Journal of Computational and Applied Mathematics 232 (2009) 187–200

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

The existence of countably many positive solutions for some nonlinear nth order m-point boundary value problems ☆

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ARTICLE INFO

Article history:
Received 13 October 2008

MSC:
34B15

Keywords:
Boundary value problem
Green's function
Krasnoselskii's fixed point theorem
Holder's inequality
Multiple positive solution

ABSTRACT

In this paper, we consider the existence of countably many positive solutions for nth-order m-point boundary value problems consisting of the equation

$$u^{(n)}(t) + a(t)f(u(t)) = 0, \quad t \in (0, 1),$$

with one of the following boundary value conditions:

$$u(0) = \sum_{i=1}^{m-2} k_i u(\xi_i), \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = 0,$$

and

$$u(0) = 0, \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} k_i u(\xi_i),$$

where $n \geq 2$, $k_i > 0$ $(i = 1, 2, \ldots, m-2)$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a(t) \in L^p[0, 1]$ for some $p \geq 1$ and has countably many singularities in $[0, \frac{1}{2})$. The associated Green's function for the nth order m-point boundary value problem is first given, and we show that there exist countably many positive solutions using Holder's inequality and Krasnoselskii's fixed point theorem for operators on a cone.

1. Introduction

The existence of positive solutions for nonlinear second order multi-point boundary value problems have been studied by several authors. We refer the reader to [1–6] and references therein. Recently, the existence of positive solutions for high order multi-point boundary value problems has been studied by some authors. For details, see, for example, [7–9]. However, the high order multi-point boundary value problems treated in the above-mentioned references do not discuss problems with singularities. For the singular case of high order multi-point boundary value problems, to the author's knowledge, no one has studied the existence of positive solutions in the case. Very recently, Kaufmann and Kosmatov [10] showed that there exist countably many positive solutions for the two-point boundary value problems, with infinitely many singularities of following form:

$$\begin{cases} -u''(t) = a(t)f(u(t)), & 0 < t < 1, \\ u(0) = 0, & u(1) = 0, \end{cases}$$

where $a(t) \in L^p[0, 1]$ for some $p \geq 1$ and has countably many singularities in $[0, \frac{1}{2})$.

☆ The project is supported by the Natural Science Foundation of Hebei Province (A2009000664) and the Foundation of Hebei University of Science and Technology (XL2006040).

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doi:10.1016/j.cam.2009.05.023

Motivated by the result of [10], in this paper we are interested in the existence of countably many positive solutions for nth-order m-point boundary value problems consisting of the equation

$$u^{(n)}(t) + a(t)f(u(t)) = 0, \quad t \in (0, 1), \tag{1.1}$$

with one of the following boundary value conditions:

$$u(0) = \sum_{i=1}^{m-2} k_i u(\xi_i), \qquad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = 0, \tag{1.2}$$

and

$$u(0) = 0, \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = \sum_{i=1}^{m-2} k_i u(\xi_i), \tag{1.3}$$

where $n \geq 2, k_i > 0\,(i = 1, 2, \ldots, m-2), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, f \in C([0, +\infty), [0, +\infty)), a(t) \in L^p[0, 1]$ for some $p \geq 1$ and has countably many singularities in $[0, \frac{1}{2})$. We show that the boundary value problems (1.1), (1.2) and (1.1), (1.3) have countably many solutions if a and f satisfy some suitable conditions. The key tool in our approach is the Holder's inequality and Krasnoselskii's fixed point theorem for operators on a cone.

We will suppose the following conditions are satisfied:

(H_1) there exists a sequence $\{t_k\}_{k=1}^{\infty}$ such that $t_{k+1} < t_k\ (k \in N), t_1 < \frac{1}{2}, \lim_{k\to\infty} t_k = t^* \geq 0$ and $\lim_{t\to t_k} a(t) = +\infty$ for all $k = 1, 2, \ldots$;

(H_2) there exists $H > 0$ such that $a(t) \geq H$ for all $t \in [t^*, 1 - t^*]$;

(H_3) there exists a $p \geq 1$ such that $a(t) \in L^p[0, 1]$;

(H_4) $f \in C([0, +\infty), [0, +\infty))$;

(H_5) $n \geq 2, k_i > 0\,(i = 1, 2, \ldots, m-2), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, 0 < \sum_{i=1}^{m-2} k_i(1 - \xi_i^{n-1}) < 1$;

(H_5') $n \geq 2, k_i > 0\,(i = 1, 2, \ldots, m-2), 0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, 0 < \sum_{i=1}^{m-2} k_i \xi_i^{n-1} < 1$.

We show that if $a(t)$ satisfies conditions (H_1)–(H_3) and if f satisfies oscillatory-like growth about a wedge, then the boundary value problem (1.1), (1.2) and (1.1), (1.3) have infinitely many solutions.

The paper is organized as follows. In Section 2, we provide some necessary background. In particular, we state a fixed point theorem due to Krasnoselskii's and Holder's inequality. In Section 3, the associated Green's function for the nth order two point boundary value problem is given and we also look at some properties of the Green's function associated with the boundary value problem. In Section 4, the associated Green's function for the nth order m-point boundary value problem is first given and we also look at some properties of the Green's function associated with the boundary value problem (1.1) and (1.2). We present the boundary value problems (1.1) and (1.2) have countably many solutions if a and f satisfy some suitable conditions. In Section 5, the associated Green's function for the nth order m-point boundary value problem is first given and we also look at some properties of the Green's function associated with the boundary value problem (1.1) and (1.3). We present the boundary value problems (1.1) and (1.3) have countably many solutions if a and f satisfy some suitable conditions. In Section 6, we present our main result as well as provide an example of a family of functions $a(t)$ that satisfy conditions (H_1)–(H_3) and two simple examples are presented to illustrate the applications of the obtained results.

2. Preliminary results

Definition 2.1. Let E be a Banach space over R. A nonempty convex closed set $K \subset E$ is said to be a cone, provided that

(i) $au \in K$ for all $u \in K$ and all $a \geq 0$;

(ii) $u, -u \in K$ implies $u = 0$.

Theorem 2.1 (*Krasnoselskii's Fixed Point Theorem*). *Let E be a Banach space and let $P \subset E$ be a cone. Assume Ω_1, Ω_2 are bounded open subsets of E such that $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that*

$$T : P \bigcap (\overline{\Omega}_2 \setminus \Omega_1) \to P$$

is a completely continuous operator such that either

(i) $\|Tu\| \leq \|u\|, u \in P \bigcap \partial\Omega_1$, *and* $\|Tu\| \geq \|u\|, u \in P \bigcap \partial\Omega_2$, *or*

(ii) $\|Tu\| \geq \|u\|, u \in P \bigcap \partial\Omega_1$, *and* $\|Tu\| \leq \|u\|, u \in P \bigcap \partial\Omega_2$.

Then T has a fixed point in $P \bigcap (\overline{\Omega}_2 \setminus \Omega_1)$.

In order to establish some of the norm inequalities in Theorem 2.1 we will need Holder's inequality. We use standard notation of $L^p[a, b]$ for the space of measurable functions such that

$$\int_0^1 |f(s)|^p\, ds < \infty,$$

where the integral is understood in the Lebesgue sense. The norm on $L^p[a,b]$, $\|\cdot\|$, is defined by

$$\|f\|_p = \left(\int_0^1 |f(s)|^p \, ds\right)^{\frac{1}{p}}.$$

Theorem 2.2 (*Holder's Inequality*). *Let $f \in L^p[a,b]$ and $g \in L^q[a,b]$, where $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then $fg \in L^1[a,b]$ and, moreover*

$$\int_0^1 |f(s)g(s)| \, ds \le \|f\|_p \|g\|_q.$$

Let $f \in L^1[a,b]$ and $g \in L^\infty[a,b]$. Then $fg \in L^1[a,b]$ and

$$\int_0^1 |f(s)g(s)| \, ds \le \|f\|_1 \|g\|_\infty.$$

3. Preliminary lemmas

We need the following lemmas. We also need some auxiliary results concerning the Green's function $g(t,s)$.

Lemma 3.1. *For $y(t) \in C[0,1]$, the boundary value problem*

$$\begin{cases} u^{(n)}(t) + y(t) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = 0 \end{cases} \tag{3.1}$$

has a unique solution

$$u(t) = -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s) \, ds + t^{n-1} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s) \, ds.$$

Proof. To this purpose, we let

$$u(t) = -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s) \, ds + At^{n-1} + \sum_{i=1}^{n-2} A_i t^i + B.$$

Since $u^{(i)}(0) = 0$ for $i = 0, 1, 2, \ldots, n-2$, we get $B = 0$ and $A_i = 0$ for $i = 1, 2, \ldots, n-2$. Now we solve for A by $u(1) = 0$, it follows that

$$-\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s) \, ds + A = 0,$$

we get

$$A = \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s) \, ds.$$

Therefore, (3.1) has a unique solution

$$u(t) = -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s) \, ds + t^{n-1} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s) \, ds.$$

The proof is complete. □

Lemma 3.2. *The Green's function for the boundary value problem*

$$\begin{cases} -u^{(n)}(t) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = 0 \end{cases} \tag{3.2}$$

is given by

$$g(t,s) = \frac{1}{(n-1)!} \begin{cases} t^{n-1}(1-s)^{n-1} - (t-s)^{n-1}, \quad 0 \le s \le t \le 1, \\ t^{n-1}(1-s)^{n-1}, \quad 0 \le t \le s \le 1. \end{cases} \tag{3.3}$$

Proof. The unique solution of (3.1) be expressed as

$$\begin{aligned} u(t) &= \frac{1}{(n-1)!}\left[\int_0^t t^{n-1}(1-s)^{n-1}y(s)\,\mathrm{d}s - \int_0^t (t-s)^{n-1}y(s)\,\mathrm{d}s + \int_t^1 t^{n-1}(1-s)^{n-1}y(s)\,\mathrm{d}s\right] \\ &= \frac{1}{(n-1)!}\left\{\int_0^t [t^{n-1}(1-s)^{n-1} - (t-s)^{n-1}]y(s)\,\mathrm{d}s + \int_t^1 t^{n-1}(1-s)^{n-1}y(s)\,\mathrm{d}s\right\}. \end{aligned}$$

The proof is complete. □

Lemma 3.3. *The Green's function $g(t,s)$ defined by (3.3) satisfies:*

(i) $g(t,s) \geq 0$ *is continuous on* $[0,1]\times[0,1]$;
(ii) $g(t,s) \leq g(\theta_1(s),s)$ *for all* $t,s \in [0,1]$ *and there exists a constant* $\widetilde{\gamma}_\tau > 0$ *for any* $\tau \in (0,\frac{1}{2})$ *such that*

$$\min_{t\in[\tau,1-\tau]} g(t,s) \geq \widetilde{\gamma}_\tau g(\theta_1(s),s) \geq \widetilde{\gamma}_\tau g(t',s), \quad \forall t',s\in[0,1], \tag{3.4}$$

where

$$\widetilde{\gamma}_\tau = \min\left\{\left(\frac{\tau}{\theta_1(s)}\right)^{n-1}, \frac{\tau}{1-\theta_1(s)}\right\},$$
$$\theta_1(s) = \frac{s}{1-(1-s)^{\frac{n-1}{n-2}}}, \quad (s<\theta_1(s)<1).$$

Proof. (i) It is obvious that $g(t,s)$ is continuous on $[0,1]\times[0,1]$.

For $0\leq s\leq t\leq 1$,

$$t^{n-1}(1-s)^{n-1} - (t-s)^{n-1} = (t-ts)^{n-1} - (t-s)^{n-1} \geq 0,$$

so, by (3.3), we have

$$g(t,s)\geq 0, \quad \forall t,s\in[0,1].$$

(ii) For fixed $0<s<1$, $g(t,s)$ obtains its maximum at $t=\theta_1(s)=\frac{s}{1-(1-s)^{\frac{n-1}{n-2}}}$, $(s<\theta_1(s)<1)$, that is

$$\max_{t\in[0,1]} g(t,s) = g(\theta_1(s),s), \quad \forall s\in[0,1].$$

Thus, we have

$$g(t,s)\leq g(\theta_1(s),s), \quad \forall t,s\in[0,1].$$

Next, we prove that (3.4) holds.

Let s be fixed, recall the properties that $g(t,s)\in C^{(n-2)}([0,1]\times[0,1])$ and that $\frac{\partial^{n-1}g(t,s)}{\partial t^{n-1}}$ is continuous on triangles $t<s$ and $s<t$. Recalling also that, as a function of t, $g(t,s)$ satisfies the boundary condition of (3.2), it follows by Rolle's theorem that there exist $0<t_{n-2}<t_{n-3}<\cdots<t_2<t_1<1$ such that $\frac{\partial^j g(t_j,s)}{\partial t^j}=0$, where $t_j=\theta_j(s)=\frac{s}{1-(1-s)^{\frac{n-1}{n-1-j}}}$, $(s<\theta_j(s)<1)$, $1\leq j\leq n-2$.

For $s<t_2<t_1$, $\frac{\partial^2 g(t,s)}{\partial t^2}<0$ on $[t_1,1]$.

Let

$$p(t) = \begin{cases} \dfrac{g(\theta_1(s),s)}{t_1^{n-1}}t^{n-1}, & 0\leq t\leq t_1, \\ \dfrac{g(\theta_1(s),s)}{t_1-1}(t-1), & t_1\leq t\leq 1. \end{cases}$$

For $t_1\leq t\leq 1$, by the negative concavity of $g(t,s)$ on $[t_1,1]$, we have

$$g(t,s)\geq p(t), \quad t\in[t_1,1].$$

For $0\leq t\leq t_1$, let $r(t)=g(t,s)-p(t)$, we consider two cases:

Case 1. If $0\leq t\leq s\leq t_1$,

$$\begin{aligned} r(t) &= \frac{1}{(n-1)!}\left[t^{n-1}(1-s)^{n-1} - \frac{t_1^{n-1}(1-s)^{n-1}-(t_1-s)^{n-1}}{t_1^{n-1}}t^{n-1}\right] \\ &\geq \frac{1}{(n-1)!}[t^{n-1}(1-s)^{n-1} - t^{n-1}(1-s)^{n-1}] = 0. \end{aligned}$$

Case 2. If $0 \le s \le t \le t_1$,

$$
\begin{aligned}
r(t) &= \frac{1}{(n-1)!}\left[t^{n-1}(1-s)^{n-1} - (t-s)^{n-1} - \frac{t_1^{n-1}(1-s)^{n-1} - (t_1-s)^{n-1}}{t_1^{n-1}} t^{n-1}\right] \\
&= \frac{1}{(n-1)!}\left[\frac{t^{n-1}}{t_1^{n-1}}(t_1-s)^{n-1} - (t-s)^{n-1}\right] \\
&= \frac{1}{(n-1)!}\left[\left(t - \frac{t}{t_1}s\right)^{n-1} - (t-s)^{n-1}\right] \ge 0.
\end{aligned}
$$

Thus, in all cases, we have

$$g(t,s) \ge p(t), \quad 0 \le t \le t_1.$$

So, there exists a constant $\widetilde{\gamma}_\tau > 0$ such that

$$g(t,s) \ge p(t) \ge \widetilde{\gamma}_\tau g(\theta_1(s), s), \quad t \in [\tau, 1-\tau], s \in [0,1].$$

Thus, the inequality (3.4) holds.

The proof is complete. □

4. Existence of positive solutions to (1.1) and (1.2)

In this section we present the boundary value problems (1.1) and (1.2) have countably many solutions if a and f satisfy some suitable conditions.

Lemma 4.1. *Suppose $\sum_{i=1}^{m-2} k_i(1-\xi_i^{n-1}) \ne 1$, then for $y(t) \in C[0,1]$, the boundary value problem*

$$
\begin{cases}
u^{(n)}(t) + y(t) = 0, \quad t \in (0,1), \\
u(0) = \sum\limits_{i=1}^{m-2} k_i u(\xi_i), \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = 0
\end{cases}
\tag{4.1}
$$

has a unique solution

$$
\begin{aligned}
u(t) = &-\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\,\mathrm{d}s + \frac{\sum\limits_{i=1}^{m-2} k_i t^{n-1} \int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s)\,\mathrm{d}s + \left(1 - \sum\limits_{i=1}^{m-2} k_i\right) t^{n-1} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)\,\mathrm{d}s}{1 - \sum\limits_{i=1}^{m-2} k_i(1-\xi_i^{n-1})} \\
&+ \frac{\sum\limits_{i=1}^{m-2} k_i \xi_i^{n-1} \int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)\,\mathrm{d}s - \sum\limits_{i=1}^{m-2} k_i \int_0^{\xi_i} \frac{(\xi_i-s)^{n-1}}{(n-1)!} y(s)\,\mathrm{d}s}{1 - \sum\limits_{i=1}^{m-2} k_i(1-\xi_i^{n-1})}.
\end{aligned}
$$

Proof. To this purpose, we let

$$u(t) = -\int_0^t \frac{(t-s)^{n-1}}{(n-1)!} y(s)\,\mathrm{d}s + At^{n-1} + \sum_{i=1}^{n-2} A_i t^i + B.$$

Since $u^{(i)}(0) = 0$ for $i = 1, 2, \ldots, n-2$, we get $A_i = 0$ for $i = 1, 2, \ldots, n-2$. Now we solve for A, B by $u(0) = \sum_{i=1}^{m-2} k_i u(\xi_i)$ and $u(1) = 0$, it follows that

$$
\begin{cases}
B = \sum\limits_{i=1}^{m-2} k_i \left[-\int_0^{\xi_i} \frac{(\xi_i - s)^{n-1}}{(n-1)!} y(s)\,\mathrm{d}s + A\xi_i^{n-1} + B\right], \\
-\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!} y(s)\,\mathrm{d}s + A + B = 0,
\end{cases}
$$

we get

$$\begin{cases} A+B=\displaystyle\int_0^1 \frac{(1-s)^{n-1}}{(n-1)!}y(s)\,\mathrm{d}s, \\ \displaystyle\sum_{i=1}^{m-2} k_i\xi_i^{n-1}A+\left(\sum_{i=1}^{m-2} k_i-1\right)B=\sum_{i=1}^{m-2} k_i\int_0^{\xi_i}\frac{(\xi_i-s)^{n-1}}{(n-1)!}y(s)\,\mathrm{d}s, \end{cases}$$

thus, we have

$$\begin{cases} A=\dfrac{\sum\limits_{i=1}^{m-2} k_i\int_0^{\xi_i}\frac{(\xi_i-s)^{n-1}}{(n-1)!}y(s)\,\mathrm{d}s+\left(1-\sum\limits_{i=1}^{m-2}k_i\right)\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)\,\mathrm{d}s}{1-\sum\limits_{i=1}^{m-2}k_i(1-\xi_i^{n-1})}, \\ B=\dfrac{\sum\limits_{i=1}^{m-2} k_i\xi_i^{n-1}\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)\,\mathrm{d}s-\sum\limits_{i=1}^{m-2}k_i\int_0^{\xi_i}\frac{(\xi_i-s)^{n-1}}{(n-1)!}y(s)\,\mathrm{d}s}{1-\sum\limits_{i=1}^{m-2}k_i(1-\xi_i^{n-1})}. \end{cases}$$

Therefore, (4.1) has a unique solution

$$\begin{aligned} u(t)&=-\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}y(s)\,\mathrm{d}s+\frac{\sum\limits_{i=1}^{m-2} k_it^{n-1}\int_0^{\xi_i}\frac{(\xi_i-s)^{n-1}}{(n-1)!}y(s)\,\mathrm{d}s+\left(1-\sum\limits_{i=1}^{m-2}k_i\right)t^{n-1}\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)\,\mathrm{d}s}{1-\sum\limits_{i=1}^{m-2}k_i(1-\xi_i^{n-1})} \\ &\quad+\frac{\sum\limits_{i=1}^{m-2} k_i\xi_i^{n-1}\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)\,\mathrm{d}s-\sum\limits_{i=1}^{m-2}k_i\int_0^{\xi_i}\frac{(\xi_i-s)^{n-1}}{(n-1)!}y(s)\,\mathrm{d}s}{1-\sum\limits_{i=1}^{m-2}k_i(1-\xi_i^{n-1})}. \end{aligned}$$

The proof is complete. □

Lemma 4.2. *Suppose* $0<\sum_{i=1}^{m-2}k_i(1-\xi_i^{n-1})<1$, *the Green's function for the boundary value problem*

$$\begin{cases} -u^{(n)}(t)=0, \quad t\in(0,1), \\ u(0)=\displaystyle\sum_{i=1}^{m-2}k_iu(\xi_i), \quad u'(0)=\cdots=u^{(n-2)}(0)=0, \quad u(1)=0 \end{cases} \tag{4.2}$$

is given by

$$G_1(t,s)=g(t,s)+\frac{1-t^{n-1}}{1-\sum\limits_{i=1}^{m-2}k_i(1-\xi_i^{n-1})}\sum_{i=1}^{m-2}k_ig(\xi_i,s), \tag{4.3}$$

where $g(t,s)$ *is defined by* (3.3).

Proof. The unique solution of (4.1) is expressed as

$$\begin{aligned} u(t)&=\frac{1}{(n-1)!}\Bigg\{\int_0^t[t^{n-1}(1-s)^{n-1}-(t-s)^{n-1}]y(s)\,\mathrm{d}s+\int_t^1 t^{n-1}(1-s)^{n-1}y(s)\,\mathrm{d}s \\ &\quad-\frac{t^{n-1}}{1-\sum\limits_{i=1}^{m-2}k_i(1-\xi_i^{n-1})}\sum_{i=1}^{m-2}k_i\left[\int_0^{\xi_i}(\xi_i^{n-1}(1-s)^{n-1}-(\xi_i-s)^{n-1})y(s)\,\mathrm{d}s+\int_{\xi_i}^1\xi_i^{n-1}(1-s)^{n-1}y(s)\,\mathrm{d}s\right] \\ &\quad+\frac{1}{1-\sum\limits_{i=1}^{m-2}k_i(1-\xi_i^{n-1})}\sum_{i=1}^{m-2}k_i\left[\int_0^{\xi_i}(\xi_i^{n-1}(1-s)^{n-1}-(\xi_i-s)^{n-1})y(s)\,\mathrm{d}s+\int_{\xi_i}^1\xi_i^{n-1}(1-s)^{n-1}y(s)\,\mathrm{d}s\right]\Bigg\} \end{aligned}$$

$$= \frac{1}{(n-1)!}\left\{\int_0^t [t^{n-1}(1-s)^{n-1} - (t-s)^{n-1}]y(s)\,ds + \int_t^1 t^{n-1}(1-s)^{n-1}y(s)\,ds \right.$$
$$\left. + \frac{1-t^{n-1}}{1-\sum_{i=1}^{m-2} k_i(1-\xi_i^{n-1})}\sum_{i=1}^{m-2} k_i\left[\int_0^{\xi_i}(\xi_i^{n-1}(1-s)^{n-1} - (\xi_i - s)^{n-1})y(s)\,ds + \int_{\xi_i}^1 \xi_i^{n-1}(1-s)^{n-1}y(s)\,ds\right]\right\}$$
$$= \int_0^1 G_1(t,s)y(s)\,ds.$$

The proof is complete. □

Lemma 4.3. *Suppose* $0 < \sum_{i=1}^{m-2} k_i(1-\xi_i^{n-1}) < 1$, *the Green's function* $G_1(t,s)$ *defined by* (4.3) *satisfies:*

(i) $G_1(t,s) \ge 0$ *is continuous on* $[0,1]\times[0,1]$;
(ii) $G_1(t,s) \le J_1(s)$ *for all* $t, s \in [0,1]$ *and there exists a constant* $\gamma_\tau > 0$ *for any* $\tau \in (0, \frac{1}{2})$ *such that*

$$\min_{t\in[\tau,1-\tau]} G_1(t,s) \ge \gamma_\tau J_1(s) \ge \gamma_\tau G_1(t',s), \quad \forall t', s \in [0,1], \tag{4.4}$$

where

$$J_1(s) = g(\theta_1(s),s) + \frac{1}{1-\sum_{i=1}^{m-2} k_i(1-\xi_i^{n-1})}\sum_{i=1}^{m-2} k_i g(\xi_i, s),$$
$$\gamma_\tau = \min\{\tau^{n-1}, 1-(1-\tau)^{n-1}\}$$
$$\le \min\left\{\left(\frac{\tau}{\theta_1(s)}\right)^{n-1}, \frac{\tau}{1-\theta_1(s)}, 1-(1-\tau)^{n-1}\right\} = \min\{\widetilde{\gamma}_\tau, 1-(1-\tau)^{n-1}\}.$$

Proof. (i) From the Lemma 3.3 and (4.3), we get $G_1(t,s) \ge 0$ is continuous on $[0,1]\times[0,1]$.

(ii) From the Lemma 3.3 and (4.3), we have

$$G_1(t,s) = g(t,s) + \frac{1-t^{n-1}}{1-\sum_{i=1}^{m-2} k_i(1-\xi_i^{n-1})}\sum_{i=1}^{m-2} k_i g(\xi_i, s),$$
$$\le g(\theta_1(s),s) + \frac{1}{1-\sum_{i=1}^{m-2} k_i(1-\xi_i^{n-1})}\sum_{i=1}^{m-2} k_i g(\xi_i, s) = J_1(s).$$

Next, we prove that (4.4) holds.

From the Lemma 3.3 and (4.3), for $t \in [\tau, 1-\tau]$, we have

$$G_1(t,s) = g(t,s) + \frac{1-t^{n-1}}{1-\sum_{i=1}^{m-2} k_i(1-\xi_i^{n-1})}\sum_{i=1}^{m-2} k_i g(\xi_i, s),$$
$$\ge \widetilde{\gamma}_\tau g(\theta_1(s),s) + \frac{1-(1-\tau)^{n-1}}{1-\sum_{i=1}^{m-2} k_i(1-\xi_i^{n-1})}\sum_{i=1}^{m-2} k_i g(\xi_i, s)$$
$$\ge \gamma_\tau\left[g(\theta_1(s),s) + \frac{1}{1-\sum_{i=1}^{m-2} k_i(1-\xi_i^{n-1})}\sum_{i=1}^{m-2} k_i g(\xi_i, s)\right]$$
$$= \gamma_\tau J_1(s)$$
$$\ge \gamma_\tau G_1(t',s),$$

for all $t' \in [0,1]$, where $\gamma_\tau = \min\{\tau^{n-1}, 1-(1-\tau)^{n-1}\}$, $\widetilde{\gamma}_\tau$ is defined by Lemma 3.3.

The proof is complete. □

We use the inequality (4.4) to define our cones. Let $E = C[0,1]$, then E is a Banach space with the norm $\|u\| = \max_{t\in[0,1]}|u(t)|$. For a fixed $\tau \in (0.\frac{1}{2})$, define the cone $P_\tau \subset E$ by

$$P_\tau = \{u \in E \mid u(t) \geq 0 \text{ on } [0,1], \text{ and } \min_{t\in[\tau,1-\tau]} u(t) \geq \gamma_\tau \|u\|\}.$$

Define the operator T_1 by

$$T_1 u(t) = \int_0^1 G_1(t,s)a(s)f(u(s))\,\mathrm{d}s, \quad 0 \leq t \leq 1. \tag{4.5}$$

Obviously, $u(t)$ is a solution of (1.1) and (1.2) if and only if $u(t)$ is a fixed point of operator T_1.

Theorem 2.1 requires that the operator T_1 be completely continuous and cone preserving. If T_1 is continuous and compact, then it is completely continuous. The next lemma shows that $T_1 : P_\tau \to P_\tau$ for $\tau \in (0.\frac{1}{2})$ and that T_1 is continuous and compact.

Lemma 4.4. *The operator T_1 is completely continuous and $T_1 : P_\tau \to P_\tau$ for each $\tau \in (0.\frac{1}{2})$.*

Proof. Fix $\tau \in (0.\frac{1}{2})$. Since $a(s)f(u(s)) \geq 0$ for all $s \in [0,1]$, $u \in P_\tau$ and since $G_1(t,s) \geq 0$ for all $t,s \in [0,1]$, then $T_1u(t) \geq 0$ for all $t \in [0,1]$, $u \in P_\tau$.

Let $u \in P_\tau$, by (4.4) and (4.5) we have

$$\begin{aligned}
\min_{t\in[\tau,1-\tau]} u(t) &= \min_{t\in[\tau,1-\tau]} \int_0^1 G_1(t,s)a(s)f(u(s))\,\mathrm{d}s \\
&\geq \int_0^1 \min_{t\in[\tau,1-\tau]} G_1(t,s)a(s)f(u(s))\,\mathrm{d}s \\
&\geq \gamma_\tau \int_0^1 G_1(t',s)a(s)f(u(s))\,\mathrm{d}s \\
&\geq \gamma_\tau T_1 u(t')
\end{aligned}$$

for all $t' \in [0,1]$. Thus

$$\min_{t\in[\tau,1-\tau]} u(t) \geq \gamma_\tau \|T_1 u\|.$$

Clearly the operator (4.5) is continuous. By the Arzela–Ascoli theorem T_1 is compact. Hence, the operator T_1 is completely continuous and the proof is complete. $\square$

For convenience, we denote

$$\Lambda_1 = \frac{1}{\max\limits_{t\in[\tau_1,1-\tau_1]} \int_{\tau_1}^{1-\tau_1} G_1(t,s)\,\mathrm{d}s \cdot H}, \qquad \Lambda_2 = \frac{1}{\|J_1\|_q \cdot \|a\|_p}.$$

Theorem 4.1. *Suppose condition* (H_1)–(H_5) *holds, let $\{\tau_k\}_{k=1}^\infty$ be such that $t_{k+1} < \tau_k < t_k$, $k = 1,2,\ldots$. Let $\{R_k\}_{k=1}^\infty$ and $\{r_k\}_{k=1}^\infty$ be such that*

$$R_{k+1} < \gamma_{\tau_k} r_k < r_k < R_k, \qquad M r_k < L R_k, \quad k = 1,2,\ldots,$$

where $M \in (\Lambda_1, +\infty)$, $L \in (0, \Lambda_2)$. Furthermore, for each natural number k, assume that f satisfies the following two growth conditions:

(H_6) *$f(u) \leq L R_k$ for all $u \in [0, R_k]$,*

(H_7) *$f(u) \geq M r_k$ for all $u \in [\gamma_{\tau_k} r_k, r_k]$.*

Then the boundary value problem (1.1) *and* (1.2) *has countably many positive solutions $\{u_k\}_{k=1}^\infty$ such that $r_k \leq \|u_k\| \leq R_k$ for each $k = 1,2,\ldots$.*

Proof. Consider the sequences $\{\Omega_{1,k}\}_{k=1}^\infty$ and $\{\Omega_{2,k}\}_{k=1}^\infty$ of open subsets of E defined by

$$\Omega_{1,k} = \{u \in E \mid \|u\| < R_k\},$$
$$\Omega_{2,k} = \{u \in E \mid \|u\| < r_k\}.$$

Let $\{\tau_k\}_{k=1}^\infty$ be as in the hypothesis and note that $t^* < t_{k+1} < \tau_k < t_k < \frac{1}{2}$, for all $k \in N$. For each $k \in N$, define the cone P_k by

$$P_k = \{u \in E \mid u(t) \geq 0 \text{ on } [0,1], \text{ and } \min_{t\in[\tau_k,1-\tau_k]} u(t) \geq \gamma_{\tau_k}\|u\|\}.$$

Fix k and let $u \in P_k \bigcap \partial\Omega_{2,k}$. For $s \in [\tau_k, 1-\tau_k]$, we have

$$\gamma_{\tau_k} r_k = \gamma_{\tau_k}\|u\| \leq \min_{s\in[\tau_k,1-\tau_k]} u(s) \leq u(s) \leq \|u\| = r_k.$$

By condition (H_7), we get

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,1]}\int_0^1 G_1(t,s)a(s)f(u(s))\,ds\\ &\geq \max_{t\in[0,1]}\int_{\tau_k}^{1-\tau_k} G_1(t,s)a(s)f(u(s))\,ds\\ &\geq \max_{t\in[0,1]}\int_{\tau_k}^{1-\tau_k} G_1(t,s)a(s)\,ds\cdot Mr_k\\ &\geq HMr_k\cdot\max_{t\in[\tau_1,1-\tau_1]}\int_{\tau_1}^{1-\tau_1} G_1(t,s)\,ds\\ &\geq r_k=\|u\|.\end{aligned}$$

Now let $u\in P_k\bigcap\partial\Omega_{1,k}$, then $u(s)\leq\|u\|=R_k$ for all $s\in[0,1]$. By condition (H_6), we get

$$\begin{aligned}\|Tu\| &= \max_{t\in[0,1]}\int_0^1 G_1(t,s)a(s)f(u(s))\,ds\\ &\leq \int_0^1 J_1(s)a(s)\,ds\cdot LR_k\\ &\leq \|J_1\|_q\|a\|_p\cdot LR_k\\ &\leq R_k=\|u\|.\end{aligned}$$

It is obvious that $0\in\Omega_{2,k}\subset\overline{\Omega}_{2,k}\subset\Omega_{1,k}$. Therefore, by Theorem 2.1, the operator T has at least one fixed point $u_k\in P_k\bigcap(\overline{\Omega}_{1,k}\setminus\Omega_{2,k})$ such that $r_k\leq\|u_k\|\leq R_k$. Since $k\in N$ was arbitrary, the proof is complete. □

5. Existence of positive solutions to (1.1) and (1.3)

In this section we deal with the boundary value problems (1.1) and (1.3). The method is just similar to what we have done in Section 4, so we omit the proof of main result of this section.

Lemma 5.1 (*Guo and Ji [9]*). *Suppose* $\sum_{i=1}^{m-2}k_i\xi_i^{n-1}\neq 1$, *then for* $y(t)\in C[0,1]$, *the boundary value problem*

$$\begin{cases}u^{(n)}(t)+y(t)=0, & t\in(0,1),\\ u(0)=0,\quad u'(0)=\cdots=u^{(n-2)}(0)=0,\quad u(1)=\displaystyle\sum_{i=1}^{m-2}k_iu(\xi_i)\end{cases}\tag{5.1}$$

has a unique solution

$$\begin{aligned}u(t) = &-\int_0^t\frac{(t-s)^{n-1}}{(n-1)!}y(s)\,ds+\frac{t^{n-1}}{1-\sum\limits_{i=1}^{m-2}k_i\xi_i^{n-1}}\int_0^1\frac{(1-s)^{n-1}}{(n-1)!}y(s)\,ds\\ &-\frac{t^{n-1}}{1-\sum\limits_{i=1}^{m-2}k_i\xi_i^{n-1}}\sum_{i=1}^{m-2}k_i\int_0^{\xi_i}\frac{(\xi_i-s)^{n-1}}{(n-1)!}y(s)\,ds.\end{aligned}$$

The proof is similar to that of Lemma 4.1 and thus is omitted.

Lemma 5.2. *Suppose* $0<\sum_{i=1}^{m-2}k_i\xi_i^{n-1}<1$, *the Green's function for the boundary value problem*

$$\begin{cases}-u^{(n)}(t)=0, & t\in(0,1),\\ u(0)=0,\quad u'(0)=\cdots=u^{(n-2)}(0)=0,\quad u(1)=\displaystyle\sum_{i=1}^{m-2}k_iu(\xi_i)\end{cases}\tag{5.2}$$

is given by

$$G_2(t,s)=g(t,s)+\frac{t^{n-1}}{1-\sum\limits_{i=1}^{m-2}k_i\xi_i^{n-1}}\sum_{i=1}^{m-2}k_ig(\xi_i,s),\tag{5.3}$$

where $g(t,s)$ *is defined by* (3.3).

The proof is similar to that of Lemma 4.2 and thus is omitted.

Lemma 5.3. *Suppose* $0 < \sum_{i=1}^{m-2} k_i\xi_i^{n-1} < 1$, *the Green's function* $G_2(t,s)$ *defined by* (5.3) *satisfies:*

(i) $G_2(t,s) \ge 0$ *is continuous on* $[0,1]\times[0,1]$*;*
(ii) $G_2(t,s) \le J_2(s)$ *for all* $t,s \in [0,1]$ *and there exists a constant* $\gamma'_\tau > 0$ *for any* $\tau \in (0,\frac{1}{2})$ *such that*

$$\min_{t\in[\tau,1-\tau]} G_2(t,s) \ge \gamma'_\tau J_2(s) \ge \gamma'_\tau G_2(t',s), \quad \forall t', s\in[0,1], \tag{5.4}$$

where

$$J_2(s) = g(\theta_1(s),s) + \frac{1}{1-\sum\limits_{i=1}^{m-2} k_i\xi_i^{n-1}} \sum_{i=1}^{m-2} k_i g(\xi_i, s),$$

$$\gamma'_\tau = \tau^{n-1} \le \min\left\{ \left(\frac{\tau}{\theta_1(s)}\right)^{n-1}, \frac{\tau}{1-\theta_1(s)}, \tau^{n-1} \right\} = \min\{\widetilde{\gamma}_\tau, \tau^{n-1}\}.$$

The proof is similar to that of Lemma 4.3 and thus is omitted.

We use the inequality (5.4) to define our cones. Let $E = C[0,1]$, then E is a Banach space with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$. For a fixed $\tau \in (0.\frac{1}{2})$, define the cone $P_\tau \subset E$ by

$$P_\tau = \{u \in E \mid u(t) \ge 0 \text{ on } [0,1], \text{ and } \min_{t\in[\tau,1-\tau]} u(t) \ge \gamma'_\tau \|u\|\}.$$

Define the operator T_2 by

$$T_2u(t) = \int_0^1 G_2(t,s)a(s)f(u(s))\,\mathrm{d}s, \quad 0 \le t \le 1. \tag{5.5}$$

Obviously, $u(t)$ is a solution of (1.1) and (1.3) if and only if $u(t)$ is a fixed point of operator T_2.

Theorem 2.1 requires that the operator T_2 be completely continuous and cone preserving. If T_2 is continuous and compact, then it is completely continuous. The next lemma shows that $T_2 : P_\tau \to P_\tau$ for $\tau \in (0.\frac{1}{2})$ and that T_2 is continuous and compact.

Lemma 5.4. *The operator* T_2 *is completely continuous and* $T_2 : P_\tau \to P_\tau$ *for each* $\tau \in (0.\frac{1}{2})$.

The proof is similar to that of Lemma 4.4 and thus is omitted.

For convenience, we denote

$$\Lambda'_1 = \frac{1}{\max\limits_{t\in[\tau_1,1-\tau_1]} \int_{\tau_1}^{1-\tau_1} G_2(t,s)\,\mathrm{d}s\cdot H}, \qquad \Lambda'_2 = \frac{1}{\|J_2\|_q\cdot\|a\|_p}.$$

Theorem 5.1. *Suppose condition* (H_1)–(H_4), (H'_5) *holds, let* $\{\tau_k\}_{k=1}^\infty$ *be such that* $t_{k+1} < \tau_k < t_k$, $k = 1,2,\ldots$. *Let* $\{R_k\}_{k=1}^\infty$ *and* $\{r_k\}_{k=1}^\infty$ *be such that*

$$R_{k+1} < \gamma'_{\tau_k} r_k < r_k < R_k, \quad Mr_k < LR_k, \quad k = 1,2,\ldots,$$

where $M \in (\Lambda'_1, +\infty)$, $L \in (0, \Lambda'_2)$. *Furthermore, for each natural number* k, *assume that* f *satisfies the following two growth conditions:*

(H'_6) $f(u) \le LR_k$ *for all* $u \in [0, R_k]$,
(H'_7) $f(u) \ge Mr_k$ *for all* $u \in [\gamma'_{\tau_k} r_k, r_k]$.

Then the boundary value problem (1.1) *and* (1.3) *has countably many positive solutions* $\{u_k\}_{k=1}^\infty$ *such that* $r_k \le \|u_k\| \le R_k$ *for each* $k = 1,2,\ldots$.

The proof is similar to that of Theorem 4.1 and thus is omitted.

6. Example

In this section, we present our main result as well as providing an example of a family of functions $a(t)$ that satisfy conditions (H_1)–(H_3), and two simple examples are presented to illustrate the applications of the obtained results.

Example 6.1. There exist many functions $a(t)$ that satisfy condition (H_1)–(H_3). For example, we consider the one parameter family of functions $a(t;\varepsilon) : [0,1] \to (0,+\infty]$ given by

$$a(t;\varepsilon) = \sum_{k=1}^\infty \frac{\chi[\omega_k,\omega_{k-1}]}{|t-t_k|^\varepsilon}, \tag{6.1}$$

where

$$t_0 = \frac{5}{16}, \quad t_k = t_0 - \sum_{i=0}^{k-1} \frac{1}{(i+2)^4}, \quad k = 1, 2, \ldots,$$
$$\omega_0 = 1, \quad \omega_k = \frac{1}{2}(t_k + t_{k+1}), \quad k = 1, 2, \ldots.$$

At first, it is easily seen that $a(t;\varepsilon) \geq a(1;\varepsilon) = (\frac{4}{3})^{\varepsilon}$, $t_1 = \frac{1}{4} < \frac{1}{2}$, $t_k - t_{k+1} = \frac{1}{(k+2)^4}$, $k = 1, 2, \ldots$, and (note that $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$)

$$t^* = \lim_{k\to\infty} t_k = \frac{5}{16} - \sum_{i=0}^{\infty} \frac{1}{(i+2)^4} = \frac{5}{16} - \left(\frac{\pi^4}{90} - 1\right) = \frac{21}{16} - \frac{\pi^4}{90} > \frac{1}{5}.$$

We claim that if $\varepsilon = \frac{1}{2}$, then $a(t;\varepsilon) \in L^1[0,1]$. Note that $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, we have

$$\begin{aligned}
\int_0^1 a(t;\varepsilon)\,\mathrm{d}t &= \int_0^1 \sum_{k=1}^{\infty} \frac{\chi[\omega_k, \omega_{k-1}]}{|t - t_k|^{\varepsilon}}\,\mathrm{d}t = \sum_{k=1}^{\infty} \int_0^1 \frac{\chi[\omega_k, \omega_{k-1}]}{|t - t_k|^{\varepsilon}}\,\mathrm{d}t \\
&= \sum_{k=1}^{\infty} \int_{\omega_k}^{\omega_{k-1}} \frac{1}{|t - t_k|^{\varepsilon}}\,\mathrm{d}t \\
&= \sum_{k=1}^{\infty} \left[\int_{\omega_k}^{t_k} \frac{1}{(t_k - t)^{\varepsilon}}\,\mathrm{d}t + \int_{t_k}^{\omega_{k-1}} \frac{1}{(t - t_k)^{\varepsilon}}\,\mathrm{d}t \right] \\
&= \sum_{k=1}^{\infty} \left[\int_{\frac{t_k + t_{k+1}}{2}}^{t_k} \frac{1}{(t_k - t)^{\varepsilon}}\,\mathrm{d}t + \int_{t_k}^{\frac{t_{k-1}+t_k}{2}} \frac{1}{(t - t_k)^{\varepsilon}}\,\mathrm{d}t \right] \\
&= \frac{1}{1-\varepsilon} \sum_{k=1}^{\infty} \left[\left(\frac{t_k - t_{k+1}}{2}\right)^{1-\varepsilon} + \left(\frac{t_{k-1} - t_k}{2}\right)^{1-\varepsilon} \right] \\
&= \frac{1}{2^{1-\varepsilon}(1-\varepsilon)} \sum_{k=1}^{\infty} \left[\frac{1}{(k+2)^{4(1-\varepsilon)}} + \frac{1}{(k+1)^{4(1-\varepsilon)}} \right] \\
&= \sqrt{2} \sum_{k=1}^{\infty} \left[\frac{1}{(k+2)^2} + \frac{1}{(k+1)^2} \right] \\
&= \sqrt{2} \left(\frac{\pi^2}{3} - \frac{9}{4} \right),
\end{aligned}$$

which implies $a(t;\varepsilon) \in L^1[0,1]$.

Next, we claim that if $\varepsilon = \frac{1}{4}$, then $a(t;\varepsilon) \in L^2[0,1]$. The argument is similar to the one above. In this case, we need the Cauchy product,

$$\sum_{k=1}^{\infty} a_k \cdot \sum_{k=1}^{\infty} b_k = \sum_{k=1}^{\infty} c_k, \tag{6.2}$$

where

$$c_k = \sum_{n=1}^{k} a_n b_{k-n+1}. \tag{6.3}$$

Note that

$$\int_0^1 a^2(t;\varepsilon)\,\mathrm{d}t = \int_0^1 \left[\sum_{k=1}^{\infty} \frac{\chi[\omega_k, \omega_{k-1}]}{|t - t_k|^{\varepsilon}} \right]^2 \mathrm{d}t, \tag{6.4}$$

we use (6.2) and (6.3) and the fact that, if $A \bigcap B = \emptyset$, then $\chi[A] \cdot \chi[B] = 0$ to simplify the integrand,

$$\left[\sum_{k=1}^{\infty} \frac{\chi[\omega_k, \omega_{k-1}]}{|t - t_k|^{\varepsilon}} \right]^2 = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{\chi[\omega_n, \omega_{n-1}]}{|t - t_n|^{\varepsilon}} \frac{\chi[\omega_{k-n+1}, \omega_{k-n}]}{|t - t_{k-n+1}|^{\varepsilon}} = \sum_{k=1}^{\infty} \frac{\chi[\omega_k, \omega_{k-1}]}{|t - t_k|^{2\varepsilon}} \quad \text{a.e.,}$$

and so (6.4) may be rewritten as

$$\begin{aligned}
\int_0^1 a^2(t;\varepsilon)\,\mathrm{d}t &= \int_0^1 \sum_{k=1}^{\infty} \frac{\chi[\omega_k,\omega_{k-1}]}{|t-t_k|^{2\varepsilon}}\,\mathrm{d}t = \sum_{k=1}^{\infty}\int_0^1 \frac{\chi[\omega_k,\omega_{k-1}]}{|t-t_k|^{2\varepsilon}}\,\mathrm{d}t \\
&= \sum_{k=1}^{\infty}\int_{\omega_k}^{\omega_{k-1}} \frac{1}{|t-t_k|^{2\varepsilon}}\,\mathrm{d}t \\
&= \sum_{k=1}^{\infty}\left[\int_{\omega_k}^{t_k} \frac{1}{(t_k-t)^{2\varepsilon}}\,\mathrm{d}t + \int_{t_k}^{\omega_{k-1}} \frac{1}{(t-t_k)^{2\varepsilon}}\,\mathrm{d}t\right] \\
&= \sum_{k=1}^{\infty}\left[\int_{\frac{t_k+t_{k+1}}{2}}^{t_k} \frac{1}{(t_k-t)^{2\varepsilon}}\,\mathrm{d}t + \int_{t_k}^{\frac{t_{k-1}+t_k}{2}} \frac{1}{(t-t_k)^{2\varepsilon}}\,\mathrm{d}t\right] \\
&= \frac{1}{1-2\varepsilon}\sum_{k=1}^{\infty}\left[\left(\frac{t_k-t_{k+1}}{2}\right)^{1-2\varepsilon} + \left(\frac{t_{k-1}-t_k}{2}\right)^{1-2\varepsilon}\right] \\
&= \frac{1}{2^{1-2\varepsilon}(1-2\varepsilon)}\sum_{k=1}^{\infty}\left[\frac{1}{(k+2)^{4(1-2\varepsilon)}} + \frac{1}{(k+1)^{4(1-2\varepsilon)}}\right] \\
&= \sqrt{2}\sum_{k=1}^{\infty}\left[\frac{1}{(k+2)^2} + \frac{1}{(k+1)^2}\right] \\
&= \sqrt{2}\left(\frac{\pi^2}{3} - \frac{9}{4}\right),
\end{aligned}$$

which implies $a(t;\varepsilon) \in L^2[0,1]$.

Example 6.2. As an example of the boundary value problems (1.1) and (1.2), we mention the boundary value problem

$$\begin{cases} u^{(3)}(t) + a(t)f(u) = 0, \quad t \in (0,1), \\ u(0) = u\left(\dfrac{1}{2}\right), \quad u'(0) = 0, \quad u(1) = 0, \end{cases} \tag{6.5}$$

where $a(t)$ is defined by (6.1) and $\varepsilon = \frac{1}{4}$,

$$f(u) = \begin{cases} \dfrac{24\times 10^{-(4k+2)} - 10^{-4(k+1)}}{\frac{1}{25}\times 10^{-(4k+2)} - 10^{-4(k+1)}}(u - 10^{-4(k+1)}) + 10^{-4(k+1)}, \quad u \in \left[10^{-4(k+1)}, \dfrac{1}{25}\times 10^{-(4k+2)}\right], \\ 24\times 10^{-(4k+2)}, \quad u \in \left[\dfrac{1}{25}\times 10^{-(4k+2)}, 10^{-(4k+2)}\right], \\ \dfrac{24\times 10^{-(4k+2)} - 10^{-4k}}{10^{-(4k+2)} - 10^{-4k}}(u - 10^{-4k}) + 10^{-4k}, \quad u \in [10^{-(4k+2)}, 10^{-4k}],\ (k = 1, 2, \ldots), \\ 10^{-4}, \quad u \in [10^{-4}, +\infty), \end{cases}$$

we notice that $n = 3$, $m = 3$, $k_1 = 1$, $\xi_1 = \frac{1}{2}$.

If we take $t_0 = \frac{5}{16}$, $t_k = t_0 - \sum_{i=0}^{k-1}\frac{1}{(i+2)^4}$, $\tau_k = \frac{1}{2}(t_k + t_{k+1})$, $k = 1, 2, \ldots$, then $\tau_1 = \frac{1}{4} - \frac{1}{2\times 3^4} < \frac{1}{4}$ and $t_{k+1} < \tau_k < t_k$, $\tau_k > \frac{1}{5}$, $\gamma_{\tau_k} = \min\{\tau_k^2, 1-(1-\tau_k)^2\} > \frac{1}{25}$, $k = 1, 2, \ldots$.

It follows from a direct calculation that

$$\begin{aligned}
\int_{\tau_1}^{1-\tau_1} G_1(t,s)\,\mathrm{d}s &> \int_{\frac{1}{4}}^{1-\frac{1}{4}} G_1(t,s)\,\mathrm{d}s \\
&= \int_{\frac{1}{4}}^{\frac{3}{4}} g(t,s)\,\mathrm{d}s + 4(1-t^2)\int_{\frac{1}{4}}^{\frac{3}{4}} g\left(\frac{1}{2},s\right)\mathrm{d}s \\
&= \frac{1}{2}\left\{\int_{\frac{1}{4}}^{t}[t^2(1-s)^2 - (t-s)^2]\,\mathrm{d}s + \int_t^{\frac{3}{4}} t^2(1-s)^2\,\mathrm{d}s\right. \\
&\quad \left. + 4(1-t^2)\left[\int_{\frac{1}{4}}^{\frac{1}{2}}\left(\frac{1}{4}(1-s)^2 - \left(\frac{1}{2}-s\right)^2\right)\mathrm{d}s + \int_{\frac{1}{2}}^{\frac{3}{4}}\frac{1}{4}(1-s)^2\,\mathrm{d}s\right]\right\}
\end{aligned}$$

$$= \frac{1}{384}(-64t^3 + 52t^2 - 12t + 23),$$

so

$$\max_{t\in[\tau_1,1-\tau_1]} \int_{\tau_1}^{1-\tau_1} G_1(t,s)\,\mathrm{d}s \geq \max_{t\in\left[\frac{1}{4},1-\frac{1}{4}\right]} \int_{\frac{1}{4}}^{1-\frac{1}{4}} G_1(t,s)\,\mathrm{d}s = \frac{359}{384\times 16} > \frac{1}{24},$$

and

$$\|J_1\|_2 = \left(\int_0^1 J_1^2(s)\,\mathrm{d}s\right)^{\frac{1}{2}} \leq \frac{\sqrt{218}}{24}, \qquad \|a\|_2 = \sqrt{\sqrt{2}\left(\frac{\pi^2}{3} - \frac{9}{4}\right)}.$$

In addition, if we take $r_k = 10^{-(4k+2)}$, $R_k = 10^{-4k}$, $M = 24$, $L = 1$, $H = (\frac{4}{3})^{\frac{1}{4}}$, then

$$a(t) \geq \left(\frac{4}{3}\right)^{\frac{1}{4}} = H, \quad t \in [t^*, 1-t^*],$$

$$R_{k+1} = 10^{-4(k+1)} < \frac{1}{25}\times 10^{-(4k+2)} < \gamma_{\tau_k} r_k < r_k = 10^{-(4k+2)} < R_k = 10^{-4k},$$

$$Mr_k = 24\times 10^{-(4k+2)} < LR_k = 1\times 10^{-4k}, \quad k = 1, 2, \ldots,$$

$$\Lambda_1 = \frac{1}{\max_{t\in[\tau_1,1-\tau_1]} \int_{\tau_1}^{1-\tau_1} G_1(t,s)\,\mathrm{d}s \cdot H} \leq \frac{1}{\frac{359}{384\times 16}\times\left(\frac{4}{3}\right)^{\frac{1}{4}}} < 24 = M,$$

$$\Lambda_2 = \frac{1}{\|J_1\|_2\cdot\|a\|_2} \geq \frac{1}{\frac{\sqrt{218}}{24}\times\sqrt{\sqrt{2}\left(\frac{\pi^2}{3}-\frac{9}{4}\right)}} > L = 1,$$

and $f(u)$ satisfies the following growth conditions:

$$f(u) \leq LR_k = 1\times 10^{-4k}, \quad u \in [0, 10^{-4k}],$$

$$f(u) \geq Mr_k = 24\times 10^{-(4k+2)}, \quad u \in \left[\frac{1}{25}\times 10^{-(4k+2)}, 10^{-(4k+2)}\right].$$

Then all the conditions of Theorem 4.1 are satisfied. Therefore, by Theorem 4.1 we know that boundary value problem (6.5) has countably many positive solutions $\{u_k\}_{k=1}^{\infty}$ such that $10^{-(4k+2)} \leq \|u_k\| \leq 10^{-4k}$ for each $k = 1, 2, \ldots$.

Example 6.3. As an example of the boundary value problems (1.1) and (1.3), we mention the boundary value problem

$$\begin{cases} u^{(3)}(t) + a(t)f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u'(0) = \cdots = u^{(n-2)}(0) = 0, \quad u(1) = u\left(\dfrac{1}{2}\right) \end{cases} \tag{6.6}$$

where $a(t)$ is defined by (6.1) and $\varepsilon = \frac{1}{4}$,

$$f(u) = \begin{cases} \dfrac{24\times 10^{-(4k+2)} - 10^{-4(k+1)}}{\frac{1}{25}\times 10^{-(4k+2)} - 10^{-4(k+1)}}(u - 10^{-4(k+1)}) + 10^{-4(k+1)}, \quad u \in \left[10^{-4(k+1)}, \dfrac{1}{25}\times 10^{-(4k+2)}\right], \\ 24\times 10^{-(4k+2)}, \quad u \in \left[\dfrac{1}{25}\times 10^{-(4k+2)}, 10^{-(4k+2)}\right], \\ \dfrac{24\times 10^{-(4k+2)} - 10^{-4k}}{10^{-(4k+2)} - 10^{-4k}}(u - 10^{-4k}) + 10^{-4k}, \quad u \in [10^{-(4k+2)}, 10^{-4k}], (k = 1, 2, \ldots), \\ 10^{-4}, \quad u \in [10^{-4}, +\infty), \end{cases}$$

we notice that $n = 3$, $m = 3$, $k_1 = 1$, $\xi_1 = \frac{1}{2}$.

If we take $t_0 = \frac{5}{16}$, $t_k = t_0 - \sum_{i=0}^{k-1}\frac{1}{(i+2)^4}$, $\tau_k = \frac{1}{2}(t_k + t_{k+1})$, $k = 1, 2, \ldots$, then $\tau_1 = \frac{1}{4} - \frac{1}{2\times 3^4} < \frac{1}{4}$ and $t_{k+1} < \tau_k < t_k$, $\tau_k > \frac{1}{5}$, $\gamma'_{\tau_k} = \tau_k^2 > \frac{1}{25}$, $k = 1, 2, \ldots$.

It follows from a direct calculation that

$$\int_{\tau_1}^{1-\tau_1} G_2(t,s)\,\mathrm{d}s > \int_{\frac{1}{4}}^{1-\frac{1}{4}} G_2(t,s)\,\mathrm{d}s$$

$$
\begin{aligned}
&= \int_{\frac{1}{4}}^{\frac{3}{4}} g(t,s)\,ds + \frac{4}{3}t^2 \int_{\frac{1}{4}}^{\frac{3}{4}} g\left(\frac{1}{2}, s\right)\,ds \\
&= \frac{1}{2}\left\{\int_{\frac{1}{4}}^{t} [t^2(1-s)^2 - (t-s)^2]\,ds + \int_{t}^{\frac{3}{4}} t^2(1-s)^2\,ds \right.\\
&\quad \left. + \frac{4}{3}t^2\left[\int_{\frac{1}{4}}^{\frac{1}{2}} \left(\frac{1}{4}(1-s)^2 - \left(\frac{1}{2}-s\right)^2\right)\,ds + \int_{\frac{1}{2}}^{\frac{3}{4}} \frac{1}{4}(1-s)^2\,ds\right]\right\} \\
&= \frac{1}{576}(-192t^3 + 244t^2 - 36t + 3),
\end{aligned}
$$

so

$$
\max_{t\in[\tau_1,1-\tau_1]} \int_{\tau_1}^{1-\tau_1} G_2(t,s)\,ds \geq \max_{t\in\left[\frac{1}{4},1-\frac{1}{4}\right]} \int_{\frac{1}{4}}^{1-\frac{1}{4}} G_2(t,s)\,ds = \frac{43}{64\times 12} > \frac{1}{24},
$$

and

$$
\|J_2\|_2 = \left(\int_0^1 J_2^2(s)\,ds\right)^{\frac{1}{2}} \leq \frac{\sqrt{29}}{18}, \quad \|a\|_2 = \sqrt{\sqrt{2}\left(\frac{\pi^2}{3} - \frac{9}{4}\right)}.
$$

In addition, if we take $r_k = 10^{-(4k+2)}$, $R_k = 10^{-4k}$, $M = 24$, $L = 1$, $H = (\frac{4}{3})^{\frac{1}{4}}$, then

$$
\begin{aligned}
&a(t) \geq \left(\frac{4}{3}\right)^{\frac{1}{4}} = H, \quad t\in[t^*, 1-t^*], \\
&R_{k+1} = 10^{-4(k+1)} < \frac{1}{25}\times 10^{-(4k+2)} < \gamma'_{\tau_k} r_k < r_k = 10^{-(4k+2)} < R_k = 10^{-4k}, \\
&Mr_k = 24\times 10^{-(4k+2)} < LR_k = 1\times 10^{-4k}, \quad k = 1, 2, \ldots, \\
&\Lambda'_1 = \frac{1}{\max\limits_{t\in[\tau_1,1-\tau_1]} \int_{\tau_1}^{1-\tau_1} G_2(t,s)\,ds\cdot H} \leq \frac{1}{\frac{43}{64\times 12}\times\left(\frac{4}{3}\right)^{\frac{1}{4}}} < 24 = M, \\
&\Lambda'_2 = \frac{1}{\|J_2\|_2\cdot\|a\|_2} \geq \frac{1}{\frac{\sqrt{29}}{18}\times\sqrt{\sqrt{2}\left(\frac{\pi^2}{3}-\frac{9}{4}\right)}} > L = 1,
\end{aligned}
$$

and $f(u)$ satisfies the following growth conditions:

$$
\begin{aligned}
&f(u) \leq LR_k = 1\times 10^{-4k}, \quad u\in[0, 10^{-4k}], \\
&f(u) \geq Mr_k = 24\times 10^{-(4k+2)}, \quad u\in\left[\frac{1}{25}\times 10^{-(4k+2)}, 10^{-(4k+2)}\right].
\end{aligned}
$$

Then all the conditions of Theorem 5.1 are satisfied. Therefore, by Theorem 5.1 we know that boundary value problem (6.6) has countably many positive solutions $\{u_k\}_{k=1}^{\infty}$ such that $10^{-(4k+2)} \leq \|u_k\| \leq 10^{-4k}$ for each $k = 1, 2, \ldots$.

Remark. In the above two examples, it is clear that the results [7–10] do not apply to the two examples. Hence, we generalize high order multi-point boundary value problems.

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