ELSEVIER Journal of Computational and Applied Mathematics 54 (1994) 79–97

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Total error in the discrete convolution backprojection algorithm in computerized tomography

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Received 12 June 1992

Abstract

The CBP algorithm in computerized tomography (CT) is a discrete realization of a well-known tool from approximation theory; namely, the approximation of a function f in $\mathbb{R}^n$ by its convolution with a (bandlimited) peaked kernel. The steps in a computer implementation (e.g., in medical CT scanners) of the algorithm evaluate an n-dimensional convolution by (a) interpolation of projection data (line integrals in a two-dimensional case), (b) a one-dimensional discrete convolution, and (c) interpolation of the convolved data, required in (d) a discrete backprojection (integration over a unit sphere). The total error in the algorithm is due to the discretization steps (a)–(d) and (e) the truncation error in the basic convolution approximation. In this work we augment the known error estimates for steps (b) and (d) with those for (a), (c) and (e) to arrive at a total error profile of the algorithm, which may be summarized as follows. In a discrete b-bandlimited CBP reconstruction f_b^{d} of f, under appropriate conditions in (a)–(e), the total error $f - f_b^{\mathrm{d}}$ is essentially of the order of $\varepsilon(f, b) = \sup_{\theta \in S^{n-1}} \int_{|\sigma| > b} |\sigma|^{n-1} |\hat{f}(\sigma\theta)| \, \mathrm{d}\sigma$, $b \to \infty$.

Keywords: CBP algorithm; Computerized tomography; Interpolation; Error analysis

Presently, the discrete convolution backprojection (CBP) algorithm is regarded as the most important, accurate, reliable and fast reconstruction method in computerized tomography (CT), especially in the medical field. Its utility ranges from electron microscopy, NMR, PET, X-ray CT, nondestructive testing, multiphase flow measurements, radar applications and so on up to the X-ray structure of supernova remnants.

Many theoretical aspects of this algorithm are well understood. These include the physical and mathematical principles in its derivation, certain error bounds and the various interrelations in the parameters of discretization and the angular resolution achieved, etc.

The present work achieves a total error analysis (hitherto lacking) for the discrete CBP algorithm by utilizing a mixed-norm set-up of practical interest. It, in particular, tries to resolve certain ambiguities in the interrelations between the orders of interpolation operations, the tie-up of the cut-off frequency with the sampling distance, the smoothness of the object of reconstruction, the role of the filter used and the accuracy desired in a reconstruction.

SSDI 0377-0427(93)E0006-E

1. The discrete convolution backprojection algorithm

For background material on CT, and various aspects of the CBP algorithm in particular, we refer to the comprehensive works [4,5,9] and the references therein.

Recollecting from [9, pp. 102–111], the CBP algorithm is a discrete implementation of the continuous convolution backprojection approximation

$$W_b * f = R^{\#}(w_b * Rf),$$

where W_b approximates the Dirac δ distribution in $\mathbb{R}^n$. Starting with a suitable radially symmetric function Φ, and taking $\hat{W}_b(\xi) = (2\pi)^{-n/2}\hat{\Phi}(|\xi|/b)$, one has

$$W_b = R^{\#}w_b, \qquad \hat{w}_b(\sigma) = \tfrac{1}{2}(2\pi)^{1/2-n}|\sigma|^{n-1}\hat{\Phi}\left(\frac{|\sigma|}{b}\right).$$

The function w_b is known as the convolving function, $\hat{\Phi}$ as the window function, W_b as the kernel function and b as the cut-off frequency of the convolution backprojection algorithm.

For an application of the convolution backprojection algorithm, the Radon transform $g = Rf$ is required at (θ_j, s_l), $j = 1,\ldots,p$, $l = -q,\ldots,q$, where $\theta_j \in S^{n-1}$, the unit sphere in the n-dimensional Euclidean space $\mathbb{R}^n$ and $s_l = hl$, $h = 1/q$. The convolution $w_b * g$ is replaced by the discrete convolution

$$w_b *^h g(\theta_j, s) = h \sum_{l=-q}^{q} w_b(s - s_l)g(\theta_j, s_l),$$

and the continuous backprojection $R^{\#}$ by a discrete backprojection

$$R_p^{\#}v(x) = \sum_{j=1}^{p} \alpha_{pj}v(\theta_j, x\cdot\theta_j).$$

With $g = Rf$, for various values of θ, the discrete convolution

$$w_b *^h g(\theta, s) = h \sum_{l=-q}^{q} w_b(s - s_l)g(\theta, s_l)$$

is evaluated for $s = s_j = hj$, $h = 1/q$, $j = -q,\ldots,q$. Writing $w(s)$ for $w_1(s)$, we have $w_b(s) = b^n w(bs)$ and so with the tie-up $bh = \vartheta\pi$, $\vartheta \in (0, 1]$, say, and using the symmetry of w, we need just the values $w(j\vartheta\pi)$, $j = 0,\ldots,q$, in order to evaluate the discrete convolutions

$$w_b *^h g(\theta, s_k) = b^n h \sum_{l=-q}^{q} w(bh(k-l))g(\theta, s_l), \quad k = -q,\ldots,q.$$

Using these, $w_b *^h g(\theta, s)$ for a required s is obtained by a suitable local interpolation $I_h(w_b *^h g)(\theta, s)$.

In PET and also in CT scans using fan-beam type of geometries, the projection data $g(\theta, s)$ is not equispaced in s. The simplest way to deal with such situations is to compute the parallel equispaced data for a standard application of the CBP algorithm by "rebinning" the data [5, pp. 172–174] through a suitable interpolation $J_\eta g$, η denoting the largest spacing between the sampled data points s. Choosing a convenient h of the order of η, we substitute the interpolated values $J_\eta g(\theta, s_l)$ for $g(\theta, s_l)$ in the above. In our treatment in the sequel, interpolation in θ is not considered. Thus, for the applicability of our results, in a divergent mode of data collection the detector spacings are assumed to be adjusted so that the rebinning does not require an interpolation in θ.

The final discrete convolution backprojection algorithm thus becomes

$$f \sim f_b^{\mathrm{d}} = R_p^{\#} I_h w_b *^h J_\eta Rf.$$

Bounds for the discretization errors $e_1 = R_p^{\#}(w_b *^h g - w_b * g)$ and $e_2 = (R_p^{\#} - R^{\#})(w_b * g)$ (see [6], [9, pp. 104–106]), which we make use of in later sections, are as follows. Let $0 \leqslant \hat{\Phi} \leqslant 1$ and $\hat{\Phi}(\xi) = 0$ for $|\xi| \geqslant 1$. If the backprojection quadrature satisfies

$$\int_{S^{n-1}} v(\theta)\,\mathrm{d}\theta = \sum_{j=1}^{p} \alpha_{pj} v(\theta_j), \quad v \in H'_{2m}, \tag{1}$$

where the α_{pj}'s are positive and where H'_{2m} denotes the space of even spherical harmonics of degree $2m$, and if

$$b \leqslant \vartheta m, \quad b \leqslant \frac{\pi}{h}, \qquad 0 < \vartheta < 1, \tag{2}$$

then for $f \in C_0^\infty(\Omega^n)$, the space of infinitely differentiable functions with support in the unit ball Ω^n of $\mathbb{R}^n$,

$$|e_1| \leqslant \tfrac{1}{2}(2\pi)^{-n/2} \varepsilon_0^*(f, b) \tag{3}$$

and

$$|e_2| \leqslant \|f\|_{L^\infty(\Omega^n)} \eta(\vartheta, m), \tag{4}$$

where

$$\varepsilon_0^*(f, b) = |S^{n-1}| \sup_{\theta \in S^{n-1}} \int_{|\sigma| \geqslant b} |\sigma|^{n-1} |\hat{f}(\sigma\theta)|\,\mathrm{d}\sigma,$$

and η admits an estimate of the form

$$0 \leqslant \eta(\vartheta, b) \leqslant c(\vartheta)\,\mathrm{e}^{-\lambda(\vartheta) b}, \tag{5}$$

for $b \geqslant B(\vartheta)$, with $\lambda(\vartheta)$, $c(\vartheta)$ and $B(\vartheta)$ being certain positive constants. Using density arguments, it follows that, so long as they make sense, the bounds for e_1 and e_2 remain valid for functions not necessarily in $C_0^\infty(\Omega^n)$. Functions f for which $\varepsilon_0^*(f, b)$ is sufficiently small are called *essentially b-bandlimited functions*. The quantity $\varepsilon(f, b) = \varepsilon_0^*(f, b)/|S^{n-1}|$ will play an important role in the sequel.

The filtering effect of interpolation, independently of the other discretization errors, has also been studied. Thus, for the B-spline interpolation [9, pp. 60, 107, 108]

$$I_h g(s) = \sum_l g(s_l) B_{1/h}(s - s_l),$$

of order k, if $b \leqslant \pi/h$,

$$R^{\#} I_h(w_b * g) = G_h * W_b * f + e_3,$$

where

$$\hat{G}_h(\xi) = (2\pi)^{-n/2} \times \begin{cases} \left(\operatorname{sinc}(\frac{1}{2}h\,|\,\xi\,|)\right)^k, & |\,\xi\,| \leqslant \pi/h, \\ 0, & \text{otherwise}, \end{cases}$$

$$\hat{e}_3(\xi) = 0, \quad |\,\xi\,| \leqslant \frac{\pi}{h},$$

$$\| e_3 \|_{L_2(\Omega^n)} \leqslant \left(\frac{2}{\pi}\right)^{k-1} h^\alpha \| f \|_{H_0^\alpha(\Omega^n)}, \quad \alpha \leqslant \tfrac{1}{2}(n-1),$$

where for functions f with $\operatorname{supp}(f) \subset \Omega^n$, the Sobolev space $H_0^\alpha(\Omega^n)$ norm is given by

$$\| f \|_{H_0^\alpha(\Omega^n)} = \int_{\mathbb{R}^n} \left(1 + |\,\xi\,|^2\right)^\alpha |\,\hat{f}(\xi)\,|^2 \, \mathrm{d}\xi.$$

For $n = 2$, certain bounds on the inherent error $f - W_b * f$, using the radial averages $f_r(x) = (1/|\,S^1\,|)\int_{S^1} f(x + r\theta)\, \mathrm{d}\theta$ of functions in $\mathbb{R}^2$, have been given in [8]. (For a follow-up and details of this approach, see [7,12,13].)

2. The Oh-oh-Lemma

Let c be a positive number, K a nondecreasing positive function and δ a positive function on $(0, c]$. Also, let φ and ϕ be positive functions on $(0, c]$, such that for all $h \in (0, 1]$, there hold

$$q(h) = \sup_{t \in (0, c]} \frac{\phi(th)\varphi(t)}{\varphi(th)\phi(t)} < \infty, \tag{6}$$

$$u(h) = \sup_{t \in (0, c]} \frac{\varphi(t)}{\varphi(th)} < \infty, \tag{7}$$

$$w(h) = \sup\{u(t) \colon t \in [h, 1]\} < \infty \tag{8}$$

and

$$\lim_{t \to 0} q(t) = 0. \tag{9}$$

With δ, K, φ and ϕ as above, in the following we prove a basic lemma of crucial interest in our subsequent error analysis of the discrete CBP algorithm. Envisaging its utility elsewhere in approximation theory and numerical analysis, it is stated in a little more generality than we actually require right now. Let $\mathbb{Z}_+$ denote the set of nonnegative integers.

Oh-oh-Lemma. *Let $t_n \in (0, c]$, $n \in \mathbb{Z}_+$, be such that*

$$0 < \alpha \leqslant \frac{t_{n+1}}{t_n} \leqslant \beta < 1,$$

where α and β are independent of n. If (6)–(9) hold and for some constant M,

$$K(t_j) \leqslant M\left\{\delta(t_n) + K(t_n)\frac{\phi(t_j)}{\phi(t_n)}\right\},$$

for all $n \in \mathbb{Z}_+$ and $n < j \in \mathbb{Z}_+$, then, as $t \to 0$,

$$K(t) = \begin{cases} \mathrm{O}(\varphi(t)), & \text{if } \delta(t) = \mathrm{O}(\varphi(t)), \\ \mathrm{o}(\varphi(t)), & \text{if } \delta(t) = \mathrm{o}(\varphi(t)). \end{cases}$$

Proof. Letting $A = \limsup_{t \to 0}\delta(t)/\varphi(t)$, for both the cases $\delta(t) = \mathrm{O}(\varphi(t))$ and $\delta(t) = \mathrm{o}(\varphi(t))$, $A < \infty$. Let B be any number such that $A < B$. Using $\lim_{t \to 0}q(t) = 0$, we can find an $a \in (0, c]$ such that

$$q(t) \leqslant \frac{1}{2M}, \quad t \in (0, a].$$

Since, $0 < \beta < 1$, for some $m \in \mathbb{Z}_+$, $\beta^m \leqslant a$. Let us define $h_0 = t_1$ and $h_n = t_{mn}$, $n \in \mathbb{Z}_+$. Then, writing $e(t) = K(t)/\varphi(t)$, for $n \in \mathbb{Z}_+$,

$$\begin{aligned} e(h_n) &\leqslant M\left(\frac{\delta(h_{n-1})}{\varphi(h_{n-1})}u\left(\frac{h_n}{h_{n-1}}\right) + e(h_{n-1})q\left(\frac{h_n}{h_{n-1}}\right)\right) \\ &\leqslant M\left(\frac{\delta(h_{n-1})}{\varphi(h_{n-1})}w(\alpha^m) + e(h_{n-1})q\left(\frac{h_n}{h_{n-1}}\right)\right). \end{aligned}$$

Now we can choose N large enough so that $\delta(h_n)/\varphi(h_n) \leqslant B$, $n \geqslant N$. Then, putting $C = MBw(\alpha^m)$, for $n \geqslant N+1$,

$$e(h_n) \leqslant C + \tfrac{1}{2}e(h_{n-1}),$$

so that for $k + 1 \in \mathbb{Z}_+$,

$$e(h_{k+N}) \leqslant 2C\left(1 - \frac{1}{2^k}\right) + \frac{e(h_N)}{2^k}.$$

Let $h \in (0, h_N]$ be arbitrary. We choose k such that $k + 1 \in \mathbb{Z}_+$ and $h_{N+k+1} < h \leqslant h_{N+k}$. Then, since K is nondecreasing,

$$\begin{aligned} K(h) &\leqslant K(h_{N+k}) \leqslant e(h_{N+k})u\left(\frac{h}{h_{N+k}}\right)\varphi(h) \leqslant e(h_{N+k})w\left(\frac{h_{N+k+1}}{h_{N+k}}\right)\varphi(h) \\ &\leqslant e(h_{N+k})w(\alpha^m)\varphi(h). \end{aligned}$$

Hence,

$$e(h) \leqslant w(\alpha^m)e(h_{N+k}) \leqslant w(\alpha^m)\left(2C + \frac{e(h_N)}{2^k}\right).$$

Since

$$k \geqslant \frac{1}{m}\frac{\ln(h_{N+1}/h)}{\ln(1/\alpha)},$$

$\limsup_{h\to 0} e(h) \leqslant 2Cw(\alpha^m) = 2M(w(\alpha^m))^2 B$.

As B is arbitrary such that $B > A$, it follows that

$$0 \leqslant \limsup_{h\to 0} e(h) \leqslant 2M(w(\alpha^m))^2 A,$$

from which, noting that $A = 0$ in the $\delta(t) = o(\varphi(t))$ case, both the assertions follow, completing the proof. □

The Oh-oh-Lemma extends an idea of [3] formalized in [2] ("Oh"-part with $\delta(t) = t^\alpha = \varphi(t)$, $\phi(t) = t^m$, $m > \alpha > 0$) and in [1] ($\phi(t) = t^m$); (see also [11, pp. 91, 92]). The main point of the lemma is the ease with which one is able to get certain degree of approximation results of very general orders ($\phi(t) = e^{-1/t}$, $t^m \log t$, $t^m/\log t$; $\varphi(t) = t^\alpha$, $t^\alpha \log t$, $t^\alpha/(\log t)^3$, $m > \alpha > 0$, any kth order modulus of smoothness $\omega_k(t)$, $k < m$, (for definition and many examples, see [14]) of any function in $L_q(-\infty, \infty)$, $1 \leqslant q \leqslant \infty$, being some examples of compatible pairs of ϕ and φ). In the present context of computerized tomography, it allows a study of window functions much more general than those satisfying a regularity condition of the type $1 - \hat{\Phi}(\xi) \asymp A \asymp |\xi|^m$, as $\xi \to 0$.

3. The tomographic spaces CT_0, CT_w, CT_φ and CT_φ^0

Let $CT_0 = CT_0(\Omega^n)$ denote the linear space of real-valued functions $f \in C_0(\Omega^n)$ (f defined and continuous in $\mathbb{R}^n$, with support in Ω^n) which are essentially bandlimited in the sense that

$$\varepsilon(f, b) = \sup_{\theta \in S^{n-1}} \int_{|\sigma| \geqslant b} |\sigma|^{n-1} |\hat{f}(\sigma\theta)| \, d\sigma \to 0, \quad \text{as } b \to \infty.$$

With the norm

$$\| f \|_0 = \sup_{\theta \in S^{n-1}} \int_{\mathbb{R}} |\sigma|^{n-1} |\hat{f}(\sigma\theta)| \, d\sigma,$$

CT_0 is a complete space of which $C_0^\infty(\Omega^n)$ is a dense subset. Since for $f \in CT_0$ the function

$$h(\theta) = \int_{\mathbb{R}} |\sigma|^{n-1} |\hat{f}(\sigma\theta)| \, d\sigma, \quad \theta \in S^{n-1},$$

is continuous on S^{n-1}, the "sup" in the above is actually a "max".

A motivation for introducing the tomographic space CT_0 is the e_1-error estimate for essentially bandlimited functions. We note that CT_0 is continuously imbedded in $C_0(\Omega^n)$ and that its norm comes quite close to the sup-norm; (for f radially symmetric and with $\hat{f}$ nonnegative, the two norms in fact differ only by a factor of $(2\pi)^{-n/2}$).

For a nonnegative measurable weight function w ($\geqslant 1$) on $\mathbb{R}^+ = (0, \infty)$, let the space CT_w ($\subset \mathrm{CT}_0$) consist of functions f for which

$$\| f \|_w = \sup_{\theta \in S^{n-1}} \int_{\mathbb{R}} | \sigma |^{n-1} w(| \sigma |) | \hat{f}(\sigma\theta) | \, \mathrm{d}\sigma < \infty.$$

Assuming the order function ϕ of the previous section to be defined on the whole of $\mathbb{R}^+$, for $f \in \mathrm{CT}_0$ and $t \in \mathbb{R}^+$, we define Peetre's K-functional $K_\phi(t; f)$ with the function parameter ϕ by

$$K_\phi(t; f) = \inf_{g \in \mathrm{CT}_w} \{ \| f - g \|_0 + \phi(t) \| g \|_w \}.$$

The w and ϕ in the above are assumed to be tied by

$$w(t) = \left[\phi\left(\frac{1}{t} \right) \right]^{-1}, \quad t > 0. \tag{10}$$

Functions $f \in \mathrm{CT}_0$ for which $K_\phi(t; f) = \mathrm{O}(\varphi(t))$, $t \to 0$, constitute the intermediate space $(\mathrm{CT}_0, \mathrm{CT}_w)_\varphi$, which for suitable ϕ and φ will characterize functions which may be reconstructed to within $\mathrm{O}(\varphi(1/b))$ by a discrete convolution backprojection algorithm with appropriate b-bandlimited filter functions. We also define $(\mathrm{CT}_0, \mathrm{CT}_w)_\varphi^0$ to be the space of functions f for which $K_\phi(t; f) = \mathrm{o}(\varphi(t))$, $t \to 0$. The spaces CT_φ and CT_φ^0 simply are abbreviations for the intermediate spaces $(\mathrm{CT}_0, \mathrm{CT}_w)_\varphi$ and $(\mathrm{CT}_0, \mathrm{CT}_w)_\varphi^0$, respectively.

4. The order functions and the filter

Let, in addition, the order function ϕ satisfy

$$v(h) = \sup_{t>0} \frac{\phi(th)}{\phi(t)} < \infty, \quad h \in (0, 1], \tag{11}$$

$$J = \sup_{h \in (0,1]} v(h) < \infty. \tag{12}$$

The function ϕ is also assumed to be related with the filter function Φ by the requirement that for $\xi \in \mathbb{R}^n$,

$$| 1 - \hat{\Phi}(\xi) | \leqslant A \psi(| \xi |), \quad | \xi | < 1, \tag{13}$$

where for $h \in (0, 1]$, we define $\psi(h) = \inf_{t \leqslant c} \phi(t)/\phi(t/h)$, and A is a constant. Moreover, for some constant B, Φ is assumed to satisfy

$$| \hat{\Phi}(\xi) | \leqslant B, \quad 0 \leqslant | \xi | \leqslant 1, \tag{14}$$

and

$$| \hat{\Phi}(\xi) | = 0, \quad | \xi | > 1. \tag{15}$$

The e_1 and e_2 estimates (3), (4) remain valid (to within a multiplication by the constant B) for the Φ's satisfying (14), (15).

While studying errors due to interpolation of projection and convolved data, respectively, we assume the following additional conditions on the function u associated with the order function φ:

$$\lim_{t \to 0} t^{l-n+1} u(t) = 0, \tag{16}$$

$$\lim_{t \to 0} t^{m} u(t) = 0, \tag{17}$$

where l is the order of interpolation for the nonequispaced projection data $g(\theta, s)$, m is the order of interpolation of the convolved data $w_b *^h g$ and n is the dimension of the problem space $\mathbb{R}^n$. Note that for any minimal choice $l = m + n - 1$. Some practical examples of l, m, φ, u, ϕ, v, ψ and Φ appear in Section 10.

We note here that $\lim_{t \to 0} t^m u(t) = 0$, for instance, implies that $(th)^m \leqslant h^m u(h) \varphi(th)/\varphi(t) = \mathrm{o}(\varphi(th))$, as $h \to 0$, for any fixed $t \in (0, c]$. It follows that $\eta(\vartheta, b) \leqslant c(\vartheta)\, \mathrm{e}^{-\lambda(\vartheta)b} = \mathrm{o}(1/b^m) = \mathrm{o}(\varphi(1/b))$, $b \to \infty$, i.e., $\eta(\vartheta, b)$ approaches zero much faster as compared to $\varphi(1/b)$.

In this study the order of approximation under consideration is $\varphi(1/b)$, as the bandwidth $b \to \infty$. In the sequel the conditions on the functions Φ, ϕ, φ etc., imposed till now, will be assumed to be satisfied. For a Φ such that $1 - \hat{\Phi}(\sigma) = \mathrm{O}(\sigma^m)$, $\sigma \to 0^+$, the choice $\phi(t) = t^m$ is appropriate for orders $\varphi(t)$ of approximation such as t^α, $t^\alpha \log(\log(1/t))$, $t^\alpha/\log(1/t)$, $\alpha < m$, any moduli of smoothness $\omega_k(t)$, $k < m$, etc. In practice the case $m = 2$ is quite common (see Section 10).

5. The intrinsic error in CBP

The intrinsic error $f - W_b * f$, arising due to the bandlimiting aspect of CBP, is independent of variations or refinements in the discretizations and sampling schemes.

Writing $f_b^c = W_b * f = R^{\#}(w_b * Rf)$, for the continuous convolution backprojection, we have $\hat{f}_b^c(\xi) = \hat{f}(\xi)\hat{\Phi}(\xi/b)$. Since

$$\| f_b^c \|_w = \sup_{\theta \in S^{n-1}} \int_{\mathbb{R}} |\sigma|^{n-1} w(|\sigma|) \left| \hat{\Phi}\left(\frac{\sigma\theta}{b}\right) \right| |\hat{f}(\sigma\theta)| \,\mathrm{d}\sigma,$$

using $\hat{\Phi}(\sigma\theta) = 0$, $|\sigma| \geqslant 1$, and $w(|\sigma|) \leqslant J w(b)$, $|\sigma| \leqslant b$, and $|\hat{\Phi}(\xi)| \leqslant B$, $0 < |\xi| \leqslant 1$, for $f \in \mathrm{CT}_0$ we have

$$\| f_b^c \|_w \leqslant BJw(b) \| f \|_0 < \infty, \tag{18}$$

which, in particular, implies that $f_b^c \in \mathrm{CT}_w$. Similarly, for $f \in \mathrm{CT}_w$, $f_b^c \in \mathrm{CT}_w$ and $\| f_b^c \|_w \leqslant B \| f \|_w$. It is also clear that $\| f_b^c \|_0 \leqslant B \| f \|_0$. Next, since, $\psi(|\sigma|/b) \leqslant (1/\phi(1/|\sigma|))\phi(1/b)$, $|\sigma| \leqslant b$, and $w(|\sigma|)/w(b) \geqslant 1/J$, if $|\sigma| \geqslant b$, for all b sufficiently large, we have

$$\begin{aligned}
\| f - f_b^c \|_0 &\leqslant \sup_{\theta \in S^{n-1}} \left\{ A \int_{|\sigma| \leqslant b} |\sigma|^{n-1} \psi\left(\frac{|\sigma|}{b}\right) |\hat{f}(\sigma\theta)| \,\mathrm{d}\sigma \right. \\
&\qquad \left. + (2\pi)^{-n/2} \int_{|\sigma| > b} |\sigma|^{n-1} |\hat{f}(\sigma\theta)| \,\mathrm{d}\sigma \right\}, \\
&\leqslant \max\left\{A, (2\pi)^{-n/2} J\right\} \phi\left(\frac{1}{b}\right) \| f \|_w = D\phi\left(\frac{1}{b}\right) \| f \|_w, \quad \text{say.}
\end{aligned}$$

For any $g \in \mathrm{CT}_w$,

$$\begin{aligned}\| f-f_b^c \|_0 &\leqslant \| f-g \|_0 + \| g-g_b^c \|_0 + \| (g-f)_b^c \|_0 \\ &\leqslant (1+B) \| f-g \|_0 + D\phi\left(\frac{1}{b}\right) \| g \|_w \\ &\leqslant \max\{1+B,\ D\}\left(\| f-g \|_0 + \phi\left(\frac{1}{b}\right) \| g \|_w \right).\end{aligned}$$

Taking the infimum over CT_w, with $E = \max\{1+B,\ D\}$, we get

$$\| f-f_b^c \|_0 \leqslant EK_\phi\left(\frac{1}{b};\ f\right), \tag{19}$$

so that $\| f-f_b^c \|_0 = \mathrm{O}(\varphi(1/b))$, if $f \in \mathrm{CT}_\varphi$, and $\| f-f_b^c \|_0 = \mathrm{o}(\varphi(1/b))$, if $f \in \mathrm{CT}_\varphi^0$.

Conversely, if $\| f-f_b^c \|_0 = \mathrm{O}(\delta(1/b))$, $b \to \infty$, we have $\| f-f_b^c \|_0 \leqslant A\delta(1/b)$, for some A and for all b sufficiently large ($b \geqslant b_0$, say). Then, since $f_b^c \in \mathrm{CT}_w$,

$$\begin{aligned}K_\phi(t;\ f) &= \inf_{g \in \mathrm{CT}_w} \{ \| f-g \|_0 + \phi(t) \| g \|_w \} \\ &\leqslant \inf_{g \in \mathrm{CT}_w} \left\{ \| f-f_b^c \|_0 + \phi(t)(\| f_b^c - g_b^c \|_w + \| g_b^c \|_w) \right\} \\ &\leqslant A\delta\left(\frac{1}{b}\right) + \phi(t) \inf_{g \in \mathrm{CT}_w} \{ BJw(b) \| f-g \|_w + B \| g \|_w \} \\ &\leqslant M\left[\delta\left(\frac{1}{b}\right) + \frac{\phi(t)}{\phi(1/b)} K_\phi\left(\frac{1}{b};\ f\right) \right],\end{aligned}$$

where $M = \max\{A,\ B,\ BJ\}$. Since this inequality is valid for all b sufficiently large and for all t, by the Oh-oh-Lemma we conclude that $K_\phi(t;\ f) = \mathrm{O}(\varphi(t))$ or $\mathrm{o}(\varphi(t))$, $t \to 0$, according as $\delta(t) = \mathrm{O}(\varphi(t))$ or $\mathrm{o}(\varphi(t))$ as $t \to 0$. Thus we have proved the following theorem.

Theorem 1 (Intrinsic error). *If Φ, φ and w satisfy the conditions* (6)–(15), *and $f \in \mathrm{CT}_0$, then*

$$\| f-f_b^c \|_0 = \mathrm{O}\left(\varphi\left(\frac{1}{b}\right)\right), \quad b \to \infty, \quad \textit{iff } f \in \mathrm{CT}_\varphi, \tag{20}$$

$$\| f-f_b^c \|_0 = \mathrm{o}\left(\varphi\left(\frac{1}{b}\right)\right), \quad b \to \infty, \quad \textit{iff } f \in \mathrm{CT}_\varphi^{\mathrm{o}}. \tag{21}$$

It may be noted that the Oh-oh-Lemma actually asserts that the $\mathrm{O}(\varphi(1/b))$ (or, $\mathrm{o}(\varphi(1/b))$) convergence even along a sequence $\{b_n\}$, satisfying the sparsity condition $0 < \alpha \leqslant b_n/b_{n+1} \leqslant \beta < 1$, implies that $f \in \mathrm{CT}_\varphi$ (respectively, CT_φ^0). In particular, the "oh"-part implies that the order of error in the "Oh"-part cannot be improved for the class CT_φ, as a whole.

6. An application of the Ramachandran–Lakshminarayanan filter

The ideal low-pass filter $\Phi = \Phi_0$ given by

$$\Phi_0(\xi) = \begin{cases} 1, & |\xi| \leqslant 1, \\ 0, & |\xi| > 1, \end{cases}$$

corresponds to the Ramachandran–Lakshminarayanan filter in $\mathbb{R}^2$. In $\mathbb{R}^n$ it corresponds to the kernel $W_b = W_b^0$ given by [9, pp. 102, 110]

$$W_b^0(x) = (2\pi)^{-n/2} b^n \frac{J_{n/2}(b|x|)}{(b|x|)^{n/2}}.$$

In this case, the two inequalities $|1 - \hat{\Phi}(\xi)| \leqslant A\psi(|\xi|)$, $|\xi| < 1$, and $|\hat{\Phi}(\xi)| \leqslant B$, $0 \leqslant |\xi| \leqslant 1$, are valid with $B = 1$ and for any choice of ψ, whatsoever. Denoting the continuous reconstruction by the Ramachandran–Lakshinarayanan filter by f_b^0, since there holds

$$\| f - f_b^0 \|_0 = \sup_{\theta \in S^{n-1}} \int_{|\sigma| \geqslant b} |\sigma|^{n-1} |\hat{f}(\sigma\theta)| \, \mathrm{d}\sigma = \varepsilon(f, b),$$

from the Intrinsic Error Theorem we immediately get the following theorem.

Theorem 2 (Characterization of intermediate CT spaces). *Let Φ, φ and w satisfy* (6)–(15) *and let $f \in \mathrm{CT}_0$. Then,*

$$f \in \mathrm{CT}_\varphi, \quad \textit{iff } \varepsilon(f, b) = \mathrm{O}\left(\varphi\left(\frac{1}{b}\right)\right), \quad b \to \infty, \tag{22}$$

$$f \in \mathrm{CT}_\varphi^0, \quad \textit{iff } \varepsilon(f, b) = \mathrm{o}\left(\varphi\left(\frac{1}{b}\right)\right), \quad b \to \infty. \tag{23}$$

At this stage we also note that from (18) we have $K_\phi(t; f) \leqslant \| f - f_b \|_0 + \phi(t)w(b)J \| f \|_0$. Let $\varepsilon > 0$ be given. Since $f \in \mathrm{CT}_0$, fixing b sufficiently large, the first term on the right can be made less than $\frac{1}{2}\varepsilon$. Having done so, since $\phi(t) \to 0$, $t \to 0$, there exists a $\delta > 0$, such that the second term too is less than $\frac{1}{2}\varepsilon$, for $0 < t < \delta$. Thus $K_\phi(t; f) \to 0$, as $t \to 0$. Hence the Intrinsic Error Theorem, the characterization of the intermediate CT spaces and (19) together imply the following corollary.

Corollary 3 (Intrinsic error). *If Φ, φ and w satisfy* (6)–(15), *and $f \in \mathrm{CT}_0$, then, as $b \to \infty$, $\| f - f_b^c \|_0 \to 0$. Moreover,*

$$\| f - f_b^c \|_0 = \mathrm{O}\left(\varphi\left(\frac{1}{b}\right)\right), \quad \textit{iff } \varepsilon(f, b) = \mathrm{O}\left(\varphi\left(\frac{1}{b}\right)\right), \quad b \to \infty, \tag{24}$$

$$\| f - f_b^c \|_0 = \mathrm{o}\left(\varphi\left(\frac{1}{b}\right)\right), \quad \textit{iff } \varepsilon(f, b) = \mathrm{o}\left(\varphi\left(\frac{1}{b}\right)\right), \quad b \to \infty. \tag{25}$$

In particular, the CT_0 norm being stronger than the sup-norm in $\mathbb{R}^n$, the convergence $f_b^c(x) \to f(x)$, for $f \in \mathrm{CT}_0$, as well as the convergence rate $f_b^c(x) - f(x) = \mathrm{O}(\varphi(1/b))$, for $f \in \mathrm{CT}_\varphi$, and $f_b^c(x) - f(x) = \mathrm{o}(\varphi(1/b))$, for $f \in \mathrm{CT}_\varphi^0$, remain valid uniformly in $x \in \Omega^n$ ($\mathbb{R}^n$).

7. Weighted Radon spaces $\mathscr{R}_\omega$

Motivated by the spaces CT_0 and CT_w, consider functions $g(\theta, s)$ defined on the unit cylinder $Z = S^{n-1} \times \mathbb{R}$, for which

$$\| g \|_\omega = \sup_{\theta \in S^{n-1}} \int_{\mathbb{R}} \omega(|\sigma|) |\hat{g}(\theta, \sigma)| \, \mathrm{d}\sigma < \infty,$$

where ω is a nonnegative weight function on $\mathbb{R}^+$ and $\hat{g}(\theta, \sigma)$ denotes the one-dimensional Fourier transform of $g(\theta, s)$, regarded as a function of s with θ fixed. Such functions (with the usual identification of functions equal a.e.) with norm $\|\cdot\|_\omega$ constitute our weighted radon space $\mathscr{R}_\omega$. (It may be noted that the present analysis, without forging any new tool, extends to a more general situation of $\| g \|_\omega = \sup_{\theta \in S^{n-1}} \| \omega(\cdot) \hat{g}(\theta, \cdot) \|$, where $\|\cdot\|$ is a monotone norm, i.e., satisfying $\| y \| \leqslant \| z \|$ if $|y(\sigma)| \leqslant |z(\sigma)|$, $\sigma \in \mathbb{R}^1$.)

If ω_0 and ω_1 ($\geqslant \omega_0$) are two weight functions, for $F \in \mathscr{R}_{\omega_0}$ we define the Peetre's K-functional by

$$K_\omega(t; F) = \inf_{g \in \mathscr{R}_1} \{ \| F - g \|_{\omega_0} + t \| g \|_{\omega_1} \}.$$

For $b > 0$, let F^b be defined through $\hat{F}^b(\theta, \sigma) = \hat{F}(\theta, \sigma)$, if $|\sigma| \leqslant b$, and zero, otherwise. Then $F^b \in \mathscr{R}_\omega$ if $F \in \mathscr{R}_\omega$ and moreover, $\| F^b \|_\omega \leqslant \| F \|_\omega$, $F \in \mathscr{R}_\omega$ for all ω, whatsoever. Let for some constant J_1, the quotient function $\tilde{\omega} = \omega_1 / \omega_0$ satisfy

$$\frac{\tilde{\omega}(|\sigma|)}{\tilde{\omega}(b)} \leqslant J_1, \quad |\sigma| \leqslant b.$$

Then,

$$\| F^b \|_{\omega_1} \leqslant J_1 \tilde{\omega}(b) \| F \|_{\omega_0}.$$

Further, if for some constant J_r,

$$\frac{\tilde{\omega}(b)}{\tilde{\omega}(|\sigma|)} \leqslant J_\mathrm{r}, \quad |\sigma| > b,$$

then

$$\begin{aligned} \| F - F^b \|_{\omega_0} &= \sup_{\theta \in S^{n-1}} \int_{|\sigma| > b} \omega_0(|\sigma|) |\hat{F}(\theta, \sigma)| \, \mathrm{d}\sigma \\ &= \sup_{\theta \in S^{n-1}} \int_{|\sigma| > b} (\tilde{\omega}(|\sigma|))^{-1} \omega_1(|\sigma|) |\hat{F}(\theta, \sigma)| \, \mathrm{d}\sigma \leqslant \frac{J_\mathrm{r}}{\tilde{\omega}(b)} \| F \|_{\omega_1}. \end{aligned}$$

Note that the J_1 and J_r inequalities hold for all b if

$$J_0 = \sup_{h \in (0,1]} \sup_{t>0} \frac{\tilde{\omega}(th)}{\tilde{\omega}(t)} < \infty, \tag{26}$$

which we assume in the sequel.

Let a nonnegative function ν on $(0, c]$ have the following properties analogous to those of φ defined earlier:

$$\tilde{u}(h) = \sup_{t \in (0,c]} \frac{\nu(t)}{\nu(th)} < \infty, \tag{27}$$

$$\tilde{w}(h) = \sup_{t \in [h,1]} \tilde{u}(t) < \infty, \tag{28}$$

$$\tilde{q}(h) = \sup_{t \in (0,c]} \frac{\tilde{\omega}(1/t)\nu(t)}{\tilde{\omega}(1/(th))\nu(th)} < \infty, \tag{29}$$

$$\lim_{t \to 0} \tilde{q}(t) = 0. \tag{30}$$

Let $\mathscr{R}_\nu = (\mathscr{R}_{\omega_0}, \mathscr{R}_{\omega_1})_\nu$ be the intermediate space of $g \in \mathscr{R}_{\omega_0}$ for which $K_\omega(1/\tilde{\omega}(1/t); g) = O(\nu(t))$, $t \to 0$, and $\mathscr{R}_\nu^0 = (\mathscr{R}_{\omega_0}, \mathscr{R}_{\omega_1})_\nu^0$ of those with $K_\omega(1/\tilde{\omega}(1/t); g) = o(\tilde{\nu}(t))$, $t \to 0$. Let

$$e(g; b) = \sup_{\theta \in S^{n-1}} \int_{|\sigma|>b} \omega_0(|\sigma|) |\hat{g}(\theta, \sigma)| \,\mathrm{d}\sigma.$$

Then, for $g \in \mathscr{R}_{\omega_1}$,

$$\begin{aligned} e(F; b) &\leqslant e(F-g; b) + e(g-g^b; b) + e\big((g-F)^b; b\big) \leqslant 2\, \| F-g \|_{\omega_0} + \frac{J_r}{\tilde{\omega}(b)} \| g \|_{\omega_1} \\ &\leqslant \max\{2, J_r\} \left[\| F-g \|_{\omega_0} + \frac{1}{\tilde{\omega}(b)} \| g \|_{\omega_1} \right], \end{aligned}$$

so that

$$e(F; b) \leqslant E K_\omega\left(\frac{1}{\tilde{\omega}(b)}; F \right).$$

Also, since

$$\begin{aligned} K_\omega\left(\frac{1}{\tilde{\omega}(1/t)}; F \right) &\leqslant \inf_{g \in \mathscr{R}_{\omega_1}} \left\{ \| F-F^b \|_{\omega_0} + \frac{1}{\tilde{\omega}(1/t)} \Big(\| (F-g)^b \|_{\omega_1} + \| g^b \|_{\omega_1} \Big) \right\} \\ &\leqslant \inf_{g \in \mathscr{R}_{\omega_1}} \left\{ e(F; b) + \frac{1}{\tilde{\omega}(1/t)} \big(J_1 \tilde{\omega}(b) \| F-g \|_{\omega_0} + \| g \|_{\omega_1} \big) \right\} \\ &\leqslant \max\{1, J_1\} \left[e(F; b) + \frac{\tilde{\omega}(b)}{\tilde{\omega}(1/t)} K_\omega\left(\frac{1}{\tilde{\omega}(b)}; F \right) \right], \end{aligned}$$

applying the Oh-oh-Lemma to the function $K_\omega(1/\tilde{\omega}(1/t); F)$ and orders $\tilde{\omega}(1/t)$ and $\nu(t)$, we get the following analogue of the characterization of the intermediate CT spaces.

Theorem 4 (Characterization of intermediate Radon spaces). *If* ω_0, ω_1 *and* ν *satisfy* (26)–(30),

$$\mathscr{R}_\nu = \left\{ F \in \mathscr{R}_{\omega_0} \colon e(F;\, b) = \mathrm{O}\left(\nu\left(\frac{1}{b}\right)\right),\ b \to \infty \right\},$$

$$\mathscr{R}_\nu^0 = \left\{ F \in \mathscr{R}_{\omega_0} \colon e(F;\, b) = \mathrm{o}\left(\nu\left(\frac{1}{b}\right)\right),\ b \to \infty \right\}.$$

This characterization, for appropriate choices of ω_0 and ω_1, will be used in the next two sections dealing with the discretization errors due to an interpolation of the projection and the convolved data.

8. Error in the interpolation of convolved data

Suppose I_h is any interpolation of order m, for the equispaced convolved data $(w_b *^h g)(\theta, s_k)$, $k = -q, \ldots, q$, satisfying

$$\| I_h G \|_C \leqslant Q_I \| G \|_C, \quad G \in C[-1, 1], \tag{31}$$

$$\| I_h G - G \|_C \leqslant C_I h^m \| G^{(m)} \|_C, \quad G \in C^m[-1, 1], \tag{32}$$

where $\|\cdot\|_C$ denotes the sup-norm on the interval $[-1, 1]$, $C^m[-1, 1]$ is the space of m-times continuously differentiable functions on $[-1, 1]$ and where Q_I and C_I are independent of h. An appropriate example of such an I_h is the m-point central difference interpolation. For the standard two-point linear interpolation used in most CBP applications we have $m = 2$.

Using [9, p.58]

$$\left(w_b *^h g\right)^\wedge(\theta, \sigma) = (2\pi)^{1/2} \hat{w}_b(\sigma) \sum_{l \in \mathbb{Z}} \hat{g}\left(\theta, \sigma - \frac{2\pi l}{h}\right),$$

and with D denoting the partial differentiation $\partial/\partial s$, since $|\hat{\Phi}(\xi)| \leqslant B$, $0 \leqslant |\xi| \leqslant 1$, and $\hat{w}_b(\sigma) = 0$ for $|\sigma| > b$, we have

$$\begin{aligned} |D^m(w_b *^h g)(\theta, s)| &\leqslant \int_{-b}^{b} |\sigma|^m |\hat{w}_b(\sigma)| \sum_{l \in \mathbb{Z}} \left| \hat{g}\left(\theta, \sigma - \frac{2\pi l}{h}\right) \right| \mathrm{d}\sigma \\ &\leqslant (2\pi)^{1/2 - n} \left(\tfrac{1}{2} B\right) \int_{-b}^{b} |\sigma|^{m+n-1} \sum_{l \in \mathbb{Z}} \left| \hat{g}\left(\theta, \sigma - \frac{2\pi l}{h}\right) \right| \mathrm{d}\sigma. \end{aligned}$$

Putting

$$\left(E_h w_b *^h g\right)(\theta, s) = \left(I_h w_b *^h g - w_b *^h g\right)(\theta, s),$$

and, using $b \leqslant \pi/h$,

$$\begin{aligned} \| E_h w_b *^h g \| &= \sup_{\theta \in S^{n-1}} \| E_h w_b *^h g(\theta, \cdot) \|_C \leqslant \sup_{\theta \in S^{n-1}} C_I h^m \| D^m w_b *^h g(\theta, \cdot) \|_C \\ &\leqslant C_I h^m \sup_{\theta \in S^{n-1}} \int_{\mathbb{R}} |\sigma|^m |\left(w_b *^h g\right)^\wedge(\theta, \sigma)| \,\mathrm{d}\sigma \\ &\leqslant (2\pi)^{1/2 - n} \left(\tfrac{1}{2} B\right) \sup_{\theta \in S^{n-1}} C_I h^m \int_{\mathbb{R}} |\sigma|^{m+n-1} |\hat{g}(\theta, \sigma)| \,\mathrm{d}\sigma. \end{aligned}$$

From this, with $\omega_1(\sigma)=\max\{|\sigma|^{m+n-1}, 1\}$, there follows

$$\| E_h w_b *^h g \| \leqslant (2\pi)^{1/2-n}\left(\tfrac{1}{2}B\right)C_I h^m \| g \|_{\omega_1}.$$

With $\omega_0(\sigma)=\max\{|\sigma|^{n-1}, 1\}$, $\tilde{\omega}(\sigma)=\max\{|\sigma|^m, 1\}$ and we have

$$\| E_h w_b *^h g \| \leqslant Q_I \| w_b *^h g \|_C \leqslant Q_I (2\pi)^{1/2-n}\left(\tfrac{1}{2}B\right) \| g \|_{\omega_0}.$$

It follows that for a general g,

$$\begin{aligned}\| E_h w_b *^h g \| &\leqslant \inf_{G\in\mathscr{R}_1}\left\{\| E_h(w_b *^h (g-G))\| + \| E_h(w_b *^h G)\|\right\}\\ &\leqslant \max\{Q_I, C_I\}(2\pi)^{1/2-n}\left(\tfrac{1}{2}B\right)\inf_{G\in\mathscr{R}_1}\left\{\| g-G\|_{\omega_0}+h^m\| G\|_{\omega_1}\right\}\\ &= \max\{Q_I, C_I\}(2\pi)^{1/2-n}\left(\tfrac{1}{2}B\right)K_\omega(h^m; g).\end{aligned}$$

Next,

$$\begin{aligned}&\sup_{\theta\in S^{n-1}}\int_{|\sigma|>b}\omega_0(|\sigma|)\,|\hat{g}(\theta,\sigma)|\,\mathrm{d}\sigma\\ &= (2\pi)^{(n-1)/2}\| f-f^b\|_0\\ &\leqslant \inf_{F\in \mathrm{CT}_w}(2\pi)^{(n-1)/2}\left\{\|(f-F)-(f-F)^b\|_0+\| F-F^b\|_0\right\}\\ &\leqslant \inf_{F\in \mathrm{CT}_w}(2\pi)^{(n-1)/2}\left\{\| f-F\|_0+J\phi\left(\frac{1}{b}\right)\| F\|_w\right\}\\ &\leqslant \inf_{F\in \mathrm{CT}_w}(2\pi)^{(n-1)/2}J\left\{\| f-F\|_0+\phi\left(\frac{1}{b}\right)\| F\|_w\right\}\\ &\leqslant (2\pi)^{(n-1)/2}JK_\phi\left(\frac{1}{b}; f\right).\end{aligned}$$

Hence, if $f\in \mathrm{CT}_\varphi$,

$$\sup_{\theta\in S^{n-1}}\int_{|\sigma|>b}\omega_0(|\sigma|)\,|\hat{g}(\theta,\sigma)|\,\mathrm{d}\sigma = \mathrm{O}\left(\varphi\left(\frac{1}{b}\right)\right), \quad b\to\infty.$$

Using $\lim_{t\to 0}t^m u(t)=0$, this and the theorem on the characterization of intermediate Radon spaces imply that $K_\omega(t^m; g)=\mathrm{O}(\varphi(t))$, $t\to 0$. Hence, from the K_ω-estimate of E_h in the above, there follows $\| I_h(w_b * g) - w_b * g\| = \mathrm{O}(\varphi(h))$, so that finally we have $e_I = |R_p^{\#}(I_h(w_b *^h g) - w_b *^h g)| = \mathrm{O}(\varphi(1/b))$, showing that the net contribution of the error due to the interpolation of the convolved data is of the order of $\varphi(1/b)$.

Similarly, $e_I=\mathrm{o}(\varphi(1/b))$, for $f\in \mathrm{CT}_\varphi^0$, and $e_I=\mathrm{o}(1)$, for $f\in\mathrm{CT}_0$ and $m>0$ (for the details of the reasoning, see the next section), as $b\to\infty$. Thus we have proved the following theorem.

Theorem 5 (Error in the interpolation of convolved data). *If the assumptions* (1), (2), (6)–(15), (17) *and* (31), (32) *hold,*

$$\left|\left(R_p^{\#}\left(I_h(w_b *^h g) - w_b *^h g\right)\right)(x)\right| = \begin{cases} \mathrm{O}(\varphi(1/b)), & \textit{if } f \in \mathrm{CT}_\varphi, \\ \mathrm{o}(\varphi(1/b)), & \textit{if } f \in \mathrm{CT}_\varphi^0, \\ \mathrm{o}(1), & \textit{if } f \in \mathrm{CT}_0, \end{cases} \tag{33}$$

uniformly in $x \in \Omega^n$.

9. Error in the interpolation of projection data

Let the minimum spacing h_{m} and the maximum spacing h_{M} between consecutive local interpolation data points satisfy $\beta \leqslant h_{\mathrm{m}}/h_{\mathrm{M}}$, where β is a certain constant. With the largest of the data spacings denoted by η (assumed to be of the order of h, i.e., $\eta/h \leqslant C$ for some constant C), let the projection data interpolation operator J_η be of order l:

$$\| J_\eta g \|_C \leqslant Q_J \| g \|_c, \quad g \in C[-1, 1], \tag{34}$$

$$\| J_\eta g - g \|_C \leqslant C_J \eta^l \| g^l \|_C, \quad g \in C^l[-1, 1], \tag{35}$$

Q_J and C_J being certain constants independent of η. Note, for example, that the Lagrange interpolation

$$J_\eta g(\theta, s) = \sum_{k=1}^{l} l_k(s) g(\theta, z_k),$$

$$l_k(s) = \prod_{j(\neq k)=1}^{l} \frac{(s - z_j)}{(z_k - z_j)}, \quad k = 1, \ldots, l,$$

based on appropriate local nodes $z_1, z_2, \ldots, z_l$, provides an lth-order interpolation.

We take $g(\theta, s)$ as the projection data for f. Putting

$$(E_\eta g)(\theta, s) = (J_\eta g - g)(\theta, s) \quad \text{and} \quad \| E_\eta g \| = \sup_{\theta \in S^{n-1}} \| E_\eta g(\theta, \cdot) \|_C,$$

we have

$$\| E_\eta g \| \leqslant (2\pi)^{-1/2} C_J \eta^l \sup_{\theta \in S^{n-1}} \int_{\mathbb{R}} |\sigma|^l \, |\hat{g}(\theta, \sigma)| \, \mathrm{d}\sigma,$$

from which, if we take $\omega_1(\sigma) = \max\{|\sigma|^l, 1\}$, there follows

$$\| E_\eta g \| \leqslant (2\pi)^{-1/2} C_J \eta^l \| g \|_{\omega_1}.$$

Also, with $\omega_0(\sigma) = 1$, for some constant H_0 (depending on β),

$$\| E_\eta g \| \leqslant H_0 \| g \|_{\omega_0}.$$

Then, $\tilde{\omega}(\sigma)=\omega_1(\sigma)/\omega_0(\sigma)=\omega_1(\sigma)=\max\{|\sigma|^l, 1\}$ and hence, for any $g\in\mathscr{R}_0$,

$$\begin{aligned}\| E_\eta g\| &\leqslant \inf_{G\in\mathscr{R}_1}\{\| E_\eta(g-G)\|+\| E_\eta(G)\|\}\\ &\leqslant \max\{H_0, (2\pi)^{-1/2}C_J\}\inf_{G\in\mathscr{R}_1}\{\| g-G\|_{\omega_0}+\eta^l\| G\|_{\omega_1}\}\\ &=\max\{H_0, (2\pi)^{-1/2}C_J\}K_\omega(\eta^l;\, g).\end{aligned}$$

Since, $\hat{g}(\theta, \sigma)=(2\pi)^{(n-1)/2}\hat{f}(\sigma\theta)$, by the projection theorem, for $f\in \mathrm{CT}_0$, we have

$$\sup_{\theta\in S^{n-1}}\int_{|\sigma|>b}|\hat{g}(\theta, \sigma)|\,\mathrm{d}\sigma\leqslant\frac{(2\pi)^{(n-1)/2}\| f\|_0}{b^{n-1}}$$

and, if $f\in\mathrm{CT}_w$,

$$\begin{aligned}\sup_{\theta\in S^{n-1}}\int_{|\sigma|>b}|\hat{g}(\theta, \sigma)|\,\mathrm{d}\sigma &\leqslant Jb^{-(n-1)}\phi\left(\frac{1}{b}\right)\sup_{\theta\in S^{n-1}}\int_{|\sigma|>b}\frac{|\sigma|^{n-1}}{\phi(1/|\sigma|)}|\hat{g}(\theta, \sigma)|\,\mathrm{d}\sigma\\ &\leqslant Jb^{-(n-1)}\phi\left(\frac{1}{b}\right)(2\pi)^{(n-1)/2}\| f\|_w.\end{aligned}$$

It follows that for any $f\in\mathrm{CT}_0$,

$$\begin{aligned}e(g;\, b) &= \sup_{\theta\in S^{n-1}}\int_{|\sigma|>b}|\hat{g}(\theta, \sigma)|\,\mathrm{d}\sigma=(2\pi)^{(n-1)/2}\| f-f^b\|_{\omega_0},\quad\text{say,}\\ &\leqslant(2\pi)^{(n-1)/2}\inf_{F\in\mathrm{CT}_w}\left\{\|(f-F)-(f-F)^b\|_{\omega_0}+\| F-F^b\|_{\omega_0}\right\}\\ &\leqslant\inf_{F\in\mathrm{CT}_w}(2\pi)^{(n-1)/2}b^{-(n-1)}\left\{\| f-F\|_0+J\phi\left(\frac{1}{b}\right)\| F\|_w\right\}\\ &\leqslant\inf_{F\in\mathrm{CT}_w}(2\pi)^{(n-1)/2}b^{-(n-1)}J\left\{\| f-F\|_0+\phi\left(\frac{1}{b}\right)\| F\|_w\right\}\\ &\leqslant(2\pi)^{(n-1)/2}b^{-(n-1)}JK_\phi\left(\frac{1}{b};\, f\right).\end{aligned}$$

With $w(s)=w_1(s)$, $bh=\vartheta\pi$, $\vartheta\in(0, 1]$, and assuming that ϑ is such that

$$\sigma_w=|w(0)|+2\sum_{j=1}^{\infty}|w(j\vartheta\pi)|<\infty, \tag{36}$$

using (1) and the positivity of the α_{pj}'s, we have

$$\begin{aligned}e_J &=\left|R_p^{\#}\left(I_h\left\{w_b *^h(J_\eta g-g)\right\}\right)\right|\leqslant\sum_{j=1}^{p}\alpha_{pj}Q_I\| E_\eta g\| h\sum_{l=-q}^{q}|w_b(lh)|\\ &\cong|S^{n-1}|b^nQ_Ih\sigma_w\| E_\eta g\|.\end{aligned}$$

At this point we may note that if N denotes the order of magnitude of an extraneous noise $N(\theta_j, s)$ in the projection data for a single projection view θ_j, the worst-case reconstruction error at a point is of order $\alpha_{pj} h b^n \sigma_{w,q} N$, where $\sigma_{w,q} = |w(0)| + 2\Sigma_{j=1}^{q} |w(j\vartheta\pi)|$. Thus σ_w provides a useful measure of a filter's sensitivity to noise. Since, for the p and q in practice, approximately $p = cq^{n-1}$, $c = \pi^{n-1}/(n-1)!$ [9, p.70], and the quadrature coefficients α_{pj} are practically of the order $1/p$ of magnitude, the noise propagation due to a single view is of order $\mathrm{O}(N\sigma_{w,q})$.

If $f \in \mathrm{CT}_\varphi$, $e(g; b) = \mathrm{O}(b^{-(n-1)}\varphi(1/b))$. Taking $\nu(t) = t^{n-1}\varphi(t)$, and using $\lim_{t\to 0} t^{l-n+1} u(t) = 0$, from the theorem on the characterization of intermediate Radon spaces, we have $g \in \mathscr{R}_\nu$ and consequently, since $\eta = \mathrm{O}(h) = \mathrm{O}(1/b)$, $e_J = \mathrm{O}(\varphi(1/b))$. Similarly, $f \in \mathrm{CT}_\varphi^0$ implies that $e_J = \mathrm{o}(\varphi(1/b))$. Moreover, if $f \in \mathrm{CT}_0$, $e(g; b) = \mathrm{o}(b^{-(n-1)})$ and then with $\varphi = 1$, since for $l > n-1$, $\lim_{t\to 0} t^{l-n+1} = 0$, we have $e_J = \mathrm{o}(1)$, $b \to \infty$. Thus we have proved the following theorem.

Theorem 6 (Error in the interpolation of projection data). *If* (1), (2), (6)–(16), (31), (32) *and* (34)–(36) *hold and* $\eta = \mathrm{O}(1/b)$, *then*

$$\left| R_p^{\#}\Big(I_h\big(w_b *^h (J_\eta g - g)\big)\Big)(x)\right| = \begin{cases} \mathrm{O}(\varphi(1/b)), & \text{if } f \in \mathrm{CT}_\varphi, \\ \mathrm{o}(\varphi(1/b)), & \text{if } f \in \mathrm{CT}_\varphi^0, \\ \mathrm{o}(1), & \text{if } f \in \mathrm{CT}_0, \end{cases} \tag{37}$$

uniformly in $x \in \Omega^n$.

It may be noted that in the above analysis the finiteness of σ_w is quite important (in practical situations it is material when q is large, i.e., when the discretization is very fine). It comes as a pleasant surprise that the usual practice of choosing $\vartheta = 1$, that is, tying b to h by $b = \pi/h$, which simplifies the expressions for the convolving coefficients (for the Ramachandran–Lakshminarayanan and the Shepp–Logan filters, see [9, pp. 109–111]), also achieves this crucial finiteness of σ_w (for these and the Hamming filter in $\mathbb{R}^2$, $\hat{\Phi}(1) \neq 0$; for cosine filter $\hat{\Phi}(1) = 0$).

Theorem 7 (Finiteness of σ_w). *If $\hat{\Phi}(\xi)$ is radially symmetric and $\hat{\Phi}(\sigma)$ is twice continuously differentiable in* $[0, 1]$, *then σ_w is finite iff $(1-\vartheta)\hat{\Phi}(1) = 0$ (i.e., σ_w is finite for all $\vartheta \in (0, 1]$, if $\hat{\Phi}(1) = 0$, and is finite only for $\vartheta = 1$, if $\hat{\Phi}(1) \neq 0$).*

Proof. For $s > 0$ and $n \geqslant 2$,

$$\begin{aligned} (2\pi)^n w(s) &= \frac{1}{2}\int_{\mathbb{R}} |\sigma|^{n-1}\hat{\Phi}(|\sigma|)\, \mathrm{e}^{\mathrm{i}s\sigma}\, \mathrm{d}\sigma = \int_0^1 \sigma^{n-1}\hat{\Phi}(\sigma)\cos(s\sigma)\, \mathrm{d}\sigma \\ &= \hat{\Phi}(1)\frac{\sin s}{s} + \hat{\Phi}'(1)\frac{\cos s}{s^2} - \frac{1}{s^2}\int_0^1 \left(\sigma^{n-1}\hat{\Phi}(\sigma)\right)''\cos(s\sigma)\, \mathrm{d}\sigma, \end{aligned}$$

from which the result follows, since for $j \in \mathbb{Z}_+$ and $\vartheta \in (0, 1)$, $\max\{|\sin(j\vartheta\pi)|, |\sin((j+1)\vartheta\pi)\} \geqslant |\sin(\frac{1}{2}\min\{\vartheta, 1-\vartheta\})|$. $\square$

It may also be appropriate to remark here that divergent ray geometries in which rebinning does require an interpolation in the θ-argument, treating the corresponding interpolation error as the noise $N(\theta_j, s)$ in the above, for an overall error bound of order $\varphi(1/b)$ we need an accuracy of order $b^{-(n-1)}\varphi(1/b)$ in the θ-interpolation.

The result on the finiteness of σ_w has an important implication: the choice $h = \vartheta\pi/b$, in a b-bandlimited approximation with $0 < \vartheta < 1$, corresponds to an oversampling. Since for popular windows (e.g., Shepp–Logan, Ramachandran–Lakshminarayanan and Hamming) the conditions of the theorem are satisfied with $\hat{\Phi}(1) \neq 0$, an oversampling corresponds to $\sigma_w = \infty$, which for very large q would manifest in a large noise propagating factor $\alpha_{w,q}$.

10. Total error in the discrete convolution backprojection

A break-up of the total error in the discrete convolution backprojection algorithm is given by

$$f - f_b^{\mathrm{d}} = f - W_b * f - \left(R_p^{\#} - R^{\#}\right)(w_b * g) - R_p^{\#}\left(I_h\left(w_b *^h g\right) - w_b *^h g\right)$$
$$- R_p^{\#}\left(w_b *^h g - w_b * g\right) - R_p^{\#} I_h\left(w_b *^h \left(J_\eta g - g\right)\right).$$

Recollecting the threads from the previous sections, under the hypotheses (1), (2), (6)–(17), (31), (32) and (34)–(36) (note the use of the sampling condition $bh = \pi$ for a satisfaction of (36) for the commonly used windows) and with $\eta = \mathrm{O}(1/b)$: (a) the order of approximation $f - W_b * f$ is dealt with in the intrinsic error corollary; (b) the backprojection quadrature error $(R_p^{\#} - R^{\#})(w_b * g)$, according to (4), (5), decays exponentially; (c) the contribution $R_p^{\#}(I_h(w_b *^h g) - w_b *^h g)$ to the error due to the interpolation of the convolutions is estimated in (33); (d) the convolution discretization error $R_p^{\#}(w_b *^h g - w_b * g)$ is given by (3); and finally, (e) an estimate of the order of error in the interpolation of the projection data $R_p^{\#}(I_h(w_b *^h (J_\eta g - g)))$ is given in (37). Combining these, from the characterization of the intermediate Radon spaces, we finally arrive at the following theorem.

Theorem 8 (Total error in the discrete CBP algorithm). *If* (1), (2), (6)–(17), (31), (32) *and* (34)–(36) *hold and* $\eta = \mathrm{O}(1/b)$, *then, as* $b \to \infty$,

$$f(x) - f_b^d(x) = \begin{cases} \mathrm{O}(\varphi(1/b)), & \text{if } f \in \mathrm{CT}_\varphi, \quad \text{i.e., } \varepsilon(f, b) = \mathrm{O}(\varphi(1/b)), \\ \mathrm{o}(\varphi(1/b)), & \text{if } f \in \mathrm{CT}_\varphi^0, \quad \text{i.e., } \varepsilon(f, b) = \mathrm{o}(\varphi(1/b)), \\ \mathrm{o}(1), & \text{if } f \in \mathrm{CT}_0, \quad \text{i.e., } \varepsilon(f, b) = \mathrm{o}(1), \end{cases}$$

uniformly in $x \in \Omega^n$.

Thus, loosely speaking, in a discrete b-bandlimited CBP reconstruction, the total error is of the order of $\varepsilon(f, b)$.

For the Shepp–Logan, Hamming and cosine filters in the planar tomography $n = 2$, the window functions for $|\xi| < 1$ equal $\operatorname{sinc}(\frac{1}{2}\pi|\xi|)$, $\alpha + (1-\alpha)\cos(\frac{1}{2}\pi|\xi|)$, $0 < \alpha < 1$, and $\cos(\frac{1}{2}\pi|\xi|)$, respectively. In these cases $\psi(h) = h^2$, so that $\phi(t) = t^2$ is appropriate (usable also for the Ramachandran–Lakshminarayanan window). Then $\upsilon(h) = h^2$ and if, as an example, we take $\varphi(t) = t^{\alpha}$, $0 < \alpha < 2$, then $u(h) = t^{-\alpha}$ and the interpolation orders m and l have to satisfy $m > \alpha$ and $l > 1 + \alpha$. For $0 < \alpha < 1$, we thus get the smallest values $m = 1$ and $l = 2$, while for $1 \leqslant \alpha < 2$, we require $m = 2$ and $l = 3$. In many experiments with the above filters (see, e.g., [7,12]; also cf. [5, pp. 67–71]) the order of error for the cross sections under consideration is found to be $O(1/b)$, which corresponds to $\alpha = 1$. This is in agreement with the findings in [10] (cf. [5, pp. 140–144], [9, p.108]) that in practical tests the broken-line interpolation ($m = 2$) is satisfactory, while the nearest-neighbour interpolation ($m = 1$) is not.

Acknowledgements

The author wishes to thank Professor Dr. Frank Natterer (Münster) whose visit to the Indian Institute of Technology in Kanpur, and a subsequent discussion on the importance of piecewise linearity of convolving functions, provided an incentive for the above general study. He is also indebted to Professor Dr. P.L. Butzer for the much needed encouragement and he is grateful to the two anonymous reviewers for their helpful suggestions.

References

[1] P.S. Avadhani, Local *Lp*-inverse theorems of a general order for linear combinations of certain linear positive operators, Ph.D. Thesis, Dept. Math., Indian Inst. Technol., Kanpur, 1982.

[2] M. Becker and R.J. Nessel, An elementary approach to inverse results in approximation theory, *J. Approx. Theory* **23** (1978) 99–103.

[3] H. Berens and G.G. Lorentz, Inverse theorem for Bernstein polynomials, *Indiana Univ. Math. J.* **21** (1972) 693–708.

[4] S.R. Deans, *The Radon Transform and Some of its Applications* (Wiley, New York, 1983).

[5] G.T. Herman, *Image Reconstruction from Projections: The Fundamentals of Computerized Tomography* (Academic Press, New York, 1980).

[6] R.M. Lewitt, R.T.H. Bates and T.M. Peters, Image reconstruction from projections II: Modified back-projection methods, *Optik* **50** (1978) 85–109.

[7] P. Munshi, Error estimates for the convolution backprojection algorithm in computerized tomography, Ph.D. Thesis, Nuclear Engrg. Technol. Program, Indian Inst. Technol., Kanpur, 1988.

[8] P. Munshi, R.K.S. Rathore, K.S. Ram and M.S. Kalra, Error estimates for tomographic inversion, *Inverse Problems* **7** (1991) 399–408.

[9] F. Natterer, *The Mathematics of Computerized Tomography* (Wiley, New York, 1986).

[10] S.W. Rowland, Computer implementation of image reconstruction formulas, in: G.T. Herman, Ed., *Image Reconstruction from Projections,* Topics Appl. Phys. **32** (Springer, New York, 1979) 7–79.

[11] H.S. Shapiro, *Smoothing and Approximation of Functions* (Van Nostrand, New York, 1969).

[12] U. Singh, On the CBP-algorithm for strip integral data in computerized tomography, Ph.D. Thesis, Dept. Math., Indian Inst. Technol., Kanpur, 1991.

[13] T. Srivastava, Convolution backprojection algorithm in computerized tomography: statistical error estimates and optimal filter, Ph.D. Thesis, Dept. Math., Indian Inst. Technol., Kanpur, 1989.

[14] A.F. Timan, *Theory of Approximation of Functions of a Real Variable* (Hindustan Publ. Corp., Delhi, 1966).