ELSEVIER

Journal of Computational and Applied Mathematics 85 (1997) 123–144

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

About semiclassical polynomials on the unit circle corresponding to the class (2,2)

A. Cachafeiro*, C. Suárez

Departamento de Matematica Aplicada, E.T.S.I.I., Universidad de Vigo, Spain

Received 12 March 1997; received in revised form 15 July 1997

Abstract

In the present paper we pose the problem of characterizing the orthogonal polynomials related to the unit circle whose moment functional $\mathscr{L}$ verify a functional relation of the following type $D(\phi\mathscr{L})+\psi\mathscr{L}=0$, where ϕ and ψ are polynomials with $\deg\phi=2$ and $\deg\psi=1$. Two different situations appear depending on whether the roots of ϕ are unimodular or not. Here, we solve the last case. Moreover, we analyse the definite positive character of the solutions.

Keywords: Orthogonal polynomials; Unit circle; Semiclassical polynomials; Regular functionals; Measures on the unit circle; Difference equations

AMS classification: 42C05

1. Introduction

In the real line the classical orthogonal polynomials are characterized by different equivalent properties among which is the following: Given the functional equation $D(\phi\mathscr{L})+\psi\mathscr{L}=0$ where ϕ and ψ are polynomials with $\deg\phi\leqslant 2$ and $\deg\psi=1$, the only regular solutions are the functionals $\mathscr{L}$ corresponding to the classical families. For these families, the explicit form of ϕ and ψ is as follows:

$D(\phi\mathscr{L})+\psi\mathscr{L}=0$	$\phi(x)$	$\psi(x)$
Hermite	1	$2x$
Laguerre	x	$x-\alpha-1$
Bessel	x^2	$-\alpha x-2$
Jacobi	$1-x^2$	$(\alpha+\beta+2)x+\alpha-\beta$

* Corresponding author. E-mail: achachafe@dma.uvigo.es.

0377-0427/97/$17.00 © 1997 Elsevier Science B.V. All rights reserved
PII S0377-0427(97)00134-9

In the present paper we pose the same problem on the unit circle $\mathbb{T}$ of characterizing the orthogonal polynomials whose moment functionals verify this type of functional equation. In this direction we want to point out the characterization of the class (1, 1) in [11, 13], that is the study of the solutions of the above functional equation with $\deg\phi=1$. Therefore, we try to obtain the regular and hermitian functionals $\mathscr{L}$ which are solutions of the functional equation with $\deg\phi=2$:

$$D((z-\alpha)(z-\beta)\mathscr{L})=(b_1z+b_0)\mathscr{L} \quad \text{with } b_1\neq 0. \tag{1.1}$$

The organization of the paper is as follows: Section 2 is devoted to some preliminary results and definitions. In Section 3 we obtain that Eq. (1.1) corresponds to one of the following cases:

$$D((z-\alpha)(\bar{\alpha}z-1)\mathscr{L})=-\mathrm{i}((1+|\alpha|^2)z-2\alpha)\mathscr{L}, \tag{1.2}$$

$$D((z-\alpha)(z-\beta)\mathscr{L})=-\mathrm{i}((\alpha+\beta)z-2\alpha\beta)\mathscr{L}. \tag{1.3}$$

We restrict ourselves to the first case for which we obtain in Theorem 4 the hermitian solutions. Finally, Section 4 contains the statement of our main results concerning the regularity and positive definite character of these solutions.

2. Preliminary results

Let $\Lambda=\operatorname{span}\{z^k,\ k\in\mathbb{Z}\}$ be the space of the Laurent polynomials and $\mathscr{L}:\Lambda\to\mathbb{C}$ be a regular and hermitian functional. If we denote the moments by $\mathscr{L}(z^n)=c_n$ for $n\in\mathbb{Z}$ we say

Definition 1. $\mathscr{L}$ is hermitian if $\forall n\geqslant 0\ c_{-n}=\overline{c_n}$.
$\mathscr{L}$ is regular (positive definite) if the principal submatrices of the moment matrix are nonsingular (positive), i.e.,

$$\forall n\geqslant 0\ \Delta_n=\det\left(\mathscr{L}(z^{i-j})\right)_{i=0,\dots,n;\,j=0,\dots,n}\neq 0(>0).$$

In any case we denote $\forall n\geqslant 0\ E_n=\Delta_n/\Delta_{n-1}$ with $\Delta_{-1}=1$ and by $\{\Phi_n\}_0^\infty$ the monic orthogonal polynomial sequence (MOPS) related to $\mathscr{L}$. It is well known that $\{\Phi_n\}_0^\infty$ satisfies for $n\geqslant 1$ the following recurrence relations:

$$\Phi_n(z)=z\Phi_{n-1}(z)+\Phi_n(0)\Phi_{n-1}^*(z), \tag{2.1}$$

$$\Phi_n^*(z)=\Phi_{n-1}^*(z)+\overline{\Phi_n(0)}z\Phi_{n-1}(z), \tag{2.2}$$

$$\Phi_n(z)=(1-|\Phi_n(0)|^2)z\Phi_{n-1}(z)+\Phi_n(0)\Phi_n^*(z), \tag{2.3}$$

$$\Phi_n^*(z)=(1-|\Phi_n(0)|^2)\Phi_{n-1}^*(z)+\overline{\Phi_n(0)}\Phi_n(z), \tag{2.4}$$

where $\Phi_n^*(z)=z^n\overline{\Phi_n}(1/z)$ is the reversed polynomial of $\Phi_n(z)$.

Next we recall some definitions and results concerning semiclassical functionals. For more details see [10, 8, 2, 13].

Definition 2. Given a regular and hermitian functional $\mathscr{L}$, we say that $\mathscr{L}$ is semiclassical if there exist polynomials $\phi(z)\neq 0$ and $\psi(z)$ such that the following functional equation holds:

$$D(\phi(z)\mathscr{L}) = \psi(z)\mathscr{L},$$

where the derivative operator is defined by

$$\forall P\in\Lambda \qquad D\mathscr{L}(P(z)) = -\mathrm{i}\mathscr{L}(zP'(z)).$$

This definition is motivated by the differential behaviour of positive measures on the unit circle with respect to the integration on the unit circle. If $\deg\phi(z)=p'$ and $\max\{p'-1,\deg[(p'-1)\phi(z)+\mathrm{i}\psi(z)]\} = q$ we say that $\mathscr{L}$ belongs to the class (p',q).

It is obvious that if $\mathscr{L}$ belongs to the class (p',q), it also belongs to the class $(p'+1,q+1)$.

Theorem 1. *If $\mathscr{L}$ is a semiclassical functional verifying $D(\phi(z)\mathscr{L}) = \psi(z)\mathscr{L}$ then it holds the following relation*:

$$z^{p'}\phi(z)(\psi(z) - \mathrm{i}z\phi'(z))^* = z^{p}(\psi(z) - \mathrm{i}z\phi'(z))\phi^*(z)$$

with $p' = \deg\phi(z)$ *and* $p = \deg(\psi(z) - \mathrm{i}z\phi'(z))$.

Theorem 2. *Let $\mathscr{L}$ be a regular functional verifying $D(\phi(z)\mathscr{L}) = \psi(z)\mathscr{L}$. Then the series $S(z) = \sum_{k=0}^{\infty}\overline{c_k}z^k$ satisfies the following differential equation*:

$$z\phi(z)S'(z) + \mathrm{i}(\psi(z) - \mathrm{i}z\phi'(z))S(z) = \zeta(z),$$

where $\zeta(z)$ is a polynomial such that $\deg\zeta(z)\leqslant\max\{p',q\}$.

Theorem 3. *Let $\mathscr{L}$ be a regular functional verifying $D(\phi(z)\mathscr{L}) = \psi(z)\mathscr{L}$. If $\mathscr{L}$ is positive definite then the series $S(z) = \sum_{k=0}^{\infty}\overline{c_k}z^k$ converges for z such that $|z|<1$ and the series $\overline{S}(z^{-1})$ converges for z such that $|z|>1$.*

3. The functional equation: Hermitian solutions

The aim of this section is to obtain that Eq. (1.1) corresponds to two different cases depending on whether the roots of ϕ are unimodular or not. Here we restrict ourselves to the last case for which we obtain in Theorem 4 the semiclassical solutions. Besides in Theorem 5 and Corollary 1, we prove that these hermitian solutions verify another functional equation which is more convenient for the study of the regularity in the next section.

Lemma 1. *If the functional equation given by* (1.1) *has a regular and hermitian solution then one of the following conditions must verify*:

(i) $\beta\overline{\alpha} = 1$ *with* $|\alpha|\neq 1$, $b_0 = 2\mathrm{i}\dfrac{\alpha}{\overline{\alpha}}$ *and* $b_1 = -\mathrm{i}\dfrac{1+|\alpha|^2}{\overline{\alpha}}$, *or*

(ii) $|\alpha| = |\beta| = 1$, $b_0 = 2\mathrm{i}\alpha\beta$ *and* $b_1 = -\mathrm{i}(\alpha+\beta)$ *with* $\alpha+\beta\neq 0$.

Proof. We denote by $\mathscr{L}$ a regular and hermitian solution of Eq. (1.1) and by c_n its moments. Without loss of any generality we assume that $\mathscr{L}$ is normalized, i.e., $c_0 = 1$.

Since $\mathscr{L}$ is a semiclassical functional, by applying Theorem 1 we get that

$$\begin{aligned}&(z-\alpha)(z-\beta)(\overline{b_0}z^2 + (\overline{b_1} - \mathrm{i}(\overline{\alpha} + \overline{\beta}))z + 2\mathrm{i})\\ &\quad = (\overline{\alpha\beta}z^2 - (\overline{\alpha} + \overline{\beta})z + 1)(-2\mathrm{i}z^2 + (\mathrm{i}(\alpha + \beta) + b_1)z + b_0)\end{aligned} \tag{3.1}$$

from which it follows by identifying coefficients that

$$b_0 = 2\mathrm{i}\alpha\beta$$

and

$$\alpha\beta\overline{b_1} - b_1 = 3\mathrm{i}(\alpha + \beta) - \mathrm{i}\alpha\beta(\overline{\alpha} + \overline{\beta}). \tag{3.2}$$

Computing $D((z-\alpha)(z-\beta)\mathscr{L})(z^n) = (b_1 z + b_0)\mathscr{L}(z^n)$ we obtain that the moments verify the difference equation

$$-\mathrm{i}nc_{n+2} + (\mathrm{i}n(\alpha + \beta) - b_1)c_{n+1} - (\mathrm{i}n\alpha\beta + b_0)c_n = 0. \tag{3.3}$$

Taking $n = 0$ and $n = -1$ in (3.3) we get the system of two linear equations in two unknowns b_0 and b_1

$$b_0 + b_1 c_1 = 0, \tag{3.4}$$

$$\overline{c_1}b_0 + b_1 = \mathrm{i}(c_1 - (\alpha + \beta) + \alpha\beta\overline{c_1}) \tag{3.5}$$

that has a unique solution

$$b_1 = \mathrm{i}\frac{(c_1 - (\alpha + \beta) + \alpha\beta\overline{c_1})}{1 - |c_1|^2} \quad \text{and} \quad b_0 = -c_1 b_1,$$

i.e.,

$$2\alpha\beta - \alpha\beta|c_1|^2 = -c_1^2 + c_1(\alpha + \beta). \tag{3.6}$$

Since $b_1 \neq 0$ then $c_1 - (\alpha + \beta) + \alpha\beta\overline{c_1} \neq 0$. Besides by taking $n = -2$ in (3.3) it is easily verified that $c_1 \neq 0$ and this implies $b_0 \neq 0$.

Computing $D((z-\alpha)(z-\beta)\mathscr{L})(z^{-n}) = (b_1 z + b_0)\mathscr{L}(z^{-n})$ and combining the hermitian character of $\mathscr{L}$ and the expression of b_0 we find

$$\mathrm{i}(n-2)\overline{\alpha\beta}c_n + (\overline{b_1} - n\mathrm{i}(\overline{\alpha} + \overline{\beta}))c_{n-1} + \mathrm{i}nc_{n-2} = 0 \tag{3.7}$$

which implies

$$\mathrm{i}n\overline{\alpha\beta}c_{n+2} + (\overline{b_1} - (n+2)\mathrm{i}(\overline{\alpha} + \overline{\beta}))c_{n+1} + \mathrm{i}(n+2)c_n = 0. \tag{3.8}$$

Eliminating c_{n+2} between this last relation and (3.3) we get

$$(\overline{b_1} - \mathrm{i}(n+2)(\overline{\alpha} + \overline{\beta}) + \mathrm{i}n\overline{\alpha\beta}(\alpha + \beta) - b_1\overline{\alpha\beta})c_{n+1} + \mathrm{i}(n+2)(1 - |\alpha\beta|^2)c_n = 0 \tag{3.9}$$

and combining relations (3.2) and (3.9) it holds

$$(n-1)(|\alpha|^2\overline{\beta}+|\beta|^2\overline{\alpha}-\overline{\alpha}-\overline{\beta})c_{n+1}+(n+2)(1-|\alpha\beta|^2)c_n=0. \tag{3.10}$$

Putting $n=1$ in (3.10) and applying that $c_1\neq 0$ we get

$$|\alpha\beta|=1. \tag{3.11}$$

On the other hand, taking $n=0$ in (3.10) and using (3.11) we obtain

$$(\overline{\beta}(|\alpha|^2-1)+\overline{\alpha}(|\beta|^2-1))c_1=0,$$

which is equivalent to

$$\overline{\beta}(|\alpha|^2-1)=\frac{1}{\alpha}(|\alpha|^2-1).$$

Therefore we find either $\overline{\beta}\alpha=1$ with $|\alpha|\neq 1$ or $|\alpha|=1$.

Next we obtain b_0 and b_1 in each of the two previous cases:

(i) If $\overline{\beta}\alpha=1$ with $|\alpha|\neq 1$, we substitute in relation (3.2) $b_1=-b_0/c_1=2i\alpha/\overline{\alpha}\Phi_1(0)$ obtaining

$$-\overline{\alpha}(\Phi_1(0))^2=|\Phi_1(0)|^2(\alpha+\Phi_1(0)(1+|\alpha|^2)). \tag{3.12}$$

Since $c_1=-\Phi_1(0)$, Eq. (3.6) becomes in this case in

$$-\overline{\alpha}(\Phi_1(0))^2=-\alpha|\Phi_1(0)|^2+\Phi_1(0)(1+|\alpha|^2)+2\alpha. \tag{3.13}$$

Eliminating $|\Phi_1(0)|^2$ between (3.12) and (3.13) it holds that $\Phi_1(0)$ satisfies the following equation:

$$\overline{\alpha}(1+|\alpha|^2)(\Phi_1(0))^3+(2|\alpha|^2+(1+|\alpha|^2)^2)(\Phi_1(0))^2+3\alpha(1+|\alpha|^2)\Phi_1(0)+2\alpha^2=0,$$

from which we conclude that $\Phi_1(0)$ must be $-2\alpha/(1+|\alpha|^2)$ or $-1/\alpha$ or $-\alpha$. For $\Phi_1(0)=-1/\alpha$ or $\Phi_1(0)=-\alpha$, by substituting in (3.13), we get to a contradiction $|\alpha|=1$.

(ii) If $|\alpha|=|\beta|=1$ proceeding in the same way as in the previous case we obtain

$$-(\Phi_1(0))^2=|\Phi_1(0)|^2(\alpha\beta+\Phi_1(0)(\alpha+\beta)) \tag{3.14}$$

by substituting $b_1=2i\alpha\beta/\Phi_1(0)$ in (3.2). Since $c_1=-\Phi_1(0)$, eliminating $|\Phi_1(0)|^2$ between (3.6) after substitution of c_1 and (3.14) we find

$$(\alpha+\beta)(\Phi_1(0))^3+(2\alpha\beta+(\alpha+\beta)^2)(\Phi_1(0))^2+3\alpha\beta(\alpha+\beta)\Phi_1(0)+2(\alpha\beta)^2=0.$$

Solving the equation we get for $\Phi_1(0)$ the following values: $-2\alpha\beta/(\alpha+\beta)$ or $-\beta$ or $-\alpha$. Since the two last values get to a contradiction because $|\Phi_1(0)|\neq 1$ we conclude the result. □

Theorem 4. *Let the functional equation*

$$D\left((z-\alpha)\left(z-\frac{1}{\overline{\alpha}}\right)\mathscr{L}\right)=(-i)\left(\frac{(1+|\alpha|^2)}{\overline{\alpha}}z-\frac{2\alpha}{\overline{\alpha}}\right)\mathscr{L} \tag{3.15}$$

with $|\alpha| \neq 1$, *and let* $\mathscr{L}$ *be a regular and hermitian solution verifying the initial conditions* $\mathscr{L}(1)=1$ *and* $\mathscr{L}(z) = 2\alpha/(1+|\alpha|^2)$.

If we denote $\mathscr{L}(z^n) = c_n$, *then for* $n \geqslant 0$

$$c_n = Au_n + (1-A)v_n,$$

where

$$u_n = \frac{\alpha^n}{1+|\alpha|^2}[(n+1)+(1-n)|\alpha|^2], \qquad v_n = \frac{1}{\bar{\alpha}^n(1+|\alpha|^2)}[(1-n)+(n+1)|\alpha|^2],$$

$A = 1-\bar{\alpha}^2\Phi_2(0)/(1-|\alpha|^4)$ *and for* $n<0$ *we define* $c_n = \overline{c_{-n}}$.

Proof. Applying (3.15) to z^n we find the difference equation satisfied by c_n:

$$n\bar{\alpha}c_{n+2} - (n+1)(1+|\alpha|^2)c_{n+1} + \alpha(n+2)c_n = 0 \tag{3.16}$$

with $c_0 = 1$ and $c_1 = 2\alpha/(1+|\alpha|^2)$.

To solve the equation we use the method of the generating function [1]. First we obtain the corresponding generating function $U(s) = \sum_{n=0}^{\infty} c_n s^n$ by solving the following differential equation:

$$\alpha s(s-\bar{\alpha})\left(s-\frac{1}{\alpha}\right)U'(s) - 2(\bar{\alpha}-\alpha s^2)U(s) = -2\bar{\alpha} - \frac{2|\alpha|^2}{1+|\alpha|^2}s.$$

Then the solution is given by

$$U(s) = (-2|\alpha|^2 s^3 + C\alpha(1+|\alpha|^2)s^2 - \frac{2\bar{\alpha}}{\alpha}(1+|\alpha|^2+|\alpha|^4)s + \frac{\bar{\alpha}^2}{\alpha}(1+|\alpha|^2)) \times \frac{1}{\alpha(1+|\alpha|^2)(s-\bar{\alpha})^2(s-\frac{1}{\alpha})^2}.$$

In order that the series converges, we need, depending on the case, $\bar{\alpha}$ or $1/\alpha$ be a root of the numerator $U(s)$.

In the first situation we conclude $C=(1+|\alpha|^2+4|\alpha|^4)/\alpha^2(1+|\alpha|^2)$ and the corresponding moments are given by

$$u_n = \frac{\alpha^n}{1+|\alpha|^2}[(n+1)+(1-n)|\alpha|^2].$$

Then the functional is regular and the MOPS is $\Phi_n(z) = z^{n-2}(z-\alpha)^2$ for $n \geqslant 2$ (see [3]).

In the second case $C = (|\alpha|^2(4+|\alpha|^2+|\alpha|^4))/\alpha^2(1+|\alpha|^2)$ and the moments are

$$v_n = \frac{1}{\bar{\alpha}^n(1+|\alpha|^2)}[(1-n)+(n+1)|\alpha|^2].$$

Then the functional is regular and the MOPS is $\Psi_n(z) = z^{n-2}(z-1/\bar{\alpha})^2$ for $n \geqslant 2$ (see [3]).

Since both solutions are linearly independent we obtain the general solution of (3.16) is

$$c_n = Au_n + Bv_n$$

and the solutions verifying the initial conditions c_0 and c_1 are

$$c_n = Au_n + (1 - A)v_n.$$

In order to compute the complex number A we use the regularity of $\mathscr{L}$ which implies that $c_2 = c_1^2 - (1 - |c_1|^2)\Phi_2(0)$. Then if we substitute in the preceding expression of c_n for $n = 2$ we obtain that $A = (1 - \bar{\alpha}^2\Phi_2(0))/(1 - |\alpha|^4)$. □

Theorem 5. *Let $\mathscr{L}$ be a linear hermitian functional verifying the functional equation* (3.15) *with the initial conditions $\mathscr{L}(1) = 1$ and $\mathscr{L}(z) = 2\alpha/(1 + |\alpha|^2)$. Then $\mathscr{L}$ also verifies the following functional equation*:

$$D\left((z - \alpha)^2\left(z - \frac{1}{\bar{\alpha}}\right)^2\mathscr{L}\right) = 2\mathrm{i}(z - \alpha)^2\left(z - \frac{1}{\bar{\alpha}}\right)^2\mathscr{L}. \tag{3.17}$$

Proof. To obtain the functional equation (3.17) we take into account some results in [3]. Indeed, if $\mathscr{L}$ is a linear hermitian functional verifying (3.17) with the initial conditions $\mathscr{L}(1) = 1$ and $\mathscr{L}(z) = 2\alpha/(1 + |\alpha|^2)$, it is easy to check that the corresponding moments $\mathscr{L}(z^n) = c_n$ verify the following difference equation:

$$\begin{aligned}(n + 2)[&\bar{\alpha}^2 c_{n+4} - 2\bar{\alpha}(1 + |\alpha|^2)c_{n+3} + (1 + |\alpha|^4 + 4|\alpha|^2)c_{n+2}\\ &-2\alpha(1 + |\alpha|^2)c_{n+1} + \alpha^2 c_n] = 0.\end{aligned} \tag{3.18}$$

On the other hand, it is easy to verify that the moments

$$c_n = Au_n + (1 - A)v_n$$

given in Theorem 4 before satisfy Eq. (3.18), and this yields the result. □

Corollary 1. *Let $\mathscr{L}$ be a linear hermitian functional verifying the functional equation* (3.15) *with the initial conditions $\mathscr{L}(1) = 1$ and $\mathscr{L}(z) = 2\alpha/(1 + |\alpha|^2)$ and let $\mathscr{L}_A$ be the linear functional given by*

$$\mathscr{L}_A(z^n) = \begin{cases} 0 & \text{for } n \neq 0, \\ \dfrac{(1 - 2\Re A)(|\alpha|^2 - 1)^3}{1 + |\alpha|^2} & \text{for } n = 0. \end{cases}$$

Then $\mathscr{L}$ also verifies the following functional equation:

$$(z - \alpha)^2\left(\frac{1}{z} - \bar{\alpha}\right)^2\mathscr{L} = \mathscr{L}_A,$$

with initial conditions $\mathscr{L}(1) = 1$, $\mathscr{L}(z) = 2\alpha/(1 + |\alpha|^2)$ and $\mathscr{L}(z^2) = (-A(|\alpha|^2 - 1)^3 + 3|\alpha|^2 - 1)/(1 + |\alpha|^2)\bar{\alpha}^2$.

Proof. Taking into account Theorem 5 we obtain that the hermitian solutions of (3.15) with the initial conditions $\mathscr{L}(1)=1$ and $\mathscr{L}(z)=2\alpha/(1+|\alpha|^2)$ verify

$$\overline{\alpha}^2 c_{n+2} - 2\overline{\alpha}(1+|\alpha|^2)c_{n+1} + (1+|\alpha|^4+4|\alpha|^2)c_n$$
$$-2\alpha(1+|\alpha|^2)c_{n-1} + \alpha^2 c_{n-2} = \begin{cases} 0 & \text{for } n \neq 0, \\ \dfrac{(1-2\Re A)(|\alpha|^2-1)^3}{1+|\alpha|^2} & \text{for } n=0, \end{cases}$$

and this implies the result. □

Therefore, in order to get the regular solutions of our problem (Eq. (3.15)), we study in the next section, the regularity of the hermitian solutions $\mathscr{L}$ of the functional equation $(z-\alpha)^2(1/z-\overline{\alpha})^2\mathscr{L}=\mathscr{L}_A$ with prescribed initial conditions, i.e., we solve what is usually called an inverse problem.

4. An inverse problem. Regular solutions

This section contains the statement of our main results: Theorems 6 and 7, in which we analyse the regularity and the positive-definite character of the solutions of the posed problem.

Lemma 2. *If $\mathscr{L}$ is a regular and hermitian solution of $(z-\alpha)^2(1/z-\overline{\alpha})^2\mathscr{L}=\mathscr{L}_A$, with $|\alpha|\neq 1$ and $\mathscr{L}(1)=1$, $\mathscr{L}(z)=2\alpha/(1+|\alpha|^2)$ and $\mathscr{L}(z^2)=(-A(|\alpha|^2-1)^3+3|\alpha|^2-1)/(1+|\alpha|^2)\overline{\alpha}^2$, then the following relations hold:*

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\frac{1}{z^j}\right) = -\frac{1}{\alpha^{j-1}}(\overline{\alpha}^2-\overline{\Phi_2(0)})\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2, \quad \forall j\geqslant 0, \tag{4.1}$$

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\frac{1}{z^j}\right) = \frac{1}{\alpha^j}(j+1)(\overline{\alpha}^2-\overline{\Phi_2(0)})\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2, \quad \forall j\geqslant 0, \tag{4.2}$$

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)z^{j+1}\right) = \alpha^j(1-\overline{\alpha}^2\Phi_2(0))\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2, \quad \forall j\geqslant 0, \tag{4.3}$$

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 z^{j+2}\right) = \alpha^j(j+1)(1-\overline{\alpha}^2\Phi_2(0))\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2, \quad \forall j\geqslant 0. \tag{4.4}$$

Proof. We denote by $K(A)$ the number $\mathscr{L}_A(1)$ and by $\mathscr{L}_0$ the Lebesgue functional. From Corollary 1 we have the following relation:

$$(z-\alpha)^2\left(\frac{1}{z}-\overline{\alpha}\right)^2\mathscr{L}=K(A)\mathscr{L}_0.$$

(1) For $j \geqslant 0$ we have

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\frac{1}{z^j}\right)=\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\left(\frac{1}{z^j}-\frac{1}{\alpha^j}+\frac{1}{\alpha^j}\right)\right)$$

$$=\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\left(\frac{\alpha^j-z^j}{\alpha^j z^j}\right)\right)+\frac{1}{\alpha^j}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\right).$$

Since for $j \geqslant 2$ $z^j-\alpha^j=(z-\alpha)\sum_{k=0}^{j-1}\alpha^k z^{j-(k+1)}$, then

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\frac{(z^j-\alpha^j)}{z^j}\right)=K(A)\mathscr{L}_0\left(\sum_{k=0}^{j-1}\alpha^k z^{-(k+1)}\right)=0.$$

Therefore for $j \geqslant 2$ it holds

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\frac{1}{z^j}\right)=\frac{1}{\alpha^j}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\right)$$

and this expression is also valid for $j=0$ and $j=1$. Next we calculate

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\right)=\mathscr{L}\left(\left(\frac{1}{z^2}-\frac{2\overline{\alpha}}{z}+\overline{\alpha}^2\right)(z-\alpha)\right)$$

$$=\overline{c_1}-2\overline{\alpha}+\overline{\alpha}^2c_1-\alpha\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\right).$$

Since $c_1=2\alpha/(1+|\alpha|^2)$, then $\overline{c_1}-2\overline{\alpha}+\overline{\alpha}^2c_1=0$ and

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\right)=-\alpha\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\right)=-\alpha(\overline{c_2}-2\overline{\alpha c_1}+\overline{\alpha}^2).$$

From $\Phi_2(0)=(c_1^2-c_2)/(1-|c_1|^2)$ we get $\overline{c_2}-2\overline{\alpha c_1}+\overline{\alpha}^2=(\overline{\alpha}^2-\overline{\Phi_2(0)})((1-|\alpha|^2)/(1+|\alpha|^2))^2$ and then

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\right)=-\alpha(\overline{\alpha}^2-\overline{\Phi_2(0)})\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2.$$

(2) Let $j \geqslant 1$. We compute

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\frac{1}{z^j}\right)=\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2z^2\left(\frac{1}{z^{j+2}}-\frac{1}{\alpha^{j+2}}+\frac{1}{\alpha^{j+2}}\right)\right)$$

$$=\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2z^2\left(\frac{\alpha^{j+2}-z^{j+2}}{z^{j+2}\alpha^{j+2}}\right)\right)+\frac{1}{\alpha^{j+2}}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2z^2\right).$$

Since

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\frac{(z^{j+2}-\alpha^{j+2})}{z^j}\right)=\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\frac{(z-\alpha)}{z^j}\left(\sum_{i=0}^{j+1}\alpha^i z^{j+1-i}\right)\right)$$

and we take into account that

$$\sum_{i=0}^{j+1} \alpha^i z^{j+1-i} = (z-\alpha)\left(\sum_{i=0}^{j}(i+1)\alpha^i z^{j-i}\right) + (j+2)\alpha^{j+1},$$

then

$$\begin{aligned}
&\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 \frac{(z^{j+2}-\alpha^{j+2})}{z^j}\right)\\
&= \mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 \frac{(z-\alpha)^2}{z^j}\left(\sum_{i=0}^{j}(i+1)\alpha^i z^{j-i}\right)\right) + \mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 \frac{(z-\alpha)}{z^j}(j+2)\alpha^{j+1}\right)\\
&= K(A)\mathscr{L}_0\left(\sum_{i=0}^{j}(i+1)\frac{\alpha^i}{z^i}\right) + (j+2)\alpha^{j+1}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 \frac{(z-\alpha)}{z^j}\right)\\
&= K(A) + (j+2)\alpha^{j+1}\left(-\frac{1}{\alpha^{j-1}}\right)(\overline{\alpha}^2 - \overline{\Phi_2(0)})\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2,
\end{aligned}$$

where the last equality follows from (1) above.

Since $A = (1-\overline{\alpha}^2\Phi_2(0))/(1-|\alpha|^4)$ then

$$K(A) = \left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2 (1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))). \tag{4.5}$$

Therefore, by substituting $K(A)$,

$$\begin{aligned}
\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 \frac{(z^{j+2}-\alpha^{j+2})}{z^j}\right) = \left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2 &(1-(j+1)|\alpha|^4 - 2\Re(\alpha^2\overline{\Phi_2(0)})\\
&+(j+2)\alpha^2\overline{\Phi_2(0)}).
\end{aligned}$$

On the other hand, $\mathscr{L}((1/z-\overline{\alpha})^2 z^2) = 1-2\overline{\alpha}c_1+\overline{\alpha}^2 c_2 = ((1-|\alpha|^2)/(1+|\alpha|^2))^2(1-\overline{\alpha}^2\Phi_2(0))$. Therefore,

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 \frac{1}{z^j}\right) = \frac{1}{\alpha^j}(j+1)(\overline{\alpha}^2 - \overline{\Phi_2(0)})\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2.$$

It is easy to see that the relation is true for $j=0$ and thus we get (2).

(3)

$$\begin{aligned}
\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 (z-\alpha)z^{j+1}\right) &= \mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 (z-\alpha)(z^{j+1}-\alpha^{j+1}+\alpha^{j+1})\right)\\
&= \mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 (z-\alpha)(z^{j+1}-\alpha^{j+1})\right) + \alpha^{j+1}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 (z-\alpha)\right).
\end{aligned}$$

For $j \geqslant 1$ it holds

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)(z-\alpha)\sum_{i=0}^{j}\alpha^i z^{j-i}\right)+\alpha^{j+1}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\right)$$
$$=K(A)\mathscr{L}_0\left(\sum_{i=0}^{j}\alpha^i z^{j-i}\right)+\alpha^{j+1}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\right).$$

By substituting $K(A)$ by expression (4.5) and using (1) for $j=0$ we finally obtain (3). Besides it is easy to check the equality for $j=0$:

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)z\right)=\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)(z-\alpha+\alpha)\right)$$
$$=K(A)\mathscr{L}_0(1)+\alpha\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\right)=(1-\overline{\alpha}^2\Phi_2(0))\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2.$$

(4)

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 z^{j+2}\right)=\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z^{j+2}-\alpha^{j+2})\right)+\alpha^{j+2}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\right)$$
$$=\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\left[(z-\alpha)\sum_{i=0}^{j}(i+1)\alpha^i z^{j-i}+(j+2)\alpha^{j+1}\right]\right)+\alpha^{j+2}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\right)$$
$$=K(A)\mathscr{L}_0\left(\sum_{i=0}^{j}(i+1)\alpha^i z^{j-i}\right)+(j+2)\alpha^{j+1}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\right)+\alpha^{j+2}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\right).$$

By using (1) and (2) for $j=0$ we get

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2 z^{j+2}\right)=(j+1)\alpha^j K(A)+(-(j+2)\alpha^{j+2}+\alpha^{j+2})(\overline{\alpha}^2-\overline{\Phi_2(0)})\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2,$$

from which, by using (4.5), the result follows. □

In the next lemma two representations of the orthogonal polynomials $\{\Phi_n(z)\}$ are given. These representations are necessary for obtaining the values $\Phi_n(\alpha)$ and $\Phi_n'(\alpha)$ in Theorem 6.

Lemma 3. *If $\mathscr{L}$ is a regular and hermitian solution of $(z-\alpha)^2(1/z-\overline{\alpha})^2\mathscr{L}=\mathscr{L}_A$ with $|\alpha|\neq 1$ and initial conditions $\mathscr{L}(1)=1$, $\mathscr{L}(z)=2\alpha/(1+|\alpha|^2)$ and $\mathscr{L}(z^2)=(-A(|\alpha|^2-1)^3+3|\alpha|^2-1)/(1+|\alpha|^2)\overline{\alpha}^2$, then $E_n\neq 0$, with*

$$E_n=\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2\left[\frac{(\overline{\alpha}^2-\overline{\Phi_2(0)})}{\alpha^{n-2}}[(n-1)\Phi_n(\alpha)-\alpha\Phi_n'(\alpha)]\right.$$
$$\left.+1+|\alpha|^4-2\Re(\overline{\alpha}^2\Phi_2(0))\right] \quad \text{for } n\geqslant 3. \tag{4.6}$$

Besides the orthogonal polynomial sequence $\{\Phi_n(z)\}$ *related to* $\mathscr{L}$ *is given by the following representations*:

$$\begin{aligned}\Phi_n(z) &= z^n + z^{n-1}\left(-2\alpha - [(n-2)\Phi_n(\alpha) - \alpha\Phi_n'(\alpha)]\frac{(\overline{\alpha}^2 - \overline{\Phi_2(0)})}{\alpha^{n-3}(1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0)))}\right)\\ &\quad + z^{n-2}\left(\alpha^2 + [(n-1)\Phi_n(\alpha) - \alpha\Phi_n'(\alpha)]\frac{(\overline{\alpha}^2 - \overline{\Phi_2(0)})}{\alpha^{n-4}(1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0)))}\right)\\ &\quad + z\Phi_n'(\alpha)\frac{1-\overline{\alpha}^2\Phi_2(0)}{1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))} + [\Phi_n(\alpha) - \alpha\Phi_n'(\alpha)]\\ &\quad \times \frac{1-\overline{\alpha}^2\Phi_2(0)}{1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))}, \quad \forall n \geqslant 3;\end{aligned} \tag{4.7}$$

$$\Phi_2(z) = z^2 - 2\frac{(\alpha + \Phi_2(0)\overline{\alpha})}{1+|\alpha|^2}z + \Phi_2(0),$$

$$\Phi_1(z) = z - \frac{2\alpha}{1+|\alpha|^2},$$

$$\begin{aligned}\Phi_n(z) &= z^n + z^{n-1}\left(-\frac{2}{\overline{\alpha}} - \left[(n-2)\overline{\alpha}\Phi_n\left(\frac{1}{\overline{\alpha}}\right) - \Phi_n'\left(\frac{1}{\overline{\alpha}}\right)\right]\frac{\overline{\alpha}^{n-2}(1-\alpha^2\overline{\Phi_2(0)})}{1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))}\right)\\ &\quad + z^{n-2}\left(\frac{1}{\overline{\alpha}^2} + \left[(n-1)\overline{\alpha}\Phi_n\left(\frac{1}{\overline{\alpha}}\right) - \Phi_n'\left(\frac{1}{\overline{\alpha}}\right)\right]\frac{\overline{\alpha}^{n-3}(1-\alpha^2\overline{\Phi_2(0)})}{1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))}\right)\\ &\quad + z\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)\frac{\overline{\alpha}^2(\alpha^2 - \Phi_2(0))}{1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))} + \left[\Phi_n\left(\frac{1}{\overline{\alpha}}\right) - \frac{1}{\overline{\alpha}}\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)\right]\\ &\quad \times \frac{\overline{\alpha}^2(\alpha^2 - \Phi_2(0))}{1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))}, \quad \forall n \geqslant 3.\end{aligned} \tag{4.8}$$

Proof. We distinguish two cases:

(i) We suppose that $K(A) \neq 0$ (or equivalently, $2\Re A \neq 1$). We can write for $n \geqslant 3$:

$$\frac{\Phi_n(z) - \Phi_n(\alpha) - \Phi_n'(\alpha)(z-\alpha)}{(z-\alpha)^2} = z^{n-2} + \sum_{k=0}^{n-3}\lambda_{nk}z^k.$$

Applying $(z-\alpha)^2(\frac{1}{z} - \overline{\alpha})^2\mathscr{L}$ to both members we get

$$\begin{aligned}&K(A)\mathscr{L}_0\left(\frac{\Phi_n(z) - \Phi_n(\alpha) - \Phi_n'(\alpha)(z-\alpha)}{(z-\alpha)^2}z^{-j}\right)\\ &\quad = \begin{cases} K(A) & \text{if } j = n-2,\\ K(A)\lambda_{nj} & \text{if } j \in \{0,\ldots,n-3\},\end{cases}\end{aligned}$$

i.e.,

$$\mathscr{L}\left(\left(\frac{1}{z}-\bar{\alpha}\right)^2\Phi_n(z)z^{-j}\right)-\Phi_n(\alpha)\mathscr{L}\left(\left(\frac{1}{z}-\bar{\alpha}\right)^2 z^{-j}\right)$$
$$-\Phi_n'(\alpha)\mathscr{L}\left(\left(\frac{1}{z}-\bar{\alpha}\right)^2(z-\alpha)z^{-j}\right)$$
$$=\begin{cases}K(A) & \text{for } j=n-2,\\ K(A)\lambda_{nj} & \text{for } j\in\{0,\dots,n-3\}.\end{cases} \tag{4.9}$$

Since

$$\mathscr{L}\left(\left(\frac{1}{z}-\bar{\alpha}\right)^2\Phi_n(z)z^{-j}\right)=\mathscr{L}(z^{-(j+2)}\Phi_n(z))$$
$$=\begin{cases}E_n & \text{for } j=n-2,\\ 0 & \text{for } j\in\{0,\dots,n-3\},\end{cases} \tag{4.10}$$

if we take into account the previous lemma we obtain for $j\in\{0,\dots,n-3\}$

$$\lambda_{nj}=-\frac{\alpha^{-j}(\bar{\alpha}^2-\overline{\Phi_2(0)})}{1+|\alpha|^4-2\Re(\bar{\alpha}^2\Phi_2(0))}[(j+1)\Phi_n(\alpha)-\alpha\Phi_n'(\alpha)]$$

and

$$\frac{\Phi_n(z)-\Phi_n(\alpha)-\Phi_n'(\alpha)(z-\alpha)}{(z-\alpha)^2}=z^{n-2}-\lambda\Phi_n(\alpha)\sum_{j=0}^{n-3}(j+1)\left(\frac{z}{\alpha}\right)^j+\lambda\alpha\Phi_n'(\alpha)\sum_{j=0}^{n-3}\left(\frac{z}{\alpha}\right)^j,$$

with

$$\lambda=\frac{(\bar{\alpha}^2-\overline{\Phi_2(0)})}{1+|\alpha|^4-2\Re(\bar{\alpha}^2\Phi_2(0))}.$$

By computing the sums

$$\sum_{j=0}^{n-3}\left(\frac{z}{\alpha}\right)^j=\frac{z^{n-2}-\alpha^{n-2}}{\alpha^{n-3}(z-\alpha)}\quad\text{and}\quad\sum_{j=0}^{n-3}(j+1)\left(\frac{z}{\alpha}\right)^j=\frac{(n-2)z^{n-1}-\alpha(n-1)z^{n-2}+\alpha^{n-1}}{\alpha^{n-3}(z-\alpha)^2}$$

we have for $n\geqslant 3$

$$\Phi_n(z)=z^n+z^{n-1}\left(-2\alpha-\lambda(n-2)\frac{\Phi_n(\alpha)}{\alpha^{n-3}}+\lambda\frac{\Phi_n'(\alpha)}{\alpha^{n-4}}\right)$$
$$+z^{n-2}\left(\alpha^2+\lambda(n-1)\frac{\Phi_n(\alpha)}{\alpha^{n-4}}-\lambda\frac{\Phi_n'(\alpha)}{\alpha^{n-5}}\right)+z(1-\lambda\alpha^2)\Phi_n'(\alpha)$$
$$+(1-\lambda\alpha^2)[\Phi_n(\alpha)-\alpha\Phi_n'(\alpha)]. \tag{4.11}$$

Taking into account that

$$1-\lambda\alpha^2=\frac{(1-\overline{\alpha}^2\Phi_2(0))}{1+|\alpha|^4-2\Re(\overline{\alpha}^2\Phi_2(0))}$$

we obtain (4.7). Besides, from (4.9) and (4.10) for $j=n-2$ and the previous lemma we get (4.6):

$$E_n=\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2(1+|\alpha|^4-2\Re(\overline{\alpha}^2\Phi_2(0)))\left((n-1)\lambda\frac{\Phi_n(\alpha)}{\alpha^{n-2}}-\lambda\frac{\Phi_n'(\alpha)}{\alpha^{n-3}}+1\right). \tag{4.12}$$

Therefore, since $\mathscr{L}$ is regular $E_1=((1-|\alpha|^2)/(1+|\alpha|^2))^2\neq 0$, $E_2=(1-|\Phi_2(0)|^2)((1-|\alpha|^2)/(1+|\alpha|^2))^2\neq 0$, and for $n\geqslant 3$

$$\left((n-1)\frac{\Phi_n(\alpha)}{\alpha^{n-2}}-\frac{\Phi_n'(\alpha)}{\alpha^{n-3}}\right)\frac{(\overline{\alpha}^2-\overline{\Phi_2(0)})}{1+|\alpha|^4-2\Re(\overline{\alpha}^2\Phi_2(0))}+1\neq 0.$$

In order to obtain $\Phi_n(z)$ we need to know $\Phi_n(\alpha)$ and $\Phi_n'(\alpha)$. To solve this problem we proceed in a similar way as before: Since for $n\geqslant 3$

$$\frac{\Phi_n(z)-\Phi_n\left(\frac{1}{\overline{\alpha}}\right)-\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)\left(z-\frac{1}{\overline{\alpha}}\right)}{(z-\frac{1}{\overline{\alpha}})^2}=z^{n-2}+\sum_{k=0}^{n-3}\mu_{nk}z^k$$

then

$$\frac{z^2\alpha^2\overline{\Phi_n}\left(\frac{1}{z}\right)-z^2\alpha^2\overline{\Phi_n\left(\frac{1}{\overline{\alpha}}\right)}+z\alpha\overline{\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)}(z-\alpha)}{(z-\alpha)^2}=\frac{1}{z^{n-2}}+\sum_{k=0}^{n-3}\overline{\mu_{nk}}\frac{1}{z^k}.$$

Applying $(z-\alpha)^2(1/z-\overline{\alpha})^2\mathscr{L}$ to both members we get

$$\begin{aligned}&\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\alpha^2\overline{\Phi_n}\left(\frac{1}{z}\right)z^{j+2}\right)-\overline{\Phi_n\left(\frac{1}{\overline{\alpha}}\right)}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\alpha^2z^{j+2}\right)\\&\quad+\overline{\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)}\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2(z-\alpha)\alpha z^{j+1}\right)\\&=\begin{cases}K(A) & \text{for } j=n-2,\\ K(A)\overline{\mu}_{nj} & \text{for } j\in\{0,\dots,n-3\}.\end{cases}\end{aligned}$$

For $j\in\{0,\dots,n-3\}$, taking into account Lemma 2 and applying that

$$\mathscr{L}\left(\left(\frac{1}{z}-\overline{\alpha}\right)^2\overline{\Phi_n}\left(\frac{1}{z}\right)z^{j+2}\right)=\begin{cases}\overline{\alpha^2}E_n & \text{for } j=n-2,\\ 0 & \text{for } j\in\{0,\dots,n-3\},\end{cases}$$

we deduce

$$\mu_{nj}=-\left[\Phi_n\left(\frac{1}{\overline{\alpha}}\right)\overline{\alpha}(j+1)-\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)\right]\overline{\alpha}^{j+1}\frac{(1-\alpha^2\overline{\Phi_2(0)})}{1+|\alpha|^4-2\Re(\overline{\alpha}^2\Phi_2(0))}.$$

Therefore, if we denote by

$$\mu=\frac{(1-\alpha^2\overline{\Phi_2(0)})}{1+|\alpha|^4-2\Re(\overline{\alpha}^2\Phi_2(0))}$$

it holds that

$$\frac{\Phi_n(z)-\Phi_n(\frac{1}{\alpha})-\Phi_n'(\frac{1}{\alpha})(z-\frac{1}{\alpha})}{(z-\frac{1}{\alpha})^2}=z^{n-2}+\mu\sum_{j=0}^{n-3}\left(-\Phi_n\left(\frac{1}{\overline{\alpha}}\right)\overline{\alpha}^{j+2}(j+1)+\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)\overline{\alpha}^{j+1}\right)z^j.$$

If we compute the sums and make some calculations we get

$$\Phi_n(z)-\Phi_n\left(\frac{1}{\overline{\alpha}}\right)-\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)\left(z-\frac{1}{\overline{\alpha}}\right)=z^{n-2}\left(z-\frac{1}{\overline{\alpha}}\right)^2-\mu\Phi_n\left(\frac{1}{\overline{\alpha}}\right)[(n-2)(\overline{\alpha}z)^{n-1}$$
$$-(n-1)(\overline{\alpha}z)^{n-2}+1]+\mu\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)((\overline{\alpha}z)^{n-2}-1)\left(z-\frac{1}{\overline{\alpha}}\right). \tag{4.13}$$

Finally, if we substitute μ in (4.13) we obtain (4.8).

(ii) Let us now assume that $K(A)=0$ $(2\Re A=1)$. Proceeding in the same way as in the previous case we obtain from (4.9) for $j\in\{0,\dots,n-3\}$:

$$(\overline{\alpha}^2-\overline{\Phi_2(0)})\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2\alpha^{-j}[(j+1)\Phi_n(\alpha)-\alpha\Phi_n'(\alpha)]=0.$$

Since $A\neq 1$ then $\Phi_2(0)\neq\alpha^2$ and the last equation implies

$$(j+1)\Phi_n(\alpha)=\alpha\Phi_n'(\alpha),\quad \forall j\in\{0,\dots,n-3\} \text{ and } n\geqslant 3.$$

Then

$$\Phi_n(\alpha)=\Phi_n'(\alpha)=0,\quad \forall n\geqslant 3.$$

For $j=n-2$ from (4.9) and (4.10) we obtain that $E_n=0$, $n\geqslant 3$ in contradiction with the regularity of $\mathscr{L}$. □

Theorem 6. *If $\mathscr{L}$ is a hermitian solution of $(z-\alpha)^2(1/z-\overline{\alpha})^2\mathscr{L}=\mathscr{L}_A$, with $|\alpha|\neq 1$ and initial conditions $\mathscr{L}(1)=1$, $\mathscr{L}(z)=2\alpha/(1+|\alpha|^2)$, $\mathscr{L}(z^2)=(-A(|\alpha|^2-1)^3+3|\alpha|^2-1)/(1+|\alpha|^2)\overline{\alpha}^2$ and $|\Phi_2(0)|\neq 1$, then $\mathscr{L}$ is regular if and only if*

$$\forall n\geqslant 3\quad \alpha^{2n-7}D_n\neq 0, \tag{4.14}$$

with

$$\alpha^{2n-7}D_n=|\alpha^2-\Phi_2(0)|^4|\alpha|^2+|1-\alpha^2\overline{\Phi_2(0)}|^4|\alpha|^{2(2n-7)}$$
$$-|\alpha^2-\Phi_2(0)|^2|1-\alpha^2\overline{\Phi_2(0)}|^2((n-2)^2|\alpha|^4-2(n-1)(n-3)|\alpha|^2+(n-2)^2)|\alpha|^{2(n-4)}. \tag{4.15}$$

Besides the orthogonal polynomial sequence $\{\Phi_n(z)\}$ *related to* $\mathscr{L}$ *is given by* (4.7) *with*

$$\Phi_n(\alpha) = \frac{(1-|\alpha|^2)^2}{\overline{\alpha}D_n}[1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))](\alpha^2 - \Phi_2(0)) \times \left(\frac{|\alpha^2 - \Phi_2(0)|^2}{\alpha^{n-4}} + |1-\alpha^2\overline{\Phi_2(0)}|^2\overline{\alpha}^{n-4}[1-n+(n-2)|\alpha|^2]\right), \tag{4.16}$$

$$\Phi_n'(\alpha) = \frac{(1-|\alpha|^2)}{\overline{\alpha}D_n}[1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))](\alpha^2 - \Phi_2(0)) \times \left(\frac{|\alpha^2 - \Phi_2(0)|^2}{\alpha^{n-3}}[(n-2)-n|\alpha|^2] - |1-\alpha^2\overline{\Phi_2(0)}|^2\overline{\alpha}^{n-3}[-n+(n-2)|\alpha|^2]\right). \tag{4.17}$$

Proof. $\Rightarrow$: Taking into account Lemma 3, first we compare coefficients between (4.7) and (4.8). If we identify the coefficients of z we get

$$\forall n \geqslant 3 \quad \overline{\alpha}^2(\alpha^2 - \Phi_2(0))\Phi_n'\left(\frac{1}{\overline{\alpha}}\right) = (1-\overline{\alpha}^2\Phi_2(0))\Phi_n'(\alpha).$$

Therefore, if $\Phi_2(0) = \alpha^2$ then $\Phi_n'(\alpha) = 0 \quad \forall n \geqslant 3$, and if $\Phi_2(0) \neq \alpha^2$ then

$$\Phi_n'\left(\frac{1}{\overline{\alpha}}\right) = \frac{(1-\overline{\alpha}^2\Phi_2(0))}{\overline{\alpha}^2(\alpha^2 - \Phi_2(0))}\Phi_n'(\alpha). \tag{4.18}$$

In the same way, identifying the independent terms

$$\overline{\alpha}^2(\alpha^2 - \Phi_2(0))\left[\Phi_n\left(\frac{1}{\overline{\alpha}}\right) - \frac{1}{\overline{\alpha}}\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)\right] = (1-\overline{\alpha}^2\Phi_2(0))[\Phi_n(\alpha) - \alpha\Phi_n'(\alpha)],$$

thus if $\Phi_2(0) = \alpha^2$ then $\Phi_n(\alpha) = 0 \ \forall n \geqslant 3$, and $\Phi_n(z) = z^{n-2}(z-\alpha)^2$. If $\Phi_2(0) \neq \alpha^2$ and we take into account (4.18) then

$$\Phi_n\left(\frac{1}{\overline{\alpha}}\right) = \frac{(1-\overline{\alpha}^2\Phi_2(0))}{\overline{\alpha}^3(\alpha^2 - \Phi_2(0))}[\overline{\alpha}\Phi_n(\alpha) + (1-|\alpha|^2)\Phi_n'(\alpha)]. \tag{4.19}$$

From the coefficients of z^{n-1} we deduce

$$(\overline{\alpha}^2 - \overline{\Phi_2(0)})\left(-\frac{(n-2)}{\alpha^{n-3}}\Phi_n(\alpha) + \frac{1}{\alpha^{n-4}}\Phi_n'(\alpha)\right) + (1-\alpha^2\overline{\Phi_2(0)}) \times \left((n-2)\overline{\alpha}^{n-1}\Phi_n\left(\frac{1}{\overline{\alpha}}\right) - \overline{\alpha}^{n-2}\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)\right) = 2\frac{(|\alpha|^2-1)}{\overline{\alpha}}[1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))]. \tag{4.20}$$

From the coefficients of z^{n-2} we get

$$(\overline{\alpha}^2 - \overline{\Phi_2(0)})\left(\frac{(n-1)}{\alpha^{n-4}}\Phi_n(\alpha) - \frac{1}{\alpha^{n-5}}\Phi_n'(\alpha)\right) - (1 - \alpha^2\overline{\Phi_2(0)})$$
$$\times\left((n-1)\overline{\alpha}^{n-2}\Phi_n\left(\frac{1}{\overline{\alpha}}\right) - \overline{\alpha}^{n-3}\Phi_n'\left(\frac{1}{\overline{\alpha}}\right)\right) = \frac{(1-|\alpha|^2)(1+|\alpha|^2)}{\overline{\alpha}^2}[1 + |\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))]. \tag{4.21}$$

We eliminate $\Phi_n'(1/\overline{\alpha})$ between (4.20) and (4.21) and we substitute $\Phi_n(1/\overline{\alpha})$ by using (4.19) obtaining

$$\left(\frac{|\alpha^2 - \Phi_2(0)|^2}{\alpha^{n-3}}[(n-1)|\alpha|^2 - (n-2)] - |1 - \alpha^2\overline{\Phi_2(0)}|^2\overline{\alpha}^{n-3}\right)\Phi_n(\alpha)$$
$$+\left(\frac{|\alpha^2 - \Phi_2(0)|^2}{\alpha^{n-4}} - |1 - \alpha^2\overline{\Phi_2(0)}|^2\overline{\alpha}^{n-4}\right)(1-|\alpha|^2)\Phi_n'(\alpha)$$
$$= \frac{-(1-|\alpha|^2)^2}{\overline{\alpha}}[1 + |\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))](\alpha^2 - \Phi_2(0)). \tag{4.22}$$

From (4.20) and (4.21) we eliminate $\Phi_n(1/\overline{\alpha})$ and taking into account (4.18) we have

$$(n-1)(n-2)(\overline{\alpha}^2 - \overline{\Phi_2(0)})\frac{(|\alpha|^2 - 1)}{\alpha^{n-3}}\Phi_n(\alpha)$$
$$+\left(\frac{\overline{\alpha}^2 - \overline{\Phi_2(0)}}{\alpha^{n-4}}[(n-1) - (n-2)|\alpha|^2] - |1 - \alpha^2\overline{\Phi_2(0)}|^2\frac{\overline{\alpha}^{n-4}}{(\alpha^2 - \Phi_2(0))}\right)\Phi_n'(\alpha)$$
$$= \frac{(|\alpha|^2 - 1)}{\overline{\alpha}}(1 + |\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))[n - (n-2)|\alpha|^2]). \tag{4.23}$$

We can consider the two last equations as a linear system in the unknowns $\Phi_n(\alpha)$ and $\Phi_n'(\alpha)$. If we denote the determinant of the matrix coefficients by D_n we know, since $\mathcal{L}$ is regular, that D_n must be different from 0 for all $n \geqslant 3$. Taking into account that

$$D_n = \frac{|\alpha^2 - \Phi_2(0)|^4}{\alpha^{2n-7}}|\alpha|^2 - |\alpha^2 - \Phi_2(0)|^2|1 - \alpha^2\overline{\Phi_2(0)}|^2\frac{\overline{\alpha}^{n-4}}{\alpha^{n-3}}$$
$$\times((n-2)^2|\alpha|^4 - 2(n-1)(n-3)|\alpha|^2 + (n-2)^2) + |1 - \alpha^2\overline{\Phi_2(0)}|^4\overline{\alpha}^{2n-7}, \tag{4.24}$$

we deduce that $\alpha^{2n-7} \in \mathbb{R}$ and we obtain (4.15). By solving the system we get the expressions of the unknowns $\Phi_n(\alpha)$ and $\Phi_n'(\alpha)$ given by (4.16) and (4.17). Finally, from (4.13) we obtain

$$\overline{\alpha}D_nE_n\left(\frac{1+|\alpha|^2}{1-|\alpha|^2}\right)^2\frac{1}{1 + |\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))}$$
$$= \overline{\alpha}D_n + \frac{\overline{\alpha}^2 - \overline{\Phi_2(0)}}{[1 + |\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))]\alpha^{n-2}}((n-1)\overline{\alpha}D_n\Phi_n(\alpha) - \alpha\Phi_n'(\alpha)\overline{\alpha}D_n).$$

If we substitute (4.16) and (4.17) and operate we get

$$\overline{\alpha} D_n E_n \left(\frac{1+|\alpha|^2}{1-|\alpha|^2}\right)^2 = [1+|\alpha|^4 - 2\Re(\overline{\alpha}^2 \Phi_2(0))]\frac{D_{n+1}}{\overline{\alpha}},$$

from which we conclude

$$\forall n \geqslant 3 \; E_n = \frac{D_{n+1}}{\overline{\alpha}^2 D_n}\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2 [1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))]$$

$$= \frac{\alpha^{2n-5} D_{n+1}}{|\alpha|^4 \alpha^{2n-7} D_n}\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2 [1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))].$$

Thus, if $\mathscr{L}$ is regular the result holds.

$\Leftarrow$: We suppose that $D_n \neq 0 \; \forall n \geqslant 3$ and we prove that the sequence of polynomials given by (4.7) is the MOPS related to $\mathscr{L}$. We shall calculate $\mathscr{L}(\Phi_n(z)z^{-k})$ distinguishing several cases since the result depends on whether the moments have positive or negative subindex. From (4.11)

$$\Phi_n(z) = z^{n-2}(z-\alpha)^2 - \frac{\lambda\Phi_n(\alpha)}{\alpha^{n-3}}[(n-2)z-(n-1)\alpha]z^{n-2}$$
$$+ \frac{\lambda\Phi_n'(\alpha)}{\alpha^{n-4}} z^{n-2}(z-\alpha) + (1-\lambda\alpha^2)\Phi_n'(\alpha)(z-\alpha) + (1-\lambda\alpha^2)\Phi_n(\alpha),$$

$$\mathscr{L}(\Phi_n(z)z^{-k}) = \mathscr{L}(z^{n-k-2}(z-\alpha)^2) - \frac{\lambda\Phi_n(\alpha)}{\alpha^{n-3}}\mathscr{L}(z^{n-k-2}[(n-2)z-(n-1)\alpha])$$
$$+ \frac{\lambda\Phi_n'(\alpha)}{\alpha^{n-4}}\mathscr{L}(z^{n-k-2}(z-\alpha)) + (1-\lambda\alpha^2)\Phi_n'(\alpha)\mathscr{L}(z^{-k}(z-\alpha))$$
$$+ (1-\lambda\alpha^2)\Phi_n(\alpha)\mathscr{L}(z^{-k}) \tag{4.25}$$

(i) If $1 \leqslant k \leqslant n-2$, from (4.2) and the hermitian character of $\mathscr{L}$, we obtain

$$\mathscr{L}(z^{n-k-2}(z-\alpha)^2) = \overline{\mathscr{L}\left(z^{-n+2+k}\left(\frac{1}{z}-\overline{\alpha}\right)^2\right)}$$
$$= \frac{1}{\overline{\alpha}^{n-2-k}}(n-1-k)(\alpha^2 - \Phi_2(0))\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2. \tag{4.26}$$

The coefficient of $\Phi_n(\alpha)$ in (4.25) is

$$\frac{\overline{\alpha}^k}{(1-|\alpha|^4)(1+|\alpha|^2)\overline{[1+|\alpha|^4 - 2\Re(\overline{\alpha}^2\Phi_2(0))]}}$$
$$\times\left[\frac{|\alpha^2-\Phi_2(0)|^2}{|\alpha|^{2(n-3)}}[-(n-2)(n-k-2)\right.$$
$$+((n-2)(n-k)+(n-1)(n-k-3))|\alpha|^2 - (n-1)(n-k-1)|\alpha|^4]$$
$$\left.+|1-\overline{\alpha}^2\Phi_2(0)|^2[(k+1)-(k-1)|\alpha|^2]\right], \tag{4.27}$$

and the coefficient of $\Phi_n'(\alpha)$ is

$$\frac{(1-|\alpha|^2)\overline{\alpha}^{k-1}}{(1-|\alpha|^4)(1+|\alpha|^2)[1+|\alpha|^4-2\Re(\overline{\alpha}^2\Phi_2(0))]} \times \left[\frac{|\alpha^2-\Phi_2(0)|^2}{|\alpha|^{2(n-4)}}[(n-k-2)-(n-k-1)|\alpha|^2]+|1-\overline{\alpha}^2\Phi_2(0)|^2[k-(k-1)|\alpha|^2]\right]. \tag{4.28}$$

Taking into account the previous expressions and the values of D_n, $\Phi_n(\alpha)$ and $\Phi_n'(\alpha)$ given by (4.15) –(4.17) we deduce that (4.25) is 0 for $1\leqslant k\leqslant n-2$ and $\forall n\geqslant 3$.

(ii) If $k=n-1$, one can proceed as in case (i). The coefficients obtained for $\Phi_n(\alpha)$ and for $\Phi_n'(\alpha)$ are the same as in the previous case.

(iii) If $k=0$, same as (ii).

(iv) If $k=n$ we compute the first addend in (4.25) from (4.4) for $j=0$. Besides, the coefficient of $\Phi_n(\alpha)$ in (4.25) is

$$\frac{\overline{\alpha}^n}{(1-|\alpha|^4)(1+|\alpha|^2)[1+|\alpha|^4-2\Re(\overline{\alpha}^2\Phi_2(0))]}\left[\frac{|\alpha^2-\Phi_2(0)|^2}{|\alpha|^{2(n-2)}}[(n-1)-(n+1)|\alpha|^2] +|1-\overline{\alpha}^2\Phi_2(0)|^2[(n+1)-(n-1)|\alpha|^2] -\frac{(n-1)(1-\alpha^2\overline{\Phi_2(0)})(\overline{\alpha}^2-\overline{\Phi_2(0)})(|\alpha|^2-1)^3}{|\alpha|^{2(n-2)}\overline{\alpha}^2}\right], \tag{4.29}$$

and the coefficient of $\Phi_n'(\alpha)$ is

$$\frac{(1-|\alpha|^2)\overline{\alpha}^{n-1}}{(1-|\alpha|^4)(1+|\alpha|^2)[1+|\alpha|^4-2\Re(\overline{\alpha}^2\Phi_2(0))]} \times \left[-\frac{|\alpha^2-\Phi_2(0)|^2}{|\alpha|^{2(n-3)}}-\frac{(1-|\alpha|^2)^2}{\overline{\alpha}^2|\alpha|^{2(n-3)}}(1-\alpha^2\overline{\Phi_2(0)})(\alpha^2-\overline{\Phi_2(0)}) +|1-\overline{\alpha}^2\Phi_2(0)|^2[n-(n-1)|\alpha|^2]\right]. \tag{4.30}$$

Next one can proceed as in case (i) obtaining that $\mathscr{L}(\Phi_n(z)z^{-n})=E_n$, where E_n is given as follows:

$$E_n=\left(\frac{1-|\alpha|^2}{1+|\alpha|^2}\right)^2[1+|\alpha|^4-2\Re(\overline{\alpha}^2\Phi_2(0))]\frac{D_{n+1}}{\overline{\alpha}^2D_n}. \qquad \square$$

Theorem 7. *The only functionals $\mathscr{L}$ which solve our problem in the positive-definite case are the following*:

(i) *If* $|\alpha|<1$ *then* $\mathscr{L}(z^n)=u_n$ $(n\geqslant 0)$, *i.e.*, $A=1$. *The MOPS related to* $\mathscr{L}$ *is given by* $\Phi_n(z)=z^{n-2}(z-\alpha)^2$ $\forall n\geqslant 2$ *and* $\Phi_1(z)=z-2\alpha/(1+|\alpha|^2)$.

(ii) *If* $|\alpha|>1$ *then* $\mathscr{L}(z^n)=v_n$ $(n\geqslant 0)$, *i.e.*, $A=0$. *The MOPS related to* $\mathscr{L}$ *is given by* $\Psi_n(z)=z^{n-2}(z-1/\overline{\alpha})^2$ $\forall n\geqslant 2$ *and* $\Psi_1(z)=z-2\alpha/(1+|\alpha|^2)$.

Proof. If $\mathscr{L}$ is positive definite the series $S(z)$ defined in Theorem 3 is

$$S(z)=\frac{1}{1+|\alpha|^2}\sum_{n=0}^{+\infty}\left[\overline{A}[(n+1)+(1-n)|\alpha|^2](\overline{\alpha}z)^n+(1-\overline{A})[(1-n)+(n+1)|\alpha|^2]\left(\frac{z}{\alpha}\right)^n\right]. \tag{4.31}$$

Since $|\alpha|\neq 1$ we distinguish two possibilities:

(i) If $|\alpha|<1$ there exists z such that $|\alpha|<|z|<1$. In this case $|\overline{\alpha}z|<1$ and the series $\sum_{n=0}^{+\infty}\overline{A}[(n+1)+(1-n)|\alpha|^2](\overline{\alpha}z)^n$ converges $\forall A\in\mathbb{C}$. On the other hand, since $|z/\alpha|>1$ the series $\sum_{n=0}^{+\infty}(1-\overline{A})[(1-n)+(n+1)|\alpha|^2](z/\alpha)^n$ diverges if $A\neq 1$. Then in order that $S(z)$ converges for all z in the unit disk, A must be equal to 1.

(ii) If $|\alpha|>1$ there exists z such that $1/|\alpha|<|z|<1$. In this case $|\alpha z|>1$ and $|z/\alpha|<1$. Therefore the series $\sum_{n=0}^{+\infty}(1-\overline{A})[(1-n)+(n+1)|\alpha|^2](z/\alpha)^n$ converges $\forall A\in\mathbb{C}$ and the series $\sum_{n=0}^{+\infty}\overline{A}[(n+1)+(1-n)|\alpha|^2](\overline{\alpha}z)^n$ diverges except for $A=0$. Then in order that $S(z)$ converges for all z in the unit disk, $1-A$ must be equal to 1. □

The orthogonal polynomials appearing in this theorem coincide with those related to rational modifications of the Lebesgue measure, i.e., orthogonal polynomials with respect to the measure $d\omega(\theta)=d\theta/2\pi|z-\alpha|^2$, $z=e^{i\theta}$, $|\alpha|<1$, which is a semiclassical measure and the corresponding induced functional belongs to the class (2,2) (see [3]).

It seems natural to ask for the existence of families of semiclassical orthogonal polynomials belonging to the class (2,2) which are different from the Bernstein-Szegő polynomials obtained in the positive definite case. The next example shows that for every complex number α, if we choose $\Phi_2(0)=\alpha^2+1/\overline{\alpha}^2$ then the functional is regular and therefore there exists the corresponding sequence of orthogonal polynomials.

Example (*A Regular case*). Let us consider $\Phi_2(0)=\alpha^2+1/\overline{\alpha}^2$ then $1-|\Phi_2(0)|^2=-(1+|\alpha|^4+|\alpha|^8)/|\alpha|^4<0$.

From (4.15) $\forall n\geqslant 3$,

$$\alpha^{2n-7}D_n=\frac{1}{|\alpha|^6}[1-|\alpha|^{2(n+2)}-|\alpha|^{n+1}(n-2)(|\alpha|^2-1)]\times[1-|\alpha|^{2(n+2)}+|\alpha|^{n+1}(n-2)(|\alpha|^2-1)].$$

Putting $z=|\alpha|$ we can consider $|\alpha|^6\alpha^{2n-7}D_n$ as a product of two polynomials in z as follows:

$$[z^{2(n+2)}+(n-2)z^{n+3}-(n-2)z^{n+1}-1][z^{2(n+2)}-(n-2)z^{n+3}+(n-2)z^{n+1}-1].$$

We look for the positive roots of the following polynomials:

$$P_{2n+4}(z) = z^{2(n+2)} + (n-2)z^{n+3} - (n-2)z^{n+1} - 1,$$

$$Q_{2n+4}(z) = z^{2(n+2)} - (n-2)z^{n+3} + (n-2)z^{n+1} - 1.$$

By applying Descartes's rules of signs [6] we deduce that the polynomial $P_{2n+4}(z)$ has exactly one positive root $z = 1$.

Next we analyse the polynomial $Q_{2n+4}(z)$:

$$Q_{2n+4}\left(\frac{1}{z}\right) = -\frac{Q_{2n+4}(z)}{z^{2n+4}},$$

hence if z is root of $Q_{2n+4}(z)$ then $1/z$ is also a root.

If $n \geqslant 4$ is even

$$R_{2n+2}(z) = \frac{Q_{2n+4}(z)}{z^2-1} = z^{2n+2} + z^{2n} + \cdots + z^{n+2} - (n-2)z^{n+1} + z^n + \cdots + z^2 + 1.$$

If $n \geqslant 4$ is odd

$$R_{2n+2}(z) = \frac{Q_{2n+4}(z)}{z^2-1} = z^{2n+2} + z^{2n} + \cdots + z^{n+3} - (n-3)z^{n+1} + z^{n-1} + \cdots + z^2 + 1.$$

Let us assume that $n \geqslant 4$ and n is even

$$zR_{2n+2}(z) - R_{2(n+1)+2}(z) = -\frac{(z^{n+3}-1)(z^{n+2}-1)}{z+1} < 0, \quad \forall z > 1$$

then

$$zR_{2n+2}(z) < R_{2n+4}(z), \quad \forall z > 1.$$

Taking into account that

$$R_{10}(z) = z^{10} + z^8 + z^6 - 2z^5 + z^4 + z^2 + 1 > 0, \quad \forall z > 1$$

we deduce that $R_{12} > zR_{10} > 0$, $\forall z > 1$.

On the other hand,

$$R_{2(n+1)+2}(z) - R_{2(n+2)+2}(z) = z^{n+2}(1-z)[z^{n+3} + z^{n+2} + \cdots + z + 1 - n] < 0 \quad \text{if } z > 1.$$

From the last expression for $n = 4$ we obtain that $R_{14} > R_{12} > 0$, $\forall z > 1$. Proceeding inductively from the recursion relation one easily sees that $R_{2n+2}(z) > 0$, $\forall z > 1$ and therefore the polynomial $Q_{2n+4}(z) \neq 0$, $\forall z > 0$, and $z \neq 1$ because if it had one real root lesser than 1 then it would had to have one other larger than 1.

References

[1] R.P. Agarwal, Difference Equations and Inequalities, Marcel Dekker, New York, 1992.

[2] M. Alfaro, F. Marcellán, Recent trends in orthogonal polynomials on the unit circle, IMACS Ann. Comput. Appl. Math. 9 (1–4) (1991) 3–14.

[3] A. Cachafeiro, C. Suárez, Orthogonal Polynomials Related to the Unit Circle and Differential-Difference Equations, Rendiconti di Matematica, Ser. VII, vol. 17, Roma, 1997.

[4] G. Freud, Orthogonal Polynomials, Pergamon Press, Oxford, 1971.
[5] Ya L. Geronimus, Polynomials Orthogonal on a circle and their Applications, Amer. Math. Soc. Transl. (1), vol. 3, Providence, RI, 1962, pp. 1–78.
[6] P. Henrici, Applied and Computational Complex Analysis, Pure and Applied Mathematics. A Wiley-Interscience Series, vol. 1, Wiley, New York, 1974.
[7] H. Levy, F. Lessman, Finite Differences Equations, Dover, New York, 1992.
[8] F. Marcellán, P. Maroni, Orthogonal polynomials on the unit circle and their derivatives, Constr. Approx. 7 (1991) 341–348.
[9] F. Marcellán, F. Peherstorfer, R. Steinbauer, Orthogonality properties of linear combinations of orthogonal polynomials, Adv. Comput. Math. 5 (1996) 281–295.
[10] P. Maroni, Prolégomènes à l'ètude des polinômes orhtogonaux semiclassiques, Annal. Mat. Pura Appl. (4) 149 (1987) 165–184.
[11] C. Suárez, About the orthogonality of derivatives of orthogonal polynomials on the unit circle, Aportaciones Matemáticas, Serie Editada por la Sociedad Matemática de México, 1996.
[12] G. Szegő, Orthogonal Polynomials, 4th ed., Amer. Math. Soc. Colloq. Publ. 23, Providence, Rhode Island, 1975.
[13] C. Tasis, Propiedades diferenciales de los polinomios ortogonales relativos a la circunferencia unidad, Doctoral Dissertation, Universidad de Cantabria, 1989.