



## The asymptotic behaviour of recurrence coefficients for orthogonal polynomials with varying exponential weights

A.B.J. Kuijlaars\*, P.M.J. Tibboel

Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200B, 3001 Leuven, Belgium

### ARTICLE INFO

#### Article history:

Received 24 September 2007

#### Keywords:

Riemann–Hilbert problems  
Recurrence coefficients  
Orthogonal polynomials  
Steepest descent analysis

### ABSTRACT

We consider orthogonal polynomials  $\{p_{n,N}(x)\}_{n=0}^{\infty}$  on the real line with respect to a weight  $w(x) = e^{-NV(x)}$  and in particular the asymptotic behaviour of the coefficients  $a_{n,N}$  and  $b_{n,N}$  in the three-term recurrence  $x\pi_{n,N}(x) = \pi_{n+1,N}(x) + b_{n,N}\pi_{n,N}(x) + a_{n,N}\pi_{n-1,N}(x)$ . For one-cut regular  $V$  we show, using the Deift–Zhou method of steepest descent for Riemann–Hilbert problems, that the diagonal recurrence coefficients  $a_{n,n}$  and  $b_{n,n}$  have asymptotic expansions as  $n \rightarrow \infty$  in powers of  $1/n^2$  and powers of  $1/n$ , respectively.

© 2009 Elsevier B.V. All rights reserved.

### 1. Introduction

We consider the asymptotic behavior of the recurrence coefficients  $a_{n,N}$  and  $b_{n,N}$  in the three-term recurrence relation

$$x\pi_{n,N}(x) = \pi_{n+1,N}(x) + b_{n,N}\pi_{n,N}(x) + a_{n,N}\pi_{n-1,N}(x)$$

for orthogonal polynomials with respect to varying exponential weights. Here  $\pi_{n,N}$  is the  $n$ th degree monic orthogonal polynomial with respect to a varying weight

$$w_N(x) = e^{-NV(x)}$$

where  $V$  is real analytic on  $\mathbb{R}$  with  $\lim_{x \rightarrow \pm\infty} \frac{V(x)}{\log(1+x^2)} = +\infty$ . Moreover,  $V$  is assumed to be one-cut regular, which means that the equilibrium measure  $d\mu_V = \psi_V(x)dx$  associated with  $V$  is supported on one interval  $[a, b]$  where it has the form

$$\psi_V(x)dx = \sqrt{(b-x)(x-a)}h(x)\chi_{[a,b]}(x)dx \quad (1.1)$$

where  $h$  is real analytic, strictly positive on  $[a, b]$ , and in addition the inequality (3.1) is strict for  $x \in \mathbb{R} \setminus [a, b]$ . See e.g. [1,2,5,11,16] for the definition of the equilibrium measure and for more information on the one-cut regular case.

Under these assumptions Deift et al. [7] proved that  $a_{n,n}$  and  $b_{n,n}$  have asymptotic expansions in powers of  $1/n$ . Their approach is based on the Deift–Zhou method of steepest descent applied to the Riemann–Hilbert problem for orthogonal polynomials of Fokas, Its, and Kitaev [12]. This method was first introduced in [9] and further developed in [6–8] and many papers since then.

The asymptotic result on the recurrence coefficients was considerably refined by Bleher and Its [2, Theorem 5.2] who showed for polynomial  $V$  that there exists  $\varepsilon > 0$  and real analytic functions  $f_{2k}(s)$ ,  $g_{2k}(s)$ ,  $k = 0, 1, \dots$ , on  $[1 - \varepsilon, 1 + \varepsilon]$  such that the asymptotic expansions

$$a_{n,N} \sim f_0\left(\frac{n}{N}\right) + \sum_{m=1}^{\infty} N^{-2m} f_{2m}\left(\frac{n}{N}\right) \quad (1.2)$$

\* Corresponding author.

E-mail addresses: [arno.kuijlaars@wis.kuleuven.be](mailto:arno.kuijlaars@wis.kuleuven.be) (A.B.J. Kuijlaars), [pieter.tibboel@wis.kuleuven.be](mailto:pieter.tibboel@wis.kuleuven.be) (P.M.J. Tibboel).

$$b_{n,N} \sim g_0 \left( \frac{n+1/2}{N} \right) + \sum_{m=1}^{\infty} N^{-2m} g_{2m} \left( \frac{n+1/2}{N} \right) \quad (1.3)$$

hold uniformly as  $n, N \rightarrow \infty$  with  $1 - \varepsilon \leq n/N \leq 1 + \varepsilon$ . These  $1/N^2$  expansions are used in [2] to prove the  $1/N^2$  expansion of the free energy (a.k.a. logarithm of the partition function or Hankel determinant) of the associated random matrix ensemble in the one-cut regular case, see also [11].

The proof of (1.2) and (1.3) in [2] is based on the Deift et al. result referred to above, in combination with so-called string equations. It is of some interest to find a proof that is based on the Riemann–Hilbert steepest descent analysis only. Here we do this for the diagonal case  $n = N$ , and we obtain the following.

**Theorem 1.1.** *Let  $V$  be real analytic and one-cut regular. Then there exist constants  $\alpha_{2m}$  and  $\beta_m$ ,  $m = 1, 2, \dots$  (depending on  $V$ ) such that  $a_{n,n}$  and  $b_{n,n}$  have the following asymptotic expansions as  $n \rightarrow \infty$ :*

$$a_{n,n} \sim \frac{(b-a)^2}{16} + \sum_{m=1}^{\infty} \frac{\alpha_{2m}}{n^{2m}}, \quad b_{n,n} \sim \frac{b+a}{2} + \sum_{m=1}^{\infty} \frac{\beta_m}{n^m}. \quad (1.4)$$

The first coefficient  $\beta_1$  in the expansion for  $b_{n,n}$  is given explicitly by

$$\beta_1 = \frac{1}{2\pi(b-a)} \left( \frac{1}{h(b)} - \frac{1}{h(a)} \right) \quad (1.5)$$

where  $h$  is the function appearing in the expression (1.1) for the equilibrium measure associated with  $V$ .

In our proof of Theorem 1.1 we follow the main lines of the steepest descent analysis of [7]. We will deduce that the odd powers in the expansion of  $a_{n,n}$  vanish from the structure of the local Airy parametrices around the endpoints. The expression (1.5) for  $\beta_1$  is new, although it is likely that it can be deduced from the approach of [2] as well. The explicit formula (1.5) shows that  $\beta_1 = 0$  if and only if  $h(a) = h(b)$ . It is very easy to construct examples of one-cut regular  $V$  such that  $h(a) \neq h(b)$  and so  $\beta_1 \neq 0$ . We have thus corrected an error in a paper of Albeverio, Pastur, and Shcherbina [1, Theorem 1, formula (1.34)] which claims that  $\beta_1 = 0$  always in the one-cut regular case.

**Example 1.2.** We may explicitly check Theorem 1.1 using Jacobi polynomials with varying parameters  $\alpha = AN$ ,  $\beta = BN$ ,  $A, B > 0$ . These polynomials are orthogonal with weight  $(1-x)^{AN}(1+x)^{BN}$  on  $[-1, 1]$ . The equilibrium measure takes the form (1.1) with

$$a, b = \frac{B^2 - A^2 \pm 4\sqrt{(1+A+B)(1+A)(1+B)}}{(2+A+B)^2} \quad (1.6)$$

and

$$h(x) = \frac{2+A+B}{2\pi(1-x^2)}, \quad (1.7)$$

see [17,15]. We are in the one-cut regular case, but for weights restricted to  $[-1, 1]$ . An analysis of the proof of Theorem 1.1, however, will show that the results (1.4) and (1.5) remain valid in this case as well.

From the explicit form of the recurrence coefficients for Jacobi polynomials, see e.g. [4,15],

$$a_{n,n} = \frac{4(1+A+B)(1+A)(1+B)}{\left( (2+A+B)^2 - \frac{1}{n^2} \right) (2+A+B)^2}$$

$$b_{n,n} = \frac{B^2 - A^2}{(2+A+B) \left( 2+A+B + \frac{2}{n} \right)},$$

it is easy to see that (1.4) holds. Using (1.6) and (1.7) we can also ascertain the validity of (1.5).

## 2. The Riemann–Hilbert problem

The Riemann–Hilbert problem for orthogonal polynomials was introduced by Fokas, Its, and Kitaev [12]. It asks for a  $2 \times 2$  matrix valued function  $Y(z)$  satisfying

$$\begin{cases} Y(z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R} \\ Y_+(x) = Y_-(x) \begin{pmatrix} 1 & e^{-NV(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x \in \mathbb{R} \\ Y(z) = \left( I + \mathcal{O}\left(\frac{1}{z}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} & \text{as } z \rightarrow \infty. \end{cases} \quad (2.1)$$

The unique solution of (2.1) is (see e.g. [5])

$$Y(z) = \begin{pmatrix} \kappa_{n,N}^{-1} p_{n,N}(z) & \frac{1}{2\pi i \kappa_{n,N}} \int_{\mathbb{R}} \frac{p_{n,N}(t) e^{-NV(t)}}{t-z} dt \\ -2\pi i \kappa_{n-1,N} p_{n-1,N}(z) & -\kappa_{n-1,N} \int_{\mathbb{R}} \frac{p_{n-1,N}(t) e^{-NV(t)}}{t-z} dt \end{pmatrix} \quad (2.2)$$

where  $p_{n,N}(x) = \kappa_{n,N} \pi_{n,N}(x)$  is the  $n$ th degree orthonormal polynomial. The recurrence coefficients are expressed as follows in terms of the solution of the Riemann–Hilbert problem (2.1), see [5,10].

**Proposition 2.1.** *Let*

$$Y(z) = \left( I + \frac{1}{z} Y_1 + \frac{1}{z^2} Y_2 + \mathcal{O}\left(\frac{1}{z^3}\right) \right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}. \quad (2.3)$$

Then

$$a_{n,N} = (Y_1)_{12} (Y_1)_{21} \quad (2.4)$$

and

$$b_{n,N} = \frac{(Y_2)_{12}}{(Y_1)_{12}} - (Y_1)_{22}. \quad (2.5)$$

For the remainder of this paper we will take  $N = n$ . We closely follow [5,7] in applying the Deift–Zhou method of steepest descent for Riemann–Hilbert problems to (2.1).

### 3. The Deift–Zhou method of steepest descent

The goal of the Deift–Zhou method of steepest descent for Riemann–Hilbert problems is to change the original problem into a problem for which the asymptotics for  $z \rightarrow \infty$  are normalised and for which all matrices, jump matrices and solutions alike, are asymptotically close to the identity matrix for large  $n$  which can be solved iteratively. The specific details and steps needed to achieve this goal shall be explained below.

#### 3.1. The first step: Transformation $Y \mapsto T$

The key aspect of the first step of the analysis is the equilibrium measure  $\mu_V$  corresponding to  $V$ . This equilibrium measure  $\mu_V$  is the unique probability measure that satisfies for some  $l$ ,

$$2 \int \log |x-y|^{-1} d\mu_V(y) + V(x) \geq l, \quad \text{for all } x \in \mathbb{R}, \quad (3.1)$$

$$2 \int \log |x-y|^{-1} d\mu_V(y) + V(x) = l, \quad \text{for all } x \in \text{supp } \mu_V. \quad (3.2)$$

For the one-cut regular case that we are considering we have that the support is one interval  $[a, b]$  and  $d\mu_V(x) = \psi_V(x)dx$  as in (1.1). In addition the inequality (3.1) is strict for  $x \in \mathbb{R} \setminus [a, b]$ .

Define

$$g(z) = \int \log(z-s) d\mu_V(s) = \int \log(z-s) \psi_V(s) ds \quad (3.3)$$

and

$$\phi(z) = \pi \int_b^z ((s-b)(s-a))^{\frac{1}{2}} h(s) ds, \quad z \in \mathbb{C} \setminus (-\infty, b] \quad (3.4)$$

$$\tilde{\phi}(z) = \pi \int_a^z ((s-b)(s-a))^{\frac{1}{2}} h(s) ds, \quad z \in \mathbb{C} \setminus [a, +\infty). \quad (3.5)$$

If we now put

$$T(z) = e^{-n(l/2)\sigma_3} Y(z) e^{-ng(z)\sigma_3} e^{n(l/2)\sigma_3}, \quad (3.6)$$

where  $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  is the third Pauli matrix, then  $T$  satisfies the Riemann–Hilbert problem

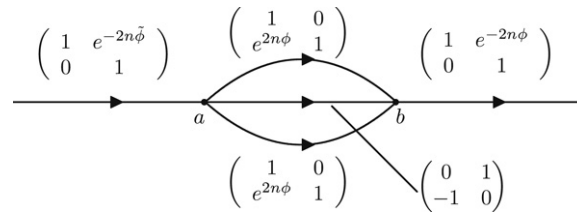


Fig. 1. Jump matrices for  $S$  after opening of the lens.

$$\begin{cases} T(z) \text{ is analytic in } \mathbb{C} \setminus \mathbb{R}, \\ T_+(x) = T_-(x)J_T(x) \text{ for } x \in \mathbb{R}, \\ T(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ as } z \rightarrow \infty, \end{cases} \quad (3.7)$$

where

$$J_T(x) = \begin{cases} \begin{pmatrix} 1 & e^{-2n\tilde{\phi}(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x < a, \\ \begin{pmatrix} e^{2n\phi_+(x)} & 1 \\ 0 & e^{2n\phi_-(x)} \end{pmatrix} & \text{for } x \in (a, b), \\ \begin{pmatrix} 1 & e^{-2n\phi(x)} \\ 0 & 1 \end{pmatrix} & \text{for } x > b. \end{cases} \quad (3.8)$$

Since the inequality in (3.1) is strict for  $x < a$  and  $x > b$  we have that  $\tilde{\phi}(x) > 0$  for  $x < a$  and  $\phi(x) > 0$  for  $x > b$ . Thus the jump matrices for  $T$  on  $(-\infty, a)$  and  $(b, \infty)$  tend to the identity matrix as  $n \rightarrow \infty$ .

### 3.2. The second step: Transformation $T \mapsto S$

The second transformation is the so-called *opening of the lens* and it is based on the factorisation

$$\begin{pmatrix} e^{2n\phi_+(x)} & 1 \\ 0 & e^{2n\phi_-(x)} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ e^{2n\phi_-(x)} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ e^{2n\phi_+(x)} & 1 \end{pmatrix} \quad (3.9)$$

of the jump matrix  $J_T$  on  $(a, b)$ . The factorisation (3.9) allows us to split the jump on  $(a, b)$  as shown in Fig. 1.

We use  $\Sigma_1$  and  $\Sigma_2$  to denote the upper and lower lips of the lens, respectively. We define  $S$  as follows:

- For  $z$  outside the lens, we put  $S = T$ .
- For  $z$  within the region enclosed by  $\Sigma_1$  and  $(a, b)$ ,

$$S = T \begin{pmatrix} 1 & 0 \\ -e^{2n\phi} & 1 \end{pmatrix}. \quad (3.10)$$

- For  $z$  within the region enclosed by  $\Sigma_2$  and  $(a, b)$ ,

$$S = T \begin{pmatrix} 1 & 0 \\ e^{2n\phi} & 1 \end{pmatrix}. \quad (3.11)$$

Then  $S$  satisfies the following Riemann–Hilbert problem:

$$\begin{cases} S(z) \text{ is analytic in } \mathbb{C} \setminus (\mathbb{R} \cup \Sigma_1 \cup \Sigma_2) \\ S_+(z) = S_-(z)J_S(z) \text{ for } z \in \mathbb{R} \cup \Sigma_1 \cup \Sigma_2 \\ S(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \rightarrow \infty \end{cases} \quad (3.12)$$

where

$$J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix} & \text{for } z \in \Sigma_1 \cup \Sigma_2, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (a, b), \\ \begin{pmatrix} 1 & e^{-2n\tilde{\phi}(z)} \\ 0 & 1 \end{pmatrix} & \text{for } z < a, \\ \begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} & \text{for } z > b. \end{cases} \quad (3.13)$$

We may (and do) assume that the lips of the lens are in the region where  $\operatorname{Re} \phi < 0$ , so that the jump matrices for  $S$  on  $\Sigma_1$  and  $\Sigma_2$  tend to the identity matrix as  $n \rightarrow \infty$ .

### 3.3. The third step: Parametrix away from endpoints

The parametrix away from the branch points is a ‘global solution’  $N(z)$  satisfying the Riemann–Hilbert problem

$$\begin{cases} N(z) \text{ is analytic in } \mathbb{C} \setminus [a, b] \\ N_+(x) = N_-(x) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ for } x \in (a, b) \\ N(z) = I + \mathcal{O}\left(\frac{1}{z}\right) \text{ for } z \rightarrow \infty \end{cases} \quad (3.14)$$

which has a solution (see [5])

$$N(z) = \begin{pmatrix} \frac{\beta(z) + \beta^{-1}(z)}{2} & \frac{\beta(z) - \beta^{-1}(z)}{2} \\ -\frac{\beta(z) - \beta^{-1}(z)}{2i} & \frac{\beta(z) + \beta^{-1}(z)}{2} \end{pmatrix} \quad (3.15)$$

where  $\beta(z) = \left(\frac{z-b}{z-a}\right)^{\frac{1}{4}}$ .

### 3.4. The fourth step: Parametrices near endpoints

Having constructed the ‘global solution’, the next step is finding ‘local solutions’ close to the endpoints  $a$  and  $b$ . Near  $b$ , the local situation is described as in the left picture of Fig. 2 with jump matrix

$$J_P(z) = J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 \\ e^{2n\phi(z)} & 1 \end{pmatrix} & \text{on } \Sigma_1 \cap U \text{ and } \Sigma_2 \cap U \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{on } (a, b) \cap U \\ \begin{pmatrix} 1 & e^{-2n\phi(z)} \\ 0 & 1 \end{pmatrix} & \text{on } (b, \infty) \cap U \end{cases}$$

where  $U$  is a (small) disc around  $b$ .

We therefore want to find a matrix function  $P$ , that solves

$$\begin{cases} P(z) \text{ is analytic on } U \setminus (\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \\ P_+(z) = P_-(z)J_P(z) \text{ on } (\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \cap U \\ P(z) = N(z) \left( I + \mathcal{O}\left(\frac{1}{n}\right) \right) \text{ as } n \rightarrow \infty \text{ uniformly for } z \in \partial U. \end{cases}$$

Then  $P(z)e^{n\phi(z)\sigma_3}$  should have constant jumps on  $(\Sigma_1 \cup \Sigma_2 \cup \mathbb{R}) \cap U$ , namely

$$(P(z)e^{n\phi(z)\sigma_3})_+ = (P(z)e^{n\phi(z)\sigma_3})_- \times \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & \text{for } z \in (\Sigma_1 \cup \Sigma_2) \cap U \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \text{for } z \in (a, b) \cap U \\ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & \text{for } z \in (b, \infty) \cap U. \end{cases}$$

Shrinking  $U$  if necessary, we have that

$$\zeta = f(z) = \left(\frac{3}{2}\phi(z)\right)^{2/3}$$

defines a conformal map from  $U$  to a convex neighbourhood of  $\zeta = 0$ . We may and do assume that the lips of the lens are taken so that  $\Sigma_1 \cap U$  is mapped into  $\arg \zeta = 2\pi/3$ , and  $\Sigma_2 \cap U$  is mapped into  $\arg \zeta = 2\pi/3$ , see Fig. 2. Denoting the

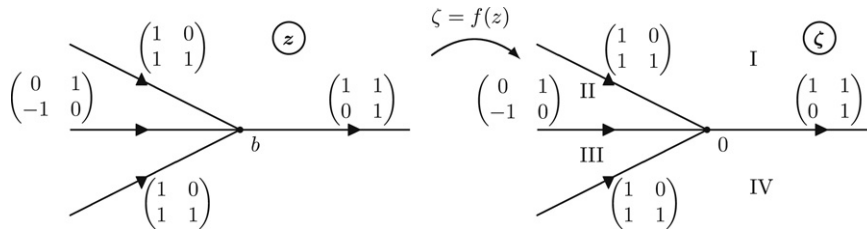


Fig. 2. Mapping of neighbourhood of  $b$  onto a neighbourhood of  $f(b) = 0$ .

sectors in the  $\zeta$ -plane by I, II, III, IV as in Fig. 2, and using the usual Airy function  $\text{Ai}(\zeta)$ , we construct the Airy model solution  $\Phi$  by

$$\Phi(\zeta) = \begin{cases} \begin{pmatrix} \text{Ai}(\zeta) & \omega \text{Ai}(\omega\zeta) \\ \text{Ai}'(\zeta) & \omega^2 \text{Ai}'(\omega\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector IV} \\ \begin{pmatrix} \text{Ai}(\zeta) & -\omega^2 \text{Ai}(\omega^2\zeta) \\ \text{Ai}'(\zeta) & -\omega \text{Ai}'(\omega^2\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector I} \\ \begin{pmatrix} -\omega \text{Ai}(\omega\zeta) & -\omega^2 \text{Ai}(\omega^2\zeta) \\ -\omega^2 \text{Ai}'(\omega\zeta) & -\omega \text{Ai}'(\omega^2\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector II} \\ \begin{pmatrix} -\omega^2 \text{Ai}(\omega^2\zeta) & \omega \text{Ai}(\omega\zeta) \\ -\omega \text{Ai}'(\omega^2\zeta) & \omega^2 \text{Ai}'(\omega\zeta) \end{pmatrix} & \text{for } \zeta \text{ in sector III} \end{cases}$$

where  $\omega = e^{2\pi i/3}$ . Then  $\Phi$  has the jump matrices in the  $\zeta$ -plane indicated on the right-hand side of Fig. 2.

Then for any analytic prefactor  $E_n(z)$  we have that

$$P(z) = E_n(z) \Phi(n^{2/3} f(z)) e^{n\phi(z)\sigma_3} \quad (3.16)$$

has the required jump matrices  $J_P$ . If we choose

$$E_n = \sqrt{\pi} N(z) \begin{pmatrix} 1 & -1 \\ -i & -i \end{pmatrix} (n^{2/3} f(z))^{\sigma_3/4} \quad (3.17)$$

then the matching condition  $P(z) = N(z)(I + \mathcal{O}(1/n))$  as  $n \rightarrow \infty$  for  $z \in \partial U$ , is satisfied as well, see e.g. [3,5,7] for further details.

A similar construction yields a parametrix  $\tilde{P}$  in a small disc  $\tilde{U}$  around  $a$ . One can see that  $\tilde{P}$  can be obtained by taking  $P$  and interchanging  $a$  and  $b$  and conjugating with  $\sigma_3$ .

### 3.5. The fifth step: Transformation $S \mapsto R$

Using the parametrices  $N$ ,  $P$ , and  $\tilde{P}$ , we define the third transformation  $S \mapsto R$  as follows

$$R(z) = \begin{cases} S(z)N(z)^{-1} & \text{for } z \in \mathbb{C} \setminus \overline{(U \cup \tilde{U})} \\ S(z)P(z)^{-1} & \text{for } z \in U \\ S(z)\tilde{P}(z)^{-1} & \text{for } z \in \tilde{U}. \end{cases} \quad (3.18)$$

Then  $R$  has no jump on  $[a, b] \setminus \overline{(U \cup \tilde{U})}$ , as the jumps of  $S$  and  $N^{-1}$  cancel out. In  $U$  and  $\tilde{U}$  the jumps of  $S$  cancel out with the jumps of  $P$  and  $\tilde{P}$ , leaving only jumps for  $R$  on the contour  $\Sigma_R$  shown in Fig. 3.

The Riemann–Hilbert problem for  $R$  is

$$\begin{cases} R(z) \text{ is analytic on } \mathbb{C} \setminus \Sigma_R \\ R_+(z) = R_-(z)J_R(z) & \text{for } z \in \Sigma_R \\ R(z) = I + \mathcal{O}\left(\frac{1}{z}\right) & \text{for } z \rightarrow \infty \end{cases}$$

where

$$J_R(z) = \begin{cases} N(z)J_S(z)N(z)^{-1} & \text{for } z \in \Sigma_R \setminus (\partial U \cup \partial \tilde{U}) \\ P(z)N(z)^{-1} & \text{for } z \in \partial U \\ \tilde{P}(z)N(z)^{-1} & \text{for } z \in \partial \tilde{U}. \end{cases}$$

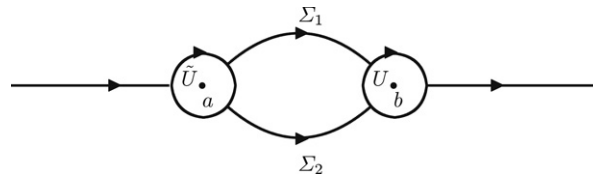


Fig. 3. Contour  $\Sigma_R$  for the Riemann–Hilbert problem for  $R$ .

The jump matrices  $J_R(z) = N(z)J_S(z)N(z)^{-1}$  tend to the identity matrix at an exponential rate as  $n \rightarrow \infty$ . The jump matrices on  $\partial U$  and  $\partial \tilde{U}$  tend to the identity matrix but at a slower rate of  $1/n$  as  $n \rightarrow \infty$ . The precise form is obtained from the asymptotic expansion of the Airy function as  $z \rightarrow \infty$ ,  $-\pi < \arg z < \pi$ , (see [13])

$$\text{Ai}(z) \sim \frac{e^{-\frac{2}{3}z^{\frac{3}{2}}}}{2\sqrt{\pi}z^{\frac{1}{4}}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(3k + \frac{1}{2})}{9^k (2k)! \Gamma(\frac{1}{2})} \frac{1}{z^{\frac{3}{2}k}} \quad (3.19)$$

and the corresponding asymptotic expansion for  $\text{Ai}'(z)$ . Using these facts in the parametrix  $P$  we find an asymptotic expansion for the jump of  $R$  on  $\partial U$

$$J_R(z) = P(z)N(z)^{-1} \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k} \Delta_k(z) \quad (3.20)$$

where

$$\begin{aligned} \Delta_k(z) = & \frac{1}{\sqrt{\pi}} \left( \frac{\Gamma(3k + \frac{1}{2})}{9^k (2k)!} - \frac{\Gamma(3k - \frac{3}{2})}{4 \cdot 9^{k-1} (2(k-1))!} \right) \frac{1}{(\frac{3}{2}\phi(z))^k} I \\ & - \frac{1}{4\sqrt{\pi}} \frac{\Gamma(3k - \frac{3}{2})}{9^{k-1} (2(k-1))!} \frac{1}{(\frac{3}{2}\phi(z))^k} \sigma_2 \quad \text{for } k \text{ even} \end{aligned} \quad (3.21)$$

and

$$\begin{aligned} \Delta_k(z) = & -\frac{\beta(z)^2}{(\frac{3}{2}\phi(z))^k} \frac{1}{2\sqrt{\pi}} \left( \frac{\Gamma(3k + \frac{1}{2})}{9^k (2k)!} - \frac{\Gamma(3k - \frac{3}{2})}{2 \cdot 9^{k-1} (2(k-1))!} \right) (\sigma_3 + i\sigma_1) \\ & - \frac{\beta(z)^{-2}}{(\frac{3}{2}\phi(z))^k} \frac{1}{2\sqrt{\pi}} \frac{\Gamma(3k + \frac{1}{2})}{9^k (2k)!} (\sigma_3 - i\sigma_1) \quad \text{for } k \text{ odd} \end{aligned} \quad (3.22)$$

where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (3.23)$$

are the Pauli matrices.

A similar expansion

$$J_R(z) = \tilde{P}(z)N(z)^{-1} \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k} \tilde{\Delta}_k(z) \quad (3.24)$$

holds for the jump matrix on  $\partial \tilde{U}$ .

As a result we find by the methods of [7], see also [14, Lemma 8.3],

**Lemma 3.1.** *There exist matrix valued functions  $R_k(z)$  with the property that for every  $l \in \mathbb{N}$ , there exist constants  $C > 0$  and  $r > 0$  such that for every  $z$  with  $|z| \geq r$ ,*

$$\left\| R(z) - I - \sum_{k=1}^l \frac{R_k(z)}{n^k} \right\| \leq \frac{C}{|z|n^{l+1}}. \quad (3.25)$$

So we write

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{1}{n^k} R_k(z). \quad (3.26)$$

From (3.26), (3.20) and (3.24) and the Riemann–Hilbert problem for  $R$ , we find an additive Riemann–Hilbert problem for  $R_k(z)$ ,

$$\begin{cases} R_k(z) \text{ is analytic on } \mathbb{C} \setminus (\partial U \cup \partial \tilde{U}) \\ R_{k+}(z) = R_{k-}(z) + \sum_{l=0}^{k-1} R_{l-}(z) \Delta_{k-l}(z) & \text{for } z \in \partial U \\ R_{k+}(z) = R_{k-}(z) + \sum_{l=0}^{k-1} R_{l-}(z) \tilde{\Delta}_{k-l}(z) & \text{for } z \in \partial \tilde{U} \\ R_k(z) = \mathcal{O}\left(\frac{1}{z}\right) & \text{as } z \rightarrow \infty \end{cases} \quad (3.27)$$

where  $R_0(z) = I$ . These Riemann–Hilbert problems can be successively solved using the Sokhotskii–Plemelj formula, or using a technique based on Laurent series expansion as in [14].

#### 4. Proof of Theorem 1.1

For the proof of (1.4) we do not need to compute the explicit forms of the  $R_k$ 's. However, we need to know that they have the following structure. Recall that the Pauli matrices are given in (3.23).

**Lemma 4.1.** *For  $k$  odd,  $R_k(z)$  is a linear combination of  $\sigma_1$  and  $\sigma_3$  and for  $k$  even,  $R_k(z)$  is a linear combination of  $I$  and  $\sigma_2$ .*

**Proof.** For  $k = 1$ , we know because of (3.27) that  $R_{1+} = R_{1-} + \Delta_1$  on  $\partial U$  and  $R_{1+} = R_{1-} + \tilde{\Delta}_1$  on  $\partial \tilde{U}$ . As  $\Delta_1, \tilde{\Delta}_1 \in \text{span}\{\sigma_1, \sigma_3\}$  on account of (3.22),  $R_1(z)$  must be a linear combination of  $\sigma_1$  and  $\sigma_3$  as well.

Let  $k \geq 1$  and once more observe (3.27). If  $k$  is odd, then again by (3.22)  $\Delta_k, \tilde{\Delta}_k \in \text{span}\{\sigma_1, \sigma_3\}$  and using induction on  $k$ , for every  $l < k$ ,  $R_{l-}(z) \Delta_{k-l}(z)$  and  $R_{l-}(z) \tilde{\Delta}_{k-l}(z)$  are products of a linear combination of  $\sigma_1$  and  $\sigma_3$  and a linear combination of  $I$  and  $\sigma_2$  (see also (3.21) and (3.22)), which results in a linear combination of  $\sigma_1$  and  $\sigma_3$ . Thus all terms in the (additive) jump for  $R_k$  on  $\partial U$  and on  $\partial \tilde{U}$  are in the span of  $\sigma_1$  and  $\sigma_3$ , and it follows that  $R_k \in \text{span}\{\sigma_1, \sigma_3\}$  if  $k$  is odd.

If  $k$  is even, then by induction, where we use again (3.21) and (3.22), we have that  $R_{l-}(z) \Delta_{k-l}(z)$  and  $R_{l-}(z) \tilde{\Delta}_{k-l}(z)$  are either products of two linear combinations of  $I$  and  $\sigma_2$  (in case  $l$  is even), or products of two linear combinations of  $\sigma_1$  and  $\sigma_3$  (in case  $l$  is odd). In both cases we find that  $R_{l-}(z) \Delta_{k-l}(z)$  and  $R_{l-}(z) \tilde{\Delta}_{k-l}(z)$  are linear combinations of  $I$  and  $\sigma_2$ , which implies that  $R_k \in \text{span}\{I, \sigma_2\}$  if  $k$  is even.  $\square$

Now we can finally prove our main result.

**Proof of Theorem 1.1.** We start from the expressions (2.4) and (2.5) for  $a_{n,n}$  and  $b_{n,n}$  in terms of the solution of the Riemann–Hilbert problem for  $Y$ . Following the transformations  $Y \mapsto T \mapsto S$ , we find that

$$a_{n,n} = (S_1)_{12} (S_1)_{21} \quad (4.1)$$

and

$$b_{n,n} = \frac{(S_2)_{12}}{(S_1)_{12}} - (S_1)_{22} \quad (4.2)$$

where  $S_1$  and  $S_2$  are the terms in the expansion of  $S(z)$  as  $z \rightarrow \infty$ ,

$$S(z) = I + \frac{1}{z} S_1 + \frac{1}{z^2} S_2 + \mathcal{O}\left(\frac{1}{z^3}\right).$$

To obtain (4.2) we use that  $g(z) = \log z + \mathcal{O}(1/z)$ , see also [10].

By (3.18), we know that  $S(z) = R(z)N(z)$  for  $|z|$  large enough, so we need the first terms in the expansions of  $N(z)$  and  $R(z)$  as  $z \rightarrow \infty$ . From (3.15) we have

$$\begin{aligned} N(z) &= \frac{\beta(z) + \beta(z)^{-1}}{2} I + \frac{\beta(z) - \beta(z)^{-1}}{2} \sigma_2 \\ &= I - \frac{(b-a)}{4} \sigma_2 \frac{1}{z} + \left( \frac{(b-a)^2}{32} I - \frac{b^2 - a^2}{8} \sigma_2 \right) \frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \end{aligned} \quad (4.3)$$



and from Lemma 4.1

$$R(z) = I + \frac{1}{z} \left( \sum_{m \text{ odd}} \frac{1}{n^m} (R_{m1\sigma_1} \sigma_1 + R_{m1\sigma_3} \sigma_3) + \sum_{m \text{ even}} \frac{1}{n^m} (R_{m1I} + R_{m1\sigma_2} \sigma_2) \right) \\ + \frac{1}{z^2} \left( \sum_{m \text{ odd}} \frac{1}{n^m} (R_{m2\sigma_1} \sigma_1 + R_{m2\sigma_3} \sigma_3) + \sum_{m \text{ even}} \frac{1}{n^m} (R_{m2I} + R_{m2\sigma_2} \sigma_2) \right) + \mathcal{O} \left( \frac{1}{z^3} \right) \quad (4.4)$$

where the constants  $R_{mjl}$ ,  $R_{mj\sigma_k}$ , for  $m \in \mathbb{N}$ ,  $j = 1, 2$ , and  $k = 1, 2, 3$  are such that  $R_{mjl}I + \sum_{k=1}^3 R_{mj\sigma_k} \sigma_k$  is the coefficient of  $z^{-j}$  in the Laurent expansion of  $R_m(z)$  around  $z = \infty$ .

Therefore, by (4.3) and (4.4),

$$S(z) = R(z)N(z) \sim I + \frac{1}{z} \left( -\frac{(b-a)}{4} \sigma_2 + \sum_{m \text{ odd}} \frac{1}{n^m} (R_{m1\sigma_1} \sigma_1 + R_{m1\sigma_3} \sigma_3) + \sum_{m \text{ even}} \frac{1}{n^m} (R_{m1I} + R_{m1\sigma_2} \sigma_2) \right) \\ + \frac{1}{z^2} \left( \frac{(b-a)^2}{32} I - \frac{b^2 - a^2}{8} \sigma_2 + \sum_{m \text{ odd}} \frac{1}{n^m} \left( \left( R_{m2\sigma_1} + i \frac{b-a}{4} R_{m1\sigma_3} \right) \sigma_1 \right. \right. \\ \left. \left. + \left( R_{m2\sigma_3} - i \frac{b-a}{4} R_{m1\sigma_1} \right) \sigma_3 \right) + \sum_{m \text{ even}} \frac{1}{n^m} \left( \left( R_{m2I} - \frac{b-a}{4} R_{m1\sigma_2} \right) I \right. \right. \\ \left. \left. + \left( R_{m2\sigma_2} - \frac{b-a}{4} R_{m1I} \right) \sigma_2 \right) \right) + \mathcal{O} \left( \frac{1}{z^3} \right) \quad (4.5)$$

which implies that

$$(S_1)_{12} \sim \frac{b-a}{4} i + \sum_{m \text{ odd}} \frac{1}{n^m} R_{m1\sigma_1} - i \sum_{m \text{ even}} \frac{1}{n^m} R_{m1\sigma_2} \quad (4.6)$$

and

$$(S_1)_{21} \sim -\frac{b-a}{4} i + \sum_{m \text{ odd}} \frac{1}{n^m} R_{m1\sigma_1} + i \sum_{m \text{ even}} \frac{1}{n^m} R_{m1\sigma_2}. \quad (4.7)$$

Inserting (4.6) and (4.7) into (4.1) then finally gives

$$a_{n,n} \sim \frac{(b-a)^2}{16} + \sum_{m=1}^{\infty} \frac{\alpha_{2m}}{n^{2m}}$$

for certain constants  $\alpha_{2m}$ .

Similar to (4.6) and (4.7) we have that  $(S_2)_{12}$  and  $(S_1)_{22}$  have asymptotic expansions in powers of  $1/n$ . From the expansion (4.5) for  $S$ , we see

$$(S_2)_{12} \sim \frac{b^2 - a^2}{8} i + \sum_{m \text{ odd}} \frac{1}{n^m} \left( \frac{b-a}{4} i R_{m1\sigma_3} + R_{m2\sigma_1} \right) + \sum_{m \text{ even}} \frac{1}{n^m} i \left( \frac{b-a}{4} R_{m1I} - R_{m2\sigma_2} \right)$$

and

$$(S_1)_{22} \sim -\sum_{m \text{ odd}} \frac{1}{n^m} R_{m1\sigma_3} + \sum_{m \text{ even}} \frac{1}{n^m} R_{m1I}.$$

From (4.2) it then follows that

$$b_{n,n} \sim \sum_{m=0}^{\infty} \frac{\beta_m}{n^m} \quad (4.8)$$

where  $\beta_0 = \frac{b+a}{2}$  and

$$\beta_1 = 2R_{11\sigma_3} - \frac{4}{b-a} i R_{12\sigma_1} + \frac{2(b+a)}{b-a} i R_{11\sigma_1}. \quad (4.9)$$

Our final task is to further evaluate the right-hand side of (4.9). As in [14], we have that  $\Delta_1$  is meromorphic in a neighbourhood of  $b$  with a pole in  $b$ . Indeed, if we write

$$\frac{\beta(z)^{-2}}{\phi(z)} = (z-b)^{-2} \sum_{m=0}^{\infty} B_m (z-b)^m, \quad B_0 = \frac{3}{2\pi h(b)}, \quad (4.10)$$

and use (3.22), then we find for  $z$  in a neighbourhood of  $b$ ,

$$\Delta_1(z) = \left( -\frac{5B_1}{144} (\sigma_3 - i\sigma_1) + \frac{7B_0}{144(b-a)} (\sigma_3 + i\sigma_1) \right) \frac{1}{z-b} - \frac{5B_0}{144} (\sigma_3 - i\sigma_1) \frac{1}{(z-b)^2} + \mathcal{O}(1). \quad (4.11)$$

Similarly, for  $z$  in a neighbourhood of  $a$ , we have

$$\frac{\beta(z)^2}{\tilde{\phi}(z)} = (z-a)^{-2} \sum_{m=0}^{\infty} A_m (z-a)^m, \quad A_0 = \frac{3}{2\pi h(a)}, \quad (4.12)$$

and

$$\tilde{\Delta}_1(z) = \left( -\frac{5A_1}{144} (\sigma_3 + i\sigma_1) - \frac{7A_0}{144(b-a)} (\sigma_3 - i\sigma_1) \right) \frac{1}{z-a} - \frac{5A_0}{144} (\sigma_3 + i\sigma_1) \frac{1}{(z-a)^2} + \mathcal{O}(1). \quad (4.13)$$

As in [14] we have that  $R_1(z)$  for  $z \in \mathbb{C} \setminus \overline{U \cup \tilde{U}}$  is equal to the sum of the Laurent parts of (4.11) and (4.13). Expanding  $R_1(z)$  as  $z \rightarrow \infty$ , we then get

$$R_1(z) = R_{11} \frac{1}{z} + R_{12} \frac{1}{z^2} + \mathcal{O}\left(\frac{1}{z^3}\right) \quad \text{as } z \rightarrow \infty,$$

where

$$\begin{aligned} R_{11} &= -\frac{5A_1}{144} (\sigma_3 + i\sigma_1) - \frac{7A_0}{144(b-a)} (\sigma_3 - i\sigma_1) - \frac{5B_1}{144} (\sigma_3 - i\sigma_1) + \frac{7B_0}{144(b-a)} (\sigma_3 + i\sigma_1) \\ R_{12} &= -\frac{5aA_1}{144} (\sigma_3 + i\sigma_1) - \frac{7aA_0}{144(b-a)} (\sigma_3 - i\sigma_1) - \frac{5bB_1}{144} (\sigma_3 - i\sigma_1) + \frac{7bB_0}{144(b-a)} (\sigma_3 + i\sigma_1) \\ &\quad - \frac{5A_0}{144} (\sigma_3 + i\sigma_1) - \frac{5B_0}{144} (\sigma_3 - i\sigma_1). \end{aligned}$$

Thus

$$R_{11\sigma_3} = -\frac{5(A_1 + B_1)}{144} - \frac{7(A_0 - B_0)}{144(b-a)}, \quad (4.14)$$

$$R_{11\sigma_1} = -i \frac{5(A_1 - B_1)}{144} + i \frac{7(A_0 + B_0)}{144(b-a)}, \quad (4.15)$$

$$R_{12\sigma_1} = -i \frac{5(aA_1 - bB_1)}{144} + i \frac{7(aA_0 + bB_0)}{144(b-a)} - i \frac{5(A_0 - B_0)}{144}. \quad (4.16)$$

Inserting (4.14)–(4.16) into (4.9), we find after straightforward calculations that  $A_1$  and  $B_1$  fully disappear and that (4.9) reduces to

$$\beta_1 = \frac{B_0 - A_0}{3(b-a)}.$$

Using the explicit formulas for  $A_0$  and  $B_0$  given in (4.10) and (4.12), we arrive at (1.5), which completes the proof of Theorem 1.1.  $\square$

## Acknowledgements

We are grateful to Pavel Bleher and Alexander Its for valuable discussions on their paper [2]. The authors are supported by FWO-Flanders projects G.0455.04 and G.0427.09, by K.U. Leuven research grants OT/04/21 and OT/08/33, and by the Belgian Interuniversity Attraction Pole P06/02. The first author is also supported by the European Science Foundation Program MISGAM, and by a grant from the Ministry of Education and Science of Spain, project code MTM2005-08648-C02-01.

## References

- [1] S. Albeverio, L. Pastur, M. Shcherbina, On the  $1/n$  expansion for some unitary invariant ensemble of random matrices, *Comm. Math. Phys.* 224 (2001) 271–305.
- [2] P.M. Bleher, A.R. Its, Asymptotics of the partition function of a random matrix model, *Ann. Inst. Fourier* 55 (2005) 1943–2000.
- [3] P.M. Bleher, A.B.J. Kuijlaars, Large  $n$  limit of Gaussian random matrices with external source, part I, *Comm. Math. Phys.* 252 (2004) 43–76.
- [4] T.S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon & Breach, New York, 1978.
- [5] P.A. Deift, *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*, in: *Courant Lecture Notes in Mathematics*, vol. 3, New York University, Courant Institute of Mathematical Sciences, New York, 2000.
- [6] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides, X. Zhou, Strong asymptotics of orthogonal polynomials with respect to exponential weights, *Comm. Pure Appl. Math.* 52 (1999) 1491–1552.
- [7] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides, X. Zhou, Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory, *Comm. Pure Appl. Math.* 52 (1999) 1335–1425.
- [8] P. Deift, S. Venakides, X. Zhou, New results in small dispersion KdV by an extension of the steepest descent method for Riemann–Hilbert problems, *Internat. Math. Res. Not.* 1997 (1997) 286–299.
- [9] P. Deift, X. Zhou, A steepest descent method for oscillatory Riemann–Hilbert problems. Asymptotics for the MKdV equation, *Ann. Math.* 137 (2) (1993) 295–368.
- [10] M. Duits, A.B.J. Kuijlaars, Painlevé I asymptotics for orthogonal polynomials with respect to a varying quartic weight, *Nonlinearity* 19 (2006) 2211–2245.
- [11] N.M. Ercolani, K.D.T-R McLaughlin, Asymptotics of the partition function for random matrices via Riemann–Hilbert techniques and applications to graphical enumeration, *Internat. Math. Res. Not.* 2003 (2003) 755–820.
- [12] A.S. Fokas, A.R. Its, A.V. Kitaev, The isomonodromy approach to matrix models in 2D quantum gravity, *Comm. Math. Phys.* 147 (1992) 395–430.
- [13] H. Hochstadt, *The Functions of Mathematical Physics*, in: *Pure and Applied Mathematics*, vol. XXIII, Wiley-Interscience, New York, 1971.
- [14] A.B.J. Kuijlaars, K.T-R. McLaughlin, W. Van Assche, M. Vanlessen, The Riemann–Hilbert approach to strong asymptotics for orthogonal polynomials on  $[-1, 1]$ , *Adv. Math.* 188 (2004) 337–398.
- [15] A.B.J. Kuijlaars, W. Van Assche, The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients, *J. Approx. Theory* 99 (1999) 167–197.
- [16] E.B. Saff, V. Totik, *Logarithmic Potentials with External Fields*, in: *Grundle Math. Wiss.*, vol. 316, Springer-Verlag, Berlin, 1997.
- [17] E.B. Saff, J.L. Ullman, R.S. Varga, Incomplete polynomials: An electrostatics approach, in: E.W. Cheney (Ed.), *Approximation Theory III*, 1980, pp. 769–782.