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An inverse problem for undamped gyroscopic systems[☆]

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ABSTRACT

Linear undamped gyroscopic systems are defined by three real matrices, $M > 0$, $K > 0$, and G ($G^T = -G$); the mass, stiffness, and gyroscopic matrices, respectively. In this paper an inverse problem is considered: given complete information about eigenvalues and eigenvectors, $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{2n-1}, \lambda_{2n}\} \in \mathbf{C}^{2n\times 2n}$ and $X = [x_1, x_2, \ldots, x_{2n-1}, x_{2n}] \in \mathbf{C}^{n\times 2n}$, where the diagonal elements of Λ are all purely imaginary, X is of full row rank n, and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \ldots, n$, find M, K and G such that $MX\Lambda^2 + GX\Lambda + KX = 0$. The solvability condition for the inverse problem and a solution to the problem are presented, and the results of the inverse problem are applied to develop a method for model updating.

1. Introduction

Free oscillations of multi dof conservative gyroscopic systems are governed by the set of second-order differential equations

$$M\ddot{q}(t) + G\dot{q}(t) + Kq(t) = 0, \tag{1.1}$$

where the vector $q(t)$ represents the generalized coordinates of the system, M, K and G are $n \times n$ real matrices. The mass matrix, M, and stiffness matrix, K, are symmetric positive definite, and the gyroscopic matrix G is always skew-symmetric (that is, $G^T = -G$).

It is well-known that all solutions of this differential equation can be obtained by solving the quadratic eigenvalue problem (QEP)

$$(\lambda^2 M + \lambda G + K)x = 0. \tag{1.2}$$

Complex numbers λ and nonzero vectors x for which this relation holds are, respectively, the eigenvalues and eigenvectors of the system. If λ_j, x_j form an eigenvalue–eigenvector pair, then there is a natural association between the diagonal matrix Λ of $2n$ eigenvalues and an $n \times 2n$ matrix X with corresponding eigenvectors as its columns. Then it is easy to see that the $2n$ columns of the matrix

$$MX\Lambda^2 + GX\Lambda + KX = 0 \tag{1.3}$$

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summarize $2n$ separate eigenvalue–eigenvector relations of type (1.2). It will be convenient to define the "λ-matrix" or "matrix polynomial",

$$L(\lambda) = \lambda^2 M + \lambda G + K. \tag{1.4}$$

It is well-understood that the quadratic eigenvalue problem of Eq. (1.4) can be studied via the linearized $2n \times 2n$ eigenvalue problem $\lambda A - B$ where

$$A := \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix}, \qquad B := \begin{bmatrix} 0 & K \\ -K & -G \end{bmatrix}. \tag{1.5}$$

Thus, if σ denotes the set of all eigenvalues (the spectrum), then

$$\sigma(\lambda^2 M + \lambda G + K) = \sigma\left(\lambda \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} - \begin{bmatrix} 0 & K \\ -K & -G \end{bmatrix}\right).$$

Also, $(\lambda^2 M + \lambda G + K)x = 0$ is equivalent to

$$\left(\lambda \begin{bmatrix} K & 0 \\ 0 & M \end{bmatrix} - \begin{bmatrix} 0 & K \\ -K & -G \end{bmatrix}\right) \begin{bmatrix} x \\ \lambda x \end{bmatrix} = 0.$$

Rotating bodies, axially moving materials, and fluid-conveying pipes are examples of gyroscopic systems. The "forward" problem is, of course, to find the eigenvalues and eigenvectors when the coefficient matrices are given, and has been studied by several authors [1–7]. Our main interest in this paper is the corresponding inverse problem: Given complete information about eigenvalues and eigenvectors, re-construct the coefficient matrices. More precisely, we consider the following problem.

Problem IP. Given a pair of matrices (Λ, X) in the form

$$\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_{2n-1}, \lambda_{2n}\} \in \mathbf{C}^{2n\times 2n} \tag{1.6}$$

and

$$X = [x_1, x_2, \ldots, x_{2n-1}, x_{2n}] \in \mathbf{C}^{n\times 2n}, \tag{1.7}$$

where the diagonal elements of Λ are all purely imaginary, X is of full row rank n, and both Λ and X are closed under complex conjugation in the sense that $\lambda_{2j} = \bar{\lambda}_{2j-1} \in \mathbf{C}$, $x_{2j} = \bar{x}_{2j-1} \in \mathbf{C}^n$ for $j = 1, \ldots, n$, find $M > 0$, $K > 0$ and skew-symmetric matrix G that satisfy the equation of (1.3). In other words, each pair (λ_t, x_t), $t = 1, \ldots, 2n$, is an eigenpair of the quadratic pencil $L(\lambda)$.

The problem of reconstruction of the physical properties from spectral data is classified as an inverse eigenvalue problem. The inverse eigenvalue problem is a diverse area full of research interests and activities. Some general reviews and extensive bibliographies can be found in [8–10]. The latest progress in solving the inverse eigenvalue problems has been detailed in the recent book in [11]. Recently, the inverse quadratic eigenvalue problem (IQEP) has received much attention. For example, based on the spectral theory of matrix polynomials, Lancaster et al. considered the IQEP of constructing real matrices M, C, and K [12,13], Hermitian matrices M, C, and K [14], and real symmetric positive definite or semidefinite matrices M, C, and K [13,15] so that the quadratic pencil $Q(\lambda) = \lambda^2 M + \lambda C + K$ has the complete information on eigenvalues and eigenvectors. The inverse spectral problems of determining real symmetric matrices C and K so that the monic quadratic pencil $Q(\lambda) = \lambda^2 I_n + \lambda C + K$ possesses the complete spectral information have been solved in [16–19]. Observe that in a large or complicated structural system, it is often impossible to measure complete spectral information due to the finite bandwidth of measuring devices. It might be more reasonable to consider an IQEP where only partial spectral information is prescribed. Chu et al. [20] considered the problem of recovering a serially linked, damped mass-spring system from two prescribed eigenpairs. Bai [21] considered the problem of determining real symmetric tridiagonal matrices C and K so that the monic quadratic pencil $Q(\lambda) = \lambda^2 I_n + \lambda C + K$ possesses partially described eigenpairs. For the IQEP with $k(k \le n)$ prescribed eigenpairs, special symmetric solutions M, C, and K with M and K being symmetric positive definite, a general solution and some particular solutions with additional eigeninformation have been derived in [22,23], respectively. For the IQEP with $k(n < k \le 2n)$ prescribed eigenpairs, the solvability theory of the problem and the general symmetric solution have been given in [24]. We note that in several recent works for gyroscopic systems, including those in [25,26], as well as Sarkissian [27], studies are undertaken toward a feedback design problem for a second-order control system. That consideration eventually leads to either a full or a partial eigenstructure assignment problem for the QEP. The proportional and derivative state feedback controller designated in these studies is capable of assigning specific eigenvalues. Nonetheless, these results cannot meet the basic requirement that the updated matrices be symmetric or skew-symmetric. In contrast to the development of IQEP for damping structural systems, however, to the best of our knowledge, few results can be found in the literature to solve the Problem IP directly.

The goal of this paper is to derive conditions on the spectral information under which the Problem IP is solvable, and provide the representation of a solution to the problem. By applying the results of the inverse problem, we develop a method for model updating problem which has important implications in the modal testing and finite element industry.

In this paper we shall adopt the following notation. $\mathbf{C}^{m\times n}$, $\mathbf{R}^{m\times n}$ denote the set of all $m\times n$ complex and real matrices, respectively. Capital letters $A, B, C, \ldots$, denote matrices, lower case letters denote column vectors, Greek letters denote scalars, $\bar{\alpha}$ denotes the conjugate of the complex number α. I_n denotes the $n\times n$ identity matrix. $A^T, A^+, R(A)$ and $N(A)$ denote the transpose, the Moore–Penrose generalized inverse, the range space and the null space of A, respectively. $P_{R(A)}$ denotes the orthogonal projector on $R(A)$, and $\|\cdot\|$ stands for the matrix Frobenius norm. We write $A>0$ if A is a real symmetric positive definite matrix.

2. Solving Problem IP

Let

$$W_{2n} = \operatorname{diag}\left\{\frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}, \ldots, \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}\right\} \in \mathbf{C}^{2n\times 2n} \tag{2.1}$$

with $i=\sqrt{-1}$. Then

$$\tilde{\Lambda} = \bar{W}_{2n}^T \Lambda W_{2n} = \operatorname{diag}\left\{\begin{bmatrix}0 & \beta_1\\ -\beta_1 & 0\end{bmatrix}, \ldots, \begin{bmatrix}0 & \beta_{2n-1}\\ -\beta_{2n-1} & 0\end{bmatrix}\right\} \in \mathbf{R}^{2n\times 2n} \tag{2.2}$$

and

$$\tilde{X} = XW_{2n} = [\sqrt{2}y_1, \sqrt{2}z_1, \ldots, \sqrt{2}y_{2n-1}, \sqrt{2}z_{2n-1}] \in \mathbf{R}^{n\times 2n}, \tag{2.3}$$

where $\beta_i = \operatorname{Im}(\lambda_i)$ (the imaginary part of the complex number λ_i), $y_i = \operatorname{Re}(x_i)$ (the real part of the complex vector x_i), $z_i = \operatorname{Im}(x_i)$ for $i = 1, 3, \ldots, 2n-1$. Thus, the equation of (1.3) can be written equivalently as

$$M\tilde{X}\tilde{\Lambda}^2 + G\tilde{X}\tilde{\Lambda} + K\tilde{X} = 0. \tag{2.4}$$

Theorem 2.1. *Let matrix pair* $(\Lambda, X) \in \mathbf{C}^{2n\times 2n}\times\mathbf{C}^{n\times 2n}$ *be given as in (1.6) and (1.7). By the unitary transformation* W_{2n}, (Λ, X) *is transformed into real matrix pair* $(\tilde{\Lambda}, \tilde{X}) \in \mathbf{R}^{2n\times 2n}\times\mathbf{R}^{n\times 2n}$ *expressed as in (2.2) and (2.3). If*

$$\tilde{X}\tilde{\Lambda}\tilde{X}^T = 0, \tag{2.5}$$

then Problem IP is solvable and a solution to Problem IP gives

$$M = (\tilde{X}\tilde{\Lambda}\tilde{\Lambda}^T\tilde{X}^T)^{-1} = -(\tilde{X}\tilde{\Lambda}^2\tilde{X}^T)^{-1}, \tag{2.6}$$

$$K = (\tilde{X}\tilde{X}^T)^{-1}, \tag{2.7}$$

$$G = M\tilde{X}\tilde{\Lambda}^3\tilde{X}^T M. \tag{2.8}$$

Proof. From (2.6)–(2.8) we have

$$\begin{aligned} M\tilde{X}\tilde{\Lambda}^2 + G\tilde{X}\tilde{\Lambda} + K\tilde{X} &= M(\tilde{X}\tilde{\Lambda}^2 + \tilde{X}\tilde{\Lambda}^3\tilde{X}^T(\tilde{X}\tilde{\Lambda}\tilde{\Lambda}^T\tilde{X}^T)^{-1}\tilde{X}\tilde{\Lambda} + \tilde{X}\tilde{\Lambda}\tilde{\Lambda}^T\tilde{X}^T(\tilde{X}\tilde{X}^T)^{-1}\tilde{X})\\ &= M\tilde{X}\tilde{\Lambda}^2(I_{2n} - \tilde{\Lambda}^T\tilde{X}^T(\tilde{X}\tilde{\Lambda}\tilde{\Lambda}^T\tilde{X}^T)^{-1}\tilde{X}\tilde{\Lambda} - \tilde{X}^T(\tilde{X}\tilde{X}^T)^{-1}\tilde{X})\\ &= M\tilde{X}\tilde{\Lambda}^2(I_{2n} - (\tilde{X}\tilde{\Lambda})^+\tilde{X}\tilde{\Lambda} - \tilde{X}^+\tilde{X})\\ &= M\tilde{X}\tilde{\Lambda}^2\left(P_{N(\tilde{X}\tilde{\Lambda})} - P_{R(\tilde{X}^T)}\right). \end{aligned}$$

It follows from $\tilde{X}\tilde{\Lambda}\tilde{X}^T = 0$ and $\operatorname{rank}(\tilde{X}\tilde{\Lambda}) = \operatorname{rank}(\tilde{X}^T)$ that $R(\tilde{X}^T) = N(\tilde{X}\tilde{\Lambda})$. Thus, $M\tilde{X}\tilde{\Lambda}^2 + G\tilde{X}\tilde{\Lambda} + K\tilde{X} \equiv 0$, as required. □

Next, we prove that for a given system there is an eigenmatrix pair such that the condition (2.5) holds. So, the hypotheses of (2.5) is with reason.

Theorem 2.2. *Let* $n\times n$ *real matrices* $M>0, K>0$ *and* $G(G^T=-G)$ *be given and* $\sigma(L(\lambda)) = \Lambda = \sigma(\lambda I_{2n} - \tilde{\Lambda})$, *as above. Then the eigenvectors of* $L(\lambda)$ *can be normalized in such a way that (2.5) holds. Also, the known matrices* M, G, K *can be written as in the form (2.6)–(2.8).*

Proof. It follows from $M>0, K>0$ that $A = \begin{bmatrix}K & 0\\ 0 & M\end{bmatrix} > 0$. Therefore, there exists a nonsingular matrix $P\in\mathbf{R}^{2n\times 2n}$ such that $P^TAP = I_{2n}$. Notice that $P^TBP = P^T\begin{bmatrix}0 & K\\ -K & -G\end{bmatrix}P$is still a skew-symmetric matrix. Then there is an orthogonal matrix $U\in\mathbf{R}^{2n\times 2n}$ such that

$$U^T(P^TBP)U = \tilde{\Lambda}.$$

Write $Q = PU$. Then we have

$$Q^TAQ = I_{2n}, \qquad Q^TBQ = \tilde{\Lambda}. \tag{2.9}$$

Let $Q = \begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix}$, $\tilde{X} \in \mathbf{R}^{n\times 2n}$. It follows from $AQ\tilde{\Lambda} = BQ$ that $\tilde{Y} = \tilde{X}\tilde{\Lambda}$ and the equation of (2.4) holds. Using the first equation of (2.9), we get $QQ^T = A^{-1}$ and hence

$$\begin{bmatrix} \tilde{X} \\ \tilde{X}\tilde{\Lambda} \end{bmatrix} \begin{bmatrix} \tilde{X}^T & \tilde{\Lambda}^T\tilde{X}^T \end{bmatrix} = \begin{bmatrix} K^{-1} & 0 \\ 0 & M^{-1} \end{bmatrix}. \tag{2.10}$$

By comparison matrix entries of Eq. (2.10), we obtain (2.5)–(2.7). Substituting (2.5)–(2.7) into (2.4) yields (2.8). □

3. An application to the model updating problem

Formulas (2.6)–(2.8) can now be further partitioned to solve the model correction problem. Consider repartitioning $\tilde{X}$, $\tilde{\Lambda}$ as

$$\tilde{X} = \begin{bmatrix} \tilde{X}_1 & \tilde{X}_2 \end{bmatrix}, \qquad \tilde{\Lambda} = \begin{bmatrix} \tilde{\Lambda}_1 & 0 \\ 0 & \tilde{\Lambda}_2 \end{bmatrix}. \tag{3.1}$$

Here, $\tilde{X}_1 \in \mathbf{R}^{n\times m}$, $\tilde{\Lambda}_1 \in \mathbf{R}^{m\times m}$ and m is even. With these partitions, equations of (2.5)–(2.8) become

$$\tilde{X}_1\tilde{\Lambda}_1\tilde{X}_1^T + \tilde{X}_2\tilde{\Lambda}_2\tilde{X}_2^T = 0, \tag{3.2}$$

$$M^{-1} = -\tilde{X}_1\tilde{\Lambda}_1^2\tilde{X}_1^T - \tilde{X}_2\tilde{\Lambda}_2^2\tilde{X}_2^T, \tag{3.3}$$

$$K^{-1} = \tilde{X}_1\tilde{X}_1^T + \tilde{X}_2\tilde{X}_2^T, \tag{3.4}$$

$$M^{-1}GM^{-1} = \tilde{X}_1\tilde{\Lambda}_1^3\tilde{X}_1^T + \tilde{X}_2\tilde{\Lambda}_2^3\tilde{X}_2^T. \tag{3.5}$$

Let the original (known) system matrices be denoted by Λ_0, X_0, M_0, K_0 and G_0. Quantities of the new or updated system are denoted by Λ, X, M, K and G. The corrections to the original system are denoted by $\Delta\Lambda$, ΔX, ΔM, ΔK and ΔG. Thus, the spectral properties of the new system are

$$\Lambda = \Lambda_0 + \Delta\Lambda, \qquad X = X_0 + \Delta X. \tag{3.6}$$

Let $\tilde{\Lambda} = \bar{W}_{2n}^T\Lambda W_{2n}$, $\tilde{X} = XW_{2n}$. If $\tilde{X}\tilde{\Lambda}\tilde{X}^T = 0$ and $\mathrm{rank}(\tilde{X}) = n$, then the equation of (3.6) provides a solution to the model correction problem when substituted into Eqs. (2.6)–(2.8). That is if Λ_0, X_0 are computed from known values of M_0, K_0 and G_0, say from a finite element model, then this model can be updated, by assigning values of $\Delta\Lambda$, ΔX to yield a new, or modified, set of coefficient matrices M, K and G.

The problem of more practical interest is to modify M, K and G by changing only part of the spectral data (see [12,15]). In this case assume that the matrices of incomplete spectral information $\Lambda_{10} \in \mathbf{C}^{m\times m}$, $X_{10} \in \mathbf{C}^{n\times m}$ are known for the first m eigenvalues and associated eigenvectors of the original system. The remainder of the spectral properties $\Lambda_{20} \in \mathbf{C}^{(2n-m)\times(2n-m)}$, $X_{20} \in \mathbf{C}^{n\times(2n-m)}$ are not changed and are unknown. Let $\tilde{X}_{10} = X_{10}W_m$, $\tilde{\Lambda}_{10} = \bar{W}_m^T\Lambda_{10}W_m$, $\tilde{X}_{20} = X_{20}W_{2n-m}$, $\tilde{\Lambda}_{20} = \bar{W}_{2n-m}^T\Lambda_{20}W_{2n-m}$. Then the original system, according to Theorem 2.2, can be written as

$$M_0^{-1} = -\tilde{X}_{10}\tilde{\Lambda}_{10}^2\tilde{X}_{10}^T - \tilde{X}_{20}\tilde{\Lambda}_{20}^2\tilde{X}_{20}^T, \tag{3.7}$$

$$K_0^{-1} = \tilde{X}_{10}\tilde{X}_{10}^T + \tilde{X}_{20}\tilde{X}_{20}^T, \tag{3.8}$$

$$M_0^{-1}G_0M_0^{-1} = \tilde{X}_{10}\tilde{\Lambda}_{10}^3\tilde{X}_{10}^T + \tilde{X}_{20}\tilde{\Lambda}_{20}^3\tilde{X}_{20}^T \tag{3.9}$$

with the condition that

$$\tilde{X}_{10}\tilde{\Lambda}_{10}\tilde{X}_{10}^T + \tilde{X}_{20}\tilde{\Lambda}_{20}\tilde{X}_{20}^T = 0. \tag{3.10}$$

Following Eq. (3.6), the new spectral matrices become $\tilde{\Lambda}_1 = \tilde{\Lambda}_{10} + \Delta\tilde{\Lambda}_1$, $\tilde{X}_1 = \tilde{X}_{10} + \Delta\tilde{X}_1$, so that the system matrices become

$$M^{-1} = -\tilde{X}_1\tilde{\Lambda}_1^2\tilde{X}_1^T - \tilde{X}_{20}\tilde{\Lambda}_{20}^2\tilde{X}_{20}^T, \tag{3.11}$$

$$K^{-1} = \tilde{X}_1\tilde{X}_1^T + \tilde{X}_{20}\tilde{X}_{20}^T, \tag{3.12}$$

$$M^{-1}GM^{-1} = \tilde{X}_1\tilde{\Lambda}_1^3\tilde{X}_1^T + \tilde{X}_{20}\tilde{\Lambda}_{20}^3\tilde{X}_{20}^T \tag{3.13}$$

with the condition that

$$\tilde{X}_1\tilde{\Lambda}_1\tilde{X}_1^T + \tilde{X}_{20}\tilde{\Lambda}_{20}\tilde{X}_{20}^T = 0. \tag{3.14}$$

By subtracting Eqs. (3.7)–(3.9) from (3.11)–(3.13), we have

Theorem 3.1. *Let $n \times n$ real matrices $M_0 > 0, K_0 > 0$ and $G_0(G_0^T = -G_0)$ be given and $\Lambda_{10} \in \mathbf{C}^{m\times m}, X_{10} \in \mathbf{C}^{n\times m}$ be known for the first m (m is even) eigenvalues and associated eigenvectors of the original system $L_0(\lambda) = \lambda^2 M_0 + \lambda G_0 + K_0$. Assume that the corrections to the original spectral data are $\Delta\Lambda_1, \Delta X_1$. Let $\tilde{X}_{10} = X_{10}W_m$, $\tilde{\Lambda}_{10} = \bar{W}_m^T \Lambda_{10} W_m$, $\tilde{X}_1 = (X_{10} + \Delta X_1)W_m$, $\tilde{\Lambda}_1 = \bar{W}_m^T(\Lambda_{10} + \Delta\Lambda_1)W_m$. If*

$$\tilde{X}_1\tilde{\Lambda}_1\tilde{X}_1^T = \tilde{X}_{10}\tilde{\Lambda}_{10}\tilde{X}_{10}^T, \tag{3.15}$$

then the updated system with the coefficient matrices

$$M = (M_0^{-1} + \Delta M)^{-1} = (I_n + M_0\Delta M)^{-1}M_0, \tag{3.16}$$

$$K = (K_0^{-1} + \Delta K)^{-1} = (I_n + K_0\Delta K)^{-1}K_0, \tag{3.17}$$

$$G = M(M_0^{-1}G_0M_0^{-1} + \Delta G)M, \tag{3.18}$$

where

$$\Delta M = \tilde{X}_{10}\tilde{\Lambda}_{10}^2\tilde{X}_{10}^T - \tilde{X}_1\tilde{\Lambda}_1^2\tilde{X}_1^T, \tag{3.19}$$

$$\Delta K = \tilde{X}_1\tilde{X}_1^T - \tilde{X}_{10}\tilde{X}_{10}^T, \tag{3.20}$$

$$\Delta G = \tilde{X}_1\tilde{\Lambda}_1^3\tilde{X}_1^T - \tilde{X}_{10}\tilde{\Lambda}_{10}^3\tilde{X}_{10}^T, \tag{3.21}$$

has the assigned partial spectral data $\Lambda_1, \tilde{X}_1$ and the remainder spectral data are the same as those of the original system.

We remark that if $\|M_0\Delta M\| \ll 1, \|K_0\Delta K\| \ll 1$, then we can avoid inverse calculations by the following admissible approximating formulas:

$$M \approx M_0 - M_0\Delta M M_0,$$
$$K \approx K_0 - K_0\Delta K K_0,$$
$$G \approx G_0 + M_0\Delta G M_0 - M_0\Delta M G_0 - G_0\Delta M M_0.$$

4. Numerical examples

Based on Theorem 2.1 we can state the following algorithm.

Algorithm 4.1 (*An Algorithm for Solving Problem IP*)**.**

(1) Input Λ, X;
(2) Compute the matrix W_{2n} according to (2.1);
(3) Compute the matrices $\tilde{\Lambda}, \tilde{X}$ according to (2.2) and (2.3);
(4) If (2.5) holds, then continue, otherwise, go to (1);
(5) According to (2.6)–(2.8) calculate M, K and G.

Example 4.1 (*Spatial Oscillations of a Particle [25]*)**.** Let a particle of mass m be connected to a ring of radius α via two springs of constant κ and free length β. Suppose that the ring is rotating with constant angular velocity $\omega = (\omega_1, \omega_2, \omega_3)^T$. Then for small oscillations about the equilibrium position, the equation of motion is

$$M_0\ddot{u} + G_0\dot{u} + K_0u = 0, \tag{4.1}$$

where

$$M_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad G_0 = \begin{bmatrix} 0 & -2\omega_3 & 2\omega_2 \\ 2\omega_3 & 0 & -2\omega_1 \\ -2\omega_2 & 2\omega_1 & 0 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} 2\omega_n^2 - \omega_2^2 - \omega_3^2 & \omega_1\omega_2 & \omega_1\omega_3 \\ \omega_1\omega_2 & 2\omega_n^2\gamma - \omega_1^2 - \omega_3^2 & \omega_2\omega_3 \\ \omega_1\omega_3 & \omega_2\omega_3 & 2\omega_n^2\gamma - \omega_1^2 - \omega_2^2 \end{bmatrix},$$

$\gamma = 1 - \beta/\alpha$, $\omega_n^2 = \kappa/m$, and u_1, u_2, u_3 are the displacements of the mass in the x-, y-, z-direction.

Let

$$\Lambda = \mathrm{diag}\{6.0860i, -6.0860i, 3.1895i, -3.1895i, 0.8878i, -0.8878i\},$$

$$X = \begin{bmatrix} -0.0000 - 0.1079i & 0.0438 + 0.0293i & -0.0877 + 0.0147i \\ -0.0000 + 0.1079i & 0.0438 - 0.0293i & -0.0877 - 0.0147i \\ 0.0078 - 0.0695i & 0.0000 - 0.1880i & 0.0518 - 0.0882i \\ 0.0078 + 0.0695i & 0.0000 + 0.1880i & 0.0518 + 0.0882i \\ -0.0279 - 0.1519i & -0.1497 + 0.1592i & 0.3579 \\ -0.0279 + 0.1519i & -0.1497 - 0.1592i & 0.3579 \end{bmatrix}^T.$$

According to Algorithm 4.1, it is calculated that the condition (2.5) holds. Using the Software "MATLAB 6.5", we can get

$$M = \begin{bmatrix} 1.0000 & -0.0000 & 0.0000 \\ -0.0000 & 1.0000 & -0.0000 \\ 0.0000 & -0.0000 & 1.0000 \end{bmatrix},$$

$$K = \begin{bmatrix} 13.0000 & 2.0000 & 1.0000 \\ 2.0000 & 7.0000 & 2.0000 \\ 1.0000 & 2.0000 & 4.0000 \end{bmatrix},$$

$$G = \begin{bmatrix} -0.0000 & -2.0000 & 4.0000 \\ 2.0000 & 0.0000 & -2.0000 \\ -4.0000 & 2.0000 & 0.0000 \end{bmatrix}.$$

Also, we can figure out: $\|MX\Lambda^2 + GX\Lambda + KX\| = 9.8156\mathrm{e}{-014}$. The resulting matrices are coincident with the results in [25] by setting $\omega = [1, 2, 1]^T$, $\omega_n = 3$, $\gamma = 1/2$.

According to Theorem 3.1, we can state the following algorithm.

Algorithm 4.2 (*An Algorithm for Solving Model UpdatingProblem*)**.**

(1) Input $M_0, K_0, G_0, \Lambda_{10}, X_{10}, \Delta\Lambda_1, \Delta X_1$;
(2) Compute the matrices $\tilde{X}_{10} = X_{10}W_m$, $\tilde{\Lambda}_{10} = \bar{W}_m^T\Lambda_{10}W_m$, $\tilde{X}_1 = (X_{10} + \Delta X_1)W_m$, $\tilde{\Lambda}_1 = \bar{W}_m^T(\Lambda_{10} + \Delta\Lambda_1)W_m$;
(3) If (3.15) holds, then continue, otherwise, go to (1);
(4) Calculate ΔM, ΔK and ΔG by (3.19)–(3.21);
(5) According to (3.16)–(3.18) calculate M, K and G.

The following example comes from [28] with $K_0(2, 2) = 6$, $K_0(3, 3) = 5$.

Example 4.2. Let

$$M_0 = \begin{bmatrix} 8 & -2 & 1 & 0 \\ -2 & 10 & 4 & 4 \\ 1 & 4 & 10 & -1.2 \\ 0 & 4 & -1.2 & 8 \end{bmatrix},$$

$$K_0 = \begin{bmatrix} 4 & -3 & 2 & 0 \\ -3 & 6 & 1 & -3 \\ 2 & 1 & 5 & -2 \\ 0 & -3 & -2 & 4 \end{bmatrix},$$

$$G_0 = \begin{bmatrix} 0 & -16 & -8 & -12 \\ 16 & 0 & -40 & -12 \\ 8 & 40 & 0 & 16 \\ 12 & 12 & -16 & 0 \end{bmatrix}$$

and

$$\Lambda_{10} = \mathrm{diag}\{7.1836i, -7.1836i, 1.1572i, -1.1572i\},$$

$$X_{10} = \begin{bmatrix} 0.3476 + 0.0507i & 0.3476 - 0.0507i & 0.3897 + 0.3057i & 0.3897 - 0.3057i \\ 0.5514 + 0.2928i & 0.5514 - 0.2928i & -0.0934 - 0.3694i & -0.0934 + 0.3694i \\ -0.5179 + 0.1645i & -0.5179 - 0.1645i & 0.1176 - 0.1233i & 0.1176 + 0.1233i \\ -0.3845 + 0.2093i & -0.3845 - 0.2093i & -0.0990 + 0.7554i & -0.0990 - 0.7554i \end{bmatrix},$$

$$\Delta\Lambda_1 = \mathrm{diag}\{-0.7004i, 0.7004i, -0.11283i, 0.11283i\},$$

$$\Delta X_1 = \begin{bmatrix} 0.0183 + 0.0027i & 0.0183 - 0.0027i & 0.0205 + 0.0161i & 0.0205 - 0.0161i \\ 0.0290 + 0.0154i & 0.0290 - 0.0154i & -0.0049 - 0.0194i & -0.0049 + 0.0194i \\ -0.0273 + 0.0087i & -0.0273 - 0.0087i & 0.0062 - 0.0065i & 0.0062 + 0.0065i \\ -0.0202 + 0.0110i & -0.0202 - 0.0110i & -0.0052 + 0.0398i & -0.0052 - 0.0398i \end{bmatrix}.$$

That is, the prescribed eigenvalues and eigenvectors are

$$\Lambda_1 = \Lambda_{10} + \Delta\Lambda_1 = \mathrm{diag}\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}, \qquad X_1 = X_{10} + \Delta X_1 = [x_1, x_2, x_3, x_4].$$

According to Algorithm 4.2, it is calculated that the condition (3.15) holds. Using the Software "MATLAB", we can figure out

$$M = \begin{bmatrix} 8.4053 & -2.1196 & 0.7783 & 0.1537 \\ -2.1196 & 11.0450 & 4.4719 & 4.4209 \\ 0.7783 & 4.4719 & 10.7468 & -1.1332 \\ 0.1537 & 4.4209 & -1.1332 & 8.7300 \end{bmatrix},$$

$$K = \begin{bmatrix} 3.8953 & -2.8299 & 2.0074 & -0.0989 \\ -2.8299 & 5.6163 & 0.9027 & -2.7188 \\ 2.0074 & 0.9027 & 4.9302 & -1.8975 \\ -0.0989 & -2.7188 & -1.8975 & 3.7709 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.0000 & -16.2285 & -8.3068 & -11.6295 \\ 16.2285 & -0.0000 & -39.6767 & -12.1871 \\ 8.3068 & 39.6767 & 0.0000 & 16.1554 \\ 11.6295 & 12.1871 & -16.1554 & -0.0000 \end{bmatrix}.$$

We define the residual as

$$\mathrm{res}(\lambda_i, x_i) = \|(\lambda_i^2 M + \lambda_i G + K)x_i\|,$$

and the numerical results shown in the following table.

(λ_i, x_i)	(λ_1, x_1)	(λ_2, x_2)	(λ_3, x_3)	(λ_4, x_4)
$\mathrm{res}(\lambda_i, x_i)$	1.4103e−012	1.4103e−012	9.6006e−014	9.6006e−014

Furthermore, we observe that the remainder spectral data of the updated system are just about those of the original system with absolute error $\|MX_{20}\Lambda_{20}^2 + GX_{20}\Lambda_{20} + KX_{20}\| \approx 4.8705\mathrm{e}{-014}$.

5. Concluding remarks

This paper has provided a solution to the inverse eigenvalue problem for undamped gyroscopic systems in $2n$ space to include the determination of all three coefficient matrices M, G and K from given spectral and modal data and the results of the inverse problem are applied to develop a method for model updating. The approach is demonstrated by two numerical examples and reasonable results are produced.

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