j=0}^{n+1} \frac{\omega_j}{\sum_{i=0}^{n+1} \omega_i a_{n+1,i}(t_i;q)} \geq 0,$$

$$\rho_2 = \begin{bmatrix} n+1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} n+1 \\ 1 \end{bmatrix} \cdot \dots \cdot \begin{bmatrix} n+1 \\ n+1 \end{bmatrix} \geq 0,$$

$$\rho_3 = q^0 \cdot q^1 \cdot \dots \cdot q^{n(n+1)/2} \geq 0$$

and

$$D = \begin{pmatrix} (1-t_0)^{n+1} & t_0(1-t_0)^n & \cdots & t_0^{n+1} \\ (1-t_1)^{n+1} & t_1(1-t_1)^n & \cdots & t_1^{n+1} \\ \vdots & \vdots & \vdots & \vdots \\ (1-t_{n+1})^{n+1} & t_{n+1}(1-t_{n+1})^n & \cdots & t_{n+1}^{n+1} \end{pmatrix}.$$

We know that the matrix D is a collocation matrix of basis functions $(1-t)^{n+1}, t(1-t)^n, \cdots, t^{n+1}$ about any points $0 \leq t_0 < t_1 < \cdots < t_n < t_{n+1} \leq 1$, and so the matrix D is a totally positive matrix, $det(D) \geq 0$, thus $det(A_{n+1}) \geq 0$, matrix A_{n+1} is a totally positive matrix. To sum up, the collocation matrix A_n is a totally positive matrix for arbitrary n.

Since $\sum_{i=0}^{n} r_{n,i}(t;q) = 1$, weighted Lupaş q-analogue of Bernstein basis function $r_{n,0}(t;q), r_{n,1}(t;q), \cdots, r_{n,n}(t;q)$ form a normalized totally positivity basis of the rational function space $\mathbb{R}_n$. □

Remark 2.1. From Theorem 2.1 and the reducibility of weighted Lupaş q-analogue of Bernstein functions of degree n, when all of weights $\omega_i = \omega \neq 0$ $(i = 0, 1, \cdots, n)$, we obtain that Lupaş q-analogue of Bernstein basis is a NTP basis. Since the curves constructed by a NTP basis have the variation diminishing property [32], Lupaş q-Bézier curves are variation diminishing, which had been proved in [26]. Consequently, they are convexity-preserving and monotonicity-preserving as well.

By means of the total positivity of weighted Lupaş q-analogue of Bernstein functions, we can gain these shape-preserving properties of weighted Lupaş q-Bézier curves in the following section.

3. Weighted Lupaş q-Bézier curves

3.1. Definition and some basic properties

Definition 3.1. Given $n+1$ vectors $\mathbf{P_i} \in \mathcal{R}^3 (i = 0, 1, \cdots, n)$ a real number $q > 0$, and real positive numbers $\omega_0, \omega_1, \cdots, \omega_n$, we define *weighted Lupaş q-Bézier curve of degree n* as

$$R(t; q) = \sum_{i=0}^{n} \mathbf{P_i} r_{n,i}(t; q), 0 \leq t \leq 1. \tag{10}$$

$\mathbf{P_i}$ $(i = 0, 1, 2, \cdots, n)$ are *control points* and can form a *control polygon* by adjacently joining up. $\omega_i \, (i = 0, 1, 2, \cdots, n)$ are *weights* which can bring more flexibility to curve modeling.

Weighted Lupaş q-Bézier curve inherits the following properties of Lupaş q-Bézier curve:

- ***Geometrical invariant and affine invariant***.
- ***Convex hull property***: $R(t; q)$ lies in the convex hull of its control polygon.
- ***The end-point interpolation property***:

$$R(0; q) = \mathbf{P_0}, R(1; q) = \mathbf{P_n}.$$

- ***q-inverse symmetry***: When $\omega_i = \omega_{n-i}$, the weighted Lupaş q-Bézier curve obtained by reversing the order of the control points is the same as the weighted Lupaş q-Bézier curve just by altering the parameter as $1/q$.

- ***Reducibility***: when all of weights $\omega_i = \omega \neq 0$ $(i = 0, 1, \cdots, n)$, formula (10) reduces to Lupaş q-Bézier curve; when $q = 1$, formula (10) reduces to classical rational Bézier curve.

Moreover, weighted Lupaş q-Bézier curve shares the end-point property of derivative with classical rational Bézier curve and Lupaş q-Bézier curve.

Theorem 3.1. The end-point property of derivative:

$$R'(0;q) = \frac{[n]\omega_1(\mathbf{P_1} - \mathbf{P_0})}{\omega_0}, R'(1;q) = \frac{[n]\omega_{n-1}(\mathbf{P_n} - \mathbf{P_{n-1}})}{\omega_n q^{n-1}}.$$

Proof. Let

$$\mathbf{R}(t;q) = \textstyle\sum_{i=0}^{n} \mathbf{P_i} r_{n,i}(t;q) = \frac{\sum_{i=0}^{n} \omega_i a_{n,i}(t;q)\mathbf{P_i}}{\sum_{i=0}^{n} \omega_i a_{n,i}(t;q)} \triangleq \frac{\mathbf{Q}(t;q)}{\mathbf{W}(t;q)},$$

where $a_{n,i}(t;q) = \begin{bmatrix} n \\ i \end{bmatrix} q^{i(i-1)/2} t^i (1-t)^{n-i}, i = 0, 1, \cdots, n$, and obviously

$$a_{n,i}(0;q) = \begin{cases} 1, & i = 0, \\ 0, & i \neq 0, \end{cases} \quad a_{n,i}(1;q) = \begin{cases} q^{n(n-1)/2}, & i = n, \\ 0, & i \neq n. \end{cases}$$

Since

$$\mathbf{R}(t;q)\mathbf{W}(t;q) = \mathbf{Q}(t;q),$$

By evaluating the derivatives on both sides, we have

$$\mathbf{R}'(t;q)\mathbf{W}(t;q) + \mathbf{R}(t;q)\mathbf{W}'(t;q) = \mathbf{Q}'(t;q).$$

Because

$$\begin{aligned} [a_{n,i}(t;q)]' &= \frac{[n]}{[i]} \cdot \begin{bmatrix} n-1 \\ i-1 \end{bmatrix} \cdot q^{i(i-1)/2} \cdot i t^{i-1} (1-t)^{n-i} \\ &- \frac{[n]}{[n-i]} \cdot (n-i) \cdot \begin{bmatrix} n-1 \\ i \end{bmatrix} \cdot q^{i(i-1)/2} t^i (1-t)^{n-i-1} \\ &\triangleq c_{n,i} a_{n-1,i-1}(t;q) - d_{n,n-i} a_{n-1,i}(t;q), \end{aligned}$$

where $c_{n,i} = \frac{[n]}{[i]} \cdot q^{i-1} \cdot i, d_{n,n-i} = \frac{[n]}{[i]} \cdot i$,
so

$$\mathbf{Q}'(t;q) = \sum_{i=0}^{n} \omega_i a'_{n,i}(t;q)\mathbf{P_i} = \sum_{i=0}^{n-1} (\omega_{i+1} c_{n,i+1} \mathbf{P_{i+1}} - \omega_i d_{n,n-i} \mathbf{P_i}) a_{n-1,i}(t;q),$$

$$\mathbf{W}^{'}(t;q)=\sum_{i=0}^{n}\omega_i a_{n,i}^{'}(t;q)=\sum_{i=0}^{n-1}(\omega_{i+1}c_{n,i+1}-\omega_i d_{n,n-i})a_{n-1,i}(t;q).$$

Therefore we obtain our results

$$R^{'}(0;q)=\frac{[n]\omega_1(\mathbf{P_1}-\mathbf{P_0})}{\omega_0},R^{'}(1;q)=\frac{[n]\omega_{n-1}(\mathbf{P_n}-\mathbf{P_{n-1}})}{\omega_n q^{n-1}}.$$

□

Because weighted Lupaş q-Bézier curve has $\{r_{n,i}(t;q)\}_{i=0}^{n}$ as its NTP basis, we obtain the following variation diminishing property.

Theorem 3.2. Weighted Lupaş q-Bézier curves are variation diminishing, which means that the number of times any straight line crosses the weighted Lupaş q-Bézier curve is no more than the number of times it crosses the control polygon.

Consequently, a weighted Lupaş q-Bézier curve is convexity-preserving and monotonicity-preserving.

3.2. Degree elevation and de Casteljau algorithm

By means of homogeneous coordinate expression, the degree elevation algorithm and de Casteljau algorithm for a weighted Lupaş q-Bézier curve are straightforward generalizations of those for a Lupaş q-Bézier curve which are investigated in [26].

Degree elevation

Let $R(t;q)=\sum_{i=0}^{n}\mathbf{P_i}r_{n,i}(t;q)=\frac{\sum_{i=0}^{n}\omega_i\mathbf{P_i}a_{n,i}(t;q)}{\sum_{i=0}^{n}\omega_i a_{n,i}(t;q)}, 0\le t\le 1$, where

$$a_{n,i}(t;q)=\begin{bmatrix}n\\i\end{bmatrix}q^{i(i-1)/2}t^i(1-t)^{n-i}, i=0,1,\cdots,n,$$

then

$$R(t;q)=\frac{\sum_{i=0}^{n+1}\omega_i^*\mathbf{P_i^*}a_{n+1,i}(t;q)}{\sum_{i=0}^{n+1}\omega_i^* a_{n+1,i}(t;q)}, i=0,1,\cdots,n+1, \tag{11}$$

where

$$\omega_i^*=\Big(1-\frac{[n+1-i]}{[n+1]}\Big)\omega_{i-1}+\frac{[n+1-i]}{[n+1]}\omega_i,$$

$$\mathbf{P_i^*}=\Big(\Big(1-\frac{[n+1-i]}{[n+1]}\Big)\omega_{i-1}\mathbf{P_{i-1}}+\frac{[n+1-i]}{[n+1]}\omega_i\mathbf{P_i}\Big)\Big/\omega_i^*.$$

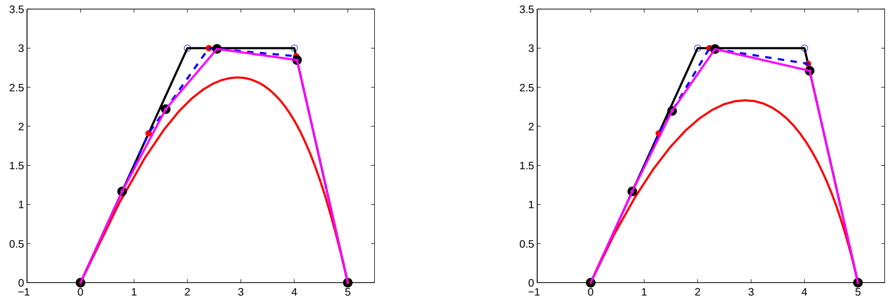

Figure 2: Degree elevation of cubic weighted Lupaş q-Bézier curves.

For $q = 0.5$, Figure 2 show degree elevation of cubic weighted Lupaş q-Bézier curve (left) with weights $\omega_0 = 1, \omega_1 = 2, \omega_2 = 1, \omega_3 = 1$ and cubic Lupaş q-Bézier curve (right).

de Casteljau algorithm

Weighted Lupaş q-Bézier curve of degree n can be written as two kinds of linear combination of two weighted Lupaş q-Bézier curves of degree $n-1$. The algorithms can be expressed as:

$$\begin{cases} \omega_i^0(t;q) \equiv \omega_i^0 \equiv \omega_i, i = 0, 1, \cdots, n, \\ \omega_i^k(t;q) = (1-t)\omega_i^{k-1}(t;q) + q^{n-k} t \omega_{i+1}^{k-1}(t;q), \\ k = 1, 2, \cdots, n, \quad i = 0, 1, \cdots, n-k. \end{cases}$$

$$\begin{cases} \mathbf{P_i}^0(t;q) \equiv \mathbf{P_i}^0 \equiv \mathbf{P_i}, i = 0, 1, \cdots, n, \\ \mathbf{P_i}^k(t;q) = \Big((1-t)\omega_i^{k-1}(t;q)\mathbf{P_i}^{k-1}(t;q) + q^{n-k} t \omega_{i+1}^{k-1}(t;q)\mathbf{P_{i+1}}^{k-1}(t;q)\Big) \Big/ \omega_i^k(t;q), \\ k = 1, 2, \cdots, n, \quad i = 0, 1, \cdots, n-k. \end{cases}$$

or

$$\begin{cases} \omega_i^0(t;q) \equiv \omega_i^0 \equiv \omega_i, i = 0, 1, \cdots, n, \\ \omega_i^k(t;q) = q^i(1-t)\omega_i^{k-1}(t;q) + q^i t \omega_{i+1}^{k-1}(t;q), \\ k = 1, 2, \cdots, n, \quad i = 0, 1, \cdots, n-k. \end{cases}$$

$$\begin{cases} \mathbf{P_i}^0(t;q) \equiv \mathbf{P_i}^0 \equiv \mathbf{P_i}, i = 0, 1, \cdots, n, \\ \mathbf{P_i}^k(t;q) = \Big(q^i(1-t)\omega_i^{k-1}(t;q)\mathbf{P_i}^{k-1}(t;q) + q^i t \omega_{i+1}^{k-1}(t;q)\mathbf{P_{i+1}}^{k-1}(t;q)\Big) \Big/ \omega_i^k(t;q), \\ k = 1, 2, \cdots, n, \quad i = 0, 1, \cdots, n-k. \end{cases}$$

then

$$R(t;q) = \frac{\sum_{i=0}^{n-1} \omega_i^1 \mathbf{P_i}^1 a_{n-1,i}(t;q)}{\sum_{i=0}^{n-1} \omega_i^1 a_{n-1,i}(t;q)} = \cdots = \frac{\sum_{i=0}^{n-k} \omega_i^k \mathbf{P_i}^k a_{n-k,i}(t;q)}{\sum_{i=0}^{n-k} \omega_i^k a_{n-k,i}(t;q)} = \cdots = \mathbf{P_0}^n(t;q). \tag{12}$$

For $q = 2$ and $t = 0.5$, Figure 3 show the de Casteljau algorithm of a cubic weighted Lupaş q-Bézier curve (left) with weights $\omega_0 = 1, \omega_1 = 2, \omega_2 = 2, \omega_3 = 1$ and a cubic Lupaş q-Bézier curve (right).

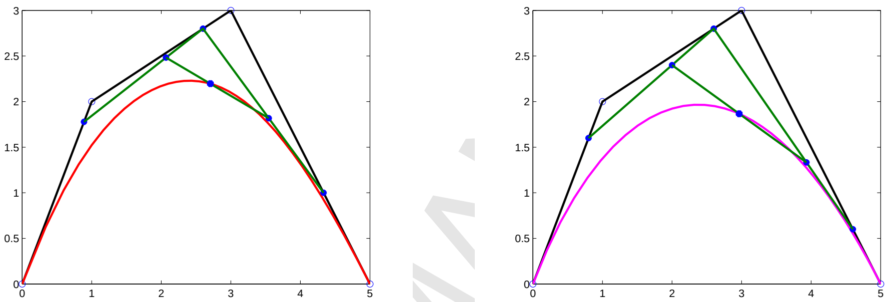

Figure 3: The de Casteljau algorithm of cubic weighted Lupaş q-Bézier curves.

4. Weights and conic section

In order to constitute NTP basis, the real weights $\omega_0, \omega_1, \cdots, \omega_n$ in the weighted Lupaş q-Bézier curve must be positive. However, except that the two end weights ω_0 and ω_n must be positive, the use of zero weights, for $i = 1, \cdots, n-1$, does not cause any embarrassment [31] and results in interesting curve shapes. Similar discussions for classical rational Bézier curves was described in [33].

For a fixed $q > 0$, $\omega_0, \omega_n > 0$, and $\omega_i \geq 0 (i = 1, \cdots, n-1)$, if ω_i increases (decreases), the point $R(t;q)$ moves closer to (farther from) the control point $\mathbf{P_i}$, hence the curve is pulled toward (pushed away from) the control point $\mathbf{P_i}$. That is

$$\lim_{\omega_i \to +\infty} R(t;q) = \begin{cases} \mathbf{P_0}, & t = 0; \\ \mathbf{P_i}, & t \in (0,1); \\ \mathbf{P_n}, & t = 1. \end{cases}$$

Figure 4 shows cubic weighted Lupaş q-Bézier curves when the weight ω_2 changes. This motivates the following investigation on the geometric meaning of the weights.

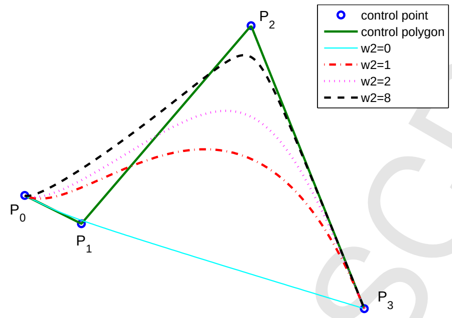

Figure 4: Cubic weighted Lupaş q-Bézier curves with different ω_2.

4.1. The geometric meaning of the weights

Given $\mathbf{P_i} \in \mathcal{R}^3 (i = 0, 1, \cdots, n)$ and $q > 0$, let us choose n fixed weights $\{\omega_j | j = 0, \cdots, i-1, i+1, \cdots, n\}$ so that $\omega_0, \omega_n > 0$ and $\omega_j \geq 0$. Then for a fixed $t \in (0, 1)$ define the following points (Figure 5):

$$\mathbf{S} := R(t; q; \omega_i = 0), \mathbf{M} := R(t; q; \omega_i = 1), \mathbf{S_i} := R(t; q; \omega_i \; arbitrary).$$

Since

$$\mathbf{S_i} = R(t; q; \omega_i) = \frac{\sum_{j=0}^{n} \omega_j \mathbf{P_j} a_{n,j}(t; q)}{\sum_{j=0}^{n} \omega_j a_{n,j}(t; q)} = \frac{\sum_{j \neq i}^{n} \omega_j \mathbf{P_j} a_{n,j}(t; q)}{\sum_{j=0}^{n} \omega_j a_{n,j}(t; q)} + \frac{\omega_i \mathbf{P_i} a_{n,i}(t; q)}{\sum_{j=0}^{n} \omega_j a_{n,j}(t; q)},$$

$$\mathbf{S} = R(t; q; 0) = \frac{\sum_{j \neq i}^{n} \omega_j \mathbf{P_j} a_{n,j}(t; q)}{\sum_{j \neq 0}^{n} \omega_j a_{n,j}(t; q)},$$

and

$$\mathbf{M} = R(t; q; 1) = \frac{\sum_{j \neq i}^{n} \omega_j \mathbf{P_j} a_{n,j}(t; q)}{\sum_{j=0}^{n} \omega_j a_{n,j}(t; q)} + \frac{\mathbf{P_i} a_{n,j}(t; q)}{\sum_{j=0}^{n} \omega_j a_{n,j}(t; q)},$$

we get

$$\mathbf{S_i} = (1-\mu)\mathbf{S} + \mu \mathbf{P_i}, \mathbf{M} = (1-\nu)\mathbf{S} + \nu \mathbf{P_i};$$

where

$$\mu = \frac{\omega_i a_{n,j}(t; q)}{\sum_{j=0}^{n} \omega_j a_{n,j}(t; q)}, 1 - \mu = \frac{\sum_{j \neq i}^{n} \omega_i a_{n,j}(t; q)}{\sum_{j=0}^{n} \omega_j a_{n,j}(t; q)},$$

$$\nu = \frac{a_{n,j}(t; q)}{\sum_{j=0}^{n} \omega_j a_{n,j}(t; q)}, 1 - \nu = \frac{\sum_{j \neq i}^{n} \omega_j a_{n,j}(t; q)}{\sum_{j=0}^{n} \omega_j a_{n,j}(t; q)}.$$

So, $\mathbf{M}$ and $\mathbf{S_i}$ lie on the same line passing through $\mathbf{S}$ and $\mathbf{P_i}$. Furthermore,

$$\begin{aligned}
\frac{|\mathbf{MP_i}|}{|\mathbf{SM}|} : \frac{|\mathbf{S_iP_i}|}{|\mathbf{SS_i}|} &= \frac{1-\nu}{\nu} : \frac{1-\mu}{\mu} \\
&= \frac{\sum_{j\neq i}^{n} \omega_j a_{n,j}(t;q)}{a_{n,j}(t;q)} : \frac{\sum_{j\neq i}^{n} \omega_j a_{n,j}(t;q)}{\omega_i a_{n,j}(t;q)} \\
&= \omega_i. \qquad (13)
\end{aligned}$$

Readers familiar with projective geometry, will recognize that (13) is the *cross-ratio*, the most important quantity of projective geometry, of the four points $\mathbf{P_i}, \mathbf{S}, \mathbf{M}, \mathbf{S_i}$ in this order. Using the identity (13), the effect of pulling becomes quite clear: as $\mathbf{S_i}$ moves toward $\mathbf{P_i}$, μ approaches 1 and thus ω_i tends to infinity as ν constant for a fixed $t \in (0, 1)$.

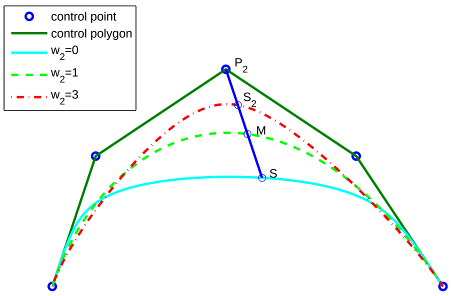

Figure 5: The geometric meaning of ω_2.

4.2. Conic sections

We use quadratic weighted Lupaş q-Bézier curves, $R(t;q) = \frac{\sum_{i=0}^{2} \omega_i a_{2,i}(t;q)\mathbf{P_i}}{\sum_{i=0}^{2} \omega_i a_{2,i}(t;q)}$, to represent conic sections, and the type of conic can be determined by looking at the denominator $\Omega_n(t;q)$ (see [34] Chapter 7), which can be written as

$$\begin{aligned}
\Omega_2(t;q) &= \sum_{i=0}^{2} \omega_i a_{2,i}(t;q) \\
&= (1-t)^2\omega_0 + [2]t(1-t)\omega_1 + qt^2\omega_2 \\
&= (\omega_0 - [2]\omega_1 + q\omega_2)t^2 + ([2]\omega_1 - 2\omega_0)t + \omega_0. \qquad (14)
\end{aligned}$$

The roots of equation (14) are

$$t_{1,2} = \frac{(2\omega_0 - [2]\omega_1) \pm \omega_1 \sqrt{[2]^2 - 4qk}}{2(\omega_0 - [2]\omega_1 + q\omega_2)}, \tag{15}$$

where $k = \omega_0\omega_2/\omega_1^2$ is the conic shape factor. It is customary to choose $\omega_0 = \omega_2 = 1$. Then if $\omega_1 = 1$, $R(t;q)$ is a parabola. Assuming $\omega_1 \neq 1$, $R(t;q)$ implies that

- if $k > \frac{[2]^2}{4q}$, then $R(t;q)$ has no real solutions, there are no points at infinity on the curve, hence it is an ellipse;
- if $k = \frac{[2]^2}{4q}$, then $R(t;q)$ has one real solution, there is one point on the curve at infinity, and it is a parabola;
- if $k < \frac{[2]^2}{4q}$, then $R(t;q)$ has two roots , the curve has two points at the infinity, and it is a hyperbola.

Expressing these conditions in terms of ω_1, we have

- $\omega_1^2 < \frac{4q}{[2]^2} \Rightarrow R(t;q)$ is an ellipse;
- $\omega_1^2 = \frac{4q}{[2]^2} \Rightarrow R(t;q)$ is a parabola;
- $\omega_1^2 > \frac{4q}{[2]^2} \Rightarrow R(t;q)$ is a hyperbola.

Figure 6 shows conic sections which are generated by quadratic weighted Lupaş q-Bézier curves with $\omega_0 = \omega_2 = 1$, ω_1 choose $-1/2, 0, 2\sqrt{2}/3$ and $\sqrt{3}$ (bottom to top), respectively. Notice that ω_1 can be zero or negative. $\omega_1 = 0$ yields a straight line segment from $\mathbf{P_0}$ to $\mathbf{P_2}$. $\omega_1 < 0$ yields the complementary arc traversed in the reverse order and the convex hull property does not hold.

5. Shape effects

Now weighted Lupaş q-Bézier curve have q and weights $\omega_i (i = 0, 1, \cdots, n)$ as its shape parameters, we have more freedom to control the shape of the curve. We illustrate weighted Lupaş q-Bézier curves with different q and weights in Figure 7, the value of q is sometimes dominant (left), while weights may be dominant with non-significant changes of q (right).

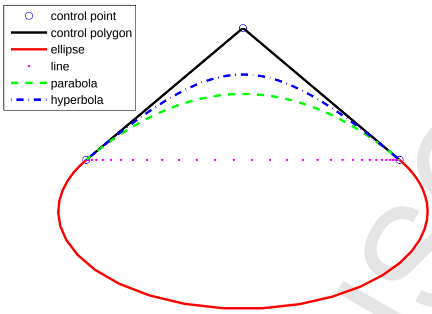

Figure 6: Conic sections generated by different ω_1.

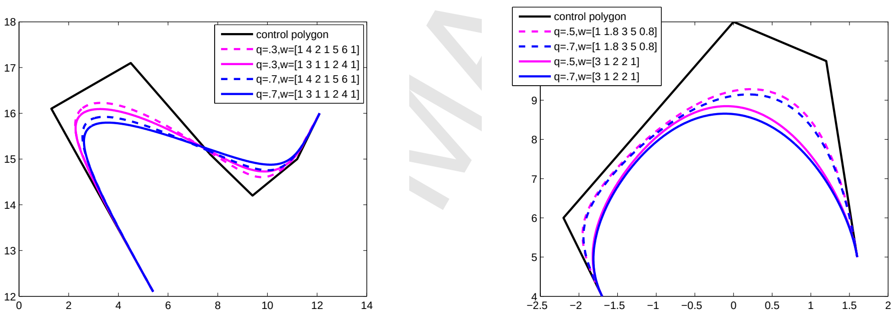

Figure 7: Weighted Lupaş q-Bézier curves with different q and weights.

According to the reducibility of weighted Lupaş q-Bézier curve, when all of the weights $\omega_i = \omega \neq 0$ $(i = 0, 1, \cdots, n)$, the curve reduces to Lupaş q-Bézier curve; when $q = 1$, the curve reduces to classical rational Bézier curve. In Figure 8, we compare classical rational Bézier curve (top left) with weighted Lupaş q-Bézier curves sharing same $q = 2$ (top right), sharing same weights $w = [1\ 5\ 7\ 5\ 1]$ (bottom left), having different values of q and weights (bottom right).

We illustrate the superiority of weighted Lupaş q-Bézier curve (right) to mimic the shape of its control polygon over rational Phillips q-Bézier curves (left) in Figure 9.

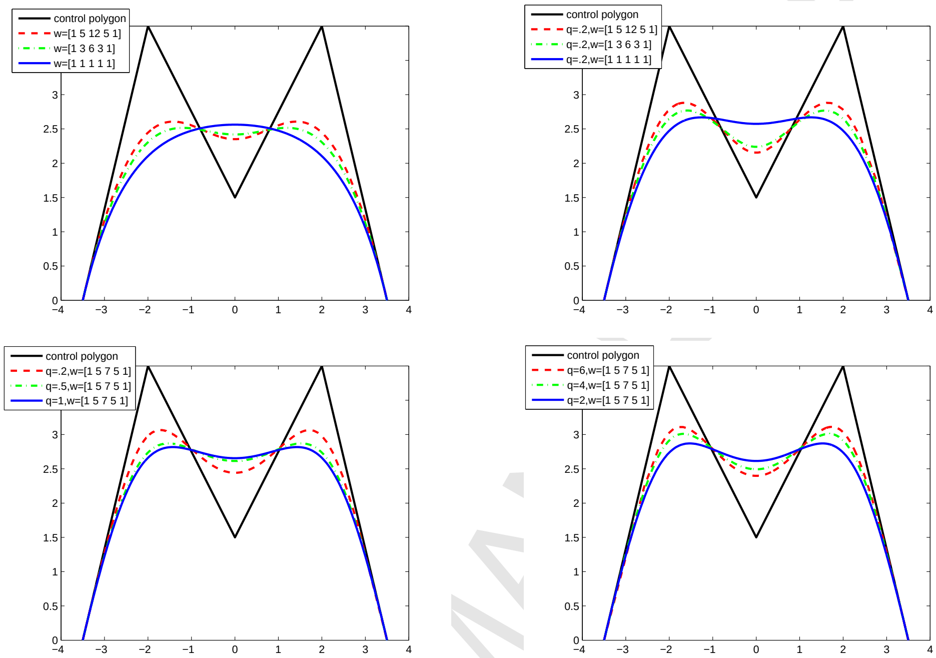

Figure 8: Comparison between classical rational Bézier curves (top left) and weighted Lupaş q-Bézier curves.

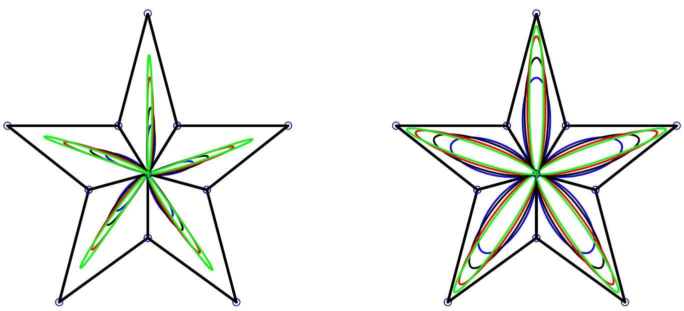

Figure 9: Comparison between rational Phillips q-Bézier curves (left) and weighted Lupaş q-Bézier curves (right).

6. Conclusions and future work

In this paper, we present a one parameter family of rational Bernstein-Bézier curves based on weighted Lupaş q-analogue of Bernstein basis. Weighted

Lupaş q-Bézier curves share many properties with classical rational Bézier curves (the case $q = 1$). They are affine invariant, lie in the convex hull of their control points, satisfy the variation diminishing property, satisfy the end-point interpolation and derivative properties. In addition, these curves possess analogous algorithms for degree elevation, recursive evaluation and can represent conic sections exactly. Numerical examples show that weighted Lupaş q-Bézier curves have more modeling flexibility than classical rational Bernstein-Bézier curves and Lupaş q-Bézier curves, and meanwhile they provide better approximations to the control polygon than rational Phillips q-Bézier curves.

Since the r^{th} derivative of a weighted Lupaş q-Bézier curve at an endpoint depends only on the $r+1$ control points near (and including) that endpoint, rather than that of rational Phillips q-Bézier curve [16] or h-Bézier curve (Pólya curves) [35] which involve all the control points, we will discuss the smooth blending of weighted Lupaş q-Bézier curves in the following paper. We also hope to construct a new generalization of B-spline based on Lupaş q-analogue of Bernstein operators.

Acknowledgement

We are very grateful to the anonymous referees for the inspiring comments and the valuable suggestions which improved our paper considerably. This work is supported by the National Natural Science Foundation of China(Nos. 61502144, 61573127), the Natural Science Foundation of Hebei Province (No. A2014205157), Training Program for Leading Talents of Innovation Teams in the Universities of Hebei Province (LJRC022) and Foundation of Hebei Normal University.

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