Journal of Computational and Applied Mathematics 233 (2010) 1969–1979

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

On a quasi-reversibility regularization method for a Cauchy problem of the Helmholtz equation

Ai-Lin Qian [a,c,*], Xiang-Tuan Xiong [b], Yu-Jiang Wu [a]

[a] Department of Mathematics, Lanzhou University, Lanzhou, Gansu, 730000, People's Republic of China
[b] Department of Mathematics, Northwest Normal University, Lanzhou, Gansu, 730070, People's Republic of China
[c] Department of Mathematics and Statistics, Xianning University, Xianning, Hubei, 437000, People's Republic of China

ARTICLE INFO

Article history:
Received 25 December 2008
Received in revised form 9 June 2009

Keywords:
Ill-posed
Cauchy problem of Helmholtz equation
Quasi-reversibility
Regularization

ABSTRACT

In this paper, we consider the Cauchy problem for the Helmholtz equation in a rectangle, where the Cauchy data is given for $y = 0$ and boundary data are for $x = 0$ and $x = \pi$. The solution is sought in the interval $0 < y \le 1$. A quasi-reversibility method is applied to formulate regularized solutions which are stably convergent to the exact one with explicit error estimates.

1. Introduction

The Helmholtz equation arises in many areas, especially in practical physical applications, such as acoustic, wave propagation and scattering, vibration of the structure, electromagnetic scattering and so on. Several numerical methods have been proposed to solve this problem, such as alternating iterative algorithm based on the boundary element method (BEM) [1], the conjugate gradient method [2], the method of fundamental solutions (MFS) [3–5,2,6,7], modified method [8]. Although there exists a vast literature on the Cauchy problem for the Helmholtz equation, to the authors' knowledge, there are much fewer papers devoted to the error estimates. The main aim is to give a regularization method and investigate the error estimates between the regularization solution and the exact one.

Consider the Cauchy problem for the Helmholtz equation in a rectangle: determine the solution $w(x, y)$ for $0 < y \le 1$ from the input data $\phi(\cdot) := w(\cdot, 0)$, $h(\cdot) := w_y(\cdot, 0)$, when $w(x, y)$ satisfies

$$\begin{aligned} &\Delta w(x, y) + k^2 w(x, y) = 0, \quad 0 < x < \pi, 0 < y < 1, \\ &w(x, 0) = \phi(x), \quad 0 \le x \le \pi, \\ &w_y(x, 0) = h(x), \quad 0 \le x \le \pi, \\ &w(0, y) = w(\pi, y) = 0, \quad 0 \le y \le 1. \end{aligned} \tag{1.1}$$

Physically, ϕ, h can only be measured, there will be measurement errors, and we would actually have as data some functions $\phi^\delta(x), h^\delta(x) \in L^2(0, \pi)$, for which

$$\|\phi^\delta - \phi\| + \|h^\delta - h\| \le \delta, \tag{1.2}$$

* Corresponding author at: Department of Mathematics and Statistics, Xianning University, Xianning, Hubei, 437000, People's Republic of China. Tel.: +86 715 8144689; fax: +86 715 8144688.
E-mail address: qianal07@lzu.cn (A.-L. Qian).

0377-0427/$ – see front matter Crown Copyright © 2009 Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2009.09.031

where the constant $\delta > 0$ represents a bound on the measurement error, $\|\cdot\|$ denotes the L^2-norm, and there exists a constant $E > 0$, such that the following a priori bound exists (e.g., say, the energy of the solution $w(x, y)$ at the right boundary $y = 1$ is finite.)

$$\|w(\cdot, 1)\| \leq E. \tag{1.3}$$

In order to solve this problem, we split the Cauchy problem into two independent Cauchy problems:

$$\begin{aligned} &\Delta u(x, y) + k^2 u(x, y) = 0, \quad 0 < x < \pi, 0 < y < 1,\\ &u(x, 0) = \phi(x), \quad 0 \leq x \leq \pi,\\ &u_y(x, 0) = 0, \quad 0 \leq x \leq \pi,\\ &u(0, y) = u(\pi, y) = 0, \quad 0 \leq y \leq 1 \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} &\Delta v(x, y) + k^2 v(x, y) = 0, \quad 0 < x < \pi, 0 < y < 1,\\ &v(x, 0) = 0, \quad 0 \leq x \leq \pi,\\ &v_y(x, 0) = h(x), \quad 0 \leq x \leq \pi,\\ &v(0, y) = v(\pi, y) = 0, \quad 0 \leq y \leq 1, \end{aligned} \tag{1.5}$$

solve them and then take $w = u + v$.
It is easy to verify that the function

$$u_m(x, y) = \frac{\sin(mx)}{m} \sinh\left(\sqrt{m^2 - k^2} y\right), \tag{1.6}$$

is the exact solution of problem (1.5) with

$$u_m(x, 0) = \phi_m(x) = \frac{\sin(mx)}{m}, \tag{1.7}$$

where $m > k$ are positive integers. Note that $\sup_{x\in(0,\pi)} |\phi_m(x)|$ tends to zero as $m \to \infty$, but $\sup_{x\in(0,\pi)} |u_m(x, y)| \to \infty (m \to \infty)$ for fixed $y > 0$. Thus Eq. (1.5) is an ill-posed problem so that it is impossible to solve using classical numerical methods and requires special techniques, i.e., regularization method to be employed. Eq. (1.4) is also an ill-posed problem.

At the other hand, separation of variables leads to the solution of problem (1.4)

$$u(x, y) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} c_n \sin(nx) \cosh\left(\sqrt{n^2 - k^2} y\right), & 0 < k < 1,\\ \displaystyle\sum_{n=1}^{[k]} c_n \sin(nx) \cosh\left(\sqrt{k^2 - n^2} y\right) + \sum_{n=[k]+1}^{\infty} c_n \sin(nx) \cosh\left(\sqrt{n^2 - k^2} y\right), & k \geq 1, \end{cases} \tag{1.8}$$

where

$$c_n = \frac{2}{\pi} \int_0^{\pi} \phi(t) \sin(nt) \mathrm{d}t. \tag{1.9}$$

The solution of problem (1.5)

$$v(x, y) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} d_n \sin(nx) \sinh\left(\sqrt{n^2 - k^2} y\right), & 0 < k < 1,\\ \displaystyle\sum_{n=1}^{[k]} d_n \sin(nx) \sinh\left(\sqrt{k^2 - n^2} y\right) + \sum_{n=[k]+1}^{\infty} d_n \sin(nx) \sinh\left(\sqrt{n^2 - k^2} y\right), & k \geq 1, \end{cases} \tag{1.10}$$

where

$$d_n = \frac{2}{\sqrt{n^2 - k^2}\pi} \int_0^{\pi} h(t) \sin(nt) \mathrm{d}t. \tag{1.11}$$

We shall use perturbation method to construct stable solutions of the problems (1.1) and then obtain error estimate.

2. Modified regularization method

For system (1.4), if $0 < k < 1$ we consider the system

$$\begin{aligned} &\Delta u^\delta(x, y) + k^2 u^\delta(x, y) - \mu^2 u^\delta_{xxyy}(x, y) = 0, \quad 0 < x < \pi, 0 < y < 1,\\ &u^\delta(x, 0) = \phi^\delta(x), \quad 0 \leq x \leq \pi,\\ &u^\delta_y(x, 0) = 0, \quad 0 \leq x \leq \pi,\\ &u^\delta(0, y) = u^\delta(\pi, y) = 0, \quad 0 \leq y \leq 1. \end{aligned} \tag{2.1}$$

Separation of variables leads to the solution

$$u^\delta(x,y)=\sum_{n=1}^{\infty}c_n^\delta\sin(nx)\cosh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right),\tag{2.2}$$

where

$$c_n^\delta=\frac{2}{\pi}\int_0^\pi\phi^\delta(t)\sin(nt)\mathrm{d}t.\tag{2.3}$$

If $k\geq 1$, we modified the exact solution u as follows:

$$u^\delta(x,y)=\sum_{n=1}^{[k]}c_n^\delta\sin(nx)\cosh\left(\sqrt{k^2-n^2}y\right)+\sum_{n=[k]+1}^{\infty}c_n^\delta\sin(nx)\cosh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right),\tag{2.4}$$

where c_n^δ is defined by (2.3).

Lemma 2.1. *Suppose u be the solution of problem (1.4) with the exact data ϕ and u^δ be the modified solution defined by (2.2) and (2.4) with the noise data ϕ^δ, let ϕ^δ satisfy $\|\phi^\delta-\phi\|\leq\delta$ and let the exact solution u at $y=1$ satisfy (1.3). If we select $\mu=\frac{1}{\ln(\frac{E}{\delta})}$, then for fixed $0<y<1$ we get the error bound*

$$\|u^\delta(\cdot,y)-u(\cdot,y)\|\leq\begin{cases}E^y\delta^{1-y}+C_1\dfrac{E}{(\ln(E/\delta))^2}, & 0<k<1,\\ \delta+E^y\delta^{1-y}+C_1\dfrac{E}{(\ln(E/\delta))^2}, & k\geq 1,\end{cases}\tag{2.5}$$

where $C_1=(\frac{4}{(1-y)e})^4+k^2(\frac{2}{(1-y)e})^2$, i.e.,

$$\|u^\delta(\cdot,y)-u(\cdot,y)\|\leq O((\ln(E/\delta))^2),\quad \textit{for } \delta\to 0.\tag{2.6}$$

For system (1.5) we consider the system

$$\begin{aligned}&\Delta v^\delta(x,y)+k^2v^\delta(x,y)-\mu^2v^\delta_{xxyy}(x,y)=0,\quad 0<x<\pi,0<y<1,\\&v^\delta(x,0)=0,\quad 0\leq x\leq\pi,\\&v^\delta_y(x,0)=h^\delta(x),\quad 0\leq x\leq\pi,\\&v^\delta(0,y)=v^\delta(\pi,y)=0,\quad 0\leq y\leq 1.\end{aligned}\tag{2.7}$$

Separation of variables leads to the solution

$$v^\delta(x,y)=\begin{cases}\displaystyle\sum_{n=1}^{\infty}d_n^\delta\sin(nx)\sinh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right), & 0<k<1,\\ \displaystyle\sum_{n=1}^{[k]}d_n^\delta\sin(nx)\sinh\left(\sqrt{k^2-n^2}y\right)+\sum_{n=[k]+1}^{\infty}d_n^\delta\sin(nx)\sinh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right), & k\geq 1,\end{cases}\tag{2.8}$$

where

$$d_n^\delta=\frac{2}{\sqrt{n^2-k^2}\pi}\int_0^\pi h^\delta(t)\sin(nt)\mathrm{d}t.\tag{2.9}$$

Lemma 2.2. *Suppose v be the solution of problem (1.5) with the exact data h and v^δ be the modified solution defined by (2.8) with the noise data h^δ, let h^δ satisfy $\|h^\delta-h\|\leq\delta$ and let the exact solution v at $y=1$ satisfy (1.3). If we select $\mu=\frac{1}{\ln(\frac{E}{\delta})}$, then for fixed $0<y<1$ we get the error bound*

$$\|v^\delta(\cdot,y)-v(\cdot,y)\|\leq\begin{cases}\dfrac{1}{2}E^y\delta^{1-y}+C_2\dfrac{E}{\left(\ln\frac{E}{\delta}\right)^2}, & 0<k<1,\\ \delta+\dfrac{1}{2}E^y\delta^{1-y}+C_2\dfrac{E}{\left(\ln\frac{E}{\delta}\right)^2}, & k\geq 1,\end{cases}\tag{2.10}$$

i.e.,

$$\|v^\delta(\cdot,y)-v(\cdot,y)\|\leq O((\ln(E/\delta))^2),\quad \textit{for } \delta\to 0.\tag{2.11}$$

Theorem 2.3. *Suppose that $w = u+v$ is the solution with exact data $[\phi, h]$ and that $w^\delta = u^\delta + v^\delta$ is the solution with measured data $[\phi_\delta, h_\delta]$.If we have a bound $\|w(\cdot, 1)\| \le E$, and the measured function satisfies $\|\phi - \phi^\delta\| + \|h - h^\delta\| \le \delta$ and if we choose $\mu = \frac{1}{\ln(E/\delta)}$, then for fixed $0 < y < 1$, we get the error bound*

$$\|w^\delta(\cdot, y) - w(\cdot, y)\| \le \begin{cases} \frac{3}{2}E^y\delta^{1-y} + (C_1 + C_2)\frac{E}{\left(\ln\frac{E}{\delta}\right)^2}, & 0 < k < 1, \\ 2\delta + \frac{3}{2}E^y\delta^{1-y} + (C_1 + C_2)\frac{E}{\left(\ln\frac{E}{\delta}\right)^2}, & k \ge 1, \end{cases} \tag{2.12}$$

i.e.,

$$\|w^\delta(\cdot, y) - w(\cdot, y)\| \le O((\ln(E/\delta))^2), \quad \text{for } \delta \to 0. \tag{2.13}$$

Proof. $\|w - w^\delta\| = \|(u+v) - (u^\delta + v^\delta)\| \le \|u - u^\delta\| + \|v - v^\delta\|$, then the theorem is straightforward by using triangle inequality and Lemmas 2.1 and 2.2. □

From Theorem 2.3, we find that w^δ is an approximation of exact solution w. The approximation error depends continuously on the measurement error for fixed $0 < y < 1$. However, as $y \to 1$, the accuracy of regularized solution becomes progressively lower. This is common in the theory of ill-posed problems, if we do not have additional conditions on the smoothness of the solution.

To retain the continuous dependence of the solution at $y = 1$, instead of (1.3), we introduce a stronger a priori assumption,

$$\left\| \left.\frac{\partial^p w(\cdot, y)}{\partial y^p}\right|_{y=1} \right\| \le E, \tag{2.14}$$

where $p > 0$ is an integer. This priori condition shows that L^2-norm of $w(\cdot, y)$'s p-order derivatives with respect to the variable y at the boundary $y = 1$ are bounded.

Theorem 2.4. *Suppose that w is given with exact data $[\phi, h]$ and that w^δ is given with measured data $[\phi^\delta, h^\delta]$. If we have an a priori bound (2.14), and the measured function $[\phi^\delta, h^\delta]$ satisfies $\|\phi - \phi^\delta\| + \|h - h^\delta\| \le \delta$. The parameter $\mu \in (0, 1)$ is chosen as*

$$\mu = \frac{1}{\ln\left(\frac{E}{\delta}\left(\ln\frac{E}{\delta}\right)^{-p}\right)}. \tag{2.15}$$

Then for $p > 0$, we get the error bound

$$\|w^\delta(\cdot, 1) - w(\cdot, 1)\| \le \begin{cases} \frac{3}{2}\frac{E}{\left(\ln\frac{E}{\delta}\right)^p} + \varepsilon, & 0 < k < 1, \\ 2\delta + \frac{3}{2}\frac{E}{\left(\ln\frac{E}{\delta}\right)^p} + \varepsilon k^2, & k \ge 1, \end{cases} \tag{2.16}$$

where $\varepsilon = \max\{3\mu^{2p/3}, \frac{3}{2}\mu^2\}E$.

Remark 2.5. Since the regularization parameter $\mu \to 0$ as the measured error $\delta \to 0$, we can easily find that, for $p > 0$, $\varepsilon \to 0(\delta \to 0)$, thus

$$\lim_{\delta\to 0} \|w(\cdot, 1) - w^\delta(\cdot, 1)\| = 0, \quad p > 0.$$

Remark 2.6. We separately consider the case $0 \le y \le 1$ and the case $y = 1$ in order to emphasize the following facts. For the case $0 \le y \le 1$, the a priori bound $\|w(\cdot, 1)\|$ is sufficient. However, for the case $y = 1$, the stronger a priori bound for $\|\frac{\partial^p w(\cdot,y)}{\partial y^p}|_{y=1}\|$ where $p > 0$ must be imposed.

Remark 2.7. For the quasi-reversibility method (2.1), the method is not unique, e.g., we can use the following method to replace (2.1):

$$\begin{aligned} &\Delta u^\delta(x, y) + k^2 u^\delta(x, y) - \mu^2 u^\delta_{xxxxyy}(x, y) = 0, \quad 0 < x < \pi, 0 < y < 1, \\ &u^\delta(x, 0) = \phi^\delta(x), \quad 0 \le x \le \pi, \\ &u^\delta_y(x, 0) = 0, \quad 0 \le x \le \pi, \\ &u^\delta(0, y) = u^\delta(\pi, y) = 0, \quad 0 \le y \le 1. \end{aligned} \tag{2.17}$$

To demonstrate the usefulness of the above method, we consider a special case (Please see Ref. [9]):

$$\begin{array}{ll} u_{xx} + u_{yy} + k^2 u = 0 & x \in \mathbb{R} \\ u(x,0) = \phi(x) & x \in \mathbb{R}, \\ \partial_y u(x,0) = 0 & x \in \mathbb{R}, \\ u(\cdot, y) \in L^2(\mathbb{R}) & y \in (0,1). \end{array} \tag{2.18}$$

We use the quasi-reversibility method:

$$\begin{array}{ll} u_{xx} + u_{yy} + k^2 u - \mu^2 u_{xxxxyy} = 0 & x \in \mathbb{R} \\ u(x,0) = \phi_\delta(x) & x \in \mathbb{R}, \\ \partial_y u(x,0) = 0 & x \in \mathbb{R}, \\ u(\cdot, y) \in L^2(\mathbb{R}) & y \in (0,1). \end{array} \tag{2.19}$$

By using Fourier transform technique with respect to variable $x \in \mathbb{R}$, we can easily obtain the regularization solution

$$\hat{u}(\xi, y) = \cosh(y\sqrt{(\xi^2 - k^2)/(1+\mu^2\xi^4)})\hat{\phi}_\delta(\xi). \tag{2.20}$$

3. Proofs of Lemmas 2.1 and 2.2, Theorem 2.4

Proof of Lemma 2.1. From (1.8), (2.2) and (2.4) we have

$$\phi(x) = u(x,0) = \sum_{n=1}^{\infty} c_n \sin(nx), \qquad \phi^\delta(x) = u^\delta(x,0) = \sum_{n=1}^{\infty} c_n^\delta \sin(nx), \tag{3.1}$$

where

$$c_n = \frac{2}{\pi}\int_0^\pi \phi(t)\sin(nt)\mathrm{d}t, \qquad c_n^\delta = \frac{2}{\pi}\int_0^\pi \phi^\delta(t)\sin(nt)\mathrm{d}t. \tag{3.2}$$

Thus the condition $\|\phi - \phi^\delta\| \le \delta$ is equivalent to

$$\frac{\pi}{2}\sum_{n=1}^{\infty}(c_n - c_n^\delta)^2 \le \delta^2. \tag{3.3}$$

For the case $0 < k < 1$, by the first equation of (1.8), the assumption $\|u(\cdot,1)\| \le E$ is equivalent to

$$\|u(\cdot,1)\|^2 = \frac{\pi}{2}\sum_{n=1}^{\infty} c_n^2 \cosh^2\left(\sqrt{n^2-k^2}\right) \le E^2. \tag{3.4}$$

By the first equation in (1.8), (2.2), (2.4), (3.3) and (3.4), we have

$$\begin{aligned} \left\|u(\cdot,y) - u^\delta(\cdot,y)\right\| &\le \left(\frac{\pi}{2}\sum_{n=1}^{\infty}(c_n - c_n^\delta)^2\cosh^2\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right)\right)^{\frac{1}{2}} \\ &\quad + \left(\frac{\pi}{2}\sum_{n=1}^{\infty} c_n^2\left(\cosh\left(\sqrt{n^2-k^2}y\right) - \cosh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right)\right)^2\right)^{\frac{1}{2}} \\ &\le \delta\sup_{n\ge1}A(n) + E\sup_{n\ge1}B(n), \end{aligned}$$

where

$$A(n) = \cosh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right), \qquad B(n) = \left|\frac{\cosh\left(\sqrt{n^2-k^2}y\right) - \cosh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right)}{\cosh\left(\sqrt{n^2-k^2}\right)}\right|. \tag{3.5}$$

We now estimate $A(n)$. Since $A(n) \le \cosh(\frac{y}{\mu}) \le \mathrm{e}^{\frac{y}{\mu}}$, so

$$\delta\sup_{n\ge1}A(n) \le \delta\mathrm{e}^{\frac{y}{\mu}} = E^y\delta^{1-y}. \tag{3.6}$$

In the following, we estimate $B(n)$. Let $\xi = \sqrt{n^2-k^2}$, $\tau = \sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}$, note that $\xi > \tau$ and

$$1-\mathrm{e}^{-r} \le r (r \ge 0), \tag{3.7}$$

we get

$$\xi^2-\tau^2 = n^2-k^2-\frac{n^2-k^2}{1+\mu^2 n^2} \le (n^2-k^2)n^2\mu^2, \tag{3.8}$$

then

$$\begin{aligned} B(n) &= \left|\frac{\cosh(\xi y)-\cosh(\tau y)}{\cosh(\xi)}\right| = \frac{(\mathrm{e}^{\xi y}-\mathrm{e}^{\tau y})-(\mathrm{e}^{\xi y}-\mathrm{e}^{\tau y})/\mathrm{e}^{(\xi+\tau)y}}{\mathrm{e}^{\xi}+\mathrm{e}^{-\xi}} \le \frac{(\mathrm{e}^{\xi y}-\mathrm{e}^{\tau y})(1-\mathrm{e}^{-(\xi+\tau)y})}{\mathrm{e}^{\xi}} \\ &\le \mathrm{e}^{-\xi(1-y)}(\xi+\tau)y(1-\mathrm{e}^{-(\xi-\tau)y}) \le (\xi^2-\tau^2)y^2\mathrm{e}^{-\xi(1-y)} = \mu^2 n^2(n^2-k^2)y^2\mathrm{e}^{-\xi(1-y)} \\ &= \mu^2 y^2 \xi^2(\xi^2+k^2)\mathrm{e}^{-\xi(1-y)} \le \xi^4\mu^2\mathrm{e}^{-\xi(1-y)}+\xi^2k^2\mu^2\mathrm{e}^{-\xi(1-y)} := a(\xi)+b(\xi). \end{aligned}$$

The function $a(\xi)$ attains its maximum

$$a_{\max}(\xi) = a\left(\frac{4}{1-y}\right) = \mu^2\left(\frac{4}{(1-y)e}\right)^4,$$

$b(\xi)$ attains its maximum

$$b_{\max}(\xi) = b\left(\frac{2}{1-y}\right) = \mu^2k^2\left(\frac{2}{(1-y)e}\right)^2.$$

Consequently, for fixed $0<y<1$,

$$B(n) \le C_1\mu^2, \tag{3.9}$$

where $C_1 = (\frac{4}{(1-y)e})^4 + k^2(\frac{2}{(1-y)e})^2$.

Hence, for fixed $0<y<1$,

$$\|u^\delta(\cdot,y)-u(\cdot,y)\| \le E^y\delta^{1-y}+C_1\frac{E}{\left(\ln\frac{E}{\delta}\right)^2}. \tag{3.10}$$

In the following, we consider the case $k \ge 1$. Note that $\|u(\cdot,1)\| \le E$ is equivalent to

$$\|u(\cdot,1)\|^2 = \frac{\pi}{2}\sum_{n=1}^{[k]} c_n^2\cosh^2\left(\sqrt{k^2-n^2}\right)+\frac{\pi}{2}\sum_{n=[k]+1}^{[\infty]} c_n^2\cosh^2\left(\sqrt{n^2-k^2}\right) \le E^2. \tag{3.11}$$

Then, by the second equation in (1.8), (2.4), (3.3) and (3.11), we have

$$\begin{aligned} \|u(\cdot,y)-u^\delta(\cdot,y)\| \le{}& \left(\frac{\pi}{2}\sum_{n=1}^{[k]}(c_n-c_n^\delta)^2\cosh^2\left(\sqrt{k^2-n^2}y\right)\right)^{\frac12}+\left(\frac{\pi}{2}\sum_{n=[k]+1}^{\infty}(c_n-c_n^\delta)^2\cosh^2\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right)\right)^{\frac12} \\ &+\left(\frac{\pi}{2}\sum_{n=[k]+1}^{\infty}c_n^2\left(\cosh\left(\sqrt{n^2-k^2}y\right)-\cosh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right)\right)^2\right)^{\frac12} \\ \le{}& \delta+\delta\sup_{n\ge[k]+1}A(n)+E\sup_{n\ge[k]+1}B(n), \end{aligned}$$

where $A(n)$ and $B(n)$ is defined by (3.5). Similar to the case $0<k<1$, we have

$$\delta\sup_{n\ge[k]+1}A(n) \le E^y\delta^{1-y}, \tag{3.12}$$

$$B(n) \le C_1\left(\ln\frac{E}{\delta}\right)^{-2}. \tag{3.13}$$

Therefore, for fixed $0<y<1$ and $k\ge1$,

$$\|u^\delta(\cdot,y)-u(\cdot,y)\| \le \delta+E^y\delta^{1-y}+C_1\frac{E}{\left(\ln\frac{E}{\delta}\right)^2}. \quad \Box \tag{3.14}$$

Proof of Lemma 2.2. Note that condition $\|h-h^\delta\| \le \delta$ gives

$$(n^2-k^2)\frac{\pi}{2}\sum_{n=1}^{\infty}(d_n-d_n^\delta)^2 \le \delta^2. \tag{3.15}$$

For the case $0<k<1$, by the first equation of (1.10), the assumption $\|v(\cdot,1)\| \le E$ is equivalent to

$$\|v(\cdot,1)\|^2 = \frac{\pi}{2}\sum_{n=1}^{\infty} d_n^2 \sinh^2\left(\sqrt{n^2-k^2}\right) \le E^2. \tag{3.16}$$

By the first equation in (1.10), (2.8), (3.15) and (3.16), we have

$$\begin{aligned}\|v(\cdot,y)-v^\delta(\cdot,y)\| &\le \left(\frac{\pi}{2}\sum_{n=1}^{\infty}(d_n-d_n^\delta)^2\sinh^2\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right)\right)^{\frac{1}{2}} \\ &\quad + \left(\frac{\pi}{2}\sum_{n=1}^{\infty}d_n^2\left(\sinh\left(\sqrt{n^2-k^2}y\right)-\sinh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right)\right)^2\right)^{\frac{1}{2}} \\ &\le \delta\sup_{n\ge1}(C(n)/\sqrt{n^2-k^2}) + E\sup_{n\ge1}D(n) \le \delta\frac{1}{\sqrt{1-k^2}}\sup_{n\ge1}(C(n)) + E\sup_{n\ge1}D(n),\end{aligned}$$

where

$$C(n) = \sinh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right), \qquad D(n) = \left|\frac{\sinh\left(\sqrt{n^2-k^2}y\right)-\sinh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right)}{\sinh\left(\sqrt{n^2-k^2}\right)}\right|. \tag{3.17}$$

We now estimate $C(n)$.Since $C(n) \le \sinh(\frac{y}{\mu}) \le \frac{1}{2}\mathrm{e}^{\frac{y}{\mu}}$, so

$$\delta\sup_{n\ge1}C(n) \le \frac{1}{2}E^y\delta^{1-y}. \tag{3.18}$$

In the following, we estimate $D(n)$. Similar to (3.12) and (3.13), note that $\xi > \tau$ and

$$\sqrt{1+\mu^2n^2} \le 1+\frac{1}{2}\mu^2n^2, \quad 1-\mathrm{e}^{-r} \le r(r\ge 0), \tag{3.19}$$

we get

$$\xi-\tau = \sqrt{n^2-k^2} - \sqrt{\frac{n^2-k^2}{1+\mu^2n^2}} \le \frac{1}{2}\mu^2n^2\sqrt{n^2-k^2}, \tag{3.20}$$

we have

$$\begin{aligned}D(n) &= \left|\frac{\sinh(\xi y)-\sinh(\tau y)}{\sinh(\xi)}\right| = \frac{(\mathrm{e}^{\xi y}-\mathrm{e}^{-\xi y})/2-(\mathrm{e}^{\tau y}-\mathrm{e}^{-\tau y})/2}{(\mathrm{e}^{\xi}-\mathrm{e}^{-\xi})/2} \\ &\le \frac{(\mathrm{e}^{\xi y}-\mathrm{e}^{-\xi y})/2-(\mathrm{e}^{\tau y}-\mathrm{e}^{-\tau y})/2}{\mathrm{e}^{\xi}/2} \quad (\text{when } n\ge 2) \\ &= \frac{(\mathrm{e}^{\xi y}-\mathrm{e}^{\tau y})+(\mathrm{e}^{-\tau y}-\mathrm{e}^{-\xi y})}{\mathrm{e}^{\xi}} \le 2\frac{\mathrm{e}^{\xi y}-\mathrm{e}^{\tau y}}{\mathrm{e}^{\xi}} = 2\mathrm{e}^{-\xi(1-y)}(1-\mathrm{e}^{-(\xi-\tau)y}) \\ &\le \mu^2n^2\sqrt{n^2-k^2}\mathrm{e}^{-\xi(1-y)} = \mu^2(\xi^2+k^2)\xi\mathrm{e}^{-\xi(1-y)} \\ &= \mu^2\xi^3\mathrm{e}^{-\xi(1-y)} + \mu^2k^2\xi\mathrm{e}^{-\xi(1-y)} := c(\xi)+d(\xi).\end{aligned}$$

The function $c(\xi) := \mu^2\xi^4\mathrm{e}^{-\xi(1-y)}$ attains its maximum

$$c_{\max}(\xi) = c\left(\frac{3}{1-y}\right) = \mu^2\left(\frac{3}{(1-y)e}\right)^3,$$

$d(\xi)$ attains its maximum

$$d_{\max}(\xi)=d\left(\frac{1}{1-y}\right)=\mu^2k^2\left(\frac{1}{(1-y)e}\right).$$

Consequently, for fixed $0<y<1$,

$$D(n)\le C_2\mu^2, \tag{3.21}$$

where $C_2=(\frac{3}{(1-y)e})^3+k^2(\frac{1}{(1-y)e})$.

Hence, for fixed $0<y<1$,

$$\|v^\delta(\cdot,y)-v(\cdot,y)\|\le\frac{1}{2}E^y\delta^{1-y}+C_2\frac{E}{(\ln\frac{E}{\delta})^2}. \tag{3.22}$$

In the following, we consider the case $k\ge 1$. Note that $\|v(\cdot,1)\|\le E$ is equivalent to

$$\|v(\cdot,1)\|^2=\frac{\pi}{2}\sum_{n=1}^{[k]}d_n^2\sinh^2\left(\sqrt{k^2-n^2}\right)+\frac{\pi}{2}\sum_{n=[k]+1}^{[\infty]}d_n^2\sinh^2\left(\sqrt{n^2-k^2}\right)\le E^2. \tag{3.23}$$

Then, by the second equation in (1.10), (2.8), (3.3) and (3.23), we have

$$\begin{aligned}\|v(\cdot,y)-v^\delta(\cdot,y)\|&\le\left(\frac{\pi}{2}\sum_{n=1}^{[k]}(d_n-d_n^\delta)^2\sinh^2\left(\sqrt{k^2-n^2}y\right)\right)^{\frac12}+\left(\frac{\pi}{2}\sum_{n=[k]+1}^{\infty}\left(d_n-d_n^\delta\right)^2\sinh^2\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right)\right)^{\frac12}\\&\quad+\left(\frac{\pi}{2}\sum_{n=[k]+1}^{\infty}d_n^2\left(\sinh\left(\sqrt{n^2-k^2}y\right)-\sinh\left(\sqrt{\frac{n^2-k^2}{1+\mu^2n^2}}y\right)\right)^2\right)^{\frac12}\\&\le\delta+\delta\sup_{n\ge[k]+1}C(n)+E\sup_{n\ge[k]+1}D(n),\end{aligned}$$

where $C(n)$ and $D(n)$ is defined by (3.17). Similar to the case $0<k<1$, we have

$$\delta\sup_{n\ge[k]+1}C(n)\le\frac{1}{2}E^y\delta^{1-y}, \tag{3.24}$$

$$D(n)\le C_2\left(\ln\frac{E}{\delta}\right)^{-2}. \tag{3.25}$$

Therefore, for fixed $0<y<1$ and $k\ge 1$,

$$\|v^\delta(\cdot,y)-v(\cdot,y)\|\le\delta+\frac{1}{2}E^y\delta^{1-y}+C_2\frac{E}{(\ln\frac{E}{\delta})^2}.\quad\square \tag{3.26}$$

Proof of Theorem 2.4. Firstly, consider $0<k<1$, from (1.8) and (2.14), we have

$$\left\|\frac{\partial^p u(\cdot,y)}{\partial y^p}\bigg|_{y=1}\right\|^2=\begin{cases}\frac{\pi}{2}\sum_{n=1}^{\infty}(c_n)^2(n^2-k^2)^p\cosh^2(n)\le E^2, & p\text{ is even},\\ \frac{\pi}{2}\sum_{n=1}^{\infty}(c_n)^2(n^2-k^2)^p\sinh^2(n)\le E^2, & p\text{ is odd}.\end{cases}$$

From (1.10) and (2.14), we have

$$\left\|\frac{\partial^p v(\cdot,y)}{\partial y^p}\bigg|_{y=1}\right\|^2=\begin{cases}\frac{\pi}{2}\sum_{n=1}^{\infty}(d_n)^2(n^2-k^2)^p\sinh^2(n)\le E^2, & p\text{ is even},\\ \frac{\pi}{2}\sum_{n=1}^{\infty}(d_n)^2(n^2-k^2)^p\cosh^2(n)\le E^2, & p\text{ is odd}.\end{cases}$$

In the following, we only discuss the case that p is even, i.e.,

$$\frac{\pi}{2}\sum_{n=1}^{\infty}(c_n)^2(n^2-k^2)^p\cosh^2(n)\le E^2,\qquad\frac{\pi}{2}\sum_{n=1}^{\infty}(d_n)^2(n^2-k^2)^p\sinh^2(n)\le E^2. \tag{3.27}$$

Since the procedure of proof is completely similar when p is odd. Note that $w = u + v$ and $w^\delta = u^\delta + v^\delta$, we have

$$\|w(\cdot,1) - w^\delta(\cdot,1)\| \le \sqrt{\frac{\pi}{2}\sum_{n=1}^{\infty}(c_n)^2\left(\cosh\left(\sqrt{n^2-k^2}\right) - \cosh\left(\frac{\sqrt{n^2-k^2}}{\sqrt{1+\mu^2 n^2}}\right)\right)^2}$$
$$+ \sqrt{\frac{\pi}{2}\sum_{n=1}^{\infty}(c_n - c_n^\delta)^2\cosh^2\left(\frac{\sqrt{n^2-k^2}}{\sqrt{1+\mu^2 n^2}}\right)} + \sqrt{\frac{\pi}{2}\sum_{n=1}^{\infty}(d_n - d_n^\delta)^2\sinh^2\left(\frac{\sqrt{n^2-k^2}}{\sqrt{1+\mu^2 n^2}}\right)}$$
$$+ \sqrt{\frac{\pi}{2}\sum_{n=1}^{\infty}(d_n)^2\left(\sinh\left(\sqrt{n^2-k^2}\right) - \sinh\left(\frac{\sqrt{n^2-k^2}}{\sqrt{1+\mu^2 n^2}}\right)\right)^2}.$$

Now the condition (3.37) and $\|\phi - \phi^\delta\| \le \delta$ and $\|h - h^\delta\| \le \delta$ lead to

$$\|w(\cdot,1) - w^\delta(\cdot,1)\| \le \sup_{n\ge1}\tilde{A}(n)E + \sup_{n\ge1}\tilde{C}(n)\delta + \sup_{n\ge1}\tilde{D}(n)\delta + \sup_{n\ge1}\tilde{B}(n)E, \tag{3.28}$$

where

$$\tilde{A}(n) = \frac{\cosh(\xi) - \cosh(\tau)}{\sqrt{n^2-k^2}^{\,p}\cosh(\xi)}, \qquad \tilde{C}(n) = \cosh(\tau),$$
$$\tilde{B}(n) = \frac{\sinh(n) - \sinh(\tau)}{\sqrt{n^2-k^2}^{\,p}\sinh(\xi)}, \qquad \tilde{D}(n) = \frac{1}{\sqrt{n^2-k^2}}\sinh(\tau).$$

We now start estimating the second and third terms on the right-hand side of (3.28). Since $\cosh(\cdot)$, $\sinh(\cdot)$ are monotone increasing functions in the interval $[0,\infty)$ and μ is chosen in Theorem 2.4, we have

$$\tilde{C}(n)\delta = \cosh\left(\sqrt{n^2-k^2}/\sqrt{1+\mu^2 n^2}\right)\delta \le \cosh(1/\mu)\delta \le e^{1/\mu}\delta = E\left(\ln\frac{E}{\delta}\right)^{-p}, \tag{3.29}$$
$$\tilde{D}(n)\delta = \frac{1}{\sqrt{n^2-k^2}}\sinh\left(\frac{\sqrt{n^2-k^2}}{\sqrt{1+\mu^2 n^2}}\right)\delta \le \frac{1}{2}\frac{1}{\sqrt{1-k^2}}e^{\frac{1}{\mu}}\delta = \frac{1}{2}\frac{1}{\sqrt{1-k^2}}\frac{E}{\left(\ln\frac{E}{\delta}\right)^p}.$$

We now consider $\tilde{A}(n) + \tilde{B}(n)$, taking the similar procedure of Lemmas 2.1 and 2.2, then

$$\tilde{A}(n) + \tilde{B}(n) \le 3(1 - e^{-(\xi-\tau)})/\xi^p. \tag{3.30}$$

For estimating (3.30), we now distinguish between two cases.

Case 1: for large values of n, i.e., for $\xi = \sqrt{n^2-k^2} \ge \frac{1}{\mu^{2/3}}$, note that $\xi \ge \tau$, we have

$$\tilde{A}(n) + \tilde{B}(n) \le 3\frac{1}{\xi^p} \le 3\mu^{\frac{2p}{3}}. \tag{3.31}$$

Case 2: for $\xi < \frac{1}{\mu^{2/3}}$, using inequalities $1 - e^r \le r (r \ge 0)$, we have

$$\tilde{A}(n) + \tilde{B}(n) \le \frac{3}{2}\mu^2\xi^{3-p}. \tag{3.32}$$

If $0 < p < 3$, from (3.32), we have

$$\tilde{A}(n) + \tilde{B}(n) \le \frac{3}{2}\mu^2\left(\frac{1}{\mu^{2/3}}\right)^{3-p} = \frac{3}{2}\mu^{\frac{2p}{3}}. \tag{3.33}$$

If $p \ge 3$, note that $n \ge 1$, from (3.32), we have

$$\tilde{A}(n) + \tilde{B}(n) \le \frac{3}{2}\mu^2 = \frac{3}{2}\mu^2. \tag{3.34}$$

Summarizing (3.31), (3.33) and (3.34), we complete the estimate of the first term and the forth term on the right-hand side of (3.28), i.e.,

$$\tilde{A}(n) + \tilde{B}(n) \le \max\left\{3\mu^{2p/3}, \frac{3}{2}\mu^2\right\}E =: \varepsilon, \quad p > 0. \tag{3.35}$$

Secondly, consider $k \geq 1$, from (1.8) and (2.14), we have

$$\begin{cases} \frac{\pi}{2}\sum_{n=1}^{[k]}(c_n)^2(k^2-n^2)^p\cosh^2\left(\sqrt{k^2-n^2}\right)+\frac{\pi}{2}\sum_{[k]+1}^{\infty}c_n^2(n^2-k^2)^p\cosh^2\left(\sqrt{n^2-k^2}\right)\leq E^2, & p \text{ is even},\\ \frac{\pi}{2}\sum_{n=1}^{[k]}(c_n)^2(k^2-n^2)^p\sinh^2\left(\sqrt{k^2-n^2}\right)+\frac{\pi}{2}\sum_{[k]+1}^{\infty}c_n^2(n^2-k^2)^p\sinh^2\left(\sqrt{n^2-k^2}\right)\leq E^2, & p \text{ is odd}.\end{cases} \tag{3.36}$$

From (1.10) and (2.14), we have

$$\begin{cases} \frac{\pi}{2}\sum_{n=1}^{[k]}(d_n)^2(k^2-n^2)^p\sinh^2\left(\sqrt{k^2-n^2}\right)+\frac{\pi}{2}\sum_{[k]+1}^{\infty}d_n^2(n^2-k^2)^p\sinh^2\left(\sqrt{n^2-k^2}\right)\leq E^2, & p \text{ is even},\\ \frac{\pi}{2}\sum_{n=1}^{[k]}(d_n)^2(k^2-n^2)^p\cosh^2\left(\sqrt{k^2-n^2}\right)+\frac{\pi}{2}\sum_{[k]+1}^{\infty}d_n^2(n^2-k^2)^p\cosh^2\left(\sqrt{n^2-k^2}\right)\leq E^2, & p \text{ is odd}.\end{cases} \tag{3.37}$$

In the following, we only discuss the case that p is even, since the procedure of proof is completely similar when p is odd. Note that $w = u + v$ and $w^\delta = u^\delta + v^\delta$, we have

$$\begin{aligned}\|w(\cdot,1)-w^\delta(\cdot,1)\| \leq{}& \sqrt{\frac{\pi}{2}\sum_{n=1}^{[k]}(c_n-c_n^\delta)^2\cosh^2\left(\sqrt{k^2-n^2}\right)}+\sqrt{\frac{\pi}{2}\sum_{n=1}^{[k]}(d_n-d_n^\delta)^2\sinh^2\left(\sqrt{k^2-n^2}\right)}\\ &+\sqrt{\frac{\pi}{2}\sum_{[k]+1}^{\infty}(c_n)^2\left(\cosh\left(\sqrt{n^2-k^2}\right)-\cosh\left(\frac{\sqrt{n^2-k^2}}{\sqrt{1+\mu^2n^2}}\right)\right)^2}\\ &+\sqrt{\frac{\pi}{2}\sum_{[k]+1}^{\infty}(c_n-c_n^\delta)^2\cosh^2\left(\frac{\sqrt{n^2-k^2}}{\sqrt{1+\mu^2n^2}}\right)}+\sqrt{\frac{\pi}{2}\sum_{[k]=1}^{\infty}(d_n-d_n^\delta)^2\sinh^2\left(\frac{\sqrt{n^2-k^2}}{\sqrt{1+\mu^2n^2}}\right)}\\ &+\sqrt{\frac{\pi}{2}\sum_{[k]+1}^{\infty}(d_n)^2\left(\sinh\left(\sqrt{n^2-k^2}\right)-\sinh\left(\frac{\sqrt{n^2-k^2}}{\sqrt{1+\mu^2n^2}}\right)\right)^2}.\end{aligned}$$

Now the condition (3.36), (3.37) and $\|\phi-\phi^\delta\| \leq \delta$ and $\|h-h^\delta\| \leq \delta$ lead to

$$\begin{aligned}\|w(\cdot,1)-w^\delta(\cdot,1)\| \leq{}& 2\delta+\sup_{n\geq[k]+1}\frac{\cosh\xi-\cosh\tau}{\xi^p\cosh\xi}E+\delta\sup_{n\geq[k]+1}\cosh\tau\\ &+\sup_{n\geq[k]+1}\frac{\sinh\xi-\sinh\tau}{\xi^p\sinh\xi}E+\delta\sqrt{\frac{1+\mu^2n^2}{n^2-k^2}}\sup_{n\geq[k]+1}\sinh\tau.\end{aligned}$$

Similar to the case $0 < k < 1$, note that $\xi \geq 1$, by (3.29), we have

$$\delta\sup_{n\geq[k]+1}\cosh\tau \leq \cosh(1/\mu)\delta \leq e^{1/\mu}\delta = E\left(\ln\frac{E}{\delta}\right)^{-p} \tag{3.38}$$

$$\delta\sqrt{\frac{1}{n^2-k^2}}\sup_{n\geq[k]+1}\sinh\tau \leq \frac{1}{2}\delta\frac{1}{\sqrt{n^2-k^2}}e^{\frac{1}{\mu}} \leq \frac{1}{2}\frac{E}{(\ln\frac{E}{\delta})^p} \tag{3.39}$$

$$\sup_{n\geq[k]+1}\frac{\cosh\xi-\cosh\tau}{\xi^p\cosh\xi}E+\sup_{n\geq[k]+1}\frac{\sinh\xi-\sinh\tau}{\xi^p\sinh\xi}E \leq 3(1-e^{-(\xi-\tau)})\xi^{-p}E. \tag{3.40}$$

Case 1: for large values of n, i.e., for $\xi = \sqrt{n^2-k^2} \geq \frac{1}{\mu^{2/3}}$, note that $\xi \geq \tau$, we have

$$\left(\sup_{n\geq[k]+1}\frac{\cosh\xi-\cosh\tau}{\xi^p\cosh\xi}+\sup_{n\geq[k]+1}\frac{\sinh\xi-\sinh\tau}{\xi^p\sinh\xi}\right)E \leq 3\xi^{-p}E = 3\mu^{\frac{2p}{3}}E. \tag{3.41}$$

Case 2: for $\xi < \frac{1}{\mu^{2/3}}$, note that $\xi > 1$, we have

$$1.\ \text{if } p \geq 3,\quad \sup_{n\geq[k]+1}\frac{\cosh\xi-\cosh\tau}{\xi^p\cosh\xi}+\sup_{n\geq[k]+1}\frac{\sinh\xi-\sinh\tau}{\xi^p\sinh\xi}E \leq \frac{3}{2}\mu^2k^2; \tag{3.42}$$

$$2.\ \text{if } 0 < p < 3, \quad \sup_{n \geq [k]+1} \frac{\cosh \xi - \cosh \tau}{\xi^p \cosh \xi} + \sup_{n \geq [k]+1} \frac{\sinh \xi - \sinh \tau}{\xi^p \sinh \xi} E \leq \frac{3}{2} \mu^{2p/3} k^2. \tag{3.43}$$

Therefore, for $k \geq 1$, by (3.41)–(3.43), the second estimate in (2.16) is satisfied. □

4. Concluding remark

In this paper, we consider the non-characteristic Cauchy problem for the Helmholtz equation. Some logarithmic stability estimates are proved. The logarithmic stability estimates are much weak. We hope to obtain the Hölder stability estimates by the spectral cut-off method. This will be studied in the forthcoming paper.

Acknowledgements

The authors would like to thank the reviewers for their very careful reading and for pointing out several mistakes as well as for their useful comments and suggestions.

References

[1] L. Marin, L. Elliott, P.J. Heggs, D.B. Ingham, D. Lesnic, X. Wen, An alternating iterative algorithm for the Cauchy problem associated to the Helmholtz equation, Comput. Meth. Appl. Mech. Eng. 192 (2003) 709–722.
[2] L. Marin, L. Elliott, P.J. Heggs, D.B. Ingham, D. Lesnic, X. Wen, Conjugate gradient-boundary element solution to the Cauchy problem for Helmholtz-type equations, Comput. Mech. 31 (2003) 367–377.
[3] T. Wei, Y.C. Hon, L. Ling, Method of fundamental solutions with regularization techniques for Cauchy problems of elliptic operators, Eng. Anal. Bound. Elem. 31 (4) (2007) 373–385.
[4] L. Marin, A meshless method for the numerical solution of the Cauchy problem associated with three-dimensional Helmholtz-type equations, Appl. Math. Comput. 165 (2) (2005) 355–374.
[5] L. Marin, L. Elliott, P.J. Heggs, D.B. Ingham, D. Lesnic, X. Wen, BEM solution for the Cauchy problem associated with Helmholtz-type equations by the Landweber method, Eng. Anal. Bound. Elem. 28 (2004) 1025–1034.
[6] L. Marin, D. Lesnic, The method of fundamental solutions for the Cauchy problem associated with two-dimensional Helmholtz-type equations, Comput. Struct. 83 (2005) 267–278.
[7] L. Marin, L. Elliott, P.J. Heggs, D.B. Ingham, D. Lesnic, X. Wen, Comparison of regularization methods for solving the Cauchy problem associated with the Helmholtz equation, Int. J. Numer. Meth. Eng. 60 (11) (2004) 1933–1947.
[8] H.H. Qin, T. Wei, Modified regularization method for the Cauchy problem of the Helmholtz equation, Appl. Math. Modell. 33 (2009) 2334–2348.
[9] X.T. Xiong, Regularization theory and algorithm for some inverse problems for parabolic differential equations, Ph.D. Dissertation, Lanzhou University, 2007 (in Chinese).