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Existence of the solution to variational inequality, optimization problem, and elliptic boundary value problem through revisited best proximity point results

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ABSTRACT

In this paper, we establish some results regarding the existence of the solution to variational inequality, optimization problem and elliptic boundary value problem in Hilbert spaces. Our strategy consists in establishing new best proximity point results in the metric spaces by introducing the concept of cyclic orbital simulative contractions. We also provide nontrivial examples to show that our results are proper generalization. Further, we improve the recent best proximity results for mappings satisfying proximal simulative conditions due to Abbas et al. (2017), Samet (2015), and Tchier et al. (2016) via new class of simulation functions. Our results unify, extend and generalize various existing results.

1. Introduction and preliminaries

Variational inequality theory is important due to the fact that it is a powerful unifying methodology for the study of equilibrium problems and providing us algorithms with accompanying convergence analysis for computational purposes. Therefore, it has attained extensive attention in recent years in the field of economics, management sciences, and so on. In fact many equilibrium problems in economics, game theory, mechanics, traffic analysis, can be transformed into variational inequality problems. Variational inequality is a formulation for solving the problem where we have to optimize a functional. The theory is derived by using the techniques of nonlinear functional analysis such as fixed point theory and the theory of monotone operators. It was first introduced by Hartman and Stampacchia [1] in 1966 in their seminal paper. Later on, it was extended to vector variational inequality problems by Giannessi [2] in 1980. Since then a great deal of research started in the area of vector variational inequality problems as a consequence of a lot of inclination of researchers towards vector optimization. Many researchers have contributed in this direction including Chen [3], Giannessi [4–6], Yang and Teo [7], Mishra and Wang [8], Chinaie et al. [9], Rezaie and Zafarani [10] and so on.

Here, in this paper, on account to get a new technique to solve variational inequality problems, optimization problems and elliptic boundary value problems in Hilbert spaces, we establish new best proximity point results in the metric spaces. A fundamental result in fixed point theory is the Banach Contraction Principle [11]. Many extensions and generalizations of this contraction are made by several authors. In 2015, Khojasteh et al. [12] gave the notation of simulation function and $\mathcal{Z}$-contraction to unify the several existing fixed point results in the literature. We denote by $\Delta(Z)$, the set of all functions $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying following conditions:

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Nomenclature

$\preceq$	Partial order
A, B	Closed non-empty subsets of X
$d(x, y)$	Distance between two points x and y
$d(x, B)$	Distance between a point x and a set B, i.e. $\inf\{d(x, y) : y \in B\}$
$\mathrm{dist}(A, B)$	Distance between sets A and B, i.e. $\inf\{d(x, y) : x \in A, y \in B\}$
$d^*(x, y)$	$d(x, y) - \mathrm{dist}(A, B)$
H	Hilbert Space
I_k	Identity operator of K
T	Mapping, operator, gradient operator, cyclic contraction, cyclic orbital simulative contraction, cyclic φ-contraction.

($\zeta 1$) $\zeta(0, 0) = 0$;
($\zeta 2$) $\zeta(t, s) < s - t$ for all $t, s > 0$;
($\zeta 3$) if $\{t_n\}$, $\{s_n\}$ are two sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ then

$$\lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0.$$

The function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying ($\zeta 1$)–($\zeta 3$) is called a simulation function [12].

Definition 1.1 (*[12]*)**.** Let (X, d) be a metric space, $T : X \to X$ an operator and $\zeta \in \Delta(Z)$. Then T is called a $\mathcal{Z}$*-contraction* with respect to ζ if it satisfies

$$\zeta(d(Tx, Ty), d(x, y)) \geq 0, \quad \text{for all} \quad x, y \in X. \tag{1.1}$$

Theorem 1.1 (*[12]*)**.** *Let (X, d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect ζ. Then T has a unique fixed point. Moreover, for every $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.*

Recently, Roldán-López-de-Hierro et al. [13] modified the notion of a simulation function by replacing ($\zeta 3$) by ($\zeta' 3$):

($\zeta' 3$) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$ and $t_n < s_n$, then

$$\lim_{n\to\infty} \sup \zeta(t_n, s_n) < 0.$$

The function $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying ($\zeta 1$)-($\zeta 2$) and ($\zeta' 3$) is called a simulation function in the sense of Roldán-López-de-Hierro. Note that every simulation function in the original sense of Khojasteh et al. [12] is also a simulation function in the sense of Roldán-López-de-Hierro et al. [13], but the converse is not true.

Example 1.1 (*[13]*)**.** Let $\zeta_4 : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by

$$\zeta_4(t, s) = \begin{cases} 2s - 2t & \text{if} \quad s < t \\ ks - t & \text{otherwise,} \end{cases}$$

where $k \in \mathbb{R}$ and $k < 1$. Then ζ_4 satisfies ($\zeta 1$), ($\zeta 2$) and ($\zeta' 3$) but does not satisfy ($\zeta 3$).

Further, in [14], Argoubi et al. noted that the condition ($\zeta 1$) was not used for the proof of Theorem 1.1. Also they noted that taking $x = y$ in (1.1), one has $\zeta(0, 0) \geq 0$ and hence, if $\zeta(0, 0) < 0$, the set of operators $T : X \to X$ satisfying (1.1) is empty. One of an interesting generalization of Banach's contraction is given by Kirk et al. in [15]. They obtained existence of unique fixed point in non-empty intersection $A \cap B$ for a mapping $T : A \cup B \to A \cup B$ satisfying $T(A) \subseteq B$, $T(B) \subseteq A$ and for some $k \in (0, 1)$

$$d(Tx, Ty) \leq kd(x, y). \tag{1.2}$$

Afterwards, Eldred and Veeramani [16] modified (1.2) to cyclic contraction in the case when $A \cap B = \emptyset$ as:

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)\mathrm{dist}(A, B), \tag{1.3}$$

where $k \in (0, 1)$. In this case, a unique best proximity point of $T : A \cup B \to A \cup B$ in A is found in a uniformly convex Banach space. Recall that a point $x \in A$ is called a best proximity point of $T : A \cup B \to A \cup B$ in A if $d(x, Tx) = \mathrm{dist}(A, B)$, when the sets A and B intersects, the best proximity point reduces to fixed point. Next, Di Bari et al. [17] generalized the cyclic contraction by introducing the cyclic Meir–Keeler contraction and proved the best proximity result for this contraction. Sanhan and Mongkolkeha [18] introduced the Berinde's cyclic contractions and proved best proximity theorems for these mappings with proximally complete property. In this direction, Hussain and Iqbal [19] introduced the concept of multivalued cyclic F-contraction and obtained best proximity theorems for these mappings with proximally complete property. In [20], Al-Thagafi and Shahzad found a best proximity point for cyclic φ-contractions which has a proximal property. Further, in [21], Du and Lakzian, gave a notion of MT-cyclic contraction and established some existence and convergence theorems of iterates of best proximity points for MT-cyclic contractions. For the theory of best proximity points via simulation functions see [22–27] and references there in and for the best proximity point results for different proximal contractions see [28–33] and references there in.

On the other side, Karpagam and Agrawal [34] generalized (1.2) by giving the notion of a cyclic orbital contraction as: for some $x \in A$ there exists $k_x \in (0, 1)$ such that

$$d(T^{2n}x, Ty) \leq k_x d(T^{2n-1}x, y), \quad n \in \mathbb{N}, \; y \in A, \tag{1.4}$$

and $T : A \cup B \to A \cup B$ is such that $T(A) \subseteq B$ and $T(B) \subseteq A$. They showed that $A \cap B$ is non-empty and T has a unique fixed point.

Continuing in this direction, in this paper, we introduce the notion of cyclic orbital simulative contraction and explore the existence of best proximity points for these contraction. Our results generalize the main results of [12,16,20,21,34–36] and [13]. We also improve the results of Abbas et al. [22], Samet [24] and Tchier et al. [25] via enriched class of simulation functions.

2. Results for cyclic orbital simulative contraction

In this section, we prove the existence of best proximity points and fixed points for cyclic orbital simulative contraction. Throughout this section, let (X, d) be a metric space, A and B be two non-empty subsets of X. Define $\mathrm{dist}(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$ and $d^*(x, y) = d(x, y) - \mathrm{dist}(A, B)$.

2.1. Best proximity point results

We start this section with the following lemma.

Lemma 2.1. *Let $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be a function such that $\zeta(t, s) < s - t$ for all $t, s > 0$. If $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then for $a \geq 0$ with $a < t_n < s_n$ following holds*

$$\lim_{n\to\infty} \zeta(t_n - a, s_n - a) \leq 0.$$

Proof. Let $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = C > 0$, then

$$\lim_{n\to\infty} \zeta(t_n - a, s_n - a) \leq \lim_{n\to\infty} (s_n - t_n) = C - C = 0. \quad \square$$

We denote by $\mathfrak{Z}$ the collection of all functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ satisfying $(\zeta 2)$ and the following:

$(\zeta 4)$ If $\{t_n\}$, $\{s_n\}$ are sequences in $(0, \infty)$ such that $t_n < s_n$ for each n and $\lim_{n\to\infty} s_n > 0$, then

$$\lim_{n\to\infty} \zeta(t_n, s_n) = 0 \quad \text{implies} \quad \lim_{n\to\infty} t_n < \lim_{n\to\infty} s_n.$$

Example 2.1. Let $\zeta_k : [0, \infty) \times [0, \infty) \to \mathbb{R}$ be defined by

$$\zeta_k(t, s) = \begin{cases} ks - t & \text{if } t, s > 0 : \; t \leq s \\ \frac{1}{k}(s - t) & \text{if } t, s > 0 : \; t > s, \\ 1 & \text{if } t = s = 0, \end{cases}$$

where $k \in (0, 1)$. Then $\zeta_k(t, s) < s - t$ for all $t, s > 0$. Let $\{t_n\}$, $\{s_n\}$ be two sequences in $(0, \infty)$ be such that $t_n < s_n$ and $\lim_{n\to\infty} s_n = C > 0$, we have

$$0 = \lim_{n\to\infty} (ks_n - t_n) = \lim_{n\to\infty} ks_n - \lim_{n\to\infty} t_n,$$

which implies

$$\lim_{n\to\infty} t_n = \lim_{n\to\infty} ks_n = kC < C = \lim_{n\to\infty} s_n.$$

Thus, $\zeta_k \in \mathfrak{Z}$.

On the other side, let $\{t_n\}$, $\{s_n\}$ be sequences in $(0, \infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\lim_{n\to\infty} (s_n - t_n) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} t_n = 0$$

and for $s_n < t_n$, we have

$$\lim_{n\to\infty} \sup \zeta_k(t_n, s_n) = \lim_{n\to\infty} \sup \left(\frac{1}{k}(s_n - t_n)\right) = 0.$$

Therefore, $(\zeta 3)$ does not hold and $\zeta_k \notin \Delta(Z)$. Also, $\zeta(0, 0) = 1 \neq 0$, hence ζ_k is not a simulation function in the sense of Roldán-López-de-Hierro.

Remark 2.1. Note that if $\zeta \in \mathfrak{Z}$, then $\zeta(t, s) < s - t < 0$ for all $0 < s \leq t$.

Definition 2.1. Let A and B be non-empty subsets of a metric space (X, d). A mapping $T : A \cup B \to A \cup B$ is said to be *cyclic orbital simulative contraction* if $T(A) \subseteq B$, $T(B) \subseteq A$ and for some $x_0 \in A$ there exists $\zeta \in \mathfrak{Z}$ such that

$$\zeta(d^*(T^{2n}x_0, Ty), d^*(T^{2n-1}x_0, y)) \geq 0, \quad n \in \mathbb{N},\ y \in A. \tag{2.1}$$

Remark 2.2.

(1) Every cyclic contraction [16] is cyclic orbital simulative contraction for $\zeta = \zeta_k$. Indeed, let T be a cyclic contraction on $A \cup B$, so for all $x, y \in A \cup B$ (1.3) holds. Then for each $n \in \mathbb{N}$, we get $d^*(T^n x, Ty) < d^*(T^{n-1}x, y)$ and $d(T^{2n}x, Ty) \leq kd(T^{2n-1}x, y) + (1 - k)\mathrm{dist}(A, B)$, which implies

$$\begin{aligned} 0 &\leq k(d(T^{2n-1}x, y) - \mathrm{dist}(A, B)) - (d(T^{2n}x, Ty) - \mathrm{dist}(A, B)) \\ &= \zeta(d^*(T^{2n}x, Ty), d^*(T^{2n-1}x, y)). \end{aligned}$$

(2) Every cyclic orbital contraction [34] is cyclic orbital simulative contraction for

$$\zeta(t, s) = \begin{cases} 1 & \text{if } s = t = 0 \\ k_x s - t & \text{if } s, t > 0 : t < s \\ 2(s - t) & \text{if } s, t > 0 : t \geq s, \end{cases}$$

where $k_x \in (0, 1)$ for some $x \in A$. Indeed, let T be cyclic orbital contraction on $A \cup B$, so for some $x \in A$ there exits $k_x \in A$ such that

$$\begin{aligned} d(T^{2n}x, Ty) &\leq k_x d(T^{2n-1}x, T) \\ &\leq k_x d(T^{2n-1}x, T) + (1 - k_x)\mathrm{dist}(A, B), \end{aligned}$$

which implies $\zeta(d^*(T^{2n}x, Ty), d^*(T^{2n-1}x, y)) \geq 0$.

But a cyclic orbital simulative contraction need neither be a cyclic orbital contraction or nor a cyclic contraction as shown in Example 2.3.

Lemma 2.2. *Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic orbital simulative contraction mapping. If $x_0 \in A$ satisfies (2.1), then for each $n \in \mathbb{N}$*

$$d^*(T^n x_0, T^{n+1}x_0) < d^*(T^{n-1}x_0, T^n x_0).$$

Proof. Let $x_0 \in A$ and $n \in \mathbb{N}$, here arises two cases:
Case:I n is even, $n = 2m$, where $m \in \mathbb{N}$
then from (2.1), we have

$$\begin{aligned} 0 &\leq \zeta(d^*(T^n x_0, T^{n+1}x_0), d^*(T^{n-1}x_0, T^n x_0)) \\ &= \zeta(d^*(T^{2m}x_0, T(T^{2m}x_0)), d^*(T^{2m-1}x_0, T^{2m}x_0)) \\ &< d^*(T^{2m-1}x_0, T^{2m}x_0) - d^*(T^{2m}x_0, T(T^{2m}x_0)). \end{aligned}$$

This implies that

$$d^*(T^{2m}x_0, T(T^{2m}x_0)) < d^*(T^{2m-1}x_0, (T^{2m}x_0)).$$

Case:II $n + 1$ is even, $n = 2m - 1$, where $m \in \mathbb{N}$, then from (2.1), we have

$$\begin{aligned} 0 &\leq \zeta(d^*(T^{n+1}x_0, T^n x_0), d^*(T^n x_0, T^{n-1}x_0)) \\ &= \zeta(d^*(T^{2m}x_0, T(T^{2m-2}x_0)), d^*(T^{2m-1}x_0, T^{2m-2}x_0)) \\ &< d^*(T^{2m-1}x_0, T^{2m-2}x_0) - d^*(T^{2m}x_0, T(T^{2m-2}x_0)). \end{aligned}$$

This implies that

$$d^*(T^{2m}x_0, T(T^{2m-2}x_0)) < d^*(T^{2m-1}x_0, (T^{2m-2}x_0)).$$

This completes the proof. □

Lemma 2.3. *Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic orbital simulative contraction mapping. If $x_0 \in A$ satisfies (2.1), then for $n \in \mathbb{N}$ there exist a sequence $\{T^n x_0\}$ in $A \cup B$ such that*

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1}x_0) = dist(A, B).$$

Proof. Let $x_0 \in A$ satisfies (2.1). Since $T(A) \subseteq B$ and $T(B) \subseteq A$, we have a sequence $T^n x_0$ in X such that $T^{2n}x_0$ contained in A and $T^{2n+1}x_0$ contained in B. If for $n \in \mathbb{N}$, $d(T^n x_0, T^{n+1}x_0) = d(T^{n-1}x_0, T^n x_0) = \mathrm{dist}(A, B)$, then proof is complete. Assume that $d(T^n x_0, T^{n+1}x_0) > \mathrm{dist}(A, B)$. From Lemma 2.2, we get $\{d^*(T^n x_0, T^{n+1}x_0)\}$ is a monotonically decreasing sequence of non-negative real numbers which is bounded below by 0. Then there exists $r \geq 0$ such that for each $n \in \mathbb{N}$, $d^*(T^n x_0, T^{n+1}x_0) \to r$ as $n \to \infty$. We assert that $r = 0$, if not then

$$\lim_{n\to\infty} d^*(T^{n-1}x_0, T^n x_0) = \lim_{n\to\infty} d^*(T^n x_0, T^{n+1}x_0) = r > 0 \tag{2.2}$$

and for each $n \in \mathbb{N}$ we have two cases:
Case:I n is even, $n = 2m$, where $m \in \mathbb{N}$
Since $d^*(T^{2m}x_0, T^{2m+1}x_0) \to r$ and $d^*(T^{2m-1}x_0, T^{2m}x_0) \to r$ as $n \to \infty$ then from (2.1) and Lemma 2.1, we have

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \zeta(d^*(T^{2m}x_0, T^{2m+1}x_0), d^*(T^{2m-1}x_0, T^{2m}x_0)) \\ &= \lim_{n\to\infty} \zeta(d(T^{2m}x_0, T^{2m+1}x_0) - \mathrm{dist}(A, B), d(T^{2m-1}x_0, T^{2m}x_0) - \mathrm{dist}(A, B)) \leq 0. \end{aligned}$$

Case:II $n + 1$ is even, $n = 2m - 1$, , where $m \in \mathbb{N}$
Since $d(T^{2m-1}x_0, T^{2m}x_0), d(T^{2m-2}x_0, T^{2m-1}x_0) \to r$ as $n \to \infty$ then from (2.1) and Lemma 2.1, we have

$$\begin{aligned} 0 &\leq \lim_{n\to\infty} \zeta(d^*(T^{2m-1}x_0, T^{2m}x_0), d^*(T^{2m-2}x_0, T^{2m-1}x_0)) \\ &= \lim_{n\to\infty} \zeta(d(T^{2m-1}x_0, T^{2m}x_0) - \mathrm{dist}(A, B), d(T^{2m-2}x_0, T^{2m-1}x_0) - \mathrm{dist}(A, B)) \leq 0. \end{aligned}$$

Hence, for each $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \zeta(d^*(T^n x_0, T^{n+1}x_0), d^*(T^{n-1}x_0, T^n x_0)) = 0,$$

from (ζ4), we get

$$\lim_{n\to\infty} d^*(T^n x_0, T^{n+1}x_0) < \lim_{n\to\infty} d^*(T^{n-1}x_0, T^n x_0),$$

which is a contradiction to (2.2). Hence $d^*(T^n x_0, T^{n+1}x_0) \to 0$ as $n \to \infty$ and consequently, $d(T^n x_0, T^{n+1}x_0) \to \mathrm{dist}(A, B)$ as $n \to \infty$. $\square$

Next we prove the existence of the best proximity point for cyclic orbital simulative contraction.

Theorem 2.1. *Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic orbital simulative contraction. For $x_0 \in A$ satisfying (2.1), if $\{T^{2n}x_0\}$ and $\{T^{2n+1}x_0\}$ have convergent subsequences in A and B respectively, then there exists $(x, y) \in A \times B$ such that $d(x, Tx) = dist(A, B)$ and $d(y, Ty) = dist(A, B)$ with $d(x, y) = dist(A, B)$.*

Proof. Since $T(A) \subseteq B$ and $T(B) \subseteq A$, we have a sequence $\{T^n x_0\}$ in $A \cup B$ such that $\{T^{2n}x_0\}$ contained in A and $\{T^{2n+1}x_0\}$ contained in B. Let $\{T^{2n_k}x_0\}$ be a subsequence of $\{T^{2n}x_0\}$ such that

$$\lim_{k\to\infty} T^{2n_k}x_0 = x \tag{2.3}$$

for some $x \in A$. Since

$$\mathrm{dist}(A, B) \leq d(x, T^{2n_k-1}x_0) \leq d(x, T^{2n_k}x_0) + d(T^{2n_k}x_0, T^{2n_k-1}x_0). \tag{2.4}$$

Letting $k \to \infty$ in (2.4) and using Lemma 2.3 with (2.3), we get

$$\lim_{k\to\infty} d(x, T^{2n_k-1}x_0) = \mathrm{dist}(A, B). \tag{2.5}$$

Now,

$$\mathrm{dist}(A, B) \leq d(T^{2n_k}x_0, Tx) \leq d(T^{2n_k-1}x_0, x), \tag{2.6}$$

so, by letting $k \to \infty$ in (2.6) with (2.5) implies $d(x, Tx) = \mathrm{dist}(A, B)$. Similarly, if $\{T^{2n_k+1}x_0\}$ be a subsequence of $T^{2n+1}x_0$ such that $T^{2n_k+1}x_0 \to y \in B$ as $k \to \infty$. We can prove that $d(y, Ty) = \mathrm{dist}(A, B)$. Moreover,

$$d(x, y) = \lim_{n\to\infty} d(T^{2n_k}x_0, T^{2n_k+1}x_0) = \mathrm{dist}(A, B). \quad \square$$

Example 2.2. Let $X = \{(x, y) : -1 \leq x, y \leq 1\}$ with metric d, defined as

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$$

for all $(x_1, y_1), (x_2, y_2) \in X$. Consider $A = \{(1, 0), (0, 1)\}$ and $B = \{(-1, 0), (0, -1)\}$. Then (X, d) is a metric space and $\operatorname{dist}(A, B) = 2$. Define $T : A \cup B \to A \cup B$ and $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $T(x, y) = (-y, -x)$ and

$$\zeta(t, s) = \begin{cases} 1 & \text{if } \ s = t = 0 \\ s - qt & \text{if } \ s, t > 0 : \ s \geq t \\ q(s - t) & \text{if } \ s, t > 0 : \ s < t, \end{cases}$$

where $q > 1$. Then $T(A) = B$, $T(B) = A$, $\zeta \in \mathfrak{Z}$ and also for $n \in \mathbb{N}$,

$$\begin{aligned} T^{2n}(1, 0) = (1, 0),\ T^{2n-1}(1, 0) = (0, -1), \qquad & T^{2n}(0, 1) = (0, 1),\ T^{2n-1}(0, 1) = (-1, 0), \\ T^{2n}(-1, 0) = (-1, 0),\ T^{2n-1}(-1, 0) = (0, 1), \qquad & T^{2n}(0, -1) = (0, -1),\ T^{2n-1}(0, -1) = (1, 0). \end{aligned}$$

Now for $(1, 0) \in A$ we have for all $n \in \mathbb{N}$

$$\begin{aligned} \zeta(d^*(T^{2n}(1, 0), T(1, 0))), d^*(T^{2n-1}(1, 0), (1, 0)) &= \zeta(d^*((1, 0), (0, -1)), d^*((0, -1), (1, 0))) \\ &= \zeta(0, 0) = 1 > 0 \end{aligned}$$

and

$$\begin{aligned} \zeta(d^*(T^{2n}(1, 0), T(0, 1)), d^*(T^{2n-1}(1, 0), (0, 1))) &= \zeta(d^*((1, 0), (-1, 0)), d^*((0, -1), (0, 1))) \\ &= \zeta(0, 0) = 1 > 0. \end{aligned}$$

So, there exist $(1, 0) \in A$ and $\zeta \in \mathfrak{Z}$ such that (2.1) holds for all $y \in A$. Hence T is cyclic orbital simulative contraction. Also $\{T^{2n}(1, 0)\}$ and $\{T^{2n+1}(1, 0)\}$ have convergent subsequences in A and B respectively. Thus, all the conditions of Theorem 2.1 hold true and there exists $(1, 0) \in A$ and $(0, -1) \in B$ such that

$$\begin{aligned} d((1, 0), T(1, 0)) = d((1, 0), (0, -1)) = 2 = \operatorname{dist}(A, B) \\ d((0, -1), T(0, -1)) = d((0, -1), (1, 0)) = 2 = \operatorname{dist}(A, B) \end{aligned}$$

with

$$d((1, 0), (0, -1)) = 2 = \operatorname{dist}(A, B).$$

On the other hand, there exists no $\zeta \in \Delta(Z)$ such that T is a Z-contraction. Indeed, for $x = (1, 0)$ and $y = (-1, 0)$

$$\zeta(d(T(1, 0), T(-1, 0)), d((1, 0), (-1, 0))) = \zeta(2, 2) < 0.$$

Example 2.3. Let $X = \{0, 1, 2, 3\}$ with metric d, given as

$$d(x, y) = \begin{cases} 0 & \text{if } \ x = y \\ 2 & \text{if } \ x, y \in \{1, 2\} \\ 1 & \text{otherwise,} \end{cases}$$

$A = \{0, 1\}$ and $B = \{2, 3\}$. Then (X, d) is a complete metric space, A and B are closed subsets of X with $\operatorname{dist}(A, B) = 1$. Define $T : A \cup B \to A \cup B$ and $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by $T0 = 3$, $T1 = 2$, $T2 = 1$, $T3 = 0$ and $\zeta = \zeta_k$. Then $T(A) = B$, $T(B) = A$, $\zeta \in \mathfrak{Z}$ and also for $n \in \mathbb{N}$, $T^{2n}0 = 0$, $T^{2n-1}0 = 3$, $T^{2n}1 = 1$, $T^{2n-1}1 = 2$, $T^{2n}2 = 2$, $T^{2n-1}2 = 1$, $T^{2n}3 = 3$ and $T^{2n-1}3 = 0$. Now for $0 \in A$ we have for all $n \in \mathbb{N}$

$$\zeta(d^*(T^{2n}0, T0), d^*(T^{2n-1}0, 0)) = \zeta(d^*(0, 3), d^*(3, 0)) = \zeta(0, 0) = 1 > 0$$

and

$$\zeta(d^*(T^{2n}0, T1), d^*(T^{2n-1}0, 1)) = \zeta(d^*(0, 2), d^*(3, 1)) = \zeta(0, 0) = 1 > 0.$$

So, there exists $0 \in A$ and $\zeta \in \mathfrak{Z}$ such that (2.1) holds for all $y \in A$. Hence T is cyclic orbital simulative contraction. Also $\{T^{2n}0\}$ and $\{T^{2n+1}0\}$ have convergent subsequences in A and B respectively. Thus, all the conditions of Theorem 2.1 hold true and there exists $0 \in A$ and $3 \in B$ such that

$$\begin{aligned} d(0, T0) = d(0, 3) = 1 = \operatorname{dist}(A, B) \\ d(3, T3) = d(3, 0) = 1 = \operatorname{dist}(A, B) \end{aligned}$$

with

$$d(0, 3) = 1 = \operatorname{dist}(A, B).$$

On the other hand, we have

$$\frac{d(T^{2n}0, T1)}{d(T^{2n-1}0, 1)} = \frac{d(0, 2)}{d(3, 1)} = 1$$

and also

$$\frac{d(T^{2n}1, T0)}{d(T^{2n-1}1, 0)} = \frac{d(1, 3)}{d(2, 0)} = 1.$$

Hence there exists no $x \in A$ such that (1.4) holds for all $y \in A$. Therefore, T is not a cyclic orbital contraction.

Also, for $1 \in A$ and $2 \in B$, we have

$$\frac{d^*(T1, T2)}{d^*(1, 2)} = \frac{d^*(2, 1)}{d^*(1, 2)} = 1,$$

so, T is not a cyclic contraction.

It is also interesting to note that for $\zeta = \zeta_k$, T is not a Z-contraction. Indeed, for $x = 0$ and $y = 1$

$$\zeta(d(T0, T1), d(0, 1)) = \zeta(1, 1) = k - 1 < 0.$$

We adopt the method of Sanhan and Mongkolkeha [18] to prove the following lemma:

Lemma 2.4. *Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic orbital simulative contraction. Then for every $x_0 \in A$ satisfying (2.1), sequence $\{T^n x_0\}$ in $A \cup B$ is bounded.*

Proof. Suppose that $x_0 \in A$ satisfies (2.1), then from Lemma 2.3 there exists a sequence $\{T^n x_0\}$ in $A \cup B$ such that $d(T^{2n}x_0, T^{2n+1}x_0)$ converges to $\text{dist}(A, B)$. So, we prove that $\{T^{2n}x_0\}$ is bounded. On contrary, assume that $\{T^{2n}x_0\}$ is not bounded then there exists $p \in \mathbb{N}$ satisfying $M < d^*(T^{2n}x_0, T^{2(n+p)+1}x_0)$ and $d^*(T^{2n}x_0, T^{2(n+p)-1}x_0) \le M$ where $M = 2p\text{dist}(A, B)$. Thus, we have

$$\begin{aligned} M &< d^*(T^{2n}x_0, T^{2(n+p)+1}x_0) \\ &= d(T^{2n}x_0, T^{2(n+p)+1}x_0) - \text{dist}(A, B) \\ &\le d(T^{2n}x_0, T^{2n+1}x_0) + d(T^{2n+1}x_0, T^{2n+2}x_0) + \cdots + d(T^{2(n+p)}x_0, T^{2(n+p)+1}x_0) \\ &\quad - \text{dist}(A, B) \\ &\le d^*(T^{2n}x_0, T^{2n+1}x_0) + d^*(T^{2n+1}x_0, T^{2n+2}x_0) + \cdots + d^*(T^{2(n+p)}x_0, T^{2(n+p)+1}x_0) \\ &\quad + (2p - 1)\text{dist}(A, B). \end{aligned} \tag{2.7}$$

Letting $n \to \infty$ in (2.7) and using Lemma 2.3, we get

$$M < M - \text{dist}(A, B),$$

which leads to contradiction. Similarly, we can prove that $\{T^{2n+1}x_0\}$ is bounded and hence $\{T^n x_0\}$ is bounded. □

Recall that a subset K of a metric space is boundedly compact if each bounded sequence in K has a subsequence converging to a point in K [16]. So, from Theorem 2.1 and Lemma 2.4 we obtain

Theorem 2.2. *Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a cyclic orbital simulative contraction. If either A or B is boundedly compact, then there exists $x \in A \cup B$ such that $d(x, Tx) = \text{dist}(A, B)$.*

2.2. Fixed point results

In this section, we deduce some fixed point results from Section 2.1.

Theorem 2.3. *Let A and B be non-empty subsets of a metric space (X, d) with $\text{dist}(A, B) = 0$ and $T : A \cup B \to A \cup B$ be a mapping satisfying $T(A) \subseteq B$ and $T(B) \subseteq A$. If for $x_0 \in A$, there exists $\zeta \in \mathcal{Z}$ such that*

$$\zeta(d(T^{2n}x_0, Ty), d(T^{2n-1}x_0, y)) \ge 0, \quad n \in \mathbb{N}, \quad y \in A. \tag{2.8}$$

Then T has a unique fixed point in $A \cap B$ provided that $\{T^{2n}x_0\}$ and $\{T^{2n+1}x_0\}$ have convergent subsequences in A and B respectively.

Proof. Since $\text{dist}(A, B) = 0$, (2.8) implies that T is a cyclic orbital simulative contraction. Hence from Theorem 2.1, we have $x \in A$, $y \in B$ with $x = y$ such that $d(x, Tx) = 0$. Consequently, $A \cap B \neq \emptyset$ and T has fixed point in $A \cap B$. For uniqueness, suppose that $u \in A \cup B$ such that $Tu = u$ with $x \neq u$, then by (2.8) and Remark 2.1, we get

$$0 \le \zeta(d(T^{2n}x, Tu), d(T^{2n-1}x, u)) = \zeta(d(x, u), d(x, u)) < 0,$$

a contradiction. Thus, $x = u$. □

Theorem 2.3 together with Lemma 2.4 gives

Theorem 2.4. *Let A and B be non-empty subsets of a metric space (X, d) with $dist(A, B) = 0$ and $T : A \cup B \to A \cup B$ be a mapping such that $T(A) \subseteq B$ and $T(B) \subseteq A$ and for $x_0 \in A$ satisfying (2.8). If A and B are boundedly compact, then T has a unique fixed point in $A \cap B$.*

Remark 2.3. Recently, in [36] Radenović showed that some fixed point results for mappings satisfying cyclical contractive conditions are equivalent to fixed point results for classical contractive mappings. It can be noted in the Proof of Theorem 2.5 of [36] that to get cyclical-type theorems from classical results, completeness of metric space and closedness of subsets A_i for all i are necessary otherwise $\left(\cap_{i=1}^{m} A_i, d\right)$ is not a complete metric space. Therefore, Theorem 2.4 cannot be obtained by using the approach of Radenović.

Lemma 2.5. *Let A and B be non-empty subsets of a metric space (X, d) such that $dist(A, B) = 0$. If $T : A \cup B \to A \cup B$ is a mapping such that $T(A) \subseteq B$, $T(B) \subseteq A$ and for $x_0 \in A$ satisfy (2.8), then there exists a Cauchy sequence $\{T^{2n}x_0\}$ in A.*

Proof. Let $x_0 \in A$ satisfy (2.8). Without loss of generality, assume that for $n \in \mathbb{N}$, $T^n x_0 \neq T^{n-1}x_0$. Given that $\text{dist}(A, B) = 0$, so for all $n \in \mathbb{N}$, from Lemma 2.3 we get

$$\lim_{n\to\infty} d(T^n x_0, T^{n+1}x_0) = 0. \tag{2.9}$$

We claim that for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$d(T^{2n}x_0, T^{2m+1}x_0) < \epsilon,$$

$n, m \geq N$. If not, then there exists two sequences $\{2n_k\}, \{2m_k + 1\} \subseteq \mathbb{N}$ such that

$$d(T^{2n_k}x_0, T^{2m_k+1}x_0) \geq \epsilon, \tag{2.10}$$

for all $k \in \mathbb{N}$. We assume that $2m_k + 1$ is a minimal index for which (2.10) holds. Then for all $k \in \mathbb{N}$

$$d(T^{2n_k}x_0, T^{2m_k-1}x_0) < \epsilon. \tag{2.11}$$

From (2.10) and (2.11), we have

$$\begin{aligned}\epsilon &\leq d(T^{2n_k}x_0, T^{2m_k+1}x_0)\\ &\leq d(T^{2n_k}x_0, T^{2m_k-1}x_0) + d(T^{2m_k-1}x_0, T^{2m_k}x_0) + d(T^{2m_k}x_0, T^{2m_k+1}x_0)\\ &< \epsilon + d(T^{2m_k-1}x_0, T^{2m_k}x_0) + d(T^{2m_k}x_0, T^{2m_k+1}x_0).\end{aligned} \tag{2.12}$$

Letting $k \to \infty$ in (2.12) and using (2.9), we get

$$\lim_{k\to\infty} d(T^{2n_k}x_0, T^{2m_k+1}x_0) = \epsilon. \tag{2.13}$$

Now by triangular inequality we have

$$\begin{aligned}d(T^{2n_k}x_0, T^{2m_k+1}x_0) \leq& d(T^{2n_k}x_0, T^{2n_k-1}x_0) + d(T^{2n_k-1}x_0, T^{2m_k}x_0)\\ &+ d(T^{2m_k}x_0, T^{2m_k+1}x_0)\end{aligned} \tag{2.14}$$

and

$$\begin{aligned}d(T^{2n_k-1}x_0, T^{2m_k}x_0) \leq& d(T^{2n_k-1}x_0, T^{2n_k}x_0) + d(T^{2n_k}x_0, T^{2m_k+1}x_0)\\ &+ d(T^{2m_k+1}x_0, T^{2m_k}x_0).\end{aligned} \tag{2.15}$$

Letting $k \to \infty$ in (2.14) and (2.15) and using (2.9) and (2.13), we get

$$\lim_{k\to\infty} d(T^{2n_k-1}x_0, T^{2m_k}x_0) = \epsilon. \tag{2.16}$$

Since $\zeta \in \mathfrak{Z}$, from Lemma 2.1, (2.13) and (2.16) give

$$0 \leq \lim_{n\to\infty} \zeta(d(T^{2n_k}x_0, T^{2m_k+1}x_0), d(T^{2n_k-1}x_0, T^{2m_k}x_0)) \leq 0. \tag{2.17}$$

(2.17) together with $(\zeta 4)$ implies

$$\lim_{n\to\infty} d(T^{2n_k}x_0, T^{2m_k+1}x_0) < \lim_{n\to\infty} d(T^{2n_k-1}x_0, T^{2m_k}x_0),$$

which leads to the contradiction. Hence $\{T^{2n}x_0\}$ is a Cauchy sequence in A. $\square$

Theorem 2.5. *Let A and B be non-empty closed subsets of a complete metric space (X, d) such that $dist(A, B) = 0$. If $T : A \cup B \to A \cup B$ is a mapping such that $T(A) \subseteq B$, $T(B) \subseteq A$ and for $x_0 \in A$ satisfy (2.8), then T has a unique fixed point in $A \cap B$.*

Proof. Let $x_0 \in A$ satisfy (2.8), then from Lemma 2.5, there exist a Cauchy sequence $\{T^{2n}x_0\}$ in A and by completeness of X and closeness of A, there exists $x^* \in A$ such that $T^{2n}x_0 \to x^*$ as $n \to \infty$. Now,

$$0 \le d(T^{2n-1}x_0, x^*) \le d(T^{2n-1}x_0, T^{2n}x_0) + d(T^{2n}x_0, x^*),$$

which implies

$$\lim_{n\to\infty} d(T^{2n-1}x_0, x^*) = 0.$$

Since $\{T^{2n-1}x_0\}$ is a sequence in B, which converges to x^* and B is closed. Hence $\{T^{2n}x_0\}$ and $\{T^{2n+1}x_0\}$ have convergent subsequences in A and B respectively. From Theorem 2.4, T has a unique fixed point in $A \cap B$. $\square$

Take $A = B = X$ in Theorem 2.5 to get

Corollary 2.1. *Let (X, d) be a complete metric space and $T : X \to X$. If there exists $\zeta \in \mathfrak{Z}$ such that T satisfies*

$$\zeta(d(Tx, Ty), d(x, y)) \ge 0$$

for all $x, y \in X$. Then T has a unique fixed point in X.

2.3. Consequences

In this section, we deduce some new and existing best proximity points results and fixed point results in the literature from Sections 2.1 and 2.2.

Corollary 2.2. *Let A and B be non-empty closed subsets of a complete metric space (X, d) and $T : A \cup B \to A \cup B$ be a mapping such that $T(A) \subseteq B$ and $T(B) \subseteq A$. If for some $x_0 \in A$ there exists $k_{x_0} \in (0, 1)$ such that*

$$d(T^{2n}x_0, Ty) \le k_{x_0} d(T^{2n-1}x_0, y) + (1 - k_{x_0}) dist(A, B), \quad n \in \mathbb{N}, \quad y \in A. \tag{2.18}$$

Then there exists $(x, y) \in A \times B$ such that $d(x, Tx) = dist(A, B)$ and $d(y, Ty) = dist(A, B)$ with $d(x, y) = dist(A, B)$.

Proof. Let $x_0 \in A$ satisfies (2.18). Then from Remark 2.2(2), there exists $\zeta \in \mathfrak{Z}$ such that T is cyclic orbital simulative contraction. Since for any $n \in \mathbb{N}$, either n or $n + 1$ is even, from (2.18) we have

$$d(T^n x_0, T^{n+1}x_0) \le k_{x_0}^n d(x_0, Tx_0) + (1 - k_{x_0}^n)\mathrm{dist}(A, B),$$

which implies

$$\sum_{n=1}^{\infty} d(T^n x_0, T^{n+1}x_0) \le d(x_0, Tx_0) \sum_{n=1}^{\infty} k_{x_0} + \mathrm{dist}(A, B) \sum_{n=1}^{\infty} (1 - k_{x_0}) \quad < \infty.$$

Thus, $T^n x_0$ is a Cauchy sequence and converges to a point in $A \cup B$. Consequently, $T^{2n}x$ and $T^{2n-1}x$ are convergent sequences in A and B respectively. Hence, the result follows from Theorem 2.1. $\square$

Remark 2.4. By considering $\mathrm{dist}(A, B) = 0$ in Corollary 2.2, we obtain Theorem 2.2 of [34] and in the light of Remark 2.2(1), Theorem 2.2 reduces to Theorem 3.4 of [16].

Corollary 2.3. *Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a mapping such that $T(A) \subseteq B$ and $T(B) \subseteq A$. If for some $x_0 \in A$ there exists a strictly increasing function $\varphi : [0, \infty) \to [0, \infty)$ such that*

$$d(T^{2n}x_0, Ty) \le d(T^{2n-1}x_0, y) - \varphi(d^*(T^{2n-1}x_0, y)), \quad n \in \mathbb{N}, \quad y \in A. \tag{2.19}$$

Then there exists $(x, y) \in A \times B$ such that $d(x, Tx) = dist(A, B)$ and $d(y, Ty) = dist(A, B)$ with $d(x, y) = dist(A, B)$ provided that $\{T^{2n}x_0\}$ and $\{T^{2n+1}x_0\}$ have convergent subsequences in A and B respectively.

Proof. Define $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\zeta(t, s) = \begin{cases} s - \varphi(s) - t & \text{if } t \le s, \\ 2(s - t) & \text{if } t > s, \end{cases}$$

then $\zeta \in \mathfrak{Z}$. Let $x_0 \in A$ satisfies (2.19), then for all $n \in \mathbb{N}$ and $y \in A$ we have

$$d(T^{2n}x_0, Ty) \le d(T^{2n-1}x_0, y).$$

This implies

$$\begin{aligned}\zeta(d^*(T^{2n}x_0, Ty), d^*(T^{2n-1}x_0, y)) &= d^*(T^{2n-1}x_0, y) - \varphi(d^*(T^{2n-1}x_0, y)) - d^*(T^{2n}x_0, Ty)\\ &= d(T^{2n-1}x_0, y) - \varphi(d^*(T^{2n-1}x_0, y)) - d(T^{2n}x_0, Ty)\\ &\geq 0.\end{aligned}$$

Hence T is cyclic orbital simulative contraction. Therefore, the result follows from Theorem 2.1. □

In next example we show that there exists some functions which satisfy (2.19) but not a cyclic φ-contraction [20]

Example 2.4. Let (X, d), A, B and $T : A \cup B \to A \cup B$ be same as given in Example 2.3. Define $\varphi : [0, \infty) \to [0, \infty)$ by $\varphi(t) = \frac{t^2}{1+t}$, then φ is a strictly increasing function. Now for $0 \in A$, $\{T^{2n}0\}$ and $\{T^{2n+1}0\}$ have convergent subsequences in A and B respectively. Also, for all $n \in \mathbb{N}$

$$\begin{aligned}d(T^{2n-1}0, 0) - \varphi(d^*(T^{2n-1}0, 0)) &= d(3, 0) - \varphi(d(3, 0) - 1)\\ &= 1 - \varphi(0) = 1\\ &= d(0, 3)\\ &= d(T^{2n}0, T0)\end{aligned}$$

and

$$\begin{aligned}d(T^{2n-1}0, 1) - \varphi(d^*(T^{2n-1}0, 1)) &= d(3, 1) - \varphi(d(3, 1) - 1)\\ &= 1 - \varphi(0) = 1\\ &= d(0, 2)\\ &= d(T^{2n}0, T1).\end{aligned}$$

This shows that for all $y \in A$, there exists $0 \in A$ which satisfy (2.19). All conditions of Corollary 2.3 hold true and T has a best proximity point.

But for $x = 0$, $y = 2$, we have

$$\frac{d(Tx, Ty)}{d(x, y) - \varphi(d(x, y)) + \varphi(\text{dist}(A, B))} = 1.$$

Thus, there exists no φ such that T is cyclic φ-contraction. Therefore Theorem 4 of [20] cannot be applied for this example.

A function $\phi : [0, \infty) \to [0, 1)$ is said to be MT-function if $\limsup_{a \to b^+} \phi(a) < 1$ for all $b \in [0, \infty)$ [37]. It is obvious that if $\phi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then ϕ is an MT-function.

Corollary 2.4. *Let A and B be non-empty subsets of a metric space (X, d) and $T : A \cup B \to A \cup B$ be a mapping such that $T(A) \subseteq B$ and $T(B) \subseteq A$. If for some $x_0 \in A$ there exists a MT-function $\phi : [0, \infty) \to [0, 1)$ such that*

$$\begin{aligned}d(T^{2n}x_0, Ty) \leq\ & d(T^{2n-1}x_0, y)\phi(d(T^{2n-1}x_0, y))\\ &+ (1 - \phi(d(T^{2n-1}x_0, y)))dist(A, B), \quad n \in \mathbb{N}, \quad y \in A.\end{aligned} \tag{2.20}$$

Then there exists $(x, y) \in A \times B$ such that $d(x, Tx) = dist(A, B)$ and $d(y, Ty) = dist(A, B)$ with $d(x, y) = dist(A, B)$ provided that $\{T^{2n}x_0\}$ and $\{T^{2n+1}x_0\}$ have convergent subsequences in A and B respectively.

Proof. Define $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\zeta(t, s) = \begin{cases} s\phi(s + c) - t & \text{if } t \leq s\\ q(s - t) & \text{if } t > s,\end{cases}$$

for any $c \geq 0$, $q > 1$ and a MT-function ϕ, then $\zeta \in \mathfrak{Z}$. Choose $c = $ dist(A,B) and let $x_0 \in A$ satisfies (2.20), then for all $n \in \mathbb{N}$ and $y \in A$ we have

$$d(T^{2n}x_0, Ty) \leq d(T^{2n-1}x_0, y).$$

This gives

$$\begin{aligned}&\zeta(d^*(T^{2n}x_0, Ty), d^*(T^{2n-1}x_0, y))\\ &= d^*(T^{2n-1}x_0, y)\phi(d^*(T^{2n-1}x_0, y) + \text{dist}(A, B)) - d^*(T^{2n}x_0, Ty)\\ &= d(T^{2n-1}x_0, y)\phi(d(T^{2n-1}x_0, y)) + (1 - \phi(d(T^{2n-1}x_0, y)))\text{dist}(A, B) - d(T^{2n}x_0, Ty)\\ &\geq 0.\end{aligned}$$

Hence T is cyclic orbital simulative contraction. Therefore, the result follows from Theorem 2.1. □

Remark 2.5. Corollary 2.4 generalizes Theorem 2.4 of [21] because every MT-cyclic contraction satisfies (2.20), but converse need not to be true in general, as shown in the example below.

Example 2.5. Let $X = \mathbb{R}$ with usual metric d, $A = [-2, -1]$ and $B = [2, 3]$, then (X, d) is metric space and $\text{dist}(A, B) = 3$. Define $T : A \cup B \to A \cup B$ and $\phi : [0, \infty) \to [0, 1)$ by

$$Tx = \begin{cases} 2 & \text{if } \ x \in [-2, -1] \\ -1 & \text{if } \ x \in [2, 3) \\ -2 & \text{if } \ x = 3 \end{cases}$$

and

$$\phi(a) = \frac{2}{3} \ \text{ for all } \ a \in [0, \infty).$$

So, ϕ is a MT-function and for $n \in \mathbb{N}$, if $x \in [-2, -1]$, then $T^{2n}x = -1$ and $T^{2n-1}x = 2$ and if $x \in [2, 3]$, then $T^{2n}x = 2$ and $T^{2n-1}x = -1$. Now for $-2 \in A$, $\{T^{2n}(-2)\}$ and $\{T^{2n+1}(-2)\}$ have convergent subsequences in A and B respectively. Also, for all $n \in \mathbb{N}$ and $y \in [-2, -1]$, we have

$$\begin{aligned} & d(T^{2n-1}(-2), y)\phi(d(T^{2n-1}(-2), y)) + (1 - \phi(d(T^{2n-1}(-2), y)))\text{dist}(A, B) \\ & = d(2, y)\phi(d(2, y)) + 3(1 - \phi(d(2, y))) \\ & = \frac{2}{3}|y - 2| + 1 \\ & \geq 3 = d(-1, 2) \\ & = d(T^{2n}(-2), Ty). \end{aligned}$$

Hence T satisfies (2.20). Thus, all conditions of Corollary 2.4 are satisfied. Note that $-1 \in A$ and $2 \in B$ such that

$$\begin{aligned} d(-1, T(-1)) &= 3 = \text{dist}(A, B) \\ d(2, T(2)) &= 3 = \text{dist}(A, B) \end{aligned}$$

with

$$d(-1, 2) = 3 = \text{dist}(A, B).$$

On the other side, for $x = -1$ and $y = 3$, we get

$$\begin{aligned} d(x, y)\phi(d(x, y)) + (1 - \phi(d(x, y)))\text{dist}(A, B) &= 4\phi(d(x, y)) + 3 - 3\phi(d(x, y)) \\ &= \phi(d(x, y)) + 3 \\ &< 4 = d(Tx, Ty). \end{aligned}$$

Thus, T is not MT-cyclic contraction and Theorem 2.4 of [21] cannot be applied.

Recall that a function $\psi : [0, \infty) \to [0, \infty)$ is said to be comparison function [38] if it satisfied the following:

(ψ1) ψ is monotone increasing;
(ψ2) $\{\psi^n(t)\}$ converges to 0 as $n \to \infty$ for all $t \geq 0$.

Corollary 2.5. *Let A and B be non-empty subsets of metric space (X, d) and $T : A \cup B \to A \cup B$ be a mapping such that $T(A) \subseteq B$ and $T(B) \subseteq A$. If for some $x_0 \in A$ there exists a comparison-function $\psi : [0, \infty) \to [0, \infty)$ such that*

$$d(T^{2n}x_0, Ty) \leq \psi(d^*(T^{2n-1}x_0, y)) - dist(A, B), \quad n \in \mathbb{N}, \quad y \in A. \tag{2.21}$$

Then there exists $(x, y) \in A \times B$ such that $d(x, Tx) = dist(A, B)$ and $d(y, Ty) = dist(A, B)$ with $d(x, y) = dist(A, B)$ provided that $\{T^{2n}x_0\}$ and $\{T^{2n+1}x_0\}$ have convergent subsequences in A and B respectively.

Proof. Define $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\zeta(t, s) = \begin{cases} \psi(s) - t & \text{if } t \leq s \\ q(s - t) & \text{if } t > s, \end{cases}$$

where ψ is comparison function and $q > 1$. Then $\zeta \in \mathfrak{Z}$. Let $x_0 \in A$ satisfies (2.21), then for all $n \in \mathbb{N}$ and $y \in A$ we have

$$d(T^{2n}x_0, Ty) \leq \psi(d^*(T^{2n-1}x_0, y)).$$

This implies

$$\zeta(d^*(T^{2n}x_0, Ty), d^*(T^{2n-1}x_0, y)) = \psi(d^*(T^{2n-1}x_0, y) - d^*(T^{2n}x_0, Ty)) \geq 0.$$

Hence T is cyclic orbital simulative contraction. Therefore, the result follows from Theorem 2.1. □

Considering $\text{dist}(A, B) = 0$ in Corollary 2.5, we obtain the following:

Corollary 2.6. *Let A and B be non-empty subsets of metric space (X, d) with $dist(A, B) = 0$ and $T : A \cup B \to A \cup B$ be a mapping such that $T(A) \subseteq B$ and $T(B) \subseteq A$. If for some $x_0 \in A$ there exists a comparison-function $\psi : [0, \infty) \to [0, \infty)$ such that*

$$d(T^{2n}x_0, Ty) \leq \psi(d(T^{2n-1}x_0, y)), \quad n \in \mathbb{N}, \quad y \in A. \tag{2.22}$$

Then T has a unique fixed point in $A \cap B$ provided that $\{T^{2n}x_0\}$ and $\{T^{2n+1}x_0\}$ have convergent subsequences in A and B respectively.

Further, From Corollary 2.6, we get following two results:

Corollary 2.7. *Let A and B be non-empty subsets of metric space (X, d) with $dist(A, B) = 0$ and $T : A \cup B \to A \cup B$ be a mapping such that $T(A) \subseteq B$, $T(B) \subseteq A$ and for some $x_0 \in A$ satisfying (2.22). If A and B are boundedly compact, then T has a unique fixed point in $A \cap B$.*

Corollary 2.8. *Let A and B be non-empty closed subsets of a complete metric space (X, d). If $T : A \cup B \to A \cup B$ is a mapping such that $T(A) \subseteq B$, $T(B) \subseteq A$ and for some $x_0 \in A$ satisfying (2.22), then T has a unique fixed point in $A \cap B$.*

Remark 2.6. Corollary 2.8 generalizes Theorem 2.1 of [35] for $m = 2$. Indeed, let f be a cyclic ψ-contraction(Definition 1.3 of [35]), then $f(A) \subseteq B, f(B) \subseteq A$. From Lemma 2.2 of [38], every c-comparison function is a comparison function, so there exists a comparison function ψ such that f satisfies (2.22) and hence the result. Also, from Theorem 2.5 of [36], Theorem 1.8 of [36] is equivalent to Theorem 2.1 of [35]. Thus, Corollary 2.8 also generalizes Theorem 1.8 of [36].

Note that Corollary 2.8 is the proof of the open problem given in [36] for $m = 2$.

3. Best proximity point results for non-self mappings

In this section, we prove some best proximity results for non-self mappings with the help of Lemma 2.1 and show that many existing results in the literature can be proved by taking $\zeta \in \mathfrak{Z}$ instead of $\zeta \in \Delta(Z)$. Let (X, d) be a metric space, A and B be nonempty subsets of X, $T : A \to B$ and g be a self mapping on A. We say that $g \in \mathcal{G}_A$ if g is continuous and $d(x, y) \leq d(gx, gy)$ for all $x, y \in A$ and $T \in \mathcal{T}_g$ if $d(Tx, Ty) \leq d(Tgx, Tgy)$ for all $x, y \in A$. In the sequel, we will use the following notations.

$$\begin{aligned}
A_0 &= \{x \in A : d(x, y) = \text{dist}(A, B), \quad \text{for some} \quad y \in B\}, \\
B_0 &= \{y \in B : d(x, y) = \text{dist}(A, B), \quad \text{for some} \quad x \in A\}, \\
d(x,B) &= \inf\{d(x, y) : y \in B\}.
\end{aligned}$$

Lemma 3.1. *Let A and B be nonempty subsets of a metric space (X, d) and $T : A \to B$ be a continuous mapping. Suppose that A_0 and B_0 are nonempty with $T(A_0) \subseteq B_0$. If every sequence $\{x_n\}$ in A_0 has a convergent subsequence in A_0, then there exists a unique element $x \in A$ such that $d(x, Tx) = dist(A, B)$.*

Proof. Let $x_0 \in A_0$, then we can choose some $x_1 \in A_0$ such that $d(x_1, Tx_0) = \text{dist}(A, B)$, because $T(A_0) \subseteq B_0$. Since $Tx_1 \in B_0$, there exists some $x_2 \in A_0$ such that $d(x_2, Tx_1) = \text{dist}(A, B)$. Continuing in this manner, we can obtain two sequences $\{x_n\}$ in A_0 and $\{Tx_n\}$ in B_0 such that $d(x_n, Tx_{n-1}) = \text{dist}(A, B)$ and $d(x_{n+1}, Tx_n) = \text{dist}(A, B)$ for all $n \in \mathbb{N}$. By hypothesis there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\lim_{k\to\infty} x_{n_k} = x$$

for some $x \in A_0$. Now

$$\begin{aligned}
\text{dist}(A, B) &\leq d(x, Tx_{n_k}) \\
&\leq d(x, x_{n_k+1}) + d(x_{n_k+1}, Tx_{n_k}) \\
&= d(x, x_{n_k+1}) + \text{dist}(A, B).
\end{aligned} \tag{3.1}$$

Letting $k \to \infty$ in (3.1), we get

$$\lim_{k\to\infty} d(x, Tx_{n_k}) = \text{dist}(A, B).$$

Continuity of T gives $d(x, Tx) = \text{dist}(A, B)$. $\square$

Lemma 3.2. *Let A and B be nonempty subsets of a complete metric space (X, d) and $T : A \to B$ be a mapping. Suppose that A_0 and B_0 are nonempty with $T(A_0) \subseteq B_0$ and there exists $\zeta \in \mathfrak{Z}$ such that for all $u_1, u_2, x_1, x_2 \in A$, T satisfy*

$$d(u_1, Tx_1) = d(u_2, Tx_2) = dist(A, B) \;\; \textit{implies} \;\; \zeta(d(Tu_1, Tu_2), d(Tx_1, Tx_2)) \geq 0. \tag{3.2}$$

If B_0 is closed then every sequence $\{x_n\}$ in A_0, we have

$$\lim_{n\to\infty} d(y, x_n) = d(y, A)$$

for some $y \in B_0$.

Proof. Let $x_0 \in A_0$, then by following the same as in the proof of Lemma 3.1, we obtain a sequence $\{x_n\}$ in A_0 and $\{Tx_n\}$ in B_0 satisfying for all $n \in \mathbb{N}$

$$d(x_n, Tx_{n-1}) = \mathrm{dist}(A, B) \quad \text{and} \quad d(x_{n+1}, Tx_n) = \mathrm{dist}(A, B). \tag{3.3}$$

Assume that $Tx_{n-1} \neq Tx_n$ for all $n \in \mathbb{N}$, then from (3.2), we get

$$\begin{aligned} 0 &\leq \zeta(d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)) \\ &< d(Tx_{n-1}, Tx_n) - d(Tx_n, Tx_{n+1}) \end{aligned}$$

and hence for all $n \in \mathbb{N}$

$$d(Tx_n, Tx_{n+1}) < d(Tx_{n-1}, Tx_n). \tag{3.4}$$

It follows from (3.4) that $\{d(Tx_n, Tx_{n+1})\}$ is a monotonically decreasing sequence of nonnegative real numbers and bounded below. So, there exists $r \geq 0$ such that $d(Tx_n, Tx_{n+1}) \to r$ as $n \to \infty$. We claim that $r = 0$. Otherwise, we have

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = \lim_{n\to\infty} d(Tx_{n-1}, Tx_n) = r > 0 \tag{3.5}$$

and from (3.4) we have

$$0 < d(Tx_n, Tx_{n+1}) < d(Tx_{n-1}, Tx_n).$$

Therefor, by using Lemma 2.1 with (3.2), we obtain

$$0 \leq \lim_{n\to\infty} \zeta(d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)) \leq 0.$$

This implies

$$\lim_{n\to\infty} \zeta(d(Tx_n, Tx_{n+1}), d(Tx_{n-1}, Tx_n)) = 0.$$

From $(\zeta 4)$, we get

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) < \lim_{n\to\infty} d(Tx_{n-1}, Tx_n),$$

which leads to the contradiction to (3.5). Hence

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0. \tag{3.6}$$

To prove $\{Tx_n\}$ is a Cauchy sequence, it is enough to prove that $\{Tx_{2n}\}$ is a Cauchy sequence in X. Suppose on contrary, then there exists $\epsilon > 0$ and two subsequences $\{Tx_{2m_k}\}$ and $\{Tx_{2n_k}\}$ of $\{Tx_{2n}\}$ with $n_k > m_k \geq k$ such that

$$d(Tx_{2m_k}, Tx_{2n_k}) \geq \epsilon. \tag{3.7}$$

Without any loss of generality, we assume that for all $k \in \mathbb{N}$, n_k is the smallest positive integer greater than m_k for which (3.7) holds, then for all $k \in \mathbb{N}$

$$d(Tx_{2m_k}, Tx_{2n_k-2}) < \epsilon. \tag{3.8}$$

From (3.7) and (3.8), we get

$$\begin{aligned} \epsilon &\leq d(Tx_{2m_k}, Tx_{2n_k}) \\ &\leq d(Tx_{2m_k}, Tx_{2n_k-2}) + d(Tx_{2n_k-2}, Tx_{2n_k-1}) + d(Tx_{2n_k-1}, Tx_{2n_k}) \\ &< \epsilon + d(Tx_{2n_k-2}, Tx_{2n_k-1}) + d(Tx_{2n_k-1}, Tx_{2n_k}). \end{aligned} \tag{3.9}$$

Letting $k \in \infty$ in (3.9) and using (3.6), we obtain

$$\lim_{k\to\infty} d(Tx_{2m_k}, Tx_{2n_k}) = \epsilon. \tag{3.10}$$

Similarly,

$$\begin{aligned} \epsilon &\leq d(Tx_{2m_k}, Tx_{2n_k}) \\ &\leq d(Tx_{2m_k}, Tx_{2m_k+1}) + d(Tx_{2m_k+1}, Tx_{2n_k+1}) + d(Tx_{2n_k+1}, Tx_{2n_k}), \end{aligned} \tag{3.11}$$

and

$$d(Tx_{2m_k+1}, Tx_{2n_k+1}) \leq d(Tx_{2m_k+1}, Tx_{2m_k}) + d(Tx_{2m_k}, Tx_{2n_k}) + d(Tx_{2n_k}, Tx_{2n_k+1}). \tag{3.12}$$

Letting $k \to \infty$ in (3.11) and (3.12), using (3.6) and (3.10), we get

$$\lim_{k\to\infty} d(Tx_{2m_k+1}, Tx_{2n_k+1}) = \epsilon. \tag{3.13}$$

Also, from (3.6) we have

$$0 < d(Tx_{2m_k+1}, Tx_{2n_k+1}) < d(Tx_{2m_k}, Tx_{2n_k}). \tag{3.14}$$

By using Lemma 2.1, it follows from (3.3), (3.10), (3.13) and (3.14) that

$$0 \le \lim_{k\to\infty} \zeta(d(Tx_{2m_k+1}, x_{2n_k+1}), d(Tx_{2m_k}, x_{2n_k})) \le 0.$$

This implies

$$\lim_{k\to\infty} \zeta(d(Tx_{2m_k+1}, x_{2n_k+1}), d(Tx_{2m_k}, x_{2n_k})) = 0,$$

which further together with $(\zeta 4)$ gives

$$\lim_{k\to\infty} d(Tx_{2m_k+1}, x_{2n_k+1}) < \lim_{k\to\infty} d(Tx_{2m_k}, x_{2n_k}),$$

a contradiction. Thus, $\{Tx_n\}$ is a Cauchy sequence in X. Since B_0 is closed subset of a complete metric space X, there exists $y \in B_0$ such that

$$\lim_{n\to\infty} Tx_n = y. \tag{3.15}$$

From (3.3), for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(y, A) \le d(y, x_{n+1}) &\le d(y, Tx_n) + d(Tx_n, x_{n+1}) \\ &= d(y, Tx_n) + \mathrm{dist}(A, B) \\ &\le d(y, Tx_n) + d(y, A). \end{aligned} \tag{3.16}$$

Letting $n \to \infty$ in (3.16) and using (3.15), we obtain

$$\lim_{n\to\infty} d(y, x_{n+1}) = d(y, A). \quad \square$$

Recall that a set B is said to be approximatively compact with respect to A if every sequence $\{y_n\}$ in B, satisfying for some $x \in A$, $d(x, y_n) \to d(x, B)$ as $n \to \infty$ has a convergent subsequence [14]. So, from Lemmas 3.1 and 3.2, we get the following:

Theorem 3.1. *Let A and B be nonempty subsets of a complete metric space (X, d) and $T : A \to B$ be a continuous mapping. Suppose that A_0 and B_0 are nonempty with $T(A_0) \subseteq B_0$ and there exists $\zeta \in \mathfrak{Z}$ such that T satisfy (3.2). If A is approximatively compact with respect to B and B_0 is closed, then there exists $x \in A$ such that $d(x, Tx) = dist(A, B)$.*

Theorem 3.2. *Let A and B be nonempty closed subsets of a complete metric space (X, d) and $T : A \to B$ be a continuous mapping. Suppose that A_0 and B_0 are nonempty with $T(A_0) \subseteq B_0$ and there exists $\zeta \in \mathfrak{Z}$ such that for all $u_1, u_2, x_1, x_2 \in A$, T satisfy*

$$d(u_1, Tx_1) = d(u_2, Tx_2) = dist(A, B) \;\; implies \;\; \zeta(d(u_1, u_2), d(x_1, x_2)) \ge 0. \tag{3.17}$$

Then there exists a unique element $x \in A$ such that $d(x, Tx) = dist(A, B)$. Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ satisfying $d(x_n, Tx_n) = dist(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$ converges to the best proximity point x of T.

Proof. Let $x_0 \in A_0$, then by following the same as in proof of Lemma 3.1, we obtain a sequence $\{x_n\}$ in A_0 such that $d(x_n, Tx_{n-1}) = \mathrm{dist}(A, B)$ and $d(x_{n+1}, Tx_n) = \mathrm{dist}(A, B)$ for all $n \in \mathbb{N}$. Assume that $d(x_{n-1}, x_n) > 0$. In other case, x_n is the best proximity point of T. From (3.17), for $\zeta \in \mathfrak{Z}$, we get

$$0 \le \zeta(d(x_n, x_{n+1}), d(x_{n-1}, x_n)) < d(x_{n-1}, x_n) - d(x_n, x_{n+1})$$

and hence for all $n \in \mathbb{N}$

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n). \tag{3.18}$$

It follows from (3.18) that $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of nonnegative real numbers and bounded below. So, there exists $r \ge 0$ such that $d(x_n, x_{n+1}) \to r$ as $n \to \infty$. By using the same arguments as in Lemma 3.2, we get $r = 0$

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

and $\{x_n\}$ is a Cauchy sequence. Rest of the proof follows from the proof of Theorem 1 of [22]. $\square$

Theorem 3.3. *Let A and B be two nonempty subsets of a complete metric space (X, d), $T : A \to B$ and $g \in \mathcal{G}_A$. Suppose that A_0 and B_0 are nonempty with $T(A_0) \subseteq B_0$, $A_0 \subseteq g(A_0)$ and there exists $\zeta \in \mathfrak{Z}$ such that T satisfy (3.17). If A_0 is closed, then there exists a unique element $x \in A$ such that $d(gx, Tx) = dist(A, B)$. Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ satisfying $d(gx_n, Tx_n) = dist(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$ converges to the best proximity point x of T.*

Proof. Let $x_0 \in A_0$. Since $T(A_0) \subseteq B_0$ and $A_0 \subseteq g(A_0)$, there exists some $x_1 \in A_0$ such that $d(gx_1, Tx_0) = \text{dist}(A, B)$ and for $x_1 \in A_0$ there exists $x_2 \in A_0$ such that $d(gx_2, Tx_1) = \text{dist}(A, B)$. Continuing in this manner, for $x_n \in A_0$, we obtain a sequence $\{x_{n+1}\}$ in A_0 such that $d(gx_{n+1}, Tx_n) = \text{dist}(A, B)$ for all $n \in \mathbb{N}$. Assume that $d(gx_{n+1}, gx_n) > 0$, if $d(gx_{n+1}, gx_n) = 0$, then $d(gx_m, Tx_m) = \text{dist}(A, B)$. Since $g \in \mathcal{G}_A$, so from (3.17), we have

$$\begin{aligned} 0 &\le \zeta(d(gx_{n+1}, gx_n), d(x_n, x_{n-1})) \\ &< d(x_n, x_{n-1}) - d(gx_{n+1}, gx_n) \\ &\le d(x_n, x_{n-1}) - d(x_{n+1}, x_n). \end{aligned} \tag{3.19}$$

It follows from (3.19) that $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of nonnegative real numbers and bounded below. So, there exists $r \ge 0$ such that $d(x_n, x_{n+1}) \to r$ as $n \to \infty$. By using the same arguments as in Lemma 3.2, we get $r = 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0$$

and $\{x_n\}$ is a Cauchy sequence in X. Rest of the proof follows from the proof of Theorem 3.1 of [25]. □

Theorem 3.4. *Let A and B be two nonempty subsets of a complete metric space (X, d), $T \in \mathcal{T}_g$ and $g \in \mathcal{G}_A$. Suppose that A_0 and B_0 are nonempty with $T(A_0) \subseteq B_0$, $A_0 \subseteq g(A_0)$ and there exists $\zeta \in \mathfrak{Z}$ such that T satisfy (3.2). If $T(A_0)$ is closed and T is injective on A, then there exists a unique element $x \in A$ such that $d(gx, Tx) = dist(A, B)$. Moreover, for any fixed element $x_0 \in A_0$, the sequence $\{x_n\}$ satisfying $d(gx_n, Tx_n) = dist(A, B)$ for all $n \in \mathbb{N} \cup \{0\}$ converges to the best proximity point x of T.*

Proof. Let $x_0 \in A_0$. By following the similar reason as in the proof of Lemma 3.1, for $x_n \in A_0$, we obtain a sequence $\{x_{n+1}\}$ in A_0 such that $d(gx_{n+1}, Tx_n) = \text{dist}(A, B)$ for all $n \in \mathbb{N}$. Assume that $d(gx_{n+1}, gx_n) > 0$, if $d(gx_{n+1}, gx_n) = 0$, then $d(gx_m, Tx_m) = \text{dist}(A, B)$. Since $g \in \mathcal{G}_A$, $T \in \mathcal{T}_g$ and T is injective, so from (3.2), we have

$$\begin{aligned} 0 &\le \zeta(d(Tgx_{n+1}, Tgx_n), d(Tx_n, Tx_{n-1})) \\ &< d(Tx_n, Tx_{n-1}) - d(Tgx_{n+1}, Tgx_n) \\ &\le d(Tx_n, Tx_{n-1}) - d(Tx_{n+1}, Tx_n). \end{aligned} \tag{3.20}$$

It follows from (3.20) that $\{d(Tx_n, Tx_{n+1})\}$ is a monotonically decreasing sequence of nonnegative real numbers and bounded below. So, there exists $r \ge 0$ such that $d(x_n, x_{n+1}) \to r$ as $n \to \infty$. By using the same arguments as in Lemma 3.2, we get $r = 0$ such that

$$\lim_{n\to\infty} d(Tx_n, Tx_{n+1}) = 0$$

and $\{x_n\}$ is a Cauchy sequence in X. Rest of the proof follows from the proof of Theorem 3.2 of [25]. □

Remark 3.1. Note that in Theorem 3.1 both A and B need not to be closed. Therefore, Theorems 3.1 and 3.2 are the refinement of Theorem 2 and Theorem 1 of [22] respectively while Theorems 3.3 and 3.4 refine Theorems 3.1 and 3.2 of [25] respectively.

Now, we endow the set X with a partial order $\preceq$. For a given $\zeta \in \mathfrak{Z}$, denote T_ζ by the set of mappings $T : A \to B$ satisfying the following conditions for every $u_1, u_2, x_1, x_2 \in A$:

(C1) $x_1 \preceq x_2$, $d(u_1, Tx_1) = d(u_2, Tx_2) = \text{dist}(A, B)$ implies $u_1 \preceq u_2$;
(C2) $x_1 \preceq x_2$, $x_1 \ne x_2$, $d(u_1, Tx_1) = d(u_2, Tx_2) = \text{dist}(A, B)$ implies $\zeta(d(u_1, u_2), M(x_1, x_2)) \ge 0$, where

$$M(x_1, x_2) = \max\left\{\frac{d(x_1, u_1)d(x_2, u_2)}{d(x_1, x_2)}, d(x_1, x_2)\right\}.$$

Theorem 3.5. *Let A and B be two nonempty subsets of a complete partially ordered metric space $(X, d, \preceq)$ and $T \in \mathrm{T}_\zeta$ for some $\zeta \in \mathfrak{Z}$. Suppose that A_0 and B_0 are nonempty with $T(A_0) \subseteq B_0$, $A_0 \subseteq g(A_0)$ and there exists $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A, B)$. If A_0 is closed and T is continuous, then there exists some $x \in A$ such that $d(x, Tx) = dist(A, B)$. In addition, if for every $(x, y) \in A_0 \times A_0$ there exists some $w \in A_0$ such that $x \preceq w$ and $y \preceq w$, then T has a unique best proximity point.*

Proof. From hypothesis, there exists $x_0, x_1 \in A_0$ such that $x_0 \preceq x_1$ with $d(x_1, Tx_0) = \text{dist}(A, B)$, so, $Tx_1 \in B_0$, which yields $d(x_2, Tx_1) = \text{dist}(A, B)$ for some $x_2 \in A_0$. Since $x_0 \preceq x_1$, condition (C1) gives that $x_1 \preceq x_2$. Continuing is the same manner, we obtain a sequence $\{x_n\}$ in A_0 such that

$$d(x_{n+1}, Tx_n) = \mathrm{dist}(A, B) \tag{3.21}$$

with $x_0 \preceq x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots$. Assume that $d(x_n, x_{n+1}) > 0$, in other case, x_n is a best proximity point of T. Since $x_n \preceq x_{n+1}$, from condition (C2) and (3.21), we get

$$0 \leq \zeta(d(x_n, x_{n+1}), M(x_{n-1}, x_n)), \tag{3.22}$$

where

$$M(x_{n-1}, x_n) = \max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\}.$$

If $\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} = d(x_n, x_{n+1})$, then

$$0 \leq \zeta(d(x_n, x_{n+1}), d(x_n, x_{n+1})) < 0,$$

a contradiction. Hence $\max\{d(x_n, x_{n+1}), d(x_{n-1}, x_n)\} = d(x_{n-1}, x_n)$, therefore, (3.22) implies

$$0 \leq \zeta(d(x_n, x_{n+1}), d(x_{n-1}, x_n)) < d(x_{n-1}, x_n) - d(x_n, x_{n+1}). \tag{3.23}$$

It follows from (3.23) that $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of nonnegative real numbers and bounded below. So, there exists $r \geq 0$ such that $d(x_n, x_{n+1}) \to r$ as $n \to \infty$. By using the same arguments as in Lemma 3.2, we get $r = 0$ and $\{x_n\}$ is a Cauchy sequence in X. Rest of the proof follows from the proof of Theorem 2.1 of [24] and uniqueness follows from Theorem 2.4 of [24]. □

A set A is $(d, \preceq)$-regular if $\{a_n\} \subset A$ is non-decreasing w.r.t $\preceq$ and $d(a_n, b) \to 0$ as $n \to \infty$ implies $b = \sup\{a_n\}$ [24].

Theorem 3.6. *Let A and B be two nonempty subsets of a complete partially ordered metric space $(X, d, \preceq)$ and $T \in \mathrm{T}_\zeta$ for some $\zeta \in \mathfrak{Z}$. Suppose that A_0 and B_0 are nonempty with $T(A_0) \subseteq B_0$, $A_0 \subseteq g(A_0)$ and there exists $x_0, x_1 \in A_0$ such that $d(x_1, Tx_0) = dist(A, B)$. If A_0 is closed and A is $(d, \preceq)$-regular, then there exists some $x \in A$ such that $d(x, Tx) = dist(A, B)$. In addition, if for every $(x, y) \in A_0 \times A_0$ there exists some $w \in A_0$ such that $x \preceq w$ and $y \preceq w$, then T has a unique best proximity point.*

Proof. By following the same approach as in Theorem 3.5, we obtain a Cauchy sequence $\{x_n\}$ in X and remaining proof follows from the proof of Theorem 2.2 of [24]. □

Remark 3.2. Since there exists functions $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ such that $\zeta \in \mathfrak{Z}$ but $\zeta \notin \Delta(Z)$ (see Example 2.1). Therefore, Theorems 3.5 and 3.6 refine Theorems 2.1 and 2.2 of [24] respectively.

4. Solution to variational inequality problem, optimization problem and elliptic boundary value problem

Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and induced norm $\|.\|$ and K be a non-empty subset of H. An element $y_0 \in K$ is known as best approximation if $\|x - y_0\| = D(x, K)$, where $D(x, K) = \inf_{y \in K} \|x - y\|$ and the metric projection is a mapping $P_K : H \to K$ such that for all $x \in H$, $P_K(x) = \{y \in K : \|x - y\| = D(x, K)\}$ [39]. This metric projection plays an important role for solving the variational inequality problem. A variational inequality problem $VI(S, K)$ can be stated as follows:

Problem 4.1. Find $u \in K$ such that $\langle Su, v - u \rangle \geq 0$ for all $v \in K$, where $S : H \to H$ is given operator and K is non-empty, closed and convex subset of Hilbert space H.

It is known that, for each $u \in H$, there exists a unique nearest point $P_K(u) \in K$ such that $\|u - P_K(u)\| \leq \|u - v\|$ for all $v \in K$ [39].

Lemma 4.1 ([39]). *Let K be convex subset of Hilbert space H, $z \in H$ and $u \in K$. Then $u = P_K(z)$ if and only if $\langle z - u, v - u \rangle \leq 0$ for all $v \in K$.*

Lemma 4.2. *Let K be convex subset of Hilbert space H and $S : K \to K$. Then $u \in K$ is a solution of $\langle Su, v - u \rangle \geq 0$ for all $v \in K$ if and only if $u = P_K(u - \lambda Su)$, $\lambda > 0$.*

Proof. Take $z = u - \lambda Su$. Then by Lemma 4.1, we get

$$\begin{aligned} u = P_K(z) = P_K(u - \lambda Su) &\Leftrightarrow \langle -\lambda Su, v - u \rangle \leq 0 \\ &\Leftrightarrow -\lambda \langle Su, v - u \rangle \leq 0 \\ &\Leftrightarrow \langle Su, v - u \rangle \geq 0. \quad \square \end{aligned}$$

Now we give some results for the solution of Problem 4.1.

Theorem 4.1. *Let K be a non-empty, closed and convex subset of a real Hilbert space. Assume that for $x_0 \in K$ there exists $\zeta \in \mathfrak{Z}$ is such that for $S : K \to K$, $P_K^* = P_K(I_K - \lambda S) : K \to K$ satisfies*

$$\zeta(\|P_K^{*2n}x_0 - P_K^* y\|, \|P_K^{*2n-1}x_0 - y\|) \geq 0, \quad n \in \mathbb{N}, \quad y \in K,$$

where I_K is the identity operator on K. Then there exists a unique element $u \in K$ such that $\langle Su, v - u\rangle \geq 0$ for all $v \in K$.

Proof. Define $T : K \to K$ by $Tx = P_K(x - \lambda Sx)$ for all $x \in K$, then T satisfies all the hypothesis of Theorem 2.5 by setting $A = B = K$ and so T has a unique fixed point u. Hence by Lemma 4.2, $u \in K$ is solution of $\langle Su, v - u\rangle \geq 0$ for all $v \in K$ if and only if u is a fixed point of T. This completes the proof. □

Corollary 4.1. *Let K be a non-empty, closed and convex subset of a real Hilbert space. Assume that for $x_0 \in K$ there exists a comparison function $\psi : [0, \infty) \to [0, \infty)$ is such that for $S : K \to K$, $P_K^* = P_K(I_K - \lambda S) : K \to K$ satisfies*

$$\|P_K^{*2n}x_0 - P_K^* y\| \leq \psi(\|P_K^{*2n-1}x_0 - y\|), \quad n \in \mathbb{N}, \quad y \in K, \tag{4.1}$$

where I_K is the identity operator on K. Then there exists a unique element $u \in K$ such that $\langle Su, v - u\rangle \geq 0$ for all $v \in K$.

Proof. Define $\zeta : [0, \infty) \times [0, \infty) \to \mathbb{R}$ by

$$\zeta(t, s) = \begin{cases} \psi(s) - t & \text{if } t \leq s \\ q(s - t) & \text{if } t > s, \end{cases}$$

where ψ is comparison function and $q > 1$. Then $\zeta \in \mathfrak{Z}$. Let $x_0 \in K$ satisfies (4.1), then for all $n \in \mathbb{N}$ and $y \in K$, we have

$$\zeta(\|P_K^{*2n}x_0 - P_K^* y\|, \|P_K^{*2n-1}x_0 - y\|) = \psi(\|P_K^{*2n-1}x_0 - y\|) - d\|P_K^{*2n}x_0 - P_K^* y\| \geq 0.$$

Hence all conditions of Theorem 4.1 hold and we get the result. □

An optimization problem can be stated as:

Problem 4.2. To find $u^* \in K$ that minimizes the function $f(u)$ with subject to $u \in K$, where $f : H \to \mathbb{R}$ is continuously differentiable function and K is closed and convex subset of H.

The optimization problems can be formulated as variational inequality problems. The relationship between an optimization problem and a variational inequality problem is given by the following Lemmas.

Lemma 4.3. *Let u^* be the solution of Problem 4.2. Then u^* is the solution of variational inequality problem $\langle \nabla f(u^*), u - u^*\rangle \geq 0$ for all $u \in K$, where ∇f denoting the gradient of f.*

Lemma 4.4. *If f is a convex function on K and u^* is a solution of $VI(\nabla f, K)$. Then u^* is the solution of Problem 4.2.*

By using Theorem 4.1, we use another approach to get the solution of following optimization problem posed in [39]:

Problem 4.3. Consider the linear system of m equations in the n unknowns $u_1, u_2, \dots, u_n$

$$\begin{aligned} a_{11}u_1 + a_{12}u_2 \quad &+ \cdots + a_{1n}u_n = b_1 \\ &\cdots \\ a_{m1}u_1 + a_{m2}u_2 \quad &+ \cdots + a_{mn}u_n = b_m, \end{aligned}$$

or $Au = b$. Find a vector $u^* = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ that minimizes the expression

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}u_j - b_i \right)^2.$$

Solution. Let $H = L_2(m)$, where $L_2(m)$ is the space of functions $u : \{1, 2, \dots, m\} \to \mathbb{R}$ with an inner product $\langle u, v\rangle = \sum_{i=1}^{n} u(i)v(i)$ and ℓ_2-norm $\|u\| = \left[\sum_{i=1}^{n} |u(i)|^2\right]^{\frac{1}{2}}$, $K = \{v \in H : v = Au, u \in \mathbb{R}^n\}$ and $f(u) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij}u_j - b_i\right)^2$, where $u = (u_1, u_2, \dots, u_n)$. Then K is convex subset of H, $f : H \to \mathbb{R}$ is continuously differentiable and also $\nabla^2 f(u) \geq 0$ for all $u \in K$, therefore f is convex on K. Hence from Lemma 4.4, Problem 4.3 may be restated as follows: "Find $u^* \in K$ that satisfies $\langle \nabla f(u^*), v - u^*\rangle \geq 0$ for all $v \in K$". So by setting $S = \nabla f$ in Theorem 4.1, we get the solution of Problem 4.3. □

Recall that an operator $T : H \to H$ is called a potential operator (or gradient operator) on H [40], if there exists a Gâteaux differentiable functional $\phi : H \to \mathbb{R}$ such that $Grad\phi(x) = Tx$, for all $x \in H$, that is for all $x, h \in H$

$$\lim_{t\to 0} \frac{\phi(x+th)-\phi(x)}{t} = \langle Tx, h\rangle.$$

Consider the functional,

$$\phi(x) = \frac{1}{2}\|x\|^2 - \int_0^1 T(sx, x)ds$$

for all $x \in H$.

Proposition 4.1 (*[41]*)**.** *The fixed points of T agree with the global minima of the functional ϕ.*

Theorem 4.2. *Let H be a real Hilbert space with $\langle . \rangle$ the scaler product, C be a non-empty, convex, closed and bounded subset of H and $\phi : H \to \mathbb{R}$ be a twice Gâteaux differentiable on H. If $\|(I' - \phi'')(u)\| \leq k$, where $k \in (0, 1)$ and $(I - \phi')(C) \subset C$. Then, ϕ has a global minimum on H. Indeed there exists a $u^* \in C$ such that*

$$\phi(u^*) = \inf_H \phi.$$

In particular, $\phi'(u^*) = 0$.

Proof. Define $T : H \to H$ by $T = I - \phi'$, then T is potential operator. Also, from mean value theorem [42], for all $u, v \in H$ there exists $\tau \in [0, 1]$ such that

$$\begin{aligned}\|Tu - Tv\| &\leq \|DT(\tau u + (1-\tau)v)(u - v)\| \\ &\leq \|DT(\tau u + (1-\tau)v)\| \|(u - v)\| \\ &= \|(I' - \phi'')(\tau u + (1-\tau)v)\| \|(u - v)\| \\ &\leq k\|(u - v)\|.\end{aligned}$$

Hence there exists $\zeta \in \mathfrak{Z}$ defined as $\zeta(t, s) = s - rt$, $r > 1$ such that T satisfies $\zeta(\|Tu - Tv\|, \|u - v\|) \geq 0$. Therefore, for $u_0 \in H$, we get $\zeta(\|T^{2n}u_0 - Tv\|, \|T^{2n-1}u_0 - v\|) \geq 0$. So, T satisfies all the hypothesis of Theorem 2.5 by setting $A = B = K$, T has a unique fixed point u^*. By Proposition 4.1, fixed point u^* is a global minimizer of ϕ on H. □

Now by using Theorem 4.2, we find a weak solution of the following elliptic boundary value problem given in [40]:

$$\begin{cases} -\Delta u = f(x, u), & x \in \Omega \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4.2}$$

where $\Omega \in \mathbb{R}^n$ is a bounded domain in a n-dimensional real space, $f, f' : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, here f' is the derivative of f with respect to its second variable.

Note that a weak solution of (4.2) is a solution of following variational problem:

$$\begin{cases} \int_\Omega \nabla u \cdot \nabla v dx - \int_\Omega f(x, u) \cdot v dx, & \text{for all} \quad v \in H_0^1(\Omega), \\ u(x) \in H_0^1(\Omega). \end{cases} \tag{4.3}$$

Theorem 4.3. *Let $\mathbb{R}^n$ be an n-dimensional real space and $\Omega \subset \mathbb{R}^n$ be a nonempty bounded domain. Assume that following assertions hold:*

(1) *for each fixed $x \in \Omega$, $f(x, y)$ is a nondecreasing function of y for $w(x) \leq y \leq W(x)$;*
(2) *for all $x \in \Omega$, $s \in \mathbb{R}$, $\|f(x, s)\| \leq k^2$ where $k \in (0, 1)$;*
(3) *$|f(x, s)| \leq c_1|s|^{\sigma_1} + d_1$, and $|f'(x, s)| \leq c_2|s|^{\sigma_2} + d_2$ for some positive constants c_1, c_2, d_1, d_2 and $0 \leq \sigma_1, \sigma_2 < \frac{N+2}{N-2}$ if $N \geq 3(0 \leq \sigma_1, \sigma_2 < \infty$ if $N = 1, 2)$.*

Then there exists $u_0 \in H_0^1(\Omega)$ which is a weak solution of problem (4.2) and $u_0 \in [w, W]$.

Proof. Consider the problem (4.3) and define $\phi : H_0^1 \to \mathbb{R}$ by

$$\phi(u) = \frac{1}{2}\|u\|^2 - \int_\Omega F(x, u)dx \text{ with } F(x, u) = \int_0^u f(x, \xi)d\xi.$$

Then by assumption (3), ϕ is twice differentiable [40,43,44]. Let $C = [v, w] = \{u \in H_0^1(\Omega) : v(x) \leq u(x) \leq w(x), \forall\, x \in \Omega\}$, here $v, w \in H_0^1(\Omega)$ are subsolution and a supersolution of problem (4.3) respectively. Then from the proof of Theorem 6 in [40], $C \subset H_0^1(\Omega)$ is a closed, convex and bounded, also $(I - \phi')(C) \subset C$. From Cauchy–Schwarz and the Poincaré inequalities and the assumption (2), by following the same steps as in the proof of Theorem 6 in [40], we obtain

$$\|(I' - \phi'')(u)\| \leq k, \quad u \in H_0^1(\Omega),$$

where $k \in (0, 1)$. Hence all the assertions of Theorem 4.2 hold and there exists $u_0 \in H_0^1(\Omega)$, which is solution of problem (4.3) and consequently, weak solution of problem (4.2). □

5. Conclusion

The motivation of the presented work is to get a new approach to the existence of the solution to variational inequality problem, optimization problem and elliptic boundary value problem via best proximity point results for newly introduced mappings, cyclic orbital simulative contractions. It is also proved that our obtained results generalize and extend many existing results in the literature and nontrivial examples are provided to verify it. In addition, we also improve some proved best proximity results for non-self mappings in the literature with the help of enriched class of simulation functions. In future, this approach may be used to find solution of best proximity problems by weakening the conditions in cyclic orbital simulative contractions. These results may be extended to multivalued functions too.

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