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Convergence properties of some block Krylov subspace methods for multiple linear systems

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Abstract

In the present paper, we give some convergence results of the global minimal residual methods and the global orthogonal residual methods for multiple linear systems. Using the Schur complement formulae and a new matrix product, we give expressions of the approximate solutions and the corresponding residuals. We also derive some useful relations between the norm of the residuals.

Keywords: Global FOM; Global GMRES; Matrix Krylov subspace; Iterative methods; Multiple linear systems; Schur complement

1. Introduction

Many applications require the solution of several sparse systems of linear equations with the same coefficient matrix and different right-hand sides

$$AX = B, \tag{1.1}$$

where A is an $n \times n$ real matrix, B and X are $n \times s$ rectangular matrices with $s \ll n$.

For nonsymmetric problems, some block Krylov subspace methods have been developed these last years; see [8–11,15,18,20,23,24] and the references therein.

In [10], we introduced a global approach. It consists of projecting the initial residual onto a matrix Krylov subspace. We derived the global full orthogonalization (Gl-FOM) method and the global generalized minimum residual (Gl-GMRES) method.

In the present paper we give some new convergence results for two classes of global Krylov subspace methods. These methods are classified in two categories: the global minimal residual (Gl-MR) methods containing all the Krylov methods that are theoretically equivalent to Gl-GMRES and the global orthogonal residual (Gl-OR) methods including the methods that are theoretically equivalent to Gl-FOM. We study the convergence behaviour of these methods without referring to any algorithm. In this work, we do not consider the numerical aspect of these methods.

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The paper is organized as follows. In Section 2, we review some properties of the Schur complement and of the Kronecker product. We also introduce a new matrix product and give some of its properties. In Section 3, we define the global minimal residual methods and the global orthogonal methods. Using the Schur complement we give new expressions of the approximations and the corresponding residuals. We also derive a relationship between the residual norms. A convergence analysis is discussed in Section 4.

2. Definitions and properties

2.1. Some Schur complement identities

We first recall the definition of the Schur complement [22] and give some of their properties; for more properties see [1,4–7,14,16].

Definition 1. Let M_1 be a matrix partitioned into four blocks

$$M_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where the submatrix D is assumed to be square and nonsingular. The Schur complement of D in M_1, denoted by (M_1/D), is defined by

$$(M_1/D) = A - BD^{-1}C.$$

If D is not a square matrix then a pseudo-Schur complement of D in M_1 can still be defined [2,5]. Let us remark that having the nonsingular submatrix D in the lower right-hand side corner of M_1 is a matter of convention. We can similarly define the following Schur complements:

$$(M_1/A) = D - CA^{-1}B,$$

$$(M_1/B) = C - DB^{-1}A,$$

$$(M_1/C) = B - AC^{-1}D.$$

Now we will give some properties of the Schur complements.

Proposition 1 (*Messaoudi [13]*). *Let us assume that the submatrix D is nonsingular, then*

$$\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \Big/ D\right) = \left(\begin{bmatrix} D & C \\ B & A \end{bmatrix} \Big/ D\right) = \left(\begin{bmatrix} B & A \\ D & C \end{bmatrix} \Big/ D\right) = \left(\begin{bmatrix} C & D \\ A & B \end{bmatrix} \Big/ D\right).$$

Proposition 2 (*Messaoudi [13]*). *Assuming that the matrix D is nonsingular and E is a matrix such that the product EA is well defined, then*

$$\left(\begin{bmatrix} EA & EB \\ C & D \end{bmatrix} \Big/ D\right) = E \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \Big/ D\right).$$

We recall the first matrix Sylvester identity. Consider the matrices K and M_3 partitioned as follows:

$$K = \begin{bmatrix} A & B & E \\ C & D & F \\ G & H & L \end{bmatrix}, \quad M_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

$$M_2 = \begin{bmatrix} B & E \\ D & F \end{bmatrix}, \quad M_3 = \begin{bmatrix} D & F \\ H & L \end{bmatrix}, \quad M_4 = \begin{bmatrix} C & D \\ G & H \end{bmatrix}.$$

Proposition 3 (*The first matrix Sylvester identity (Messaoudi [13])*). *If the matrices D and M_3 are square and nonsingular, then*

$$(K/M_3) = ((K/D)/(M_3/D)) = (M_1/D) - (M_2/D)(M_3/D)^{-1}(M_4/D).$$

2.2. The Kronecker product and the $\diamond$ product

For two matrices Y and Z in $\mathbb{R}^{n\times s}$, we define the inner product $\langle Y, Z\rangle_F = \mathrm{tr}(Y^{\mathrm{T}}Z)$ where $\mathrm{tr}(Y^{\mathrm{T}}Z)$ denotes the trace of the matrix $Y^{\mathrm{T}}Z$. The associated norm is the Frobenius norm denoted by $\|.\|_F$. A system of vectors (matrices) of $\mathbb{R}^{n\times s}$ is said to be F-orthonormal if it is orthonormal with respect to $\langle ., .\rangle_F$. For $Y = [y_{i,j}] \in \mathbb{R}^{n\times s}$, we denote by $\mathrm{vec}(Y)$ the vector of $\mathbb{R}^{ns}$ defined by $\mathrm{vec}(Y) = [y(., 1)^{\mathrm{T}}, y(., 2)^{\mathrm{T}}, \ldots, y(., s)^{\mathrm{T}}]^{\mathrm{T}}$ where $y(., j)$, $j = 1, \ldots, s$, is the jth column of Y. $A \otimes B = [a_{i,j}B]$ denotes the Kronecker product of the matrices A and B. For this product, we have the following properties [12]:

(1) $(A \otimes B)^{\mathrm{T}} = A^{\mathrm{T}} \otimes B^{\mathrm{T}}$.
(2) $(A \otimes B)(C \otimes D) = (AC \otimes BD)$.
(3) If A and B are nonsingular matrices of dimension $n \times n$ and $p \times p$, respectively, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$.
(4) If A and B are $n \times n$ and $p \times p$, matrices, then
$\det(A \otimes B) = \det(A)^p \det(B)^n$ and $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\,\mathrm{tr}(B)$.
(5) $\mathrm{vec}(ABC) = (C^{\mathrm{T}} \otimes A)\,\mathrm{vec}(B)$.
(6) $\mathrm{vec}(A)^{\mathrm{T}}\,\mathrm{vec}(B) = \mathrm{trace}(A^{\mathrm{T}}B)$.

In the following we introduce a new product denoted by $\diamond$ and defined as follows:

Definition 2. Let $A = [A_1, A_2, \ldots, A_p]$ and $B = [B_1, B_2, \ldots, B_l]$ be matrices of dimension $n \times ps$ and $n \times ls$, respectively, where A_i and B_j ($i = 1, \ldots, p$; $j = 1, \ldots, l$) are $n \times s$ matrices. Then the $p \times l$ matrix $A^{\mathrm{T}} \diamond B$ is defined by:

$$A^{\mathrm{T}} \diamond B = \begin{pmatrix} \langle A_1, B_1\rangle_F & \langle A_1, B_2\rangle_F & \ldots & \langle A_1, B_l\rangle_F \\ \langle A_2, B_1\rangle_F & \langle A_2, B_2\rangle_F & \ldots & \langle A_2, B_l\rangle_F \\ \vdots & \vdots & \vdots & \vdots \\ \langle A_p, B_1\rangle_F & \langle A_p, B_2\rangle_F & \ldots & \langle A_p, B_l\rangle_F \end{pmatrix}.$$

Remarks.

(1) If $s = 1$ then $A^{\mathrm{T}} \diamond B = A^{\mathrm{T}}B$.
(2) If $s = 1$, $p = 1$ and $l = 1$, then setting $A = u \in \mathbb{R}^n$ and $B = v \in \mathbb{R}^n$, we have $A^{\mathrm{T}} \diamond B = u^{\mathrm{T}}v \in \mathbb{R}$.
(3) The matrix $A = [A_1, A_2, \ldots, A_p]$ is F-orthonormal if and only $A^{\mathrm{T}} \diamond A = I_p$.
(4) If $X \in \mathbb{R}^{n\times s}$, then $X^{\mathrm{T}} \diamond X = \|X\|_F^2$.

It is not difficult to show the following properties satisfied by the product $\diamond$.

Proposition 4. *Let $A, B, C \in \mathbb{R}^{n\times ps}$, $D \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{p\times p}$ and $\alpha \in \mathbb{R}$. Then we have*

(1) $(A + B)^{\mathrm{T}} \diamond C = A^{\mathrm{T}} \diamond C + B^{\mathrm{T}} \diamond C$.
(2) $A^{\mathrm{T}} \diamond (B + C) = A^{\mathrm{T}} \diamond B + A^{\mathrm{T}} \diamond C$.
(3) $(\alpha A)^{\mathrm{T}} \diamond C = \alpha(A^{\mathrm{T}} \diamond C)$.
(4) $(A^{\mathrm{T}} \diamond B)^{\mathrm{T}} = B^{\mathrm{T}} \diamond A$.
(5) $(DA)^{\mathrm{T}} \diamond B = A^{\mathrm{T}} \diamond (D^{\mathrm{T}}B)$.

(6) $A^{\mathrm{T}} \diamond (B(L \otimes I_s)) = (A^{\mathrm{T}} \diamond B)L$.
(7) $\|A^{\mathrm{T}} \diamond B\|_F \leqslant \|A\|_F \|B\|_F$.

Proposition 5. *Let $A \in \mathbb{R}^{n\times ps}$, $B \in \mathbb{R}^{n\times ks}$, $C \in \mathbb{R}^{k\times p}$, $D \in \mathbb{R}^{k\times k}$ and $E \in \mathbb{R}^{n\times s}$. If the matrix D is nonsingular then*

$$E^{\mathrm{T}} \diamond \left(\begin{bmatrix} A & B \\ C \otimes I_s & D \otimes I_s \end{bmatrix} \Big/ (D \otimes I_s) \right) = \left(\begin{bmatrix} E^{\mathrm{T}} \diamond A & E^{\mathrm{T}} \diamond B \\ C & D \end{bmatrix} \Big/ D \right).$$

Proof. From the definition of the Schur complement and the relation (2) of Proposition 4, we obtain

$$\begin{aligned} E^{\mathrm{T}} \diamond \left(\begin{bmatrix} A & B \\ C \otimes I_s & D \otimes I_s \end{bmatrix} \Big/ (D \otimes I_s) \right) &= E^{\mathrm{T}} \diamond A - E^{\mathrm{T}} \diamond [B(D \otimes I_s)^{-1}(C \otimes I_s)] \\ &= E^{\mathrm{T}} \diamond A - E^{\mathrm{T}} \diamond [B(D^{-1}C \otimes I_s)]. \end{aligned}$$

Therefore, using the relation (6) of Proposition 4, it follows that

$$E^{\mathrm{T}} \diamond A - E^{\mathrm{T}} \diamond [B(D^{-1}C \otimes I_s)] = E^{\mathrm{T}} \diamond A - (E^{\mathrm{T}} \diamond B)D^{-1}C = \left(\begin{bmatrix} E^{\mathrm{T}} \diamond A & E^{\mathrm{T}} \diamond B \\ C & D \end{bmatrix} \Big/ D \right). \qquad \square$$

2.3. The global QR factorization

Next, we present the global Gram–Schmidt process. Let $\mathscr{Z} = [Z_1, Z_2, \ldots, Z_k]$ be a matrix of k blocks with $Z_i \in \mathbb{R}^{n\times s}$, $i = 1, \ldots, k$. The global Gram–Schmidt algorithm allows us to generate a new F-orthonormal matrix $\mathscr{Q} = [Q_1, Q_2, \ldots, Q_k]$ such that $\mathrm{span}\{Q_1, \ldots, Q_k\} = \mathrm{span}\{Z_1, \ldots, Z_k\}$ with $\langle Q_i, Q_i\rangle_F = 1$ and $\langle Q_i, Q_j\rangle_F = 0$ if $i \neq j$. The algorithm is described as follows:

Algorithm 1. (*The modified global Gram–Schmidt algorithm*)

(1) $R = (r_{i,j}) = 0$.
(2) $r_{1,1} = \|Z_1\|_F$.
(3) $Q_1 = Z_1/r_{1,1}$.
(4) For $i = 2, \ldots, k$

  $Q = Z_i$,
  for $j = 1, \ldots, i-1$
    $r_{j,i} = \langle Q, Z_j\rangle_F$,
    $Q = Q - r_{j,i} Z_j$
  end j
  $r_{i,i} = \|Q\|_F$ and $Q_i = Q/r_{i,i}$.
 End i.

Proposition 6. *Let $\mathscr{Z} = [Z_1, Z_2, \ldots, Z_k]$ be an $n \times ks$ matrix with $Z_i \in \mathbb{R}^{n\times s}$, for $i = 1, \ldots, k$. Then applying Algorithm 1, the matrix $\mathscr{Z}$ can be factored as*

$$\mathscr{Z} = \mathscr{Q}(R \otimes I_s),$$

where $Q = [Q_1, \ldots, Q_k]$ is an $n \times ks$ F-orthonormal matrix satisfying $\mathscr{Q}^{\mathrm{T}} \diamond \mathscr{Q} = I_k$ and R is an upper triangular $k \times k$ matrix.

Proof. If Z_i is the ith column of the matrix $\mathscr{Z}$, then from Algorithm 1, we have

$$\begin{aligned} Z_i &= \sum_{j=1}^{i} r_{j,i} Q_j \\ &= \sum_{j=1}^{i} Q_j (r_{j,i} \otimes I_s) \\ &= [Q_1, Q_2, \ldots, Q_i] \left[\begin{pmatrix} r_{1,i} \\ r_{2,i} \\ \vdots \\ r_{i,i} \end{pmatrix} \otimes I_s \right]. \end{aligned}$$

If $R_i = [r_{1,i}, \ldots, r_{i,i}, 0, \ldots, 0]^{\mathrm{T}}$ is the ith column of the upper triangular matrix $R = [R_1, \ldots, R_k]$, then

$$Z_i = [Q_1, \ldots, Q_k](R_i \otimes I_s), \quad i = 1, \ldots, k.$$

Therefore, $\mathscr{Z}$ can be factored as

$$\mathscr{Z} = \mathscr{Q}(R \otimes I_s) \quad \text{with } \mathscr{Q}^{\mathrm{T}} \diamond Q = I_k. \qquad \square$$

Note that $\mathscr{Q}^{\mathrm{T}} \diamond \mathscr{Z} = \mathscr{Q}^{\mathrm{T}} \diamond (\mathscr{Q}(R \otimes I_s))$. Hence using Proposition 4, it follows that $\mathscr{Q}^{\mathrm{T}} \diamond \mathscr{Z} = R$.

3. Global OR-type and global MR-type methods

3.1. A global OR-type method

Let $\mathbf{K}_k(A, V) = \operatorname{span}\{V, AV, \ldots, A^{k-1}V\}$ denotes the matrix Krylov subspace of $\mathbb{R}^{n\times s}$ spanned by the matrices $V, AV, \ldots, A^{k-1}V$ where V is an $n \times s$ matrix. Note that $Z \in \mathbf{K}_k(A, V)$ means that

$$Z = \sum_{i=1}^{k} \alpha_i A^{i-1} V, \quad \alpha_i \in \mathbb{R}, \; i = 1, \ldots, k.$$

Now consider the block linear system of equations (1.1) and let X_0 be an initial $n \times s$ matrix with the corresponding residual $R_0 = B - AX_0$. At step k, a global OR-type method generates approximation X_k^{OR} such that

$$X_k^{OR} - X_0 = Z_k \in \mathbf{K}_k(A, R_0) \tag{3.1}$$

and the residual R_k^{OR} satisfying the orthogonality relation

$$R_k^{OR} = R_0 - AZ_k \perp_F \mathbf{K}_k(A, R_0), \tag{3.2}$$

where the notation $\perp_F$ means the orthogonality with respect to the matrix scalar product $\langle .,. \rangle_F$ Note that R_k^{OR} is obtained by projecting R_0 onto $A\mathbf{K}_k(A, R_0)$ along the F-orthogonal of the Krylov subspace $\mathbf{K}_k(A, R_0)$. If $\mathscr{P}_k^{OR}$ denotes the projector onto $A\mathbf{K}_k(A, R_0)$ along $\mathbf{K}_k(A, R_0)^{\perp}$, then from the Galerkin condition (3.2), we have

$$R_k^{OR} = R_0 - \mathscr{P}_k^{OR} R_0. \tag{3.3}$$

The relation (3.1) implies

$$X_k^{OR} = X_0 + [R_0, AR_0, \ldots, A^{k-1}R_0](\omega \otimes I_s),$$

where $\omega = [\omega_1, \ldots, \omega_k]^{\mathrm{T}}$. Then the residual R_k^{OR} is expressed as

$$R_k^{OR} = R_0 - [AR_0, A^2R_0, \ldots, A^kR_0](\omega \otimes I_s). \tag{3.4}$$

The parameters ω_i, $i=1,\ldots,k$ are determined from the orthogonality condition (3.2) which is equivalent to

$$\langle R_k^{OR}, A^i R_0\rangle_F = 0 \quad \text{for } i=0,\ldots,k-1. \tag{3.5}$$

Let $\mathscr{K}_k = [R_0, AR_0, \ldots, A^{k-1}R_0]$ and $\mathscr{W}_k = A\mathscr{K}_k$. Then from (3.4) and (3.5) we deduce that

$$(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)\omega = \mathscr{K}_k^{\mathrm{T}} \diamond R_0. \tag{3.6}$$

We have the following results:

Theorem 1. *Assume that the matrix $\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k$ is nonsingular. Then the approximate solution X_k^{OR} and the corresponding residual R_k^{OR} are expressed as the following Schur complements*:

$$X_k^{OR} = \left(\begin{bmatrix} X_0 & -\mathscr{K}_k \\ (\mathscr{K}_k^{\mathrm{T}} \diamond R_0)\otimes I_s & (\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)\otimes I_s \end{bmatrix} \Big/ (\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)\otimes I_s \right) \tag{3.7}$$

and

$$R_k^{OR} = \left(\begin{bmatrix} R_0 & \mathscr{W}_k \\ (\mathscr{K}_k^{\mathrm{T}} \diamond R_0)\otimes I_s & (\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)\otimes I_s \end{bmatrix} \Big/ (\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)\otimes I_s \right). \tag{3.8}$$

Proof. At step k, the iterate X_k^{OR} is given by $X_k^{OR} = X_0 + \mathscr{K}_k(\omega\otimes I_s)$ where ω is determined from (3.6). As the $k\times k$ matrix $\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k$ is nonsingular, then $\omega = (\mathbf{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)^{-1}(\mathscr{K}_k^{\mathrm{T}} \diamond R_0)$. Therefore,

$$\begin{aligned} X_k^{OR} &= X_0 + \mathscr{K}_k[((\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)^{-1}(\mathscr{K}_k^{\mathrm{T}} \diamond R_0))\otimes I_s] \\ &= X_0 + \mathscr{K}_k[((\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)^{-1}\otimes I_s)(\mathscr{K}_k^{\mathrm{T}} \diamond R_0\otimes I_s)] \\ &= X_0 + \mathscr{K}_k[((\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)\otimes I_s)^{-1}(\mathscr{K}_k^{\mathrm{T}} \diamond R_0\otimes I_s)]. \end{aligned}$$

This shows that X_k^{OR} can be expressed as the Schur complement given by (3.7). The proof of (3.8) can be done in a similar way. □

Theorem 2. *Assume that at step k, the matrix $\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k$ is nonsingular. Then the norm of the kth residual R_k^{OR} is given by*

$$\|R_k^{OR}\|_F^2 = \frac{\det(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{K}_k)}{\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)^2}, \tag{3.9}$$

where $\det(X)$ denotes the determinant of the square matrix X.

Proof. Note that since R_k^{OR} is an $n\times s$ matrix, we have $\|R_k^{OR}\|_F^2 = (R_k^{OR})^{\mathrm{T}} \diamond R_k^{OR}$. Using (3.4) and (3.5) we obtain $(R_k^{OR})^{\mathrm{T}} \diamond R_k^{OR} = (R_0 - \mathscr{W}_k(\omega\otimes I_s))^{\mathrm{T}} \diamond R_k^{OR}$. The orthogonality condition (3.5) implies

$$(R_k^{OR})^{\mathrm{T}} \diamond R_k^{OR} = -\omega_k(A^kR_0)^{\mathrm{T}} \diamond R_k^{OR}. \tag{3.10}$$

Let us first compute $(A^kR_0)^{\mathrm{T}} \diamond R_k^{OR}$. Using (3.8) and Proposition 5, we obtain

$$(A^kR_0)^{\mathrm{T}} \diamond R_k^{OR} = \left(\begin{bmatrix} (A^kR_0)^{\mathrm{T}} \diamond R_0 & (A^kR_0)^{\mathrm{T}} \diamond \mathscr{W}_k \\ \mathscr{K}_k^{\mathrm{T}} \diamond R_0 & \mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k \end{bmatrix} \Big/ \mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k \right).$$

Then, using Proposition 1, it follows that

$$(A^kR_0)^{\mathrm{T}} \diamond R_k^{OR} = \left(\begin{bmatrix} \mathscr{K}_k^{\mathrm{T}} \diamond R_0 & \mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k \\ (A^kR_0)^{\mathrm{T}} \diamond R_0 & (A^kR_0)^{\mathrm{T}} \diamond \mathscr{W}_k \end{bmatrix} \Big/ \mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k \right). \tag{3.11}$$

Now, as $\mathscr{K}_{k+1} = [R_0, \mathscr{W}_k]$ and $\mathscr{K}_{k+1} = [\mathscr{K}_k, A^kR_0]$, (3.11) can be expressed as

$$(A^kR_0)^{\mathrm{T}} \diamond R_k^{OR} = (\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1}/\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k).$$

Therefore, as $(A^k R_0)^{\mathrm{T}} \diamond R_k^{OR}$ is a scalar, it follows that

$$(A^k R_0)^{\mathrm{T}} \diamond R_k^{OR} = (-1)^k \frac{\det(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})}{\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)}. \tag{3.12}$$

On the other hand, ω_k can be computed, from (3.6) by the Cramer rule, as

$$\omega_k = (-1)^{k-1} \frac{\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{K}_k)}{\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)}. \tag{3.13}$$

Therefore, using (3.12) and (3.13) in (3.10), the result follows. □

3.2. A global MR-type method

A global-MR type method constructs, at step k, the approximation X_k^{MR} satisfying the following two relations:

$$X_k^{MR} - X_0 \in \mathbf{K}_k(A, R_0) \quad \text{and} \quad R_k^{MR} \perp_F \mathbf{K}_k(A, AR_0).$$

From these two relations, we obtain

$$X_k^{MR} = X_0 + \mathscr{K}_k(\alpha \otimes I_s) \tag{3.14}$$

and

$$R_k^{MR} = R_0 - \mathscr{W}_k(\alpha \otimes I_s), \tag{3.15}$$

where α is such that

$$(\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k)\alpha = \mathscr{W}_k^{\mathrm{T}} \diamond R_0. \tag{3.16}$$

If $\mathscr{P}_k^{MR}$ denotes the F-orthogonal projector onto the matrix Krylov subspace $\mathbf{K}_k(A, AR_0)$, then the residual R_k^{MR} can be expressed as $R_k^{MR} = R_0 - \mathscr{P}_k^{MR} R_0$. As we are dealing with an orthogonal projection method onto the Krylov subspace $\mathbf{K}_k(A, AR_0)$, we have the minimization property

$$\|R_k^{MR}\|_F = \min_{Z \in \mathbf{K}_k(A, R_0)} \|R_0 - AZ\|_F.$$

The next results show that X_k^{MR} and R_k^{MR} could be expressed as Schur complements.

Theorem 3. *Assume that* $\det(\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k) \neq 0$. *Then the approximate solution X_k^{MR} and the corresponding residual R_k^{MR} are expressed as the following Schur complements*:

$$X_k^{MR} = \left(\begin{bmatrix} X_0 & -\mathscr{K}_k \\ (\mathscr{W}_k^{\mathrm{T}} \diamond R_0) \otimes I_s & (\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k) \otimes I_s \end{bmatrix} \Big/ (\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k) \otimes I_s \right) \tag{3.17}$$

and

$$R_k^{MR} = \left(\begin{bmatrix} R_0 & \mathscr{W}_k \\ (\mathscr{W}_k^{\mathrm{T}} \diamond R_0) \otimes I_s & (\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k) \otimes I_s \end{bmatrix} \Big/ (\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k) \otimes I_s \right). \tag{3.18}$$

Proof. Using (3.14), (3.15) and (3.16) we get the results. □

In the following result, we give an expression of the residual norm of the MR method.

Theorem 4. *If* $\det(\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k) \neq 0$, *then we have*

$$\|R_k^{MR}\|_F^2 = \frac{\det(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})}{\det(\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k)}. \tag{3.19}$$

Proof. We have

$$\begin{aligned}\|R_k^{MR}\|_F^2 &= (R_k^{MR})^{\mathrm{T}} \diamond R_k^{MR}\\ &= (R_k^{MR})^{\mathrm{T}} \diamond (R_0 - \mathscr{W}_k(\alpha \otimes I_s))\\ &= (R_k^{MR})^{\mathrm{T}} \diamond R_0 - ((R_k^{MR})^{\mathrm{T}} \diamond \mathscr{W}_k)\alpha\\ &= (R_k^{MR})^{\mathrm{T}} \diamond R_0\\ &= R_0^{\mathrm{T}} \diamond R_k^{MR}\\ &= R_0^{\mathrm{T}} \diamond \left(\begin{bmatrix} R_0 & \mathscr{W}_k \\ (\mathscr{W}_k^{\mathrm{T}} \diamond R_0) \otimes I_s & (\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k) \otimes I_s \end{bmatrix} \Big/ (\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k) \otimes I_s \right).\end{aligned}$$

So, applying Proposition 5 we get

$$\begin{aligned}\|R_k^{MR}\|_F^2 &= \left(\begin{bmatrix} R_0^{\mathrm{T}} \diamond R_0 & R_0^{\mathrm{T}} \diamond \mathscr{W}_k \\ \mathscr{W}_k^{\mathrm{T}} \diamond R_0 & \mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k \end{bmatrix} \Big/ \mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k \right)\\ &= ((\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})/\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k)\end{aligned}$$

and as $\|R_k^{MR}\|_F$ is a scalar then we get the result. □

3.3. Some relations between the residual norms

We give some relations between the residual norms for two successive iterates and also between the residual norms for the global OR and the global MR methods.

Theorem 5. *Let R_k^{OR} and R_k^{MR} be the residuals corresponding to the kth iterates produced by the global OR and the global MR type methods, respectively. Then*

(1)

$$\frac{\|R_k^{MR}\|_F}{\|R_{k-1}^{MR}\|_F} = \sqrt{1 - c_k^2},$$

(2)

$$\|R_k^{MR}\|_F = c_k \|R_k^{OR}\|_F \text{ and}$$

(3)

$$\frac{\|R_k^{OR}\|_F}{\|R_{k-1}^{OR}\|_F} = \left(\frac{c_{k-1}}{c_k}\right)\sqrt{1 - c_k^2},$$

where

$$c_k^2 = \frac{\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)^2}{\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{K}_k)\det(\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k)}.$$

Proof. From (3.19), we have $(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1}/\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k)$. Using the fact that $\mathscr{W}_k = [\mathscr{W}_{k-1}, A^k R_0]$ and $\mathscr{K}_{k+1} = [R_0, \mathscr{W}_{k-1}, A^k R_0]$, we obtain

$$\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1} = \begin{bmatrix} R_0^{\mathrm{T}} \diamond R_0 & R_0^{\mathrm{T}} \diamond \mathscr{W}_{k-1} & R_0^{\mathrm{T}} \diamond A^k R_0 \\ \mathscr{W}_{k-1}^{\mathrm{T}} \diamond R_0 & \mathscr{W}_{k-1}^{\mathrm{T}} \diamond \mathscr{W}_{k-1} & \mathscr{W}_{k-1}^{\mathrm{T}} \diamond A^k R_0 \\ (A^k R_0)^{\mathrm{T}} \diamond R_0 & (A^k R_0)^{\mathrm{T}} \diamond \mathscr{W}_{k-1} & (A^k R_0)^{\mathrm{T}} \diamond A^k R_0 \end{bmatrix}.$$

Using Proposition 3, we get

$$\|R_k^{MR}\|_F^2 = \left(\begin{bmatrix} R_0^{\mathrm T} \diamond R_0 & R_0^{\mathrm T} \diamond \mathscr{W}_{k-1} \\ \mathscr{W}_{k-1}^{\mathrm T} \diamond R_0 & \mathscr{W}_{k-1}^{\mathrm T} \diamond \mathscr{W}_{k-1} \end{bmatrix} \Big/ \mathscr{W}_{k-1}^{\mathrm T} \diamond \mathscr{W}_{k-1}\right)$$
$$- \left(\begin{bmatrix} R_0^{\mathrm T} \diamond \mathscr{W}_{k-1} & R_0^{\mathrm T} \diamond A^k R_0 \\ \mathscr{W}_{k-1}^{\mathrm T} \diamond \mathscr{W}_{k-1} & \mathscr{W}_{k-1}^{\mathrm T} \diamond A^k R_0 \end{bmatrix} \Big/ \mathscr{W}_{k-1}^{\mathrm T} \diamond \mathscr{W}_{k-1}\right) (\mathscr{W}_k^{\mathrm T} \diamond \mathscr{W}_k / \mathscr{W}_{k-1}^{\mathrm T} \diamond \mathscr{W}_{k-1})^{-1}$$
$$\times \left(\begin{bmatrix} \mathscr{W}_{k-1}^{\mathrm T} \diamond R_0 & \mathscr{W}_{k-1}^{\mathrm T} \diamond \mathscr{W}_{k-1} \\ (A^k R_0)^{\mathrm T} \diamond R_0 & (A^k R_0)^{\mathrm T} \diamond \mathscr{W}_{k-1} \end{bmatrix} \Big/ \mathscr{W}_{k-1}^{\mathrm T} \diamond \mathscr{W}_{k-1}\right).$$

Then

$$\|R_k^{MR}\|_F^2 = \|R_{k-1}^{MR}\|^2 - (\mathscr{K}_k^{\mathrm T} \diamond \mathscr{W}_k / \mathscr{W}_{k-1}^{\mathrm T} \diamond \mathscr{W}_{k-1})(\mathscr{W}_k^{\mathrm T} \mathscr{W}_k / \mathscr{W}_{k-1}^{\mathrm T} \diamond \mathscr{W}_{k-1})^{-1} (\mathscr{W}_k^{\mathrm T} \diamond \mathscr{K}_k / \mathscr{W}_{k-1}^{\mathrm T} \diamond \mathscr{W}_{k-1}).$$

Developing this expression, we obtain

$$\frac{\|R_k^{MR}\|_F^2}{\|R_{k-1}^{MR}\|_F^2} = 1 - c_k^2,$$

where

$$c_k^2 = \frac{\det(\mathscr{K}_k^{\mathrm T} \diamond \mathscr{W}_k)^2}{\det(\mathscr{K}_k^{\mathrm T} \diamond \mathscr{K}_k)\det(\mathscr{W}_k^{\mathrm T} \diamond \mathscr{W}_k)}$$

which shows the relation (1) of the theorem.

To show the relation (2), we use (3.9) and (3.19). The last expression of the theorem is obtained from (1) and (2). □

If $s = 1$, the results of Theorem 5 coincide with the results given in [21]. Using the GMRES and the FOM algorithms, a similar theorem was also derived in [3] when $s = 1$.

4. Convergence analysis of the global OR and the global MR methods

In this section, we give some convergence results for the global OR and the global MR methods. Applying the global $\mathscr{Q}R$ factorization to $\mathscr{K}_{k+1}$ and $\mathscr{K}_k$, we get

$$\mathscr{K}_{k+1} = \mathscr{Q}_{k+1}(R_{k+1} \otimes I_s) \quad \text{and} \quad \mathscr{K}_k = \mathscr{Q}_k(R_k \otimes I_s), \tag{4.1}$$

with $\mathscr{Q}_{k+1} \in \mathbb{R}^{n\times(k+1)s}$, $R_{k+1} \in \mathbb{R}^{(k+1)\times(k+1)}$, $\mathscr{Q}_k \in \mathbb{R}^{n\times ks}$ and $R_k \in \mathbb{R}^{k\times k}$. $\mathscr{Q}_{k+1}$ and $\mathscr{Q}_k$ are F-orthonormal (orthonormal with respect to the $\diamond$ product); R_{k+1} and R_k are two upper triangular matrices. Note that

$$\mathscr{K}_{k+1}\left(\begin{bmatrix} 0_{s\times ks} \\ I_{ks} \end{bmatrix}\right) = A\mathscr{K}_k. \tag{4.2}$$

Then using (4.1) and (4.2) we get

$$\mathscr{Q}_{k+1}(R_{k+1} \otimes I_s)\begin{bmatrix} 0_{s\times ks} \\ I_{ks} \end{bmatrix} = A\mathscr{Q}_k(R_k \otimes I_s). \tag{4.3}$$

Hence applying the $\diamond$ product (with $\mathscr{Q}_{k+1}^{\mathrm T}$) to (4.3) and using the assertion (6) of Proposition 4 it follows that

$$(\mathscr{Q}_{k+1}^{\mathrm T} \diamond A\mathscr{Q}_k)R_k = R_{k+1}\begin{bmatrix} 0_{1\times k} \\ I_k \end{bmatrix}. \tag{4.4}$$

Multiplying both sides of (4.4) from the right by R_k^{-1} it follows that

$$(\mathscr{Q}_{k+1}^{\mathrm T} \diamond A\mathscr{Q}_k) = R_{k+1}\begin{bmatrix} 0_{1\times k} \\ I_k \end{bmatrix} R_k^{-1}. \tag{4.5}$$

Let $\bar{H}_k$ be the $(k+1)\times k$ matrix defined by $\bar{H}_k = \mathcal{Q}_{k+1}^{\mathrm{T}} \diamond A\mathcal{Q}_k$. Then as R_{k+1} and R_k are upper triangular matrices, it follows that $\bar{H}_k$ is an upper Hessenberg matrix. If H_k denotes the $k\times k$ matrix obtained from $\bar{H}_k$ by deleting it's last row, H_k is also an upper Hessenberg matrix given by

$$H_k = \mathcal{Q}_k^{\mathrm{T}} \diamond A\mathcal{Q}_k. \tag{4.6}$$

Using the fact that $\mathcal{Q}_{k+1} = [\mathcal{Q}_k, Q_{k+1}]$ we obtain

$$\bar{H}_k = \begin{bmatrix} H_k \\ Q_{k+1}^{\mathrm{T}} \diamond A\mathcal{Q}_k \end{bmatrix}. \tag{4.7}$$

Therefore, from (4.3), (4.4) and (4.7) we deduce the following relation:

$$A\mathcal{Q}_k = \mathcal{Q}_k(H_k \otimes I_s) + h_{k+1,k} Q_{k+1} E_k^{\mathrm{T}}, \tag{4.8}$$

where $E_k^{\mathrm{T}} = [0_s, \ldots, 0_s, I_s]$ and $h_{k+1,k} = Q_{k+1} \diamond A\mathcal{Q}_k = (r_{k+1,k+1})/(r_{k,k})$.

Theorem 6. *At step k, let R_k^{MR} and R_k^{OR} be the residual produced by the global MR and the global OR methods, respectively. Then we have*

(1)

$$\frac{\|R_k^{MR}\|_F^2}{\|R_{k-1}^{MR}\|_F^2} = \frac{\det(\bar{H}_{k-1}^{\mathrm{T}} \bar{H}_{k-1})}{\det(\bar{H}_k^{\mathrm{T}} \bar{H}_k)} h_{k+1,k}^2.$$

(2)

$$\frac{\|R_k^{OR}\|_F^2}{\|R_{k-1}^{OR}\|_F^2} = \frac{\det(H_{k-1}^{\mathrm{T}} H_{k-1})}{\det(H_k^{\mathrm{T}} H_k)} h_{k+1,k}^2.$$

Proof. (1) Applying the global *QR* factorization to the matrix $\mathcal{K}_k$, the product $\mathcal{W}_k^{\mathrm{T}} \diamond \mathcal{W}_k = (A\mathcal{K}_k)^{\mathrm{T}} \diamond (A\mathcal{K}_k)$ is expressed as

$$\mathcal{W}_k^{\mathrm{T}} \diamond \mathcal{W}_k = (A\mathcal{Q}_k(R_k \otimes I_s))^{\mathrm{T}} \diamond (A\mathcal{Q}_k(R_k \otimes I_s)). \tag{4.9}$$

Then using Proposition 4 and the definition of $\bar{H}_k$, we obtain

$$\mathcal{W}_k^{\mathrm{T}} \diamond \mathcal{W}_k = R_k^{\mathrm{T}} \bar{H}_k^{\mathrm{T}} \bar{H}_k R_k. \tag{4.10}$$

Similarly, we also have

$$\mathcal{K}_k^{\mathrm{T}} \diamond \mathcal{K}_k = R_k^{\mathrm{T}} R_k. \tag{4.11}$$

From Theorem 2, the ratio of two successive global MR residual norms is given by

$$\frac{\|R_k^{MR}\|_F^2}{\|R_{k-1}^{MR}\|_F^2} = \frac{\det(\mathcal{K}_{k+1}^{\mathrm{T}} \diamond \mathcal{K}_{k+1}) \det(\mathcal{W}_{k-1}^{\mathrm{T}} \diamond \mathcal{W}_{k-1})}{\det(\mathcal{W}_k^{\mathrm{T}} \diamond \mathcal{W}_k) \det(\mathcal{K}_k^{\mathrm{T}} \diamond \mathcal{K}_k)}. \tag{4.12}$$

Therefore, using (4.10) and (4.11) in (4.12), we obtain

$$\frac{\|R_k^{MR}\|_F^2}{\|R_{k-1}^{MR}\|_F^2} = \frac{\det(\bar{H}_{k-1}^{\mathrm{T}} \bar{H}_{k-1})}{\det(\bar{H}_k^{\mathrm{T}} \bar{H}_k)} \frac{\det(R_{k+1})^2 \det(R_{k-1})^2}{\det(R_k)^4} \tag{4.13}$$

Now, as $R_{k+1} = ([\begin{smallmatrix} R_k \\ 0_{1\times k} \end{smallmatrix}], r_{k+1})$, we get

$$\frac{\|R_k^{MR}\|_F^2}{\|R_{k-1}^{MR}\|_F^2} = \frac{\det(\bar{H}_{k-1}^{\mathrm{T}} \bar{H}_{k-1})}{\det(\bar{H}_k^{\mathrm{T}} \bar{H}_k)} \frac{r_{k+1,k+1}^2}{r_{k,k}^2}.$$

Hence, using the fact that $h_{k+1,k}^2 = (r_{k+1,k+1}^2)/r_{k,k}^2$ the result follows.

The relation (2) can be proved in the same manner. □

We notice that Theorem 6 is a generalization of a result given in [21] for the case $s=1$. Now we will give another important result.

Theorem 7. *At step k, let R_k^{MR} and R_k^{OR} be the residual produced by the global MR and the global OR methods, respectively. Then we have*

(1)

$$\|R_k^{MR}\|_F^2 = \frac{1}{e_1^{\mathrm{T}}(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})^{-1} e_1^{\mathrm{T}}}.$$

(2)

$$\|R_k^{OR}\|_F^2 = \frac{e_{k+1}^{\mathrm{T}}(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})^{-1} e_{k+1}^{\mathrm{T}}}{(e_1^{\mathrm{T}}(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})^{-1} e_{k+1}^{\mathrm{T}})^2}.$$

Proof. For R_k^{MR} we have

$$\|R_k^{MR}\|_F^2 = (\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1} / \mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k) = \frac{\det(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})}{\det(\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k)},$$

we have also $\mathscr{K}_{k+1} = [R_0, AR_0, \ldots, A^k R_0] = [R_0, \mathscr{W}_k]$, then

$$\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1} = \begin{bmatrix} R_0^{\mathrm{T}} \\ \mathscr{W}_k^{\mathrm{T}} \end{bmatrix} \diamond [R_0, \mathscr{W}_k] = \begin{bmatrix} R_0^{\mathrm{T}} \diamond R_0 & R_0^{\mathrm{T}} \diamond \mathscr{W}_k \\ \mathscr{W}_k^{\mathrm{T}} \diamond R_0 & \mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k \end{bmatrix},$$

so we get

$$e_1^{\mathrm{T}}(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})^{-1} e_1 = \frac{\det(\mathscr{W}_k^{\mathrm{T}} \diamond \mathscr{W}_k)}{\det(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})} = \frac{1}{\|R_k^{MR}\|_F^2}.$$

For R_k^{OR} we have

$$\|R_k^{OR}\|_F^2 = \frac{\det(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1}) \det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{K}_k)}{\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)^2},$$

as $\mathscr{K}_{k+1} = [R_0, \mathscr{W}_{k-1}, A^k R_0]$ then we get

$$\begin{aligned} \mathscr{K}_{k+1} \diamond \mathscr{K}_{k+1} &= \begin{bmatrix} R_0^{\mathrm{T}} \\ \mathscr{W}_{k-1}^{\mathrm{T}} \\ (A^k R_0)^{\mathrm{T}} \end{bmatrix} \diamond [R_0, \mathscr{W}_{k-1}, A^k R_0] \\ &= \begin{bmatrix} R_0^{\mathrm{T}} \diamond R_0 & R_0^{\mathrm{T}} \diamond \mathscr{W}_{k-1} & R_0^{\mathrm{T}} \diamond A^k R_0 \\ \mathscr{W}_{k-1}^{\mathrm{T}} \diamond R_0 & \mathscr{W}_{k-1}^{\mathrm{T}} \diamond \mathscr{W}_{k-1} & \mathscr{W}_{k-1}^{\mathrm{T}} \diamond A^k R_0 \\ (A^k R_0)^{\mathrm{T}} \diamond R_0 & (A^k R_0)^{\mathrm{T}} \diamond \mathscr{W}_{k-1} & (A^k R_0)^{\mathrm{T}} \diamond A^k R_0 \end{bmatrix}, \end{aligned}$$

so we have

$$e_1^{\mathrm{T}}(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})^{-1} e_{k+1} = (-1)^k \frac{\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)}{\det(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})}$$

and

$$e_{k+1}^{\mathrm{T}}(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})^{-1} e_{k+1} = \frac{\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{K}_k)}{\det(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})}.$$

Then we get

$$\frac{e_{k+1}^{\mathrm{T}}(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})^{-1}e_{k+1}}{(e_1^{\mathrm{T}}(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})^{-1}e_{k+1})^2} = \frac{\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{K}_k)\det(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})}{\det(\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{W}_k)^2} = \|R_k^{OR}\|_F^2. \qquad \square$$

Note that since $\|R_0^{MR}\|_F^2 = e_1^{\mathrm{T}}(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})e_1$, then by using the Kantorovich inequality we obtain the following result.

Theorem 8.

$$1 \geqslant \frac{\|R_k^{MR}\|_F}{\|R_0^{MR}\|_F} \geqslant 2\,\frac{\sqrt{\kappa(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})}}{(1+\kappa(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1}))},$$

where $\mathscr{K}_{k+1}$ is the global Krylov matrix and $\kappa(Z)$ denotes the condition number of the matrix Z.

This means that there is no convergence as long as the Krylov basis is well-conditioned.

Example. We consider the multiple linear system $A_n X = B$, where

$$A_n = \begin{pmatrix} 0 & \dots & \dots & \dots & 0 & 1 \\ 1 & \ddots & \ddots & \ddots & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & 0 \\ 0 & \dots & \dots & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ \vdots & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix}.$$

For this example, $\kappa(A_n) = 1$. Now, if $x_0 = 0$ then for $k = 1, \dots, n-1$, we have

$$(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1}) = 2I_{k+1}, \quad \kappa(\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1}) = 1 \quad \text{and} \quad \|R_k^{MR}\|_F^2 = 2.$$

Hence we obtain the solution at the nth iteration. If we apply the standard GMRES to each right-hand side linear system, we will also obtain a stagnation until the last iteration.

If we change the right-hand side as

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & 1 & \dots & 1 \end{pmatrix}^{\mathrm{T}},$$

then for $k \leqslant n-1$, we have

$$\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1} = \begin{pmatrix} n+1 & n & \dots & \dots & n \\ n & n+1 & \ddots & \dots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & n+1 & n \\ n & \dots & \dots & n & n+1 \end{pmatrix},$$

$$\frac{\|R_k^{MR}\|^2}{\|R_0^{MR}\|^2} = \frac{(k+1)n+1}{(kn+1)(n+1)}$$

and

$$\frac{\|R_k^{OR}\|^2}{\|R_0^{OR}\|^2} = \frac{((k+1)n+1)(kn+1)}{n^2(n+1)}.$$

If we apply the standard GMRES [19] to the linear systems $A_n x^{(1)} = b^{(1)}$ and $A_n x^{(2)} = b^{(2)}$, where $b^{(i)}$, $i = 1, 2$, is the ith column of the rectangular matrix B, then we have stagnation for the first linear system i.e.: $\|r_k^{(1)}\|_2 = 1,\ k = 1, \ldots, n-1$ and $\|r_n^{(1)}\|_2 = 0$. We have convergence at the first step for the second linear system.

We will give now some comparisons between the global GMRES for solving the multiple linear system (1.1) and the standard GMRES applied to each single linear system $Ax^{(i)} = b^{(i)}$.

Theorem 9. *Let $K_{i,k}$, $i = 1, \ldots, s$ be the Krylov matrix defined by*

$$K_{i,k} = [r_0^{(i)}, Ar_0^{(i)}, \ldots, A^{k-1} r_0^{(i)}] \quad \textit{where } r_0^{(i)} = b^{(i)} - Ax_0^{(i)}.$$

Then

$$\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{K}_k = \sum_{i=1}^{s} K_{i,k}^{\mathrm{T}} K_{i,k}.$$

When applying the GMRES to the s right-hand side linear systems separately, it is well known [21] that $\|r_k^{(i)}\|_2^2 = 1/(e_1 (K_{i,k+1}^{\mathrm{T}} K_{i,k+1})^{-1} e_1)$. We have proved that when applying the global MR method to the multiple linear system (1.1), we obtain $\|R_k^{MR}\|_F^2 = 1/(e_1 (\mathscr{K}_{k+1}^{\mathrm{T}} \diamond \mathscr{K}_{k+1})^{-1} e_1)$.

Theorem 10. *If $\|r_k^{(i)}\|_2 \neq 0$, $\forall i \in \{1, \ldots, s\}$, then*

$$s \min_{1 \leqslant i \leqslant s} \|r_k^{(i)}\|_2^2 \leqslant \frac{s^2}{\sum_{i=1}^{s} (1/\|r_k^{(i)}\|_2^2)} \leqslant \|R_k^{MR}\|_F^2.$$

Proof. The first inequality is obvious.

Since each matrix $K_{i,k}^{\mathrm{T}} K_{i,k}$ is positive semidefinite, then using Theorems 9 and 6.2 of [17], we obtain

$$\sum_{i=1}^{s} (K_{i,k}^{\mathrm{T}} K_{i,k})^{-1} \geqslant s^2 \left(\sum_{i=1}^{s} K_{i,k}^{\mathrm{T}} K_{i,k} \right)^{-1} = s^2 (\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{K}_k)^{-1},$$

where $C \geqslant D$, means that C and D are two symmetric matrices of the same size such that $C - D$ is positive semidefinite. Then we have

$$\sum_{i=1}^{s} e_1^{\mathrm{T}} (K_{i,k}^{\mathrm{T}} K_{i,k})^{-1} e_1 \geqslant s^2 e_1^{\mathrm{T}} (\mathscr{K}_k^{\mathrm{T}} \diamond \mathscr{K}_k)^{-1} e_1,$$

which implies

$$\sum_{i=1}^{s} \frac{1}{\|r_k^{(i)}\|_2^2} \geqslant \frac{s^2}{\|R_k^{MR}\|_F^2}. \qquad \square$$

5. Conclusion

We presented in this paper some convergence results of two block Krylov subspace methods without referring to any algorithm. We introduced a new matrix product and gave some of its properties. This new product helped us to derive new expressions of the approximations and the corresponding residual norms. Some relations between residual norms were also obtained.

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