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Summing one- and two-dimensional series related to the Euler series[1]

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Abstract

We present results for some infinite series appearing in Feynman diagram calculations, many of which are similar to the Euler series. These include both one-dimensional and two-dimensional series. Most of these series can be expressed in terms of $\zeta(2)$, $\zeta(3)$, the Catalan constant G and $\mathrm{Cl}_2(\pi/3)$ where $\mathrm{Cl}_2(\theta)$ is Clausen's function. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

When calculating radiative corrections in Quantum Field Theory, one will encounter multi-dimensional Feynman integrals [12]. These often present considerable mathematical challenges. Several methods are available for such calculations. Unfortunately, the evaluation of such integrals is a tedious task and a simple result cannot always be found.

A powerful method for doing Feynman integrals consists in using Mellin transforms [17, 19]. This approach is particularly useful when one wants to expand the result in powers of logarithms of the kinematical variables. One may thus factorize the integrands to be left with a number of complex contour integrals. These can in turn be evaluated by means of residue calculus. Upon calculating

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these contour integrals, the result will be expressed as infinite series over one or more summation variables. For typical applications to Quantum Field Theory, see, e.g., [9].

As a result of Feynman integrals, the Riemann zeta function [1, 2]:

$$\zeta(z)=\sum_{n=1}^{\infty}\frac{1}{n^z}$$

appears frequently. In particular, we shall need its values for 2 and 3:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\zeta(2)=\frac{\pi^2}{6}=1.644\,934\ldots,$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3}=\zeta(3)=1.202\,057\ldots.$$

In Feynman integrals, these constants often appear as results of integrations where polylogarithms are involved.

The above-mentioned method, using Mellin transforms, has been applied to Feynman integrals appearing in two-loop studies of Bhabha scattering [4, 5]. In recent work [20], the transcendental constants appear when summing series instead of resulting from integrations.

A large number of the series encountered are not found in the familiar tables. This is the case for some of the one-dimensional series, and in particular for the two-dimensional ones. The purpose of this article is to present results for some of the series encountered. These can be expressed in terms of a few constants, including $\zeta(2)$, $\zeta(3)$ and the Catalan constant G. In addition, the constant $\mathrm{Cl}_2(\pi/3)$ appears frequently. It seems to us that few of these results are known, in particular none of those involving Clausen's function. Some basic properties of Clausen's function are given in Appendix A. Similar results, including some of those given here, have been presented in [3, 6, 7, 10, 13, 22, 23]. An excellent article on triple Euler series can be found in [8]. Here, the reader will also find an appendix written by D. Broadhurst on the connection between Euler series, quantum field theory and knot theory. For further references on such series, we refer the reader to [15].

Before we turn to the evaluation of these sums, we note that in this work we will interchange integrations, differentiations, sums and limits at will. In general, care should be exercised in doing so. For the calculations shown here, all these interchanges are allowed.

2. One-dimensional series

Let us consider the family of series

$$\sum_{n=1}^{\infty}\frac{1}{n^2}[\gamma+\psi(1+kn)]=\sum_{n=1}^{\infty}\sum_{j=1}^{kn}\frac{1}{n^2 j},$$

where γ is the Euler constant, k is a positive integer and $\psi(z)$ is the logarithmic derivative of the gamma function (see Appendix A). For $k=1$, this reduces to one of the double Euler series. Our family of series can all be expressed in terms of a rational multiple of $\zeta(3)$ and a finite sum over Clausen's function [16]. This is true also for the corresponding alternating series:

Theorem 1.

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(1+kn)] = \left(\frac{k^2}{2} + \frac{3}{2k}\right)\zeta(3) + \pi\sum_{j=1}^{k-1} j\,\mathrm{Cl}_2\left(\frac{2\pi j}{k}\right), \tag{1}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}[\gamma + \psi(1+kn)] = \left(\frac{k^2}{2} - \frac{9}{8k}\right)\zeta(3) + \pi\sum_{j=1}^{k-1} j\,\mathrm{Cl}_2\left(\frac{2\pi j}{k} + \frac{\pi}{k}\right), \tag{2}$$

for $k = 1, 2, 3, \ldots$, where the sums over j are understood to be zero when $k = 1$.

For low values of k, the sums over Clausen's function can be expressed in terms of $\mathrm{Cl}_2(\pi/3)$ and the Catalan constant G. Explicit results are given in Appendix B.

Proof. We start with the proof of (1). By using the integral representation (A.1) of the psi function, we can rewrite the series as

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(1+kn)] = \sum_{n=1}^{\infty} \frac{1}{n^2}\int_0^1 \mathrm{d}t\,\frac{1-t^{kn}}{1-t} = \int_0^1 \mathrm{d}t\,\frac{\zeta(2) - \mathrm{Li}_2(t^k)}{1-t},$$

where we have interchanged integration and summation, thus enabling us to express the sum as an integral by using (A.4). By the factorization formula (A.7), the argument of the dilogarithm can be linearized, yielding

$$\int_0^1 \mathrm{d}t\,\frac{\zeta(2) - k\sum_{j=1}^{k}\mathrm{Li}_2(\omega^j t)}{1-t},$$

where $\omega = \mathrm{e}^{2\pi\mathrm{i}/k}$. If one tries to calculate this integral term by term, one will see that each term is divergent although the sum converges. For the purpose of splitting up the integral, we introduce a regulator as follows:

$$\begin{aligned}\int_0^1 \mathrm{d}t\,\frac{\zeta(2) - k\sum_{j=1}^{k}\mathrm{Li}_2(\omega^j t)}{1-t} &= \lim_{x\to 1^-}\int_0^1 \mathrm{d}t\,\frac{\zeta(2) - k\sum_{j=1}^{k}\mathrm{Li}_2(\omega^j t)}{1-xt} \\ &= \lim_{x\to 1^-}\left[\int_0^1 \mathrm{d}t\,\frac{\zeta(2)}{1-xt} - k\sum_{j=1}^{k}\int_0^1 \mathrm{d}t\,\frac{\mathrm{Li}_2(\omega^j t)}{1-xt}\right].\end{aligned}$$

The first of these integrals is trivial, while for those under the sum we will need the result for

$$I_2(x,a) = \int_0^1 \mathrm{d}t\,\frac{\mathrm{Li}_2(at)}{1-xt} \tag{3}$$

in the limit $x \to 1^-$. This integral is studied in Appendix C. Using the result from Eq. (C.4), we get

$$\lim_{x\to 1^-}\left[-\frac{\log(1-x)}{x}\zeta(2)+k\sum_{j=1}^{k}\left\{\frac{\log(1-x)}{x}\mathrm{Li}_2(\omega^j)+S_{1,2}(\omega^j)+\mathrm{Li}_3(\omega^j)\right\}\right].$$

By using the factorization formula (A.7), we get

$$\lim_{x\to 1^-}\left[-\frac{\log(1-x)}{x}\zeta(2)+\frac{\log(1-x)}{x}\mathrm{Li}_2(1)+k\sum_{j=1}^{k}S_{1,2}(\omega^j)+\frac{1}{k}\mathrm{Li}_3(1)\right]$$

$$=\frac{1}{k}\zeta(3)+k\sum_{j=1}^{k}S_{1,2}(\omega^j),$$

where the divergent parts cancel and we may let $x\to 1^-$. To proceed further, we note that $\sum_{j=1}^{k}S_{1,2}(\omega^j)=\sum_{j=1}^{k}S_{1,2}(\omega^{-j})$, since an inversion of the argument simply corresponds to performing the sum in reverse order. Thus,

$$\frac{1}{k}\zeta(3)+k\sum_{j=1}^{k}S_{1,2}(\omega^j)=\frac{1}{k}\zeta(3)+\frac{k}{2}\sum_{j=1}^{k}\left[S_{1,2}(\omega^j)+S_{1,2}\left(\frac{1}{\omega^j}\right)\right]$$

$$=\frac{1}{k}\zeta(3)+\frac{k}{2}\sum_{j=1}^{k}\left[\mathrm{Li}_3(\omega^j)-\frac{1}{6}\log^3(-\omega^j)-\log(-\omega^j)\mathrm{Li}_2(\omega^j)+\zeta(3)\right],$$

where we have used the identity (A.10). Next, we may use the factorization formula (A.7) and the fact that $\log(-\omega^j)=\mathrm{i}\pi(2j/k-1)$. This enables us to follow our convention that $\log(-1)=\mathrm{i}\pi$. Thus, we get

$$\left(\frac{k^2}{2}+\frac{3}{2k}\right)\zeta(3)+\frac{k}{2}\sum_{j=1}^{k}\left[\frac{\mathrm{i}\pi^3}{6}\left(\frac{2j}{k}-1\right)^3-\mathrm{i}\pi\left(\frac{2j}{k}-1\right)\mathrm{Li}_2(\omega^j)\right].$$

The result must be real. Thus, we may drop all the imaginary parts which eventually will cancel, and we are left with

$$\left(\frac{k^2}{2}+\frac{3}{2k}\right)\zeta(3)+\pi\sum_{j=1}^{k}j\,\mathrm{Im}\{\mathrm{Li}_2(\omega^j)\}=\left(\frac{k^2}{2}+\frac{3}{2k}\right)\zeta(3)+\pi\sum_{j=1}^{k}j\,\mathrm{Cl}_2\left(\frac{2\pi j}{k}\right)$$

$$=\left(\frac{k^2}{2}+\frac{3}{2k}\right)\zeta(3)+\pi\sum_{j=1}^{k-1}j\,\mathrm{Cl}_2\left(\frac{2\pi j}{k}\right).$$

In the first step we have used the fact that on the unit circle, the imaginary part of the dilogarithm is Clausen's function, which vanishes when the argument is an integer multiple of π. This completes the proof of the first part of the theorem.

For the alternating series (2) we will use a similar procedure. By using the integral representation (A.1) for the psi function and performing the sum over n, we are left with

$$\int_0^1 \mathrm{d}t \frac{\mathrm{Li}_2(-1) - \mathrm{Li}_2[(\phi t)^k]}{1-t},$$

where $\phi^k = -1$. We now follow the same procedure as in the proof of (1), except that we let the sum in the factorization formula run from 0 to $k-1$. Hence, we arrive at

$$\frac{1}{k}\mathrm{Li}_3(-1) + k\sum_{j=0}^{k-1} S_{1,2}(\phi\omega^j).$$

Performing the sum in reverse order simply corresponds to the substitution $\omega^j \to \omega^{-j}$. Combining this with the fact that we could equally well have introduced ϕ^{-1} instead of ϕ, we find that we will get

$$-\frac{3}{4k}\zeta(3) + \frac{k}{2}\sum_{j=0}^{k-1}\left[S_{1,2}(\phi\omega^j) + S_{1,2}\left(\frac{1}{\phi\omega^j}\right)\right]. \tag{4}$$

We use the fact that $\log(-\phi\omega^j) = \mathrm{i}\pi[(2j+1)/k - 1]$, which preserves the convention $\log(-1) = \mathrm{i}\pi$, to get

$$\left(\frac{k^2}{2} - \frac{9}{8k}\right)\zeta(3) + \frac{k}{2}\sum_{j=0}^{k-1}\left[\frac{\mathrm{i}\pi^3}{6}\left(\frac{2j+1}{k} - 1\right)^3 - \mathrm{i}\pi\left(\frac{2j+1}{k} - 1\right)\mathrm{Li}_2(\phi\omega^j)\right].$$

Again, we know that the result must be real, and we drop all the imaginary parts. Thus,

$$\left(\frac{k^2}{2} - \frac{9}{8k}\right)\zeta(3) + \pi\sum_{j=1}^{k-1} j\,\mathrm{Cl}_2\left(\frac{2\pi j}{k} + \frac{\pi}{k}\right).$$

This completes the proof of the second part of the theorem. □

An immediate corollary of this theorem is:

Corollary 1.

$$\sum_{n=1}^{\infty}\frac{1}{n^2}[\gamma + \psi(kn)] = \left(\frac{k^2}{2} + \frac{1}{2k}\right)\zeta(3) + \pi\sum_{j=1}^{k-1} j\,\mathrm{Cl}_2\left(\frac{2\pi j}{k}\right), \tag{5}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}[\gamma + \psi(kn)] = \left(\frac{k^2}{2} - \frac{3}{8k}\right)\zeta(3) + \pi\sum_{j=1}^{k-1} j\,\mathrm{Cl}_2\left(\frac{2\pi j}{k} + \frac{\pi}{k}\right), \tag{6}$$

for $k = 1, 2, 3, \ldots$, where the sum over j vanishes for $k = 1$.

Proof. This corollary follows immediately from the theorem by using the recurrence formula (A.3) for the psi function. □

For certain low values of k, the sums over Clausen's function may be simplified. Thus, we may state exact and compact results for a considerable number of series. Such results are collected in Appendix B.

In the Theorem, n^2 appears in the denominator of the summand. For higher powers of n, part of the same procedure can be carried out when summing the corresponding series. However, for the higher powers, no simple result appears known for the sum over Nielsen's functions on the unit circle. We refer to Appendix D for more details.

We now turn to some other series.

Series 1.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{[\Gamma(n)]^2}{\Gamma(2n)} = -\frac{8}{3}\zeta(3) + \frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right). \tag{7}$$

Proof. We start by using the duplication formula for the gamma function to get a hypergeometric series,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{[\Gamma(n)]^2}{\Gamma(2n)} &= 2\sum_{n=1}^{\infty} \frac{1}{n^2} \frac{\Gamma(\frac{1}{2})\Gamma(n)}{\Gamma(\frac{1}{2}+n)} \left(\frac{1}{4}\right)^n \\ &= {}_4F_3\left(1,1,1,1;\frac{3}{2},2,2;\frac{1}{4}\right) = \int_0^1 \mathrm{d}t\, {}_3F_2\left(1,1,1;\frac{3}{2},2;\frac{1}{4}t\right).\end{aligned} \tag{8}$$

In the last step we have used the integral representation for the function ${}_4F_3$, Eq. (A.21). By using the identity (7.4.2.353) in [21] and afterwards making the substitution $\frac{1}{2}\sqrt{t} = \sin\frac{1}{2}u$ we get

$$4\int_0^1 \frac{\mathrm{d}t}{t} \arcsin^2 \frac{\sqrt{t}}{2} = 2\int_0^{\pi/3} \frac{\mathrm{d}u}{2\tan\frac{1}{2}u} u^2.$$

We will use integration by parts to decompose this integral. We get

$$2u^2 \log\left(2\sin\frac{u}{2}\right)\Big|_0^{\pi/3} - 4\int_0^{\pi/3} \mathrm{d}u\, u \log\left(2\sin\frac{u}{2}\right) = -4\int_0^{\pi/3} \mathrm{d}u\, u \log\left(2\sin\frac{u}{2}\right).$$

Invoking Eqs. (6.52) and (6.46) of [16], we get[2]

$$-4\left\{\zeta(3) - \frac{\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \mathrm{Cl}_3\left(\frac{\pi}{3}\right)\right\} = \frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) - \frac{8}{3}\zeta(3). \qquad \square$$

Series 2.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{[\Gamma(n)]^2}{\Gamma(2n)} = -\frac{4}{5}\zeta(3). \tag{9}$$

[2] There is a misprint in Eq. (6.52) of [16]. The correct result is $\int_0^{\theta} \mathrm{d}\theta \log(2\sin\frac{\theta}{2}) = \zeta(3) - \theta\,\mathrm{Cl}_2(\theta) - \mathrm{Cl}_3(\theta)$.

Proof. We may rewrite this series as

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \frac{[\Gamma(n)]^2}{\Gamma(2n)} = -2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \frac{[n!]^2}{(2n)!} = -\frac{4}{5}\zeta(3),$$

where Eq. (0.1) in Ch. 9 of [3] gives us the sum of this series. □

3. Two-dimensional series

A large number of two-dimensional series encountered in Feynman integral calculations can also be summed analytically. Some results for such series will be presented here along with proofs, many of which are based on the results of our theorem. The simpler ones are evaluated by summing over one variable and recognizing the result as a one-dimensional series for which we know the sum. For the other series, we will need to use integration- or differentiation methods to analytically evaluate the sum.

When summing the simpler two-dimensional series, we will often find the following result useful:

$$\sum_{n=1}^{\infty} \frac{1}{n+a} \frac{1}{n+b} = \frac{1}{(1+a)(1+b)} \, {}_3F_2(1, 1+a, 1+b; 2+a, 2+b; 1)$$

$$= \frac{1}{b-a}[\psi(1+b) - \psi(1+a)], \quad a \neq b. \tag{10}$$

The last step follows by using Eq. (7.4.4.33) in [21].

Another useful result is

$$\sum_{n=1}^{\infty} \frac{\Gamma(n+k)}{\Gamma(1+n+2k)} = \frac{\Gamma(1+k)}{\Gamma(2+2k)} \, {}_2F_1(1, 1+k; 2+2k; 1) = \frac{\Gamma(k)}{\Gamma(1+2k)}. \tag{11}$$

This follows immediately by recognizing the series as a hypergeometric function and using (A.22).

Series 3.

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{k}{n(1+k)^2(n+k)} = \zeta(3). \tag{12}$$

Proof. We start by summing over n, using (10), to get

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{k}{n(1+k)^2(n+k)} = \sum_{k=1}^{\infty} \frac{1}{(1+k)^2}[\gamma + \psi(1+k)]$$

$$= \sum_{k=2}^{\infty} \frac{1}{k^2}[\gamma + \psi(k)] = \sum_{k=1}^{\infty} \frac{1}{k^2}[\gamma + \psi(k)] = \zeta(3),$$

where we have made use of (B.1) in the last step. □

Series 4.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(n+2k)}=\frac{3}{4}\zeta(3). \tag{13}$$

Proof. Also here we start by summing over n, using (10), to get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(n+2k)}=\sum_{k=1}^{\infty}\frac{1}{k^2}[\psi(1+2k)-\psi(1+k)]=\frac{3}{4}\zeta(3),$$

where we have made use of (B.2) and (B.4) in the last step. □

Series 5.

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(n+2k)}=\frac{5}{4}\zeta(3). \tag{14}$$

Proof. This follows as an immediate corollary from the previous result. We rewrite the sum as

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(n+2k)}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^3}+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(n+2k)}$$
$$=\frac{1}{2}\zeta(3)+\frac{3}{4}\zeta(3)=\frac{5}{4}\zeta(3). \qquad \square$$

This result can also be found with a different proof in Eq. (1.2) of [23] by using the identity (29) to relate these series.

Series 6.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}=\frac{1}{2}\zeta(2). \tag{15}$$

Proof. We start by summing over n, using (11), to get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{1}{2}\zeta(2). \qquad \square$$

Series 7.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!k}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}=\frac{1}{2}\zeta(3). \tag{16}$$

Proof. The sum over n is the same as above. Thus,

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!k}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^3}=\frac{1}{2}\zeta(3). \qquad \square$$

Series 8.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma+\psi(k)]=\frac{1}{2}\zeta(3). \tag{17}$$

Proof. Again we start by summing over n, using (11), to get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma+\psi(k)]=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}[\gamma+\psi(k)]=\frac{1}{2}\zeta(3),$$

where we have made use of (B.1) in the last step. $\square$

Series 9.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma+\psi(1+k)]=\zeta(3). \tag{18}$$

Proof. This result follows as an immediate corollary of the two previous results by using the recurrence relation (A.3). $\square$

Series 10.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma+\psi(1+2k)]=\frac{11}{8}\zeta(3). \tag{19}$$

Proof. We start once more by summing over n, using (11), to get

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma+\psi(1+2k)]=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}[\gamma+\psi(1+2k)]=\frac{11}{8}\zeta(3),$$

where we have made use of (B.4) in the last step. $\square$

We now turn to similar series involving n in the argument of the psi function. Here, the results (10) and (11) are not sufficient. We will require the use of integration and differentiation methods throughout the rest of this section.

Series 11.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma+\psi(n)]=\frac{7}{8}\zeta(3). \tag{20}$$

Proof. Relation (A.1) is used to write

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma+\psi(n)]$$
$$=\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}\int_0^1 \mathrm{d}t\frac{1-t^{n-1}}{1-t}$$
$$=\frac{1}{2}\int_0^1\frac{\mathrm{d}t}{1-t}\sum_{k=1}^{\infty}\left\{\frac{1}{k^2}-\frac{1}{k(1+2k)}{}_2F_1(1,1+k;2+2k;t)\right\},$$

where we have now interchanged the order of integration and summation, thereby being able to perform the sum over n. Next, we apply the integral representation (A.18) for ${}_2F_1$ before summing over k. Thus, we get

$$\frac{1}{2}\int_0^1\frac{\mathrm{d}t}{1-t}\sum_{k=1}^{\infty}\left\{\frac{1}{k^2}-\frac{1}{k}\int_0^1\frac{\mathrm{d}s}{1-ts}\left[\frac{(1-s)^2}{1-ts}\right]^k\right\}$$
$$=\frac{1}{2}\int_0^1\frac{\mathrm{d}t}{1-t}\left\{\zeta(2)+\int_0^1\frac{\mathrm{d}s}{1-ts}\log\left[1-\frac{(1-s)^2}{1-ts}\right]\right\}$$
$$=\frac{1}{2}\int_0^1\frac{\mathrm{d}t}{1-t}\left\{\zeta(2)+\int_0^1\frac{\mathrm{d}s}{1-ts}[\log s+\log(2-t-s)-\log(1-ts)]\right\}$$
$$=\frac{1}{2}\int_0^1\frac{\mathrm{d}t}{1-t}\left\{\zeta(2)+\int_0^1\frac{\mathrm{d}s\log s}{1-ts}+\int_0^1\frac{\mathrm{d}s\log(2-t-s)}{1-ts}-\int_0^1\frac{\mathrm{d}s\log(1-ts)}{1-ts}\right\}.$$

Using Eqs. (3.12.1) and (3.14.5) of [11] along with Eq. (A.3.1.3) of [16] to perform the integration over s, we get

$$\frac{1}{2}\int_0^1\frac{\mathrm{d}t}{1-t}\left\{\zeta(2)-\frac{1}{t}\left[\mathrm{Li}_2(t)+\frac{1}{2}\log^2\left(\frac{1-t}{t}\right)-\frac{1}{2}\log^2\left(\frac{1}{t}\right)\right.\right.$$
$$\left.\left.+\mathrm{Li}_2(1-t)-\mathrm{Li}_2[(1-t)^2]-\frac{1}{2}\log^2(1-t)\right]\right\}.$$

The identity (2.2.1) of [11] is used to rewrite this as

$$\frac{1}{2}\int_0^1\frac{\mathrm{d}t}{1-t}\left\{\zeta(2)+\frac{1}{t}[\mathrm{Li}_2[(1-t)^2]+2\log(t)\log(1-t)-\zeta(2)]\right\}$$
$$=\frac{1}{2}\int_0^1\frac{\mathrm{d}t}{t}\left\{\zeta(2)+\frac{1}{1-t}[\mathrm{Li}_2(t^2)+2\log(t)\log(1-t)-\zeta(2)]\right\}$$
$$=\int_0^1\frac{\mathrm{d}t}{t(1-t)}\log(t)\log(1-t)+\frac{1}{2}\int_0^1\frac{\mathrm{d}t}{t(1-t)}[\mathrm{Li}_2(t^2)-t\zeta(2)],$$

where we now have substituted $t \to 1-t$ in the second step. By expanding in partial fractions, we arrive at

$$\int_0^1 \frac{dt}{t}\log(t)\log(1-t) + \int_0^1 \frac{dt}{1-t}\log(t)\log(1-t)$$

$$+\frac{1}{2}\int_0^1 \frac{dt}{t}[\mathrm{Li}_2(t^2) - t\zeta(2)] + \frac{1}{2}\int_0^1 \frac{dt}{1-t}[\mathrm{Li}_2(t^2) - t\zeta(2)]$$

$$= 2\int_0^1 \frac{dt}{t}\log(t)\log(1-t) + \int_0^1 \frac{dt}{t}[\mathrm{Li}_2(t) + \mathrm{Li}_2(-t)] - \frac{1}{2}\zeta(2)$$

$$+\int_0^1 \frac{dt}{1-t}\left[\mathrm{Li}_2(t) + \mathrm{Li}_2(-t) - \frac{1}{2}\zeta(2)\right] + \frac{1}{2}\zeta(2).$$

The first two integrals have been combined by substituting $t \to 1-t$ in the second one. Identity (2.2.8) of [11] has been used to transform the two last integrals. In order to proceed, the first two of these integrals are evaluated using Eqs. (3.6.21), (3.8.13) and (3.8.15) of [11]. The result is

$$\frac{9}{4}\zeta(3) + \int_0^1 \frac{dt}{1-t}[\mathrm{Li}_2(t) - \zeta(2)] + \int_0^1 \frac{dt}{1-t}\left[\mathrm{Li}_2(-t) + \frac{1}{2}\zeta(2)\right] = \frac{7}{8}\zeta(3),$$

where Eqs. (3.8.9) and (3.8.11) of [11] have been used[3] in the last step. □

Series 12.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma + \psi(n+1)] = \frac{5}{4}\zeta(3). \tag{21}$$

Proof. The recurrence relation (A.3) for the psi function is used to write

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma + \psi(n+1)]$$

$$= \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma + \psi(n)] + \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}\frac{1}{n}$$

$$= \frac{7}{8}\zeta(3) + \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k(1+2k)}\,{}_3F_2(1,1,1+k;2,2+2k;1),$$

[3] *Note*: There is a sign error in Eq. (3.8.11) of [11]. The correct result is $\int_0^1 \frac{dt}{1-t}\left[\mathrm{Li}_2(-t) + \frac{1}{2}\zeta(2)\right] = \frac{5}{8}\zeta(3)$.

where now the sum over n has been performed in the last term. By using Eq. (7.4.4.40) of [21], we get

$$\frac{7}{8}\zeta(3)+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}\{\psi(1+2k)-\psi(1+k)\}=\frac{5}{4}\zeta(3),$$

where in the last step, Eqs. (B.2) and (B.4) have been used. □

Series 13.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma+\psi(n+k)]=\frac{3}{2}\zeta(3). \tag{22}$$

Proof. We can rewrite the sum as

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma+\psi(n+k)]\\
&\quad=\gamma\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k+x)}{\Gamma(1+n+2k)}\\
&\quad=\frac{\gamma}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{k=1}^{\infty}\frac{\Gamma(1+k+x)\Gamma(2k)}{k!\Gamma(2+2k)}\,{}_2F_1(1,1+k+x;2+2k;1)\\
&\quad=\frac{\gamma}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{k=1}^{\infty}\frac{\Gamma(1+k+x)\Gamma(2k)}{k!(k-x)\Gamma(1+2k)}\\
&\quad=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left[\frac{1}{k}+\gamma+\psi(1+k)\right]\\
&\quad=\frac{1}{2}\zeta(3)+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}[\gamma+\psi(1+k)]=\frac{3}{2}\zeta(3),
\end{aligned}$$

where the result (B.2) has been used in the last step. □

Series 14.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma+\psi(1+n+2k)]=\frac{15}{8}\zeta(3). \tag{23}$$

Proof. We can rewrite this series as

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}[\gamma+\psi(1+n+2k)]$$

$$=\gamma\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k)}+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{\Gamma(2k)\Gamma(n+k)}{\Gamma(1+n+2k-x)}$$

$$=\frac{\gamma}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{k=1}^{\infty}\frac{\Gamma(2k)}{\Gamma(2+2k-x)}{}_2F_1(1,1+k;2+2k-x;1)$$

$$=\frac{\gamma}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{k=1}^{\infty}\frac{\Gamma(2k)}{(k-x)\Gamma(1+2k-x)}$$

$$=\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}\left[\frac{1}{k}+\gamma+\psi(1+2k)\right]$$

$$=\frac{1}{2}\zeta(3)+\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^2}[\gamma+\psi(1+2k)]=\frac{15}{8}\zeta(3),$$

where the result (B.4) has been used in the last step. □

Series 15.

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k}{n!k!}\frac{1}{n+k}\frac{[\Gamma(n+k)]^3}{\Gamma(2n+2k)}=\frac{8}{3}\zeta(3)-\frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right). \tag{24}$$

Proof. First, we use the duplication formula for the gamma function and perform the sum over n to get a hypergeometric function,

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k}{n!k!}\frac{1}{n+k}\frac{[\Gamma(n+k)]^3}{\Gamma(2n+2k)}$$

$$=2\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{(-1)^k}{n!k!}\frac{1}{n+k}\frac{[\Gamma(n+k)]^2\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+n+k)}\left(\frac{1}{4}\right)^{n+k}$$

$$=2\sum_{k=1}^{\infty}\frac{\Gamma(k)\Gamma(\frac{1}{2})}{k^2\Gamma(\frac{1}{2}+k)}{}_3F_2\left(k,k,k;k+1,k+\frac{1}{2};\frac{1}{4}\right)\left(-\frac{1}{4}\right)^k.$$

By using the integral representation (A.19), we are able to perform the sum over k,

$$2\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\; s^{k-1}t^{k-1}(1-t)^{-\frac{1}{2}}\left(1-\frac{1}{4}st\right)^{-k}\left(-\frac{1}{4}\right)^k$$

$$=2\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\frac{1}{st\sqrt{1-t}}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{-st}{4-st}\right)^k$$

$$=-2\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\frac{1}{st\sqrt{1-t}}\log\left(1+\frac{st}{4-st}\right)$$

$$=2\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\frac{1}{st\sqrt{1-t}}\log\left(1-\frac{st}{4}\right).$$

Next, we integrate over s, and thereafter introduce the series representation of the dilogarithm:

$$-2\int_0^1 \mathrm{d}t\frac{1}{t\sqrt{1-t}}\mathrm{Li}_2\left(\frac{t}{4}\right)=-2\int_0^1 \mathrm{d}t\frac{1}{t\sqrt{1-t}}\sum_{n=1}^{\infty}\frac{t^n}{n^2}\left(\frac{1}{4}\right)^n$$

$$=-2\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1}{4}\right)^n\int_0^1 \mathrm{d}t\; t^{-1+n}(1-t)^{-1/2}$$

$$=-2\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+n)}\left(\frac{1}{4}\right)^n=\frac{8}{3}\zeta(3)-\frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right).$$

In the last two steps, we performed the integration over t and arrived at a series encountered in the proof of Series 1. □

Series 16.

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n!k!}\frac{1}{n+k}\frac{[\Gamma(n+k)]^3}{\Gamma(2n+2k)}=-\frac{41}{24}\zeta(3)-\frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+\frac{\pi^2}{4}\log 2+2\pi G. \tag{25}$$

Proof. The sum over n is the same as for the previous case,

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n!k!}\frac{1}{n+k}\frac{[\Gamma(n+k)]^3}{\Gamma(2n+2k)}$$

$$=2\sum_{k=1}^{\infty}\frac{\Gamma(k)\Gamma(\frac{1}{2})}{k^2\Gamma(k+\frac{1}{2})}{}_3F_2\left(k,k,k;k+1,k+\frac{1}{2};\frac{1}{4}\right)\left(\frac{1}{4}\right)^k.$$

By using the integral representation (A.19) of ${}_3F_2$, we are able to perform the sum over k,

$$
\begin{aligned}
&2\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\; s^{k-1}t^{k-1}(1-t)^{-\frac{1}{2}}\left(1-\frac{1}{4}st\right)^{-k}\left(\frac{1}{4}\right)^k\\
&\quad=2\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\frac{1}{st\sqrt{1-t}}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{st}{4-st}\right)^k\\
&\quad=-2\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\frac{1}{st\sqrt{1-t}}\log\left(1-\frac{st}{4-st}\right)\\
&\quad=2\int_0^1 \mathrm{d}s\int_0^1 \mathrm{d}t\frac{1}{st\sqrt{1-t}}\left\{\log\left(1-\frac{st}{4}\right)-\log\left(1-\frac{st}{2}\right)\right\}\\
&\quad=-2\int_0^1 \mathrm{d}t\frac{1}{t\sqrt{1-t}}\,\mathrm{Li}_2\left(\frac{t}{4}\right)+2\int_0^1 \mathrm{d}t\frac{1}{t\sqrt{1-t}}\,\mathrm{Li}_2\left(\frac{t}{2}\right).
\end{aligned}
$$

The first of these two integrals is known from the previous proof, while the second one is calculated in the same way as we did for the first one. Thus, we get

$$
\begin{aligned}
&\frac{8}{3}\zeta(3)-\frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+2\int_0^1 \mathrm{d}t\frac{1}{t\sqrt{1-t}}\sum_{n=1}^{\infty}\frac{t^n}{n^2}\left(\frac{1}{2}\right)^n\\
&\quad=\frac{8}{3}\zeta(3)-\frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+2\sum_{n=1}^{\infty}\frac{1}{n^2}\left(\frac{1}{2}\right)^n\int_0^1 \mathrm{d}t\,t^{-1+n}(1-t)^{-1/2}\\
&\quad=\frac{8}{3}\zeta(3)-\frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+2\sum_{n=1}^{\infty}\frac{1}{n^2}\frac{\Gamma(n)\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+n)}\\
&\quad=\frac{8}{3}\zeta(3)-\frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+2\;{}_4F_3\left(1,1,1,1;\frac{3}{2},2,2;\frac{1}{2}\right)\\
&\quad=\frac{8}{3}\zeta(3)-\frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+2\int_0^1 \mathrm{d}t\;{}_3F_2\left(1,1,1;\frac{3}{2},2;\frac{1}{2}t\right),
\end{aligned}
$$

where we have used the integral representation for ${}_4F_3$. Next, we use Eq. (7.4.2.353) of [21] to get

$$
\frac{8}{3}\zeta(3)-\frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right)+4\int_0^1\frac{\mathrm{d}t}{t}\arcsin^2\sqrt{\frac{t}{2}}.
$$

The substitution $\sqrt{\frac{1}{2}t} = \sin\frac{1}{2}w$ yields

$$\frac{8}{3}\zeta(3) - \frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + 2\int_0^{\pi/2} \frac{\mathrm{d}w}{2\tan\frac{1}{2}w} w^2.$$

To evaluate this integral, we integrate by parts to get

$$\begin{aligned}
&\frac{8}{3}\zeta(3) - \frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + 2w^2\log\left[2\sin\frac{w}{2}\right]\Bigg|_0^{\pi/2} - 4\int_0^{\pi/2}\mathrm{d}w\, w\log\left[2\sin\frac{w}{2}\right] \\
&\quad = \frac{8}{3}\zeta(3) - \frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{\pi^2}{4}\log 2 - 4\int_0^{\pi/2}\mathrm{d}w\; w\log\left[2\sin\frac{w}{2}\right] \\
&\quad = \frac{8}{3}\zeta(3) - \frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{\pi^2}{4}\log 2 - \frac{35}{8}\zeta(3) + 2\pi G \\
&\quad = -\frac{41}{24}\zeta(3) - \frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{\pi^2}{4}\log 2 + 2\pi G,
\end{aligned}$$

by using Eq. (6.52) of [16] in the last step. □

Series 17.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{1}{n(n+k)}\frac{\Gamma(n+k)\Gamma(n+2k)}{\Gamma(2n+2k)} = \frac{47}{12}\zeta(3) - \frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right). \tag{26}$$

To be able to prove this result, we will need the following lemma:

Lemma 1.

$$\sum_{n=1}^{k}\frac{1}{n}\frac{\Gamma(2k-n)}{\Gamma(1+k-n)} = \frac{\Gamma(2k)}{\Gamma(1+k)}\left[\psi(2k) - \psi(k)\right] \tag{27}$$

Proof of the Lemma. First, we rewrite the series as

$$\lim_{y\to 0}\frac{1}{\Gamma(1-k+y)}\sum_{n=1}^{k}\frac{1}{n}\frac{\Gamma(2k-n-y)\Gamma(1-k+y)}{\Gamma(1+k-n)}. \tag{28}$$

The combination of gamma functions inside the sum is a beta function which we will represent as an integral,

$$\frac{\Gamma(2k-n-y)\Gamma(1-k+y)}{\Gamma(1+k-n)} = \int_0^1 \mathrm{d}t\, t^{-1+2k-n-y}(1-t)^{-k+y},$$

where now $k-1<\mathrm{Re}\,y<2k-n$ for the integral to converge. The factor $1/n$ is also represented as an integral,

$$\frac{1}{n}=\int_0^1 \mathrm{d}x\, x^{n-1}.$$

The sum in (28) can then be rewritten as

$$\sum_{n=1}^{k}\frac{1}{n}\frac{\Gamma(2k-n-y)\Gamma(1-k+y)}{\Gamma(1+k-n)}$$

$$=\sum_{n=1}^{k}\int_0^1 \mathrm{d}x\, x^{n-1}\int_0^1 \mathrm{d}t\, t^{-1+2k-n-y}(1-t)^{-k+y}$$

$$=\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}t\, t^{-2+2k-y}(1-t)^{-k+y}\sum_{n=1}^{k}\left(\frac{x}{t}\right)^{n-1}$$

$$=\int_0^1 \mathrm{d}x\int_0^1 \mathrm{d}t\, t^{-2+2k-y}(1-t)^{-k+y}\left[\frac{1-(x/t)^k}{1-(x/t)}\right]$$

$$=\int_0^1 \mathrm{d}x\, x^{k-1}\int_0^1 \mathrm{d}t\, t^{-1+k-y}(1-t)^{-k+y}\left[\frac{1-(t/x)^k}{1-(t/x)}\right].$$

For the purpose of integrating over t, we would like to split the integral into two parts, according to the two terms in the numerator. In order to do so, avoiding the singularity from the denominator, we introduce a shift $\mathrm{i}\varepsilon$ in the denominator. Thus,

$$\int_0^1 \mathrm{d}x\, x^{k-1}\int_0^1 \mathrm{d}t\ t^{-1+k-y}(1-t)^{-k+y}\left[\frac{1-(t/x)^k}{1-(t/x)+\mathrm{i}\varepsilon}\right]$$

$$=\int_0^1 \mathrm{d}x\, x^{k-1}\left\{\Gamma(k-y)\Gamma(1-k+y){}_2F_1\left(1,k-y;1;\frac{1}{x}-\mathrm{i}\varepsilon\right)\right.$$

$$\left.-x^{-k}\frac{\Gamma(2k-y)\Gamma(1-k+y)}{\Gamma(1+k)}{}_2F_1\left(1,2k-y;1+k;\frac{1}{x}-\mathrm{i}\varepsilon\right)\right\}.$$

The two hypergeometric functions may be combined using the transformation formula (7.3.1.6) of [21] with $a=1$, $b=1-k$, $c=2-2k+y$ and $z=x+\mathrm{i}\varepsilon$. Thus, we may let $\varepsilon\to 0$, to get

$$\frac{\Gamma(1-k+y)\Gamma(2k-1-y)}{\Gamma(k)}\int_0^1 \mathrm{d}x\, {}_2F_1(1,1-k;2-2k+y;x)$$

$$=\frac{\Gamma(1-k+y)\Gamma(2k-1-y)}{\Gamma(k),}\,{}_3F_2(1,1,1-k;2,2-2k+y;1)$$

$$= \frac{\Gamma(1-k+y)\Gamma(2k-y)}{\Gamma(1+k)}[\psi(1-2k+y)-\psi(1-k+y)]$$

$$= \frac{\Gamma(1-k+y)\Gamma(2k-y)}{\Gamma(1+k)}\{\psi(2k-y)-\psi(k-y)+\pi\cot[\pi(2k-y)]-\pi\cot[\pi(k-y)]\}$$

$$= \frac{\Gamma(1-k+y)\Gamma(2k-y)}{\Gamma(1+k)}[\psi(2k-y)-\psi(k-y)],$$

where in the first two steps we have used Eqs. (7.2.3.9) and (7.4.4.40) of [21]. We have also used the reflection formula, Eq. (6.3.7) of [1], for the psi function. Thus, we have shown that

$$\sum_{n=1}^{k}\frac{1}{n}\frac{\Gamma(2k-n-y)}{\Gamma(1+k-n)} = \frac{\Gamma(2k-y)}{\Gamma(1+k)}[\psi(2k-y)-\psi(k-y)],$$

valid for $k-1<\operatorname{Re} y<k$. By analytic continuation, the result can be extended to all values of y, except $y=k,k+1,k+2,\ldots$. By letting $y\to 0$, we find that

$$\sum_{n=1}^{k}\frac{1}{n}\frac{\Gamma(2k-n)}{\Gamma(1+k-n)} = \frac{\Gamma(2k)}{\Gamma(1+k)}[\psi(2k)-\psi(k)],$$

and the lemma has been proven. □

Proof of Series 17. First, we rewrite the series as

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k!}\frac{1}{n(n+k)}\frac{\Gamma(n+k)\Gamma(n+2k)}{\Gamma(2n+2k)}$$

$$= \sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{n(n+k)}\frac{\Gamma(n+k)\Gamma(n+2k)}{\Gamma(2n+2k)} - \sum_{k=1}^{\infty}\frac{1}{k^2}\frac{[\Gamma(k)]^2}{\Gamma(2k)}.$$

The second of these two series is already known (cf. Series 1), whereas for the two-dimensional one, we will rewrite it using the identity

$$\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}a_{n,k} = \sum_{k=1}^{\infty}\sum_{n=1}^{k}a_{n,k-n}, \tag{29}$$

which is easily derived from identities given in Ch. 4.1 of [14]. We get

$$\sum_{k=1}^{\infty}\sum_{n=1}^{k}\frac{1}{nk}\frac{\Gamma(k)\Gamma(2k-n)}{\Gamma(2k)\Gamma(1+k-n)} - \frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{8}{3}\zeta(3).$$

By the lemma, this equals

$$\sum_{k=1}^{\infty}\frac{1}{k^2}[\psi(2k)-\psi(k)] - \frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right) + \frac{8}{3}\zeta(3) = \frac{47}{12}\zeta(3) - \frac{4\pi}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right),$$

where we have used (B.1) and (B.3). Thus, the proof is complete. □

It is interesting to note that the sums that appear in moderate-order Feynman integral calculations apparently can all be expressed in terms of known constants. This suggests that such integrals in some sense are of limited "complexity". For a qualification of this statement, we refer to the appendix of [8].

Appendix A. Some special functions and identities

Below, we collect some definitions and properties of special functions that are frequently used in the proofs.

A.1. The psi function

The psi function is defined as the logarithmic derivative of the gamma function, $\psi(z)=(\mathrm{d}/\mathrm{d}z)\log\Gamma(z)=\Gamma'(z)/\Gamma(z)$, and has the following integral representation (cf. Eq. (6.3.22) of [1]):

$$\gamma+\psi(z)=\int_0^1 \mathrm{d}t\,\frac{1-t^{z-1}}{1-t}, \tag{A.1}$$

where $\gamma=0.577\,216\ldots$ is Euler's constant, and $\psi(1)=-\gamma$. From the integral representation, one immediately finds that in the case of positive integer arguments, we have

$$\gamma+\psi(n)=\sum_{j=1}^{n-1}\frac{1}{j}. \tag{A.2}$$

The psi function satisfies the following recurrence relation (see Eq. (6.3.5) of [1]),

$$\psi(1+z)=\psi(z)+\frac{1}{z}. \tag{A.3}$$

A.2. Polylogarithms

Each of the polylogarithm functions can be represented as a series,

$$\mathrm{Li}_n(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^n}, \quad |z|<1, \quad n=0,1,2,\ldots, \tag{A.4}$$

or as an integral,

$$\mathrm{Li}_n(z)=\frac{(-1)^{n-1}}{(n-2)!}\int_0^1 \mathrm{d}t\,\frac{\log^{n-2}(t)\log(1-tz)}{t}, \quad n=2,3,4,\ldots \tag{A.5}$$

$$=\int_0^z \mathrm{d}t\,\frac{\mathrm{Li}_{n-1}(t)}{t}, \quad n=1,2,3,\ldots, \tag{A.6}$$

with $\mathrm{Li}_n(1)=\zeta(n)$, $n=2,3,4,\ldots$, and $\mathrm{Li}_n(-1)=(1/2^{n-1}-1)\zeta(n)$, $n=2,3,4,\ldots$. The following factorization formula (Eq. (7.41) of [16]) is valid for the polylogarithms:

$$\mathrm{Li}_n(t^k)=k^{n-1}\sum_{j=1}^{k}\mathrm{Li}_n(\omega^j t), \quad n=1,2,3,\ldots, \tag{A.7}$$

where $\omega=\mathrm{e}^{2\pi i/k}$. These and further properties of the polylogarithms and related functions can be found in [16, 11].

A.3. Nielsen's generalized polylogarithms

The functions $S_{n,p}(z)$ are Nielsen's generalized polylogarithms. We will make use of these functions for $p=2$, in which case they are given by the following integral representations [11]:

$$S_{n,2}(z)=\frac{(-1)^{n-1}}{2(n-1)!}\int_0^1 \mathrm{d}t\,\frac{\log^{n-1}(t)\log^2(1-zt)}{t}, \quad n=1,2,3,\ldots \tag{A.8}$$

$$=\int_0^z \mathrm{d}t\,\frac{S_{n-1,2}(t)}{t}, \quad n=1,2,3,\ldots, \tag{A.9}$$

where it is understood that $S_{0,2}(z)=\frac{1}{2}\log^2(1-z)$. We shall be needing the following identity for $S_{1,2}(z)$:

$$S_{1,2}(z)+S_{1,2}\left(\frac{1}{z}\right)=\mathrm{Li}_3(z)-\frac{1}{6}\log^3(-z)-\log(-z)\mathrm{Li}_2(z)+\zeta(3), \tag{A.10}$$

given in Eq. (2.2.15) of [11]. We will use the convention $\log(-1)=i\pi$. Thus, the formula is seen to be valid also for $z=1$. Further identities regarding the functions $S_{n,p}(z)$ can be found in [11]. Nielsen's original work is found in [18].

A.4. Clausen's functions

Each of Clausen's functions can be represented as a series (see [16]),

$$\mathrm{Cl}_{2n}(\theta)=\sum_{k=1}^{\infty}\frac{\sin k\theta}{k^{2n}},$$

$$\mathrm{Cl}_{2n-1}(\theta)=\sum_{k=1}^{\infty}\frac{\cos k\theta}{k^{2n-1}}, \tag{A.11}$$

where n can be any positive integer. Clausen's functions are seen to be periodic with period 2π.

We shall in this article need only the functions $\mathrm{Cl}_1(\theta)$, $\mathrm{Cl}_2(\theta)$ and $\mathrm{Cl}_3(\theta)$. From the definition, we see that the function $\mathrm{Cl}_2(\theta)$ is antisymmetric, $\mathrm{Cl}_2(-\theta)=-\mathrm{Cl}_2(\theta)$. When the argument is an integer multiple of π, the function vanishes, $\mathrm{Cl}_2(k\pi)=0$.

On the unit circle, the imaginary part of the dilogarithm is Clausen's function $\mathrm{Cl}_2(\theta)$,

$$\mathrm{Im}\{\mathrm{Li}_2(\mathrm{e}^{\mathrm{i}\theta})\} = \mathrm{Cl}_2(\theta). \tag{A.12}$$

This function satisfies the factorization formula

$$\tfrac{1}{2}\mathrm{Cl}_2(2\theta) = \mathrm{Cl}_2(\theta) - \mathrm{Cl}_2(\pi - \theta). \tag{A.13}$$

The Clausen function $\mathrm{Cl}_2(\theta)$ has its maximum value for $\theta = \pi/3$, $\mathrm{Cl}_2(\pi/3) = 1.014\,942\ldots$. Furthermore,

$$\mathrm{Cl}_2\left(\frac{2\pi}{3}\right) = \frac{2}{3}\mathrm{Cl}_2\left(\frac{\pi}{3}\right), \qquad \mathrm{Cl}_2\left(\frac{\pi}{2}\right) = G, \tag{A.14}$$

where $G = 0.915\,966\ldots$ is the Catalan constant.

A.5. Hypergeometric functions

We will frequently use the series definitions of the following hypergeometric functions:

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}, \tag{A.15}$$

$${}_3F_2(a,b,c;d,e;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{z^n}{n!}, \tag{A.16}$$

$${}_4F_3(a,b,c,d;e,f,g;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n(d)_n}{(e)_n(f)_n(g)_n}\frac{z^n}{n!}, \tag{A.17}$$

where $(a)_n$ is the Pochhammer symbol,

$$(a)_n \equiv \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1).$$

It is evident that these functions are all symmetric with respect to the "numerator" or the "denominator" arguments. All these series converge for $|z| < 1$. Conditions for convergence on the unit circle are given in Ch. 7 of [21].

These functions all have various integral representations. We shall be using the following ones:

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \mathrm{d}t\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}, \tag{A.18}$$

$${}_3F_2(a,b,c;d,e;z) = \frac{\Gamma(d)\Gamma(e)}{\Gamma(a)\Gamma(b)\Gamma(d-a)\Gamma(e-b)} \int_0^1 \int_0^1 \mathrm{d}t_1\, \mathrm{d}t_2 \tag{A.19}$$

$$\times t_1^{a-1} t_2^{b-1} (1-t_1)^{d-a-1} (1-t_2)^{e-b-1} (1-t_1 t_2 z)^{-c}$$

$$= \frac{\Gamma(e)}{\Gamma(c)\Gamma(e-c)} \int_0^1 \mathrm{d}t\, t^{c-1}(1-t)^{e-c-1} {}_2F_1(a,b;d;zt), \tag{A.20}$$

$${}_4F_3(a,b,c,d;e,f,g;z) = \frac{\Gamma(g)}{\Gamma(g-d)} \int_0^1 \mathrm{d}t\, t^{d-1}(1-t)^{g-d-1}\, {}_3F_2(a,b,c;e,f;zt). \tag{A.21}$$

Conditions for convergence of these integrals are also given in Ch. 7 of [21].

In the case when $z = 1$, we get

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{A.22}$$

Appendix B. Explicit results for one-dimensional series

The finite sums in the Theorem and Corollary of Section 2 may be further simplified for small values of k. One may then use properties of the Clausen function given in Appendix A to reduce the sums to a single Clausen function. Here, we collect the results for such series involving the psi function:

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(n)] = \zeta(3), \tag{B.1}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(1+n)] = 2\zeta(3), \tag{B.2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(2n)] = \tfrac{9}{4}\zeta(3), \tag{B.3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(1+2n)] = \tfrac{11}{4}\zeta(3), \tag{B.4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(3n)] = \tfrac{14}{3}\zeta(3) - \tfrac{2}{3}\pi \mathrm{Cl}_2(\tfrac{1}{3}\pi), \tag{B.5}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(1+3n)] = 5\zeta(3) - \tfrac{2}{3}\pi \mathrm{Cl}_2(\tfrac{1}{3}\pi), \tag{B.6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(4n)] = \tfrac{65}{8}\zeta(3) - 2\pi G, \tag{B.7}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(1+4n)] = \tfrac{67}{8}\zeta(3) - 2\pi G, \tag{B.8}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(6n)] = \tfrac{217}{12}\zeta(3) - \tfrac{16}{3}\pi \mathrm{Cl}_2(\tfrac{\pi}{3}), \tag{B.9}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(1+6n)] = \tfrac{73}{4}\zeta(3) - \tfrac{16}{3}\pi \mathrm{Cl}_2(\tfrac{\pi}{3}), \tag{B.10}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}[\gamma + \psi(n)] = \tfrac{1}{8}\zeta(3), \tag{B.11}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}[\gamma + \psi(1+n)] = -\tfrac{5}{8}\zeta(3), \tag{B.12}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}[\gamma + \psi(2n)] = \tfrac{29}{16}\zeta(3) - \pi G, \tag{B.13}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}[\gamma + \psi(1+2n)] = \tfrac{23}{16}\zeta(3) - \pi G, \tag{B.14}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}[\gamma + \psi(3n)] = \tfrac{35}{8}\zeta(3) - 2\pi \mathrm{Cl}_2(\tfrac{\pi}{3}), \tag{B.15}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}[\gamma + \psi(1+3n)] = \tfrac{33}{8}\zeta(3) - 2\pi \mathrm{Cl}_2(\tfrac{\pi}{3}). \tag{B.16}$$

At least four of these results were probably known by Euler, in particular the results of Eqs. (B.1), (B.2), (B.11) and (B.12). A review of the history of this type of series, known as Euler series, can be found in [3].

Appendix C. The integral $I_p(x,a)$

In the proof of the theorem in Section 2, we will need a result for the integral (3) in the limit $x \to 1^-$, which is

$$I_p(x,a) = \int_0^1 \mathrm{d}t\, \frac{\mathrm{Li}_p(at)}{1-xt} \tag{C.1}$$

for $p=2$. This integral is encountered for arbitrary positive integers p when studying the generalization of our theorem in Appendix D. This section will be devoted to the study of this integral in the appropriate limit $x \to 1^-$. Integrating by parts, we obtain

$$I_p(x,a) = -\frac{1}{x}\log(1-x)\mathrm{Li}_p(a) + \frac{1}{x}\int_0^1 \frac{\mathrm{d}t}{t}\log(1-xt)\mathrm{Li}_{p-1}(at).$$

Here, the singular part (as $x \to 1^-$) has been isolated, and we may set $x = 1$ in the remaining integral to get

$$I_p(x,a) = -\frac{1}{x}\log(1-x)\mathrm{Li}_p(a) - \Lambda_{p-1}(a),$$

where

$$\Lambda_\nu(a) = -\int_0^1 \frac{\mathrm{d}t}{t}\log(1-t)\mathrm{Li}_\nu(at). \tag{C.2}$$

Here, we will need the following result:

$$\Lambda_\nu(a) = S_{\nu,2}(a) + \mathrm{Li}_{\nu+2}(a), \tag{C.3}$$

which can be proved by induction. By noting that $\mathrm{Li}_0(z) = z/(1-z)$, we find from Eqs. (3.12.7) and (2.2.5) of [11] that the result holds for $\nu = 0$. Now, suppose the result is valid for $\nu - 1$. Consider

$$\Lambda_\nu(a) = -\int_0^1 \frac{\mathrm{d}t}{t}\log(1-t)\mathrm{Li}_\nu(at) = -\int_0^1 \frac{\mathrm{d}t}{t}\log(1-t)\int_0^1 \frac{\mathrm{d}y}{y}\mathrm{Li}_{\nu-1}(ayt),$$

which follows by Eq. (2.1.7) of [11]. By interchanging the order of integration and invoking our assumption, we may perform the t integration to get

$$\Lambda_\nu(a) = \int_0^1 \frac{\mathrm{d}y}{y}[S_{\nu-1,2}(ay) + \mathrm{Li}_{\nu+1}(ay)] = S_{\nu,2}(a) + \mathrm{Li}_{\nu+2}(a)$$

by once again using Eq. (2.1.7) of [11]. By induction, the result (C.3) holds for all non-negative integers ν.

Thus, we conclude that

$$I_p(x,a) = -\frac{1}{x}\log(1-x)\mathrm{Li}_p(a) - S_{p-1,2}(a) - \mathrm{Li}_{p+1}(a) \tag{C.4}$$

in the limit $x \to 1^-$.

Appendix D. Generalizations

One may foresee the need for generalizations of the results of the theorem, (1) and (2), to powers n^{-p} instead of n^{-2}. Let us therefore consider the sum

$$\sigma_p(k) = \sum_{n=1}^{\infty} \frac{1}{n^p}[\gamma + \psi(1+kn)] = \sum_{n=1}^{\infty}\sum_{j=1}^{kn} \frac{1}{n^p j}, \tag{D.1}$$

where $p=2,3,4,\dots$ and $k=1,2,3,\dots$. Introducing the integral representation for the psi function, and the regulator like in Section 2, we find

$$\sigma_p(k)=\sum_{n=1}^{\infty}\frac{1}{n^p}\int_0^1 \mathrm{d}t\,\frac{1-t^{kn}}{1-t}=\lim_{x\to 1^-}\int_0^1 \mathrm{d}t\,\frac{\zeta(p)-\mathrm{Li}_p(t^k)}{1-xt}.$$

For the polylogarithm, we use the factorization formula (A.7) to get

$$\sigma_p(k)=\lim_{x\to 1^-}\left[-\frac{1}{x}\log(1-x)\zeta(p)-k^{p-1}\sum_{j=1}^{k}I_p(x,\omega^j)\right],$$

where we have introduced the integral

$$I_p(x,a)=\int_0^1 \mathrm{d}t\,\frac{\mathrm{Li}_p(at)}{1-xt}.$$

This integral was calculated in Appendix C. Using this result and invoking the factorization formula (A.7), we note that the singular parts cancel. Thus, the result is

$$\sum_{n=1}^{\infty}\frac{1}{n^p}[\gamma+\psi(1+kn)]=k^{p-1}\sum_{j=1}^{k}[S_{p-1,2}(\omega^j)+\mathrm{Li}_{p+1}(\omega^j)]$$

$$=\frac{1}{k}\zeta(p+1)+k^{p-1}\sum_{j=1}^{k}S_{p-1,2}(\omega^j).$$

As in the case with $p=2$, it is only the real part of the Nielsen function that enters,

$$S_{m,2}(\omega^j)+S_{m,2}(\omega^{-j}).$$

We have not been able to find any simple expression for this real part for $m\geqslant 2$. Thus, we leave the result as

$$\sum_{n=1}^{\infty}\frac{1}{n^p}[\gamma+\psi(1+kn)]=\frac{1}{k}\zeta(p+1)+k^{p-1}\sum_{j=1}^{k}S_{p-1,2}(\omega^j). \tag{D.2}$$

Similarly, for the generalization of the alternating series (2), we find

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^p}[\gamma+\psi(1+kn)]=\frac{1}{k}\left(\frac{1}{2^p}-1\right)\zeta(p+1)+k^{p-1}\sum_{j=1}^{k}S_{p-1,2}(\omega^{j+1/2}). \tag{D.3}$$

When $p=1$, only the alternating series converges. We have been able to prove the following result:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} [\gamma + \psi(1+kn)] = -\frac{1}{4}\left(k+\frac{1}{k}\right)\zeta(2) + \frac{1}{2}\sum_{j=1}^{k}\left\{\mathrm{Cl}_1\left(\frac{2\pi j}{k}+\frac{\pi}{k}\right)\right\}^2, \tag{D.4}$$

and the immediate corollary thereof,

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} [\gamma + \psi(kn)] = \frac{1}{4}\left(\frac{1}{k}-k\right)\zeta(2) + \frac{1}{2}\sum_{j=1}^{k}\left\{\mathrm{Cl}_1\left(\frac{2\pi j}{k}+\frac{\pi}{k}\right)\right\}^2. \tag{D.5}$$

Proof. By applying (D.3) we find that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} [\gamma + \psi(1+kn)] &= -\frac{1}{2k}\zeta(2) + \sum_{j=0}^{k-1} S_{0,2}(\omega^{j+1/2}) \\ &= -\frac{1}{2k}\zeta(2) + \frac{1}{2}\sum_{j=0}^{k-1} \{\mathrm{Li}_1(\omega^{j+1/2})\}^2\end{aligned}$$

by making use of the fact that $S_{0,2}(z) = \frac{1}{2}\{\mathrm{Li}_1(z)\}^2$. On the unit circle, we know from [16] that $\mathrm{Li}_1(\theta) = \mathrm{Cl}_1(\theta) + \mathrm{i}\,\mathrm{Gl}_1(\theta)$. By retaining only the real parts of the expression, we arrive at

$$-\frac{1}{2k}\zeta(2) + \frac{1}{2}\sum_{j=0}^{k-1}\left\{\left[\mathrm{Cl}_1\left(\frac{2\pi j}{k}+\frac{\pi}{k}\right)\right]^2 - \left[\mathrm{Gl}_1\left(\frac{2\pi j}{k}+\frac{\pi}{k}\right)\right]^2\right\}.$$

The function $\mathrm{Gl}_1(\theta)$ is a well-known Fourier series equal to the 2π-periodic extension of $\frac{1}{2}(\pi - \theta)$, $0<\theta<2\pi$. This fact is used to perform the finite sum over the part containing $\mathrm{Gl}_1(\theta)$,

$$\begin{aligned}&-\frac{1}{2k}\zeta(2) - \frac{\pi^2}{8}\sum_{j=0}^{k-1}\left(1-\frac{2j}{k}-\frac{1}{k}\right)^2 + \frac{1}{2}\sum_{j=0}^{k-1}\left\{\mathrm{Cl}_1\left(\frac{2\pi j}{k}+\frac{\pi}{k}\right)\right\}^2 \\ &= -\frac{1}{4}\left(k+\frac{1}{k}\right)\zeta(2) + \frac{1}{2}\sum_{j=1}^{k}\left\{\mathrm{Cl}_1\left(\frac{2\pi j}{k}+\frac{\pi}{k}\right)\right\}^2. \quad \square\end{aligned}$$

By using the fact that $\mathrm{Cl}_1(\theta) = -\log|2\sin\frac{1}{2}\theta|$, we may now state the following results:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} [\gamma + \psi(n)] = \frac{1}{2}\log^2 2, \tag{D.6}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} [\gamma + \psi(1+n)] = -\frac{1}{2}\zeta(2) + \frac{1}{2}\log^2 2, \tag{D.7}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} [\gamma + \psi(2n)] = -\frac{3}{8}\zeta(2) + \frac{1}{4}\log^2 2, \tag{D.8}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} [\gamma + \psi(1+2n)] = -\tfrac{5}{8}\zeta(2) + \tfrac{1}{4}\log^2 2, \tag{D.9}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} [\gamma + \psi(3n)] = -\tfrac{2}{3}\zeta(2) + \tfrac{1}{2}\log^2 2, \tag{D.10}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} [\gamma + \psi(1+3n)] = -\tfrac{5}{6}\zeta(2) + \tfrac{1}{2}\log^2 2. \tag{D.11}$$

For higher values of k, the sums do not turn out to be this simple.

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