Journal of Computational and Applied Mathematics 242 (2013) 28–40

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Nonlinear periodic solutions for isothermal magnetostatic atmospheres

A.H. Khater [a,*], D.K. Callebaut [b], A.H. Bhrawy [a,c], M.A. Abdelkawy [a]

[a] Mathematics Department, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
[b] Department of Physics, CDE, University of Antwerp, B-2610 Antwerp, Belgium
[c] Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah, Saudi Arabia

ARTICLE INFO

Article history:
Received 20 November 2010
Received in revised form 19 August 2012

Keywords:
Nonlinear equations
Magnetohydrostatic equilibria
Grad–Shafranov equation
Bäcklund transformations
Solar atmosphere

ABSTRACT

The equations of magnetohydrostatic equilibria for a plasma in a gravitational field are investigated analytically. For equilibria with one ignorable spatial coordinate, the equations reduce to a single nonlinear elliptic equation for the magnetic potential known as the Grad–Shafranov equation. Specifying the arbitrary function in the latter equation, yields a nonlinear elliptic equation. Analytical nonlinear periodic solutions of this elliptic equation are obtained for the case of an isothermal atmosphere in a uniform gravitational field: e.g. a model for the solar atmosphere. We obtained several classes of exact solutions of five nonlinear evolution equations (Liouville, sinh–Poisson, double sinh–Poisson, sine–Poisson and double sine–Poisson) using the generalized tanh method. Moreover, the Bäcklund transformations are used to generate further new classes of solutions. The final results may be used to investigate some models in solar physics.

1. Introduction

The equations of magnetostatic (MS) equilibria have been used extensively to model solar magnetic structures in many articles. Heyvaerts et al. studied the general properties for a series of two-dimensional magnetohydrostatic configurations in [1]. Two types of mechanisms of nonlinear force-free magnetic fields have been treated by Low [2]. Magnetostatic field problems are studied in [3–6], where in [7], Lerche and Low study a class of astrophysical magnetic fields nonlinear problems. Khater et al. find exact soliton solutions in strongly relativistic cold plasma. Moreover magnetohydrodynamic problems studied in [8–11]. The force balance in these models consists of a balance between the pressure gradient force, the Lorentz $\mathbf{J}\wedge\mathbf{B}$ force (with $\mathbf{j}$ = electric current density, $\mathbf{B}$ = magnetic induction) and the gravitational force. The temperature distribution in the atmosphere is, in general, determined from the energy transport equation. However, in many models, the temperature distribution is specified a priori, and direct reference to the energy equation is eliminated. The remaining equations for the system are an equation of state for the gas (e.g., the dependence of the gas pressure on density and temperature) and the steady-state Maxwell's equations.

Many models of MS equilibria assume that one of the spatial coordinates is ignorable, see for example the articles [12–18], leading to simple analytical models in terms of linear or nonlinear equations for the magnetic potential A. Generalizations of this equation for steady-state MHD flows with one ignorable coordinate have been obtained by Tsinganos [9] and Low [19] both having reduced the MS equilibrium problem to one of solving a single PDE involving two scalar potentials describing the magnetic field. This development does not require the existence of an ignorable coordinate in the system and arises from a local compatibility condition for the magnetic field [20–22]. However, the nonlinear problem has been solved in several cases (see, for instance [17,18,23–25]).

* Corresponding author.
E-mail address: khater_ah@yahoo.com (A.H. Khater).

0377-0427/$ – see front matter © 2012 Published by Elsevier B.V.
doi:10.1016/j.cam.2012.10.012

In this article, we present a set of exact analytical periodic solutions for the Liouville, sinh, double sinh, sine and double sine–Poisson equations modelling isothermal MS atmosphere using the generalized tanh and Jacobi elliptic function methods [26,27]. Moreover, the Bäcklund transformations are used to generate further new classes of solutions.

2. The generalized tanh method

The key idea in this method is to use the solution of a Riccati equation to replace the tanh function in the tanh method. In what follows, the method will be reviewed briefly. Consider a given nonlinear evolution equation (NLEE)

$$G(u, u_x, u_y, u_t, u_{xy}, \ldots) = 0. \tag{1}$$

We like to know whether travelling waves (or stationary waves) are solutions of Eq. (1). The first step is to unite the independent variables x, y and t into one particular variable through the definition

$$\zeta = x + \alpha y, \qquad u(x, y) = U(\zeta),$$

and change (1) to an ordinary differential equation (ODE)

$$G(U, U', U'', U''', \ldots) = 0. \tag{2}$$

Our main goal is to derive exact or at least approximate solutions, if possible, for these ODEs. For this purpose, we introduce a new variable

$$\psi = \psi(\zeta),$$

which is a solution of the Riccati equation

$$\psi' = k + \psi^2. \tag{3}$$

Then we try the following series expansion as a solution of Eq. (1):

$$u(x, y) = U(\zeta) = \sum_{i=0}^{m} a_i \psi^i. \tag{4}$$

The parameter m is determined by balancing the linear term(s) of highest order with the nonlinear one(s) of highest order. Normally m is a positive integer, so that an analytical solution in closed form may be obtained. Substituting Eq. (3) into Eq. (4) and comparing the coefficients of each power of ψ in both sides, we get an overdetermined system of nonlinear algebraic equations with respect to $k, a_0, a_1, \ldots$. One may solve this over-determined system e.g. by means of the computer algebra system. By using the results obtained, we can derive several types of solutions:

(i) for $k < 0$

$$\psi = \begin{cases} -\sqrt{-k}\coth(\sqrt{-k}\zeta), \\ -\sqrt{-k}\tanh(\sqrt{-k}\zeta), \end{cases} \tag{5}$$

(ii) for $k = 0$

$$\psi = \frac{-1}{\zeta + c}, \tag{6}$$

(iii) for $k > 0$

$$\psi = \begin{cases} -\sqrt{k}\cot(\sqrt{k}\zeta), \\ \sqrt{k}\tan(\sqrt{k}\zeta). \end{cases} \tag{7}$$

Another advantage of the Riccati equation (3) is that the sign of k can be used to exactly judge the amount and types of the travelling wave solution of Eq. (1).

3. Basic equations and problem formulation

The equations used to describe MS atmosphere consists of the force balance equation

$$\underline{j} \wedge \underline{B} - \nabla P - \rho \nabla \Phi = 0, \tag{8}$$

where $\underline{j}$ and $\underline{B}$ are the electric current density and the magnetic induction, respectively coupled with Maxwell's equations

$$\mu \underline{j} = \nabla \wedge \underline{B}, \tag{9}$$

$$\nabla \cdot \underline{B} = 0, \tag{10}$$

where P, ρ, μ and Φ are the gas pressure, the mass density, the magnetic permeability and the gravitational potential, respectively. We assume that the temperature is uniform in space and the plasma is an ideal gas with equation of state

$$P = \rho R_0 T_0, \tag{11}$$

where R_0 is the gas constant and T_0 is the uniform temperature.

Consider a system of Cartesian coordinates (x, y, z), in which x is an ignorable coordinate and z measuring height, then the magnetic induction $\underline{B}$ may be written as

$$\underline{B} = \nabla A \wedge e_x + B_x e_x = \left(B_x, \frac{\partial A}{\partial z}, -\frac{\partial A}{\partial y}\right), \tag{12}$$

where $A(y, z)$ and $B_x(y, z)$ are the magnetic flux function and the x-component of $\underline{B}$, respectively. Note that the form Eq. (12) for $\underline{B}$ ensures that $\nabla \cdot \underline{B} = 0$. Since $\underline{B} \cdot \nabla A = 0$, $A(y, z)$ is constant along the magnetic lines of force. We restrict our attention to isothermal atmospheres in a uniform gravitational field ($\Phi = gz$), in which $B_x = 0$ and using the ideal gas law equation (11) to relate the pressure and the density to the uniform temperature T_0 of the atmosphere, Eq. (8) then requires that the pressure and the density have the form [25]

$$P(y, z) = P(A)e^{-z/h}, \tag{13}$$

$$\rho(y, z) = \frac{1}{gh} P(A)e^{-z/h}, \tag{14}$$

where $h = R_0 T_0/g$ is the (constant) scale height, and $P(A)$ is an arbitrary function of one variable to describe the variation of pressure across the magnetic lines of force at some constant height. Substituting Eqs. (9), (12)–(14) into Eq. (8), one gets [7,13]

$$\nabla^2 A + f(A)e^{-z/h} = 0, \tag{15}$$

where

$$f(A) = \mu \frac{dP}{dA}. \tag{16}$$

Subject to suitable boundary conditions on A, Eq. (15) may be solved for A in a given domain if the functional form $P(A)$ is prescribed in some suitable manner [12,13]. Eq. (15) is Ampere's law, which has been put in a form that relates the magnetic field to the plasma distribution through the equation for mechanical equilibrium.

The term $f(A)$ is, in general, nonlinear in A raising nontrivial question of existence, uniqueness, and regularity of solutions to boundary value problems based on Eq. (15). Rigorous and general mathematical results on these questions for Eq. (15) in the nonlinear regime have been obtained and discussed [1,28]. The absence of a regular solution may be interpreted to imply that electric current sheets are unavoidable. Eq. (16) gives

$$P(A) = P_0 + \frac{1}{\mu} \int f(A)dA. \tag{17}$$

Substituting Eq. (17) into Eqs. (13) and (14), we get

$$P(y, z) = \left(P_0 + \frac{1}{\mu} \int f(A)dA\right) e^{-z/h}, \tag{18}$$

$$\rho(x, z) = \frac{1}{gh} \left(P_0 + \frac{1}{\mu} \int f(A)dA\right) e^{-z/h}, \tag{19}$$

where P_0 is constant. Take the conformal transformation [23]

$$x_1 + ix_2 = e^{-z/l} e^{iy/l}. \tag{20}$$

Eq. (15) reduces to

$$\frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_2^2} + l^2 f(A)e^{\left(\frac{2}{l} - \frac{1}{h}\right)z} = 0. \tag{21}$$

In the following sections we will consider three cases for $f(A)$.

4. Liouville equation

Let us assume that $f(A)$ has the form [12,13]:

$$f(A) = -\alpha^2 A_0 e^{-2A/A_0}, \tag{22}$$

where α^2 and A_0 are constants. Hence

$$P(y,z) = \left(P_0 + \frac{\alpha^2 A_0^2}{2\mu} e^{-2A/A_0}\right) e^{-z/h}. \tag{23}$$

The term involving P_0 represents a plane-stratified component of the atmosphere. The corresponding form of Eq. (21) is given by using Eqs. (21) and (22) as

$$\nabla^2 A/A_0 = \alpha^2 l^2 e^{-2A/A_0 + \left(\frac{2}{l} - \frac{1}{h}\right)z}, \tag{24}$$

where $\nabla^2 = \frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_2^2}$.

Solutions of Eq. (24) have been obtained previously in cases [21,22]. If we set P_0, removing the plane-parallel component of the atmosphere, this is the well-known model for an infinite vertical sheet of diffuse plasma suspended by bowed magnetic field-lines.

To solve (24), we are looking for the solution where A is periodic in y with period $2\pi l$ [12,13]

$$A(y + 2\pi l, z) = A(y,z), \tag{25}$$

which corresponds to an array of plasma condensations or current filaments that are arranged periodically in the y-direction. These condensations are to be of finite extent vertically. Hence in the far region $z \to \pm\infty$, the field is required to be horizontal and uniform. The following boundary conditions apply

$$\lim_{z\to\pm\infty} \underline{B} = B_{\pm}\hat{y}, \tag{26}$$

where $B_{\pm}$ and $\hat{y}$ stand as usual are the constant field strengths and the unit vector in the y direction. Eq. (24) is a nonlinear elliptic PDE and one cannot take for granted that boundary conditions Eqs. (25) and (26) admit a solution and, where a solution exists, that it is unique. Let us set

$$A/A_0 = z/L + \omega(y,z), \tag{27}$$

where L is a constant. Eq. (24) becomes

$$\nabla^2\omega - \alpha^2 l^2 e^{-2\omega - \left(\frac{2}{L} + \frac{1}{h} - \frac{2}{l}\right)z} = 0. \tag{28}$$

Let us identify the period l by:

$$2/l = 2/L + 1/h. \tag{29}$$

Note that under the transformation (20), and take $l > 0$, we have transformed the region $0 \le y \le 2\pi l$, $0 \le z \le \infty$ into the entire $x_1 - x_2$ plane with origin $x_1 = x_2 = 0$ corresponding to $z \to \infty$ and the region $x_1^2 + x_2^2 \to 1$ corresponding to $z \to -\infty$. Noting that in the limit of an infinite period l as $l \to \infty$, Eq. (29) implies $2/L = -1/h$, and one recovers the single-structure solutions [22,23].

Eq. (28) transforms into a Liouville type equation

$$\frac{\partial^2\omega}{\partial x_1^2} + \frac{\partial^2\omega}{\partial x_2^2} - \alpha^2 l^2 e^{-2\omega} = 0. \tag{30}$$

Taken the transformation

$$e^{-2v} = u.$$

Eq. (30) tends to

$$(u_{x_1})^2 + (u_{x_2})^2 - u u_{x_1 x_1} - u u_{x_2 x_2} - 2\alpha^2 l^2 u^3 = 0. \tag{31}$$

Using the generalized tanh method we find:

$$U = a_0[1 - \tanh^2 \sqrt{-k}\zeta], \quad k < 0. \tag{32}$$

Substituting Eq. (32) into Eq. (27), yields

$$A/A_0 = \left(\frac{1}{l} - \frac{1}{2h}\right) z - \frac{1}{2}\ln(a_0) - \ln\left\{\sec h\left[\sqrt{-k}\left(\cos\left(\frac{y}{l}\right) - \nu\sin\left(\frac{y}{l}\right)\right)e^{-z/l}\right]\right\}, \tag{33}$$

$$\begin{aligned}\underline{B}/A_0 \ = \ &\left\{0, \left(\frac{1}{l} - \frac{1}{2h}\right) - \frac{\sqrt{-k}}{l}e^{-z/l}(\cos(y/l) - \nu\sin(y/l)) \times \tanh[e^{-z/l}\sqrt{-k}(\cos(y/l) - \nu\sin(y/l))],\right.\\ &\left.\frac{\sqrt{-k}}{l}e^{-z/l}(\sin(y/l) + \nu\cos(y/l)) \times \tanh[e^{-z/l}\sqrt{-k}(\cos(y/l) - \nu\sin(y/l))]\right\},\end{aligned} \tag{34}$$

$$P = P_0 e^{-z/h} + \frac{\alpha^2 A_0^2 a_0}{2\mu} e^{-2z/l}\sec h^2\left[e^{-z/l}\sqrt{-k}\left(\cos\left(\frac{y}{l}\right) - \nu\sin\left(\frac{y}{l}\right)\right)\right]. \tag{35}$$

The magnetic and plasma pressures are given by

$$\begin{aligned}P_m \ = \ &\frac{A_0^2}{2\mu}\left\{\left(\frac{1}{l} - \frac{1}{2h}\right)^2 - \frac{1+\nu^2}{l^2}ke^{-2z/l} \times \tanh^2[e^{-z/l}\sqrt{-k}(\cos(y/l) - \nu\sin(y/l))]\right.\\ &\left.-\frac{2}{l}\left(\frac{1}{l} - \frac{1}{2h}\right)e^{-z/l}\sqrt{-k}(\cos(y/l) - \nu\sin(y/l)) \times \tanh[e^{-z/l}\sqrt{-k}(\cos(y/l) - \nu\sin(y/l))]\right\},\end{aligned} \tag{36}$$

$$P = \frac{\alpha^2 A_0^2 a_0}{2\mu} e^{-2z/l}\sec h^2\left[e^{-z/l}\sqrt{-k}\left(\cos\left(\frac{y}{l}\right) - \nu\sin\left(\frac{y}{l}\right)\right)\right]. \tag{37}$$

We consider the following subcases.

(i) The subcase in which $l = 2h$. In this subcase, $\underline{B} = 0$ as $z \to \infty$.
(ii) The subcase in which $l > 2h$ (e.g. $l = 3h$). In this subcase, $\underline{B}/A_0 = -\frac{1}{6h}\hat{y}$ as $z \to \infty$. Hence as l increases above $2h$, a nonzero field appears at $z \to \infty$ of negative sign.
(iii) The subcase in which $l < 2h$ (e.g. $l = h$). In this subcase, $\underline{B}/A_0 = \frac{1}{2h}\hat{y}$ as $z \to \infty$. Hence as l decreases below $2h$, a nonzero field appears at $z \to \infty$ of positive sign.

The magnetic and plasma pressures are displayed in Fig. 1(a), (b) for the subcase (ii) respectively, with values of parameters listed in their captions. In Fig. 2(a), (b) are given: (a) the magnetic field lines (contours of A) and (b) the associated density enhancement, with values of parameters listed in their captions, respectively.

	Case 1	Case 2	Case 3
Parametric	$l = h$	$l = 2h$	$l = 3h$
Ranges	$\alpha > 0$	$\alpha > 0$	$1 + \alpha^2 \geq \frac{l^2}{4h^2}$
Field at $z \to \infty$	$\frac{1}{2h}\hat{y}$	0	$-\frac{1}{6h}\hat{y}$

5. Sinh–Poisson equation

Let us assume that $f(A)$ has the form

$$f(A) = -\frac{\lambda^2}{4}\left(\frac{A_0}{h}\right)\sinh(\tilde{A}), \tag{38}$$

where $\tilde{A} = A/(hA_0)$ is a dimensionless form of A, λ is a dimensionless constant, Eqs. (18) and (38) give

$$P(y, z) = \left(P_0 - \frac{\lambda^2 A_0^2}{4\mu}\cosh\tilde{A}\right)e^{-z/h}. \tag{39}$$

The corresponding form of Eq. (21) is given by using Eqs. (38) and (21) as

$$\frac{\partial^2\tilde{A}}{\partial x_1^2} + \frac{\partial^2\tilde{A}}{\partial x_2^2} = \lambda^2\sinh(\tilde{A}), \tag{40}$$

where $l = 2h$. Taken the transformation

$$e^{\tilde{A}} = u \quad \text{where } \sinh(\tilde{A}) = \frac{e^{\tilde{A}} - e^{-\tilde{A}}}{2}.$$

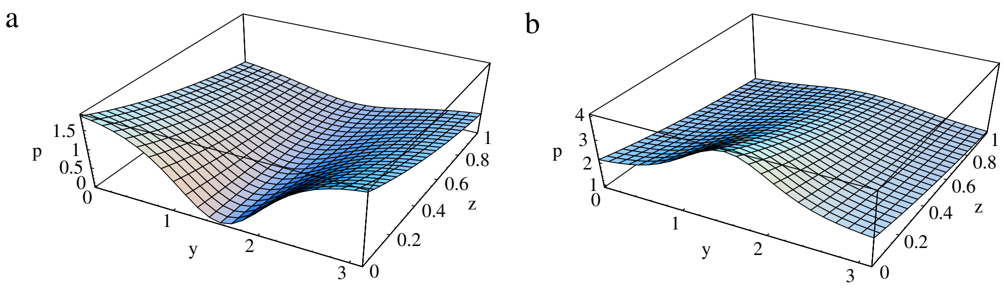

Fig. 1. (a) The magnetic pressures for $l = 2h$, $0 \le z \le 1$ and $0 \le y \le \pi$ for Eq. (32). (b) The plasma pressure (pressure enhancement) for $l = 2h$, $0 \le z \le 1$ and $0 \le y \le \pi$ for Eq. (33).

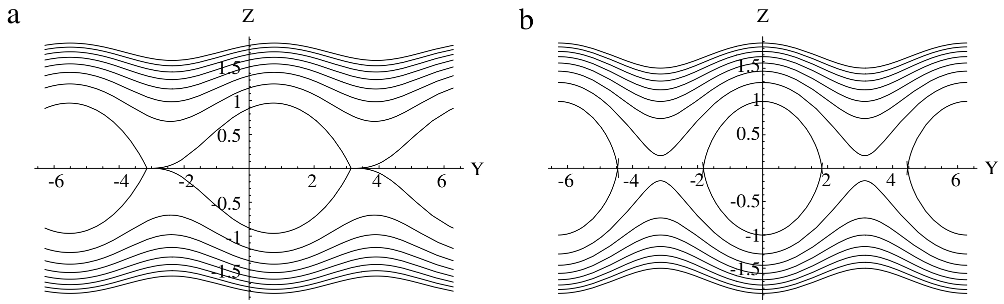

Fig. 2. (a) The magnetic field lines (contours of A) for $\alpha = 1, l = 2h$ and $-2\pi \le y \le 2\pi$ for Eq. (32). (b) The associated density enhancement for $\alpha = 1, l = 2h$ and $-2\pi \le y \le 2\pi$ for Eq. (33).

Eq. (40) tends to

$$2(u_{x_1})^2 + 2(u_{x_2})^2 - 2uu_{x_1x_1} - 2uu_{x_2x_2} + \lambda^2(u^3 - u) = 0. \tag{41}$$

Using the generalized tanh method we find that, the solution of Eq. (41) reads ($k > 0$)

$$A/A_0 = 2h \ln\Big[\tan e^{-z/l}\sqrt{k}(\cos(y/l) - \nu \sin(y/l))\Big]. \tag{42}$$

The associated magnetic induction and pressure are given by

$$\begin{aligned}\underline{B}/A_0 = \Big\{&0, -\sqrt{k}e^{-z/2h}\Big[(\cos(y/2h) - \nu \sin(y/2h)) \times \sec[\sqrt{k}e^{-z/2h}(\cos(y/2h) - \nu \sin(y/2h))]\\ &\times \operatorname{cosec}[e^{-z/2h}\sqrt{k}(\cos(y/2h) - \nu \sin(y/2h))]\Big], -\sqrt{k}e^{-z/2h}\Big[(-\sin(y/2h) - \nu \cos(y/2h))\\ &\times \sec[e^{-z/2h}\sqrt{k}(\cos(y/2h) - \nu \sin(y/2h))] \times \operatorname{cosec}[e^{-z/2h}\sqrt{k}(\cos(y/2h) - \nu \sin(y/2h))]\Big]\Big\},\end{aligned} \tag{43}$$

$$P = P_0 e^{-z/h} - \frac{\lambda^2 A_0^2}{8\mu} e^{-z/h}[\tan^2(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu \sin y/2h)) + 1]. \tag{44}$$

The magnetic and plasma pressures are given by

$$P_m = ke^{-z/h}(1 + \nu^2)\operatorname{cosec}^2[e^{-z/2h}\sqrt{k}(\cos y/2h - \nu \sin y/2h)] \times \sec^2[e^{-z/2h}\sqrt{k}(\cos y/2h - \nu \sin y/2h)], \tag{45}$$

$$P = -\frac{\lambda^2 A_0^2}{8\mu} e^{-z/h}[\tan^2(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu \sin y/2h)) + 1]. \tag{46}$$

The magnetic and plasma pressures are displayed in Fig. 3(a), (b) respectively, with values of parameters listed in their captions. Fig. 4(a), (b) give (a) the magnetic field lines (contours of A) and (b) the associated density enhancement respectively, with values of parameters listed in their captions.

6. Double sinh–Poisson equation

Let us assume that $f(A)$ has the form

$$f(A) = -\frac{\lambda^2}{4}\left(\frac{A_0}{h}\right)(\sinh(\tilde{A}) + \sinh(2\tilde{A})).$$

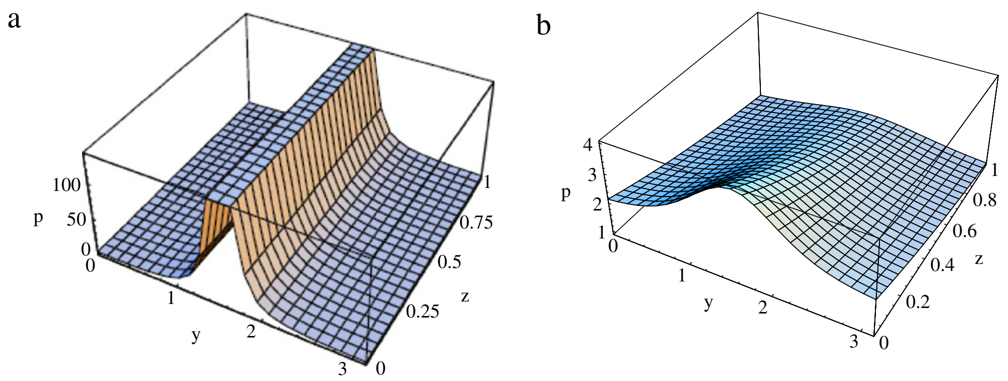

Fig. 3. (a) The magnetic pressures for $0 \le z \le 1$ and $0 \le y \le \pi$ for Eq. (44). (b) The plasma pressure (pressure enhancement) for $0 \le z \le 1$ and $0 \le y \le \pi$ for Eq. (45).

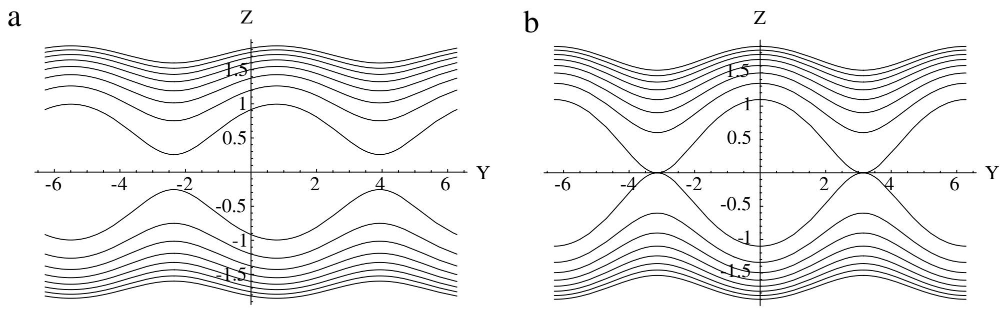

Fig. 4. (a) The magnetic field lines (contours of A) for $\alpha = 1$, and $-2\pi \le y \le 2\pi$ for Eq. (44). (b) The associated density enhancement for $\alpha = 1$, and $-2\pi \le y \le 2\pi$ for Eq. (45).

Eq. (18) gives

$$\frac{\partial^2 \tilde{A}}{\partial x_1^2} + \frac{\partial^2 \tilde{A}}{\partial x_2^2} = \lambda^2(\sinh(\tilde{A}) + \sinh(2\tilde{A})), \tag{47}$$

where $l = 2h$. Proceeding as in the previous case and using the generalized tanh method, we find that the associated magnetic induction and pressure are given by

$$\begin{aligned} \underline{B}/A_0 = \Bigg\{ & 0, -\sqrt{\frac{k}{12}} e^{-z/2h}\left[(\cos(y/2h) - \nu \sin(y/2h)) \times \frac{\sec^2[\sqrt{k} e^{-z/2h}(\cos(y/2h) - \nu\sin(y/2h))]}{\tan[e^{-z/2h}\sqrt{k}(\cos(y/2h) - \nu \sin(y/2h))] - 1}\right], \\ & \sqrt{\frac{k}{12}} e^{-z/2h}\left[(\sin(y/2h) + \nu \cos(y/2h)) \times \frac{\sec^2[\sqrt{k} e^{-z/2h}(\cos(y/2h) - \nu\sin(y/2h))]}{\tan[e^{-z/2h}\sqrt{k}(\cos(y/2h) - \nu \sin(y/2h))] - 1}\right]\Bigg\}, \end{aligned} \tag{48}$$

$$\begin{aligned} P = {} & P_0 e^{-z/h} - \frac{\lambda^2 A_0^2}{128\mu} e^{-z/h} \times \Big[9\tan^2(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu \sin y/2h)) \\ & - 6\tan(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h)) + 10\cot(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu \sin y/2h)) \\ & + 7\cot^2(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu \sin y/2h))\Big]. \end{aligned} \tag{49}$$

The magnetic and plasma pressures are given by

$$P_m = \frac{k}{4} e^{-z/h}(1+\nu^2)\frac{\sec^4[e^{-z/2h}\sqrt{k}(\cos y/2h - \nu \sin y/2h)]}{(\tan[e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h)] - 1)^2}, \tag{50}$$

$$\begin{aligned} P = {} & -\frac{\lambda^2 A_0^2}{128\mu} e^{-z/h} \times \Big[9\tan^2(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h)) + 7\cot^2(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu \sin y/2h)) \\ & - 6\tan(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu \sin y/2h)) + 10\cot(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h))\Big]. \end{aligned} \tag{51}$$

7. Sine-Poisson equation

Let us assume that $f(A)$ has the form

$$f(A) = -\frac{\lambda^2}{4}\left(\frac{A_0}{h}\right)\sin(\tilde{A}), \tag{52}$$

where

$$\tilde{A} = A/(hA_0), \tag{53}$$

is a dimensionless form of A, λ is a dimensionless constant, Eqs. (18) and (52) give

$$P(y,z) = \left(P_0 + \frac{\lambda^2 A_0^2}{4\mu}\cos\tilde{A}\right)e^{-z/h}. \tag{54}$$

The corresponding form of Eq. (21) is given by using Eqs. (21) and (52) as

$$\frac{\partial^2\tilde{A}}{\partial x_1^2} + \frac{\partial^2\tilde{A}}{\partial x_2^2} = \lambda^2\sin(\tilde{A}), \tag{55}$$

where $l = 2h$. Taken the transformation

$$e^{i\tilde{A}} = u \quad \text{where } \sin(\tilde{A}) = \frac{e^{i\tilde{A}} - e^{-i\tilde{A}}}{2i}.$$

Eq. (55) tends to

$$2(u_{x_1})^2 + 2(u_{x_2})^2 - 2uu_{x_1x_1} - 2uu_{x_2x_2} + \lambda^2(u^3 - u) = 0. \tag{56}$$

Using the generalized tanh method the magnetic and plasma pressures are given by

$$P_m = ke^{-z/2h}(1+\nu^2)\mathrm{cosec}^2[e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h)] \times \sec^2[e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h)], \tag{57}$$

$$P = \frac{\lambda^2 A_0^2}{4\mu}e^{-z/h}\frac{\tan^4[e^{-z/2h}\sqrt{k}(\cos(y/2h) - \nu\sin(y/2h))] + 1}{2\tan^2[e^{-z/2h}\sqrt{k}(\cos(y/2h) - \nu\sin(y/2h))]}. \tag{58}$$

8. Double sine–Poisson equation

Let us assume that $f(A)$ has the form

$$f(A) = -\frac{\lambda^2}{4}\left(\frac{A_0}{h}\right)(\sin(\tilde{A}) + \sin(2\tilde{A})).$$

Eq. (18) gives

$$\frac{\partial^2\tilde{A}}{\partial x_1^2} + \frac{\partial^2\tilde{A}}{\partial x_2^2} = \lambda^2(\sin(\tilde{A}) + \sin(2\tilde{A})), \tag{59}$$

where $l = 2h$. Proceeding as in the previous case and using the generalized tanh method, we find that the magnetic and plasma pressures are given by

$$P_m = \frac{k}{4}e^{-z/h}(1+\nu^2)\frac{\sec^4[e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h)]}{(\tan[e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h)] - 1)^2}, \tag{60}$$

$$\begin{aligned} P \;=\; -\frac{\lambda^2 A_0^2}{128\mu}e^{-z/h} \times \Big[&9\tan^2(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h)) - 6\tan(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h)) \\ &+ 10\cot(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h)) + 7\cot^2(e^{-z/2h}\sqrt{k}(\cos y/2h - \nu\sin y/2h))\Big]. \end{aligned} \tag{61}$$

9. The Ablowitz–Kaup–Newell–Segur system and Bäcklund transformations

The Bäcklund transformation (BT) technique is one of the direct methods for generating a new solution of a NLEE from a known solution of that equation [29]. Previously, Konno and Wadati [30], for example, had derived some BTs for the NEEs of the AKNS class. These BTs explicitly express the new solutions in terms of the known solutions of the NEEs and the corresponding wave functions, which are solutions of the associated Ablowitz–Kaup–Newell–Segur (AKNS) system. The AKNS system is a linear eigenvalue problem in the form of a system of first-order PDEs. Therefore, the problem of obtaining new solutions by BTs is equivalent to obtaining the wave function. It is known that many NEEs can be derived from the AKNS system

$$\phi_X = \mathbf{P}\phi, \qquad \phi_Y = \mathbf{Q}\phi, \tag{62}$$

where

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} \eta & q \\ r & -\eta \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} A' & B \\ C & -A' \end{pmatrix}. \tag{63}$$

Here η is a parameter, which is independent of X and Y while q and r are functions of X and Y, we will have the following system of PDEs for the unknowns ϕ_1 and ϕ_2:

$$\begin{aligned} \phi_{1X} &= \eta\phi_1 + q\phi_2, \\ \phi_{2X} &= r\phi_1 - \eta\phi_2, \\ \phi_{1Y} &= A'\phi_1 + B\phi_2, \\ \phi_{2Y} &= C\phi_1 - A'\phi_2, \end{aligned} \tag{64}$$

P and **Q** must satisfy the integrability condition

$$\begin{aligned} &-A'_X + qC - rB = 0, \\ &q_Y - B_X + 2\eta B - 2qA' = 0, \\ &r_Y - C_X + 2rA' - 2\eta C = 0. \end{aligned} \tag{65}$$

By a suitable choice of r, A', B, and C in (63) we can obtain many NEEs which q must satisfy. Konno and Wadati [30] introduced the function

$$\Gamma = \frac{\phi_1}{\phi_2}, \tag{66}$$

for each of the NEEs, they derived a BT with the form

$$U' = U + f(\Gamma, \eta) \tag{67}$$

where U' is the new solution generated from the old solution U. For use of the sequel, we list the NEEs and their corresponding BTs in the following.

The known travelling-wave solution of the NLEEs takes the form

$$q = q(\varpi); \quad \varpi = X - \kappa Y \ (\kappa \text{ is constant}). \tag{68}$$

Suppose that the components q and r of the matrix P are functions of ϖ

$$q = q(\varpi), \qquad r = r(\varpi),$$

then the components A', B and C of the matrix Q determined by Eqs. (64) are also functions of ϖ. We solve the system (65) by applying the method of characteristics. Eq. (64) possesses the following characteristic:

$$\frac{dY}{-r} = \frac{dX}{C} = \frac{d\phi_2}{\frac{1}{2}(C_X - r_Y)\phi_2}. \tag{69}$$

Integrating we find that

$$\phi_2 = k_2(C + \kappa r)^{\frac{1}{2}}, \tag{70}$$

$$-Y + k_1 = \int \frac{r}{C + \kappa r} d\varpi, \tag{71}$$

where k_1, k_2 are integration constants. Denote

$$\sigma(\varpi) = \int \frac{r}{C + \kappa r} d\varpi. \tag{72}$$

From (70) and (72), we obtain the general solution of Eq. (64):

$$\phi_2 = (C + \kappa r)^{\frac{1}{2}} f(\varepsilon), \tag{73}$$

where

$$\varepsilon = \sigma(\varpi) + Y,$$

and $f(\varepsilon)$ is a differentiable function of ε. Substituting (73) into (64) gives the general solution of ϕ_1

$$\phi_1 = (C + \kappa r)^{\frac{-1}{2}}[f'_\varepsilon + (A' + \kappa\eta)f], \tag{74}$$

where $f'_\varepsilon = \frac{df}{d\varepsilon}$. To determine the function f, from (71), (74) and the first equation in (64) we find that f must satisfy the following second order ODE:

$$f''_{\varepsilon\varepsilon} - \widetilde{\beta}f = 0, \tag{75}$$

where $\widetilde{\beta}$ is a constant defined by

$$\widetilde{\beta} = (A' + \kappa\eta)^2 + (B + \kappa q)(C + \kappa r), \tag{76}$$

will have the following three different solutions:

$$f = c_1\varepsilon + c_2 \qquad \widetilde{\beta} = 0, \tag{77}$$

$$f = c_1 \sinh\delta(\varepsilon + c_2) \quad \widetilde{\beta} > 0,\ \delta^2 = \widetilde{\beta}, \tag{78}$$

$$f = c_1 \sin\delta(\varepsilon + c_2) \quad \widetilde{\beta} < 0,\ \delta^2 = -\widetilde{\beta}, \tag{79}$$

where c_1 and c_2 are integration constants. Substituting these solutions into (74) and (73), respectively, we obtain the corresponding different solutions of the system (81)–(82):

for $\widetilde{\beta} = 0$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}(C + \kappa r)^{\frac{-1}{2}}[(A' + \kappa\eta)(c_1\varepsilon + c_2) + c_1]\\ (c_1\varepsilon + c_2)(C + \kappa r)^{\frac{1}{2}}\end{pmatrix}, \tag{80}$$

for $\widetilde{\beta} > 0$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}c_1(C + \kappa r)^{\frac{-1}{2}}[(A' + \kappa\eta)(\sinh\delta(\varepsilon + c_2)) + \delta\cosh\delta(\varepsilon + c_2)]\\ (c_1 \sinh\delta(\varepsilon + c_2))(C + \kappa r)^{\frac{1}{2}}\end{pmatrix}, \tag{81}$$

for $\widetilde{\beta} < 0$

$$\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}c_1(C + \kappa r)^{\frac{-1}{2}}[(A' + \kappa\eta)(\sin\delta(\varepsilon + c_2)) + \delta\cos\delta(\varepsilon + c_2)]\\ (c_1 \sin\delta(\varepsilon + c_2))(C + \kappa r)^{\frac{1}{2}}\end{pmatrix}. \tag{82}$$

These results (80)–(82) are valid for any NLEE contained in the AKNS system (94) − (63), provided that they meet the assumption (87). Now we apply the results obtained here to get solutions of the following.

1. The Liouville equation.
   Take the transformation

$$X = \frac{\alpha\lambda}{\sqrt{2}}(ix_1 - x_2), \qquad Y = \frac{\alpha\lambda}{\sqrt{2}}(ix_1 + x_2) \quad \text{and} \quad U(X, Y) = -2w(x_1, x_2). \tag{83}$$

   Eq. (30) becomes

$$U_{XY} = e^U, \tag{84}$$

$$\mathbf{P} = \begin{pmatrix}\eta & \frac{1}{2}U_X\\ \frac{1}{2}U_X & -\eta\end{pmatrix}, \qquad \mathbf{Q} = \frac{1}{4\eta}\begin{pmatrix}e^U & -e^U\\ e^U & -e^U\end{pmatrix}, \tag{85}$$

$$U' = U - 2\ln\left(\frac{\Gamma + 1}{\Gamma - 1}\right). \tag{86}$$

Then from (30), (32), (63), (83) and (85) we find the following

$$C = A' = -\frac{1}{B} = \frac{a_0}{4\eta}\left[1 - \tanh^2\left(\sqrt{-k}\left(\cos\left(\frac{y}{l}\right) - \nu\sin\left(\frac{y}{l}\right)\right)e^{-z/l}\right)\right], \quad k < 0, \tag{87}$$

$$q = r = \gamma_1 \tanh\left(\sqrt{-k}\left(\cos\left(\frac{y}{l}\right) - \nu\sin\left(\frac{y}{l}\right)\right)e^{-z/l}\right), \quad \gamma_1 = \frac{\sqrt{-k}(\nu - i)}{\sqrt{2}\alpha\lambda}. \tag{88}$$

$$\sigma(\varpi) = \frac{1}{\eta\kappa\gamma_1}\left(2\gamma_1\varpi + \frac{2}{\sqrt{a_0^2 + 4\eta^2\kappa^2\gamma_1^2}}\tanh^{-1}\left(\frac{a_0 + 2\eta\kappa\gamma_1 e^{2\gamma_1\varpi}}{\sqrt{a_0^2 + 4\eta^2\kappa^2\gamma_1^2}}\right)\right). \tag{89}$$

Then using Eqs. (66), (72), (80), (81), (86)–(89) we find the new solution.

2. The sinh–Poisson equation.

Take the transformation

$$X = \frac{\lambda}{2}(x_1 + ix_2), \qquad Y = \frac{\sqrt{\lambda}}{2}(x_1 - ix_2) \quad \text{and} \quad U(X, Y) = \widetilde{A}(x_1, x_2). \tag{90}$$

Eq. (40) becomes

$$U_{XY} = \sinh U, \tag{91}$$

$$\mathbf{P} = \begin{pmatrix} \eta & \frac{1}{2}U_X \\ \frac{1}{2}U_X & -\eta \end{pmatrix}, \qquad \mathbf{Q} = \frac{1}{4\eta}\begin{pmatrix} \cosh U & -\sinh U \\ \sinh U & -\cosh U \end{pmatrix}, \tag{92}$$

$$U' = U - 4\tanh^{-1}(\Gamma). \tag{93}$$

Then from (43), (63), (72), (80) and (82) we find the following

$$\begin{aligned} C = -B &= \frac{1}{4\eta}\sinh\left(\ln\left[\tan^2\left(\sqrt{-k}\left(\cos\left(\frac{y}{l}\right) - \nu\sin\left(\frac{y}{l}\right)\right)e^{-z/l}\right)\right]\right), \\ A' &= \frac{1}{4\eta}\sinh\left(\ln\left[\tan^2\left(\sqrt{-k}\left(\cos\left(\frac{y}{l}\right) - \nu\sin\left(\frac{y}{l}\right)\right)e^{-z/l}\right)\right]\right) \quad k < 0, \end{aligned} \tag{94}$$

$$q = r = \frac{\gamma_2\sec^2(\gamma_2(X - \kappa Y))}{\tan(\gamma_2(X - \kappa Y))}, \quad \gamma_2 = \frac{1 + \nu i}{\lambda}. \tag{95}$$

$$\sigma(\varpi) = \frac{2\eta(8\eta\varpi\kappa\gamma_2^2 + \ln(\cos 2\gamma_2\varpi) + 4\eta\kappa\gamma_2\sin 2\gamma_2\varpi)}{1 + 16\eta^2\kappa^2\gamma_2^2}. \tag{96}$$

Then using Eqs. (66), (72), (80) — (72), (82)–(86) we find the new solution.

3. The sine–Poisson equation.

Take the transformation

$$X = \frac{\lambda}{2}(x_1 + ix_2), \qquad Y = \frac{\sqrt{\lambda}}{2}(x_1 - ix_2) \quad \text{and} \quad U(X, Y) = \widetilde{A}(x_1, x_2). \tag{97}$$

Eq. (66) becomes

$$U_{XY} = \sin U, \tag{98}$$

$$\mathbf{P} = \begin{pmatrix} \eta & -\frac{1}{2}U_X \\ \frac{1}{2}U_X & -\eta \end{pmatrix}, \qquad \mathbf{Q} = \frac{1}{4\eta}\begin{pmatrix} \cos U & \sin U \\ \sin U & -\cos U \end{pmatrix}, \tag{99}$$

$$U' = U - 4\tanh^{-1}(\Gamma). \tag{100}$$

Then from (63), (72), (80) and (89) we find the following

$$C = B = \frac{1}{4\eta}\sin\left(\cos^{-1}\left[\frac{1}{2}\tan^2[e^{-z/2h}\sqrt{k}(\cos(y/2h) - \nu\sin(y/2h))] + \frac{1}{2}\cot^2[e^{-z/2h}\sqrt{k}(\cos(y/2h) - \nu\sin(y/2h))]\right]\right),$$
$$A' = \frac{1}{4\eta}\left[\frac{1}{2}\tan^2[e^{-z/2h}\sqrt{k}(\cos(y/2h) - \nu\sin(y/2h))] + \frac{1}{2}\cot^2[e^{-z/2h}\sqrt{k}(\cos(y/2h) - \nu\sin(y/2h))]\right] \quad k < 0, \tag{101}$$

$$q = -r = \frac{8\gamma_2\cot(2\gamma_2\varpi)\csc^2(2\gamma_2\varpi)}{\sqrt{2 - \cot^4(\gamma_2\varpi) - \tan^4(\gamma_2\varpi)}}. \tag{102}$$

$$\sigma(\varpi) = \frac{2\eta(8\eta\varpi\kappa\gamma_2^2 + \ln(\cos 2\gamma_2\varpi) + 4\eta\kappa\gamma_2\sin 2\gamma_2\varpi)}{1 + 16\eta^2\kappa^2\gamma_2^2}. \tag{103}$$

Then using Eqs. (66), (72), (80) − (72), (101)–(103) we find the new solution.

10. Conclusions

In this paper, we have investigated isothermal MS atmospheric models with one ignorable coordinate x of a Cartesian coordinate system xyz in which the distributed current is either with $j_x = -\alpha^2 A_0 e^{-2A/A_0 - z/h}$, once with $j_x = -(\lambda A_0/4h) \times (\sinh\tilde{A} + \sinh 2\tilde{A})e^{-z/h}$, once with $j_x = -(\lambda A_0/4h)\sin\tilde{A}e^{-z/h}$ and $j_x = -(\lambda A_0/4h) \times (\sin\tilde{A} + \sin 2\tilde{A})e^{-z/h}$. The underlying elliptic equation governing the force balance perpendicular to both $\underline{B}$ and e_x reduced to the Liouville, sinh, double sinh, sine and double sine–Poisson equations respectively. The main interest of this paper is five classes of nonlinear MS solutions that are obtained analytically by exploring the generalized tanh-function method and the Jacobi elliptic function, namely, the solutions corresponding to the particular choice of the pressure profile, given in terms of the magnetic flux function A by Eq. (15). Moreover the Bäcklund transformations are used to generate new classes of solutions.

Acknowledgements

The authors would like to thank the reviewers for their constructive comments and suggestions to improve the quality of the article.

References

[1] J. Heyvaerts, J.M. Larsy, M. Schatzman, P. Witomsky, Blowing up of two-dimensional magnetohydrostatic equilibria by an increase of electric current or pressure, Astron. Astrophys. 111 (1982) 104.
[2] B.C. Low, Nonlinear force-free magnetic fields, Rev. Geophys. Space Phys. 20 (1982) 145.
[3] B.C. Low, A.I. Hundhausen, E.G. Zwibel, Nonlinear periodic solutions for the isothermal magnetostatic atmosphere, Phys. Fluids 26 (1983) 2731.
[4] A.H. Khater, D.K. Callebaut, W. Malfliet, E.S. Kamel, Bäcklund transformations and exact solutions for some nonlinear evolution equations in solar magnetostatic models, J. Comput. Appl. Math. 140 (2002) 435–467.
[5] Y. Chu, Y. Cao, X. He, M. Luo, Asymptotic boundary conditions with immersed finite elements for interface magnetostatic/electrostatic field problems with open boundary, Comput. Phys. Comm. 182 (2011) 2331–2338.
[6] G.M. Webb, G.P. Zank, Application of the Sine–Poisson equation in solar magnetostatics, Sol. Phys. 127 (1990) 229.
[7] I. Lerche, B.C. Low, Some nonlinear problems in astrophysics, Physica D 4 (1982) 293.
[8] M. Nunez, Two-dimensional MHD equilibria in cold gravitating plasmas, IMA J. Appl. Math. 75 (2010) 333–342.
[9] K. Tsinganos, Magnetohydrodynamic equilibrium. I—exact solutions of the equations, Astrophys. J. 245 (1981) 764.
[10] K. Tsinganos, Magnetohydrodynamic equilibrium. II—general integrals of the equations with one ignorable coordinate, Astrophys. J. 252 (1982) 775.
[11] K. Tsinganos, Magnetohydrodynamic equilibrium-part four—nonequilibrium of nonsymmetric hydrodynamic topologies, Astrophys. J. 259 (1982) 832.
[12] J.W. Dungey, A family of solutions of the magneto-hydrostatic problem in a conducting atmosphere in a gravitational field, Mon. Not. R. Astron. Soc. 113 (1953) 180.
[13] B.C. Low, Nonisothermal magnetostatic equilibria in a uniform gravity field. I. Mathematical formulation, Astrophys. J. 197 (1975) 251.
[14] J. Birn, H. Goldstein, K. Schindler, A theory of the onset of solar eruptive processes, Sol. Phys. 57 (1978) 81.
[15] E.G. Zweibel, A.J. Hundhausen, Magnetostatic atmospheres—a family of isothermal solutions, Sol. Phys. 76 (1982) 261.
[16] R.-H. Zhu, H.-Y. Peng, M.-H. Zhang, Y.-Q. Chen, Magnetostatic modes of a ferromagnetic superlattice in an oblique applied field, Physica B 404 (2009) 2086–2090.
[17] G.M. Webb, Isothermal magnetostatic atmospheres. II—similarity solutions with current proportional to the magnetic potential cubed, Astrophys. J. 327 (1988) 933.
[18] G.M. Webb, G.P. Zank, Application of the Sine–Poisson equation in solar magnetostatics, Sol. Phys. 127 (1990) 229.
[19] B.C. Low, Three-dimensional structures of magnetostatic atmospheres. I. Theory, Astrophys. J. 293 (1985) 31.
[20] E.N. Parker, Photospheric flow and stellar winds, Astrophys. J. 251 (1981) 266.
[21] E.N. Parker, Interplanetary Dynamical Processes, Wiley, New York, 1963.

[22] E.N. Parker, Cosmical Magnetic Fields, Oxford University Press, New York, 1979.
[23] R. Kippenhahn, A. Schluter, Eine theorie der solaren filamente, Z. Astrophys. 43 (1957) 36.
[24] I. Lerche, B.C. Low, On the equilibrium of a cylindrical plasma supported horizontally by magnetic fields in uniform gravity, Sol. Phys. 67 (1980) 229.
[25] B.C. Low, Evolving force-free magnetic fields. I—the development of the pre are stage, Astrophys. J. 212 (1977) 234.
[26] W. Malfiet, Solitary wave solutions of nonlinear wave equations, Amer. J. Phys. 60 (1992) 650.
[27] B.R. Duffy, E.J. Parkes, Travelling solitary wave solutions to a seventh-order generalized KdV equation, Phys. Lett. A 214 (1996) 271.
[28] T. Amari, J.J. Aly, in: B.M. Haish, M. Rodono (Eds.), Proc. IAU Colloquium No 104-Solar and Stellar Flares, Kluwer, Dordrecht, 1988.
[29] C. Rogers, W.E. Shadwick, Bäcklund Transformations and their Applications, Academic Press, NewYork, 1982.
[30] K. Konno, M. Wadati, Simple derivation of Bäcklund transformation from Riccati form of inverse method, Progr. Theoret. Phys. 53 (1975) 1652.