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# A fully discrete $C^0$ interior penalty Galerkin approximation of the extended Fisher–Kolmogorov equation

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ABSTRACT

A fully discrete $C^0$ interior penalty finite element method is proposed and analyzed for the Extended Fisher–Kolmogorov (EFK) equation $u_t + \gamma \Delta^2 u - \Delta u + u^3 - u = 0$ with appropriate initial and boundary conditions, where $\gamma$ is a positive constant. We derive a regularity estimate for the solution $u$ of the EFK equation that is explicit in $\gamma$ and as a consequence we derive *a priori* error estimates that are robust in $\gamma$.



## 1. Introduction

We study a fully discrete $C^0$ interior penalty method for the fourth order parabolic Extended Fisher–Kolmogorov (EFK) equation:

$$\frac{\partial u}{\partial t} + \gamma \Delta^2 u - \Delta u + \phi(u) = 0 \quad \text{in } \Omega \times (0, T], \tag{1.1}$$

$$\frac{\partial u}{\partial n} = \gamma \frac{\partial \Delta u}{\partial n} = 0 \quad \text{on } \partial\Omega \times (0, T], \tag{1.2}$$

$$u(x, 0) = u_0(x), \quad x \in \Omega \tag{1.3}$$

where $\Omega \in \mathbb{R}^d$ $(d = 1, 2, 3)$ is a bounded domain with convex polyhedral boundary $\partial\Omega$, $T > 0$, $\gamma$ is a positive constant, $u_0$ is a given function of $x \in \mathbb{R}^d$ and $\phi(u) = u^3 - u$. The nonlinear function $\phi(u) = \Psi'(u)$ for $\Psi(u) = \frac{1}{4}(u^2 - 1)^2$. Specific assumptions on the initial data $u_0$ will be given later in the course of the paper. Here and throughout, $\Delta$ denotes the Laplacian. When $\gamma = 0$ in (1.1), we obtain the Fisher–Kolmogorov equation that occurs in the study of front propagation [1,2] into unstable states. The model problem (1.1)–(1.3) is proposed in [3,4] as an extension of the Fisher–Kolmogorov equation for the study of spatial patterns in bistable systems. When $\gamma$ is small ($\gamma \leq 1/8$), it is observed in [4,5] that the solutions of EFK equations are similar to the FK equation but lead to smooth fronts. However when $\gamma > 1/8$, it is possible to distinguish the solutions of these equations [5]. The study of the dynamics of the EFK equation can be found in [6,5,7].

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From the numerical point of view, a conforming finite element method has been proposed in [8] for the EFK equation (1.1) and the error analysis has been discussed. In this article, we propose and study the $C^0$ interior penalty method for (1.1). In the past few years, $C^0$ interior penalty methods [9,10] have became an attractive alternative for fourth order problems since the design of quasi-optimal $C^0$ interior penalty methods is straightforward [11,12]. During the past few years, fully discontinuous Galerkin methods have also became attractive for fourth order problems [13–17] although they involve a larger number of degrees of freedom than the $C^0$ interior penalty method. $C^0$ interior penalty methods are designed based on a standard Lagrange finite element space and a mesh dependent weak formulation involving jumps of the normal derivative across the inter-element boundaries. Since the standard Lagrange finite element spaces are designed for second order problems, they are naturally suitable for singularly perturbed fourth order problems. In [18], a $C^0$ interior penalty method is analyzed for a singularly perturbed fourth order elliptic problem and proved to be robust with respect to the small perturbation parameter. In this article, we extend the results in [18] to study a fully discrete $C^0$ interior penalty method for the EFK equation (1.1) involving a singularly perturbed fourth order term (when $\gamma$ is small). We establish the stability of the numerical solution and derive *a priori* error estimates which are robust in $\gamma$ (depend on a lower order polynomial in $\gamma^{-1}$). To accomplish this, we establish a regularity estimate for the solution $u$ of (1.1) that is explicit in $\gamma$ and derive stability bounds for an elliptic projection of $u$.

The rest of the article is organized as follows. In Section 2, we derive *a priori* bounds for the solution of the EFK equation. Therein, we propose our numerical method and prove the existence and uniqueness of a discrete solution. Moreover, we derive stability estimates for both the weak and the discrete solution. In Section 3, we derive some regularity of the weak solution. In Section 4, *a priori* error estimates that are robust in $\gamma$ are derived. Finally, we present conclusions in Section 5.

## 2. Existence and uniqueness results

In this section, we present a fully discrete $C^0$ interior penalty method and show the existence and uniqueness of the discrete solution. We derive some *a priori* bounds for the solution of the EFK equation and its discrete counterpart.

Let $V = \{v \in H^2(\Omega) : \partial v/\partial n = 0 \text{ on } \partial\Omega\}$. Denote the $L_2(\Omega)$ inner-product by $(\cdot,\ \cdot)$ and the norm by $\|\cdot\|$.

The weak form of (1.1)–(1.3) is to find $u(\cdot, t) \in V$, $t \in [0, T]$ such that

$$(u_t, v) + \gamma(\Delta u, \Delta v) + (\nabla u, \nabla v) + (\phi(u), v) = 0, \quad \forall v \in V, \tag{2.1}$$

$$u = u_0 \quad \text{at } t = 0. \tag{2.2}$$

The following lemma on the norm equivalence is useful in our analysis.

**Lemma 2.1.** *There exist two positive constants $C_1$ and $C_2$ which may depend on $\Omega$ such that*

$$C_1\|v\|_{H^2(\Omega)} \leq \left(\|\Delta v\|^2 + \|v\|^2_{H^1(\Omega)}\right)^{1/2} \leq C_2\|v\|_{H^2(\Omega)}, \quad \forall v \in V. \tag{2.3}$$

**Proof.** Let $v \in V$. Then it is obvious that $\left(\|\Delta v\|^2 + \|v\|^2_{H^1(\Omega)}\right)^{1/2} \leq C_2\|v\|_{H^2(\Omega)}$.

To prove the other way, note that $v \in V$ satisfies the following elliptic problem:

$$-\Delta v = -\Delta v \quad \text{in } \Omega,$$
$$\frac{\partial v}{\partial n} = 0 \quad \text{on } \partial\Omega.$$

Appealing to the elliptic regularity theory for the second order homogeneous Neumann problem on convex polyhedral domains [19,20], there exists some positive constant $C$ depending on $\Omega$ such that

$$\|\dot{v}\|_{H^2(\Omega)/R} \leq C\left(\|\Delta v\| + \|\partial v/\partial n\|_{H^{1/2}(\partial\Omega)}\right) = C\|\Delta v\|, \tag{2.4}$$

where $\dot{v}$ denotes the equivalence class of $v$ in the quotient space $H^2(\Omega)/R$ equipped with the norm

$$\|\dot{v}\|_{H^2(\Omega)/R} = \inf_{c\in R}\|v + c\|_{H^2(\Omega)} = \inf_{c\in R}\left(\|v + c\|^2_{L_2(\Omega)} + \|\nabla v\|^2_{L_2(\Omega)} + |v|^2_{H^2(\Omega)}\right)^{1/2}. \tag{2.5}$$

From (2.4) and (2.5), we find

$$|v|_{H^2(\Omega)} \leq C\|\Delta v\|.$$

This completes the proof. □

We derive some *a priori* bounds for $u$ which are explicit in terms of the parameter $\gamma$. Throughout the article, $C$ and $C(T)$ denote generic positive constants which are independent of the constant $\gamma$.

**Theorem 2.2.** *Let $u_0 \in H^2(\Omega)$ be such that*

$$\gamma \|\Delta u_0\|^2 + \|u_0\|^2_{H^1(\Omega)} + 2(\Psi(u_0), 1) \le M_1, \tag{2.6}$$

*for some $M_1 > 0$. Then, the solution $u$ of (2.1)–(2.2) satisfies*

$$\|u\|^2_{L_\infty\left(0,T;H^1(\Omega)\right)} \le C(T)M_1, \tag{2.7}$$

$$\gamma \|u\|^2_{L_\infty\left(0,T;H^2(\Omega)\right)} \le C(T)M_1, \tag{2.8}$$

$$\|u\|^2_{H^1\left(0,T;L_2(\Omega)\right)} \le M_1/2. \tag{2.9}$$

**Proof.** We set $v = u$ in (2.1) and find

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \gamma\|\Delta u\|^2 + \|\nabla u\|^2 + (\phi(u), u) = 0.$$

This implies

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 \le \|u\|^2.$$

By integrating the above inequality from 0 to $t$ for $t \in (0, T]$, we derive

$$\|u(t)\|^2 \le 2\int_0^t \|u(s)\|^2 ds + \|u_0\|^2 \le 2\int_0^T \|u(s)\|^2 ds + \|u_0\|^2.$$

Using Gronwall's lemma, we find for $t \in (0, T]$ that

$$\|u(t)\|^2 \le C(T)\|u_0\|^2 \le C(T)M_1. \tag{2.10}$$

This establishes the $L_\infty(0, T; L_2(\Omega))$ bound for $u$. Next, we set $v = u_t$ in (2.1) and find

$$\|u_t\|^2 + \frac{\gamma}{2}\frac{d}{dt}\|\Delta u\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \frac{d}{dt}(\Psi(u), 1) = 0.$$

By integrating from 0 to $t$ for $t \in (0, T]$, we obtain

$$\int_0^t 2\|u_t(s)\|^2 ds + \gamma\|\Delta u(t)\|^2 + \|\nabla u(t)\|^2 + 2(\Psi(u(t)), 1) = \gamma\|\Delta u_0\|^2 + \|\nabla u_0\|^2 + 2(\Psi(u_0), 1).$$

Using (2.6), we deduce for $t \in (0, T]$ that

$$\|\nabla u(t)\|^2 + \gamma\|\Delta u(t)\|^2 \le M_1,$$

and

$$\|u\|^2_{H^1\left(0,T;L_2(\Omega)\right)} \le M_1/2,$$

which together with (2.10) and Lemma 2.1 completes the proof. □

In the following theorem, we discuss on the existence and uniqueness of a solution $u$ for (2.1)–(2.2). The proof is based on the Galerkin procedure, compactness arguments and *a priori* estimates in Theorem 2.2. In [8], existence and uniqueness of a solution for the EFK equation with Dirichlet and Navier boundary conditions are discussed. Since the boundary conditions in our case are different, in the proof we need the result in Lemma 2.1 to prove that the sequence of solutions in the Galerkin procedure is bounded in the $H^2(\Omega)$ norm. Nevertheless, the rest of the proof is similar to the ones in [8,21–23] and hence we sketch the main steps of it.

**Theorem 2.3.** *Let $u_0 \in V$. Then, there exists a unique solution $u(\cdot, t) \in V$ to (2.1)–(2.2).*

**Proof** (*Uniqueness:*)**.** Let $u$ and $w$ defined on $\Omega \times [0, T]$ be two solutions of (2.1)–(2.2) and denote $z = u - w$. Then $z$ satisfies

$$(z_t, v) + \gamma(\Delta z, \Delta v) + (\nabla z, \nabla v) + (\phi(u) - \phi(w), v) = 0, \quad \forall v \in V,$$
$$z = 0 \quad \text{at } t = 0.$$

Setting $v = z$ in the above, we obtain

$$\frac{1}{2}\frac{d}{dt}\|z\|^2 + \gamma\|\Delta z\|^2 + \|\nabla z\|^2 = (\phi(w) - \phi(u), u - w).$$

Using the definition of $\phi$, we derive

$$(\phi(w)-\phi(u),u-w)=\|u-w\|^2-((w^2+uw+u^2),(u-w)^2)\le\|u-w\|^2,$$

where we have used the fact that $(w^2+uw+u^2,(u-w)^2)\ge\frac{1}{2}(u^2+w^2,(u-w)^2)\ge 0$. Therefore

$$\frac{1}{2}\frac{d}{dt}\|z\|^2+\gamma\|\Delta z\|^2+\|\nabla z\|^2\le\|z\|^2.$$

By integrating from 0 to $t$ for $t\in(0,T]$, we find

$$\frac{1}{2}\|z(t)\|^2+\int_0^t(\gamma\|\Delta z\|^2+\|\nabla z\|^2)ds\le\int_0^t\|z\|^2ds.$$

An appeal to Gronwall's lemma completes the proof of uniqueness. To prove the existence, let $\{w_j\}_{j=1}^{\infty}$ be an orthogonal basis of $V$ and set $V_m=\text{span}\{w_j\}_{j=1}^{m}$. Define a finite dimensional problem of finding $u_m(\cdot,t)\in V_m$ such that

$$u_m(\cdot,t)=\sum_{j=1}^{m}c_j(t)w_j$$

$$(u_{mt},v)+\gamma(\Delta u_m,\Delta v)+(\nabla u_m,\nabla v)+(\phi(u_m),v)=0\quad\forall v\in V_m, \tag{2.11}$$

$$u_m(\cdot,0)=\Pi_m u_0(\cdot), \tag{2.12}$$

where $\Pi_m:V\to V_m$ is a projection such that $\|\Pi_m v-v\|_{H^2(\Omega)}\to 0$ as $m\to 0$. By Picard's theorem, there exists a unique solution $u_m$ for each $m$. As in Theorem 2.2, we find by setting $v=u_m$ and $v=u_{mt}$ in (2.11) that

$$\|u_m\|_{L_\infty\left(0,T;L_2(\Omega)\right)}\le C \tag{2.13}$$

$$\gamma\|u_m\|_{L_\infty\left(0,T;H^2(\Omega)\right)}\le C \tag{2.14}$$

and

$$\|u_m\|_{H^1\left(0,T;L_2(\Omega)\right)}\le C. \tag{2.15}$$

Using a Sobolev embedding theorem [19] and (2.14), we find that

$$\gamma\|u_m\|_{L_\infty\left(0,T;L_\infty(\Omega)\right)}\le C(T)M_1. \tag{2.16}$$

By compactness [23,22], there exists $u\in L_\infty\big(0,T;H^2(\Omega)\big)\cap H^1\big(0,T;L_2(\Omega)\big)$ such that

$$\begin{aligned}&u_m(\cdot,t)\to u(\cdot,t)\quad\text{strongly in } H^1(\Omega)\\&u_m(\cdot,t)\to u(\cdot,t)\quad\text{weakly in } H^2(\Omega)\\&u_{mt}\to u_t\quad\text{weakly in } L_2\big(0,T;L_2(\Omega)\big).\end{aligned}$$

Since

$$\begin{aligned}\|\phi(u_m)-\phi(u)\|&=\|(u_m^3-u^3)-(u_m-u)\|\le\|u_m^3-u^3\|+\|u_m-u\|\\&\le\|(u^2+uu_m+u_m^2)(u-u_m)\|+\|u_m-u\|\\&\le C\left(\|u\|^2_{L_\infty(\Omega)}+\|u_m\|^2_{L_\infty(\Omega)}+1\right)\|u-u_m\|\to 0,\end{aligned}$$

the rest of the proof follows. □

Next in order to present the numerical method, to establish the existence and uniqueness of the discrete solution and its stability bounds, we introduce some notations and preliminary results.

### 2.1. Notations

Let $\mathcal{T}_h$ be a regular simplicial subdivision of $\Omega$. We denote the set of all interior edges/faces of $\mathcal{T}_h$ by $\mathcal{E}_h^i$, the set of boundary edges/faces by $\mathcal{E}_h^b$, and define $\mathcal{E}_h=\mathcal{E}_h^i\cup\mathcal{E}_h^b$. Let $h_\tau=\text{diam}\,\tau$ and $h=\max\{h_\tau:\tau\in\mathcal{T}_h\}$. The diameter of any edge/face $e\in\mathcal{E}_h$ will be denoted by $h_e$. We define the Sobolev space $H^s(\Omega,\mathcal{T}_h)$ (for $s\ge 0$) associated with the subdivision $\mathcal{T}_h$ as follows:

$$H^s(\Omega,\mathcal{T}_h)=\{v\in L_2(\Omega):\ v|_\tau\in H^s(\tau),\ \forall\,\tau\in\mathcal{T}_h\}.$$

The finite element space we use in the article is defined by

$$V_h=\{v\in C^0(\Omega):\ v|_\tau\in\mathbb{P}_2(\tau),\ \forall\,\tau\in\mathcal{T}_h\},$$

where $\mathbb{P}_2(D)$ is the space of polynomials of degree less than or equal to 2 restricted to the set $D$. It is clear that $V_h \subset H^1(\Omega) \cap H^s(\Omega, \mathcal{T}_h)$ for any positive integer $s$.

For any $e \in \mathcal{E}_h^i$, there are two elements $\tau_+$ and $\tau_-$ such that $e = \partial\tau_+ \cap \partial\tau_-$. Let $n_-$ be the unit normal of $e$ pointing from $\tau_-$ to $\tau_+$. For any $v \in H^2(\Omega, \mathcal{T}_h)$, we define the jump of the normal derivative of $v$ on $e$ by

$$[\![\nabla v]\!] = \nabla v_+|_e \cdot n_+ + \nabla v_-|_e \cdot n_-$$

where $v_\pm = v|_{\tau_\pm}$. For any $v \in H^s(\Omega, \mathcal{T}_h)$ $(s > 5/2)$, we define the mean of the Laplacian of $v$ across $e$ by

$$\{\{\Delta v\}\} = \frac{1}{2}\left(\Delta v_+|_e + \Delta v_-|_e\right).$$

For notational convenience, we also define jump and average on the boundary edges. For any $e \in \mathcal{E}_h^b$, there is a element $\tau \in \mathcal{T}_h$ such that $e = \partial\tau \cap \partial\Omega$. Let $n_e$ be the unit normal of $e$ that points outside $\tau$. For any $v \in H^2(\tau)$, we set on $e$

$$[\![\nabla v]\!] = \nabla v \cdot n_e,$$

and for any $v \in H^s(\Omega, \mathcal{T}_h)$ $(s > 5/2)$, we set

$$\{\{\Delta v\}\} = \Delta v.$$

Define

$$\begin{aligned}\mathcal{A}_h(w, v) = &\sum_{\tau\in\mathcal{T}_h} \int_\tau \Delta w \Delta v\, dx - \sum_{e\in\mathcal{E}_h} \int_e \{\{\Delta w\}\}[\![\nabla v]\!]\, ds \\ &- \sum_{e\in\mathcal{E}_h} \int_e \{\{\Delta v\}\}[\![\nabla w]\!]\, ds + \sum_{e\in\mathcal{E}_h} \int_e \frac{\eta}{h_e}[\![\nabla w]\!][\![\nabla v]\!]\, ds,\end{aligned} \tag{2.17}$$

where $\eta > 0$ is a real number. Let $N$ be some positive integer, $k = T/N$ and $t_n = kn$ for $0 \le n \le N$. Define for any set of functions $\{w_i\}_{i\ge 0}$,

$$\partial w_n = \frac{w_n - w_{n-1}}{k} \quad \text{for } n \ge 1.$$

Define the following semi-norm for $v \in H^s(\Omega, \mathcal{T}_h)$ for $s > 5/2$:

$$\|v\|_h^2 = \left(\sum_{\tau\in\mathcal{T}_h} \|\Delta v\|_{L_2(\tau)}^2 + \sum_{e\in\mathcal{E}_h} \int_e \frac{h_e}{\eta}\{\{\Delta v\}\}^2\, ds + \sum_{e\in\mathcal{E}_h} \int_e \frac{\eta}{h_e}[\![\nabla v\,]\!]^2\; ds\right)$$

**Remark 2.4.** Note that the above semi-norm is well defined only for $v \in H^s(\Omega, \mathcal{T}_h)$ for $s > 5/2$ [9,24].

We refer to [9] for a proof of the following lemma.

**Lemma 2.5.** *It holds that*

$$\mathcal{A}_h(w, v) \le C\|w\|_h\|v\|_h, \quad \forall w,\ v \in H^3(\Omega, \mathcal{T}_h).$$

*For sufficiently large $\eta$, it holds that*

$$C\|v\|_h^2 \le \mathcal{A}_h(v, v), \quad \forall v \in V_h.$$

### *2.2. $C^0$ interior penalty method*

The fully discrete $C^0$ interior penalty method is to find $U_n \in V_h (n \ge 1)$ such that

$$(\partial U_n, v) + \gamma\mathcal{A}_h(U_n, v) + (\nabla U_n, \nabla v) + (\phi(U_n), v) = 0, \quad \forall v \in V_h, \tag{2.18}$$

$$U_0 = \Pi_h u_0, \tag{2.19}$$

where $\Pi_h : V \to V_h$ is a projection which can be either a nodal interpolation or an elliptic projection.

Below, we establish the existence and uniqueness of the fully discrete solution.

**Theorem 2.6.** *Let $k < 1/2$. Then for $1 \le n \le N$, there exists a unique solution $U_n \in V_h$ to (2.18)–(2.19).*

**Proof.** To prove this by recursion, we suppose that there exists a unique solution $U_{n-1}$ (since for $n = 1$, we know $U_0$ from the initial condition) and prove the existence of a unique $U_n$ using the Brouwer fixed point theorem. For this, we define a map $S_h : V_h \to V_h$ i.e., $p = S_h(q) \in V_h$ for $q \in V_h$, by

$$(p, v) + k\gamma \mathcal{A}_h(p, v) + k(\nabla p, \nabla v) = k((1 - q^2)p, v) + (U_{n-1}, v), \quad \forall v \in V_h. \tag{2.20}$$

We will show that the map $S_h$ is well-defined. Set $v = p$ in (2.20) and find

$$\|p\|^2 + k\gamma \mathcal{A}_h(p, p) + k\|\nabla p\|^2 \le k\|p\|^2 + \|U_{n-1}\| \, \|p\|,$$

where we have used the fact that $-(q^2 p, p) \le 0$. This implies

$$\left(\frac{1}{2} - k\right) \|p\|^2 + k\gamma \mathcal{A}_h(p, p) + k\|\nabla p\|^2 \le \frac{1}{2}\|U_{n-1}\|^2.$$

It is obvious that for $k < 1/2$, the map $S_h$ is well defined and

$$\|p\| \le \frac{1}{\sqrt{1 - 2k}} \|U_{n-1}\|. \tag{2.21}$$

Let $r = \frac{1}{\sqrt{1-2k}} \|U_{n-1}\|$ and define $B_r = \{v_h \in V_h : \|v_h\| \le r\}$. Then from (2.21), it is clear that $S_h$ maps $B_r$ into itself. In order to prove that $S_h$ is continuous, we let $p_0 = S_h(q_0)$ and $p = S_h(q)$ for $q_0,\ q \in B_r$. From the definition (2.20) of $S_h$, we note

$$(p - p_0, v) + k\gamma \mathcal{A}_h(p - p_0, v) + k(\nabla(p - p_0), \nabla v) = k((1 - q^2)(p - p_0), v) + k((q_0^2 - q^2)p_0, v) \quad \forall v \in V_h,$$

and set $v = p - p_0$ to obtain

$$\|p - p_0\|^2 + k\gamma \mathcal{A}_h(p - p_0, p - p_0) + k\|\nabla(p - p_0)\| \le k\big((1 - q^2)(p - p_0), (p - p_0)\big) + k((q_0^2 - q^2)p_0, p - p_0).$$

On noting that $-\big(q^2(p - p_0), (p - p_0)\big) \le 0$ and using Holder's inequality, we find that

$$\left(\frac{1}{2} - k\right) \|p - p_0\|^2 \le \frac{k^2}{2} \|q + q_0\|^2_{L_\infty(\Omega)} \|p_0\|^2_{L_\infty(\Omega)} \|q - q_0\|^2.$$

This implies

$$\|p - p_0\|^2 \le \left(\frac{k^2}{1 - 2k}\right) \|q + q_0\|^2_{L_\infty(\Omega)} \|p_0\|^2_{L_\infty(\Omega)} \|q - q_0\|^2.$$

Since $q$, $q_0$ and $p_0 \in B_r$, the use of the inverse inequality implies

$$\|q + q_0\|^2_{L_\infty(\Omega)} \|p_0\|^2_{L_\infty(\Omega)} \le Ch_{\min}^{-2d} \|U_{n-1}\|^4 \tag{2.22}$$

and hence

$$\|p - p_0\|^2 \le C \left(\frac{k^2 h_{\min}^{-2d}}{1 - 2k}\right) \|U_{n-1}\|^4 \, \|q - q_0\|^2,$$

where $h_{\min} = \min\{h_\tau : \tau \in \mathcal{T}_h\}$. Therefore for given $k$ and $h_{\min}$, the map $S_h$ is continuous in the ball $B_r$. Now the Brouwer fixed point theorem completes the proof of existence.

To prove uniqueness, suppose there exist two solutions $U$ and $W$ at $t = t_n$. Then $U$ and $W$ satisfy

$$(U - W, v) + k\gamma \mathcal{A}_h(U - W, v) + k(\nabla(U - W), \nabla v) = k(U - U^3 - W + W^3, v), \quad \forall v \in V_h.$$

Set $v = U - W$ in the above to arrive at

$$\begin{aligned} \|U - W\|^2 + k\gamma \mathcal{A}_h(U - W, U - W) + k\|\nabla(U - W)\|^2 &= k\|U - W\|^2 - k(U^3 - W^3, U - W) \\ &\le k\|U - W\|^2 - k((U^2 + UW + W^2), (U - W)^2) \\ &\le k\|U - W\|^2, \end{aligned}$$

where we have used the fact that $(a^2 + ab + b^2) \ge \frac{1}{2}(a^2 + b^2) \ge 0$. Now uniqueness follows since $k < 1/2$. This completes the proof. $\square$

In the following, we derive stability bounds for the fully discrete solution.

**Lemma 2.7.** *Let $k < 1/2$. Then, there is a constant $C$ independent of $h$, $k$ and $\gamma$ such that*

$$\gamma \|U_n\|_h^2 + \|\nabla U_n\|^2 \le C \left(\gamma \|U_0\|_h^2 + \|\nabla U_0\|^2 + (\Psi(U_0), 1)\right),$$

*and*

$$\|U_n\|^2 \le C(T)\|U_0\|^2,$$

*for all* $1 \le n \le N$.

**Proof.** Let $1 \le i \le n \le N$ and set $v = U_i - U_{i-1}$ in (2.18) with $n = i$. Then

$$\begin{aligned}&\frac{1}{k}\|U_i - U_{i-1}\|^2 + \frac{\gamma}{2}\left[\mathcal{A}_h(U_i, U_i) - \mathcal{A}_h(U_{i-1}, U_{i-1}) + \mathcal{A}_h(U_i - U_{i-1}, U_i - U_{i-1})\right] \\ &\quad + \frac{1}{2}\left[(\nabla U_i, \nabla U_i) - (\nabla U_{i-1}, \nabla U_{i-1}) + (\nabla(U_i - U_{i-1}), \nabla(U_i - U_{i-1}))\right] + (\phi(U_i), U_i - U_{i-1}) = 0. \end{aligned} \tag{2.23}$$

Using Taylor's theorem and the fact that $\Psi'' \ge -1$, we note that

$$\begin{aligned}(\phi(U_i), U_i - U_{i-1}) &= (\Psi'(U_i), U_i - U_{i-1}) \\ &\ge (\Psi(U_i) - \Psi(U_{i-1}), 1) - \frac{1}{2}\|U_i - U_{i-1}\|^2.\end{aligned}$$

Using this in (2.23),

$$\begin{aligned}&\frac{1}{k}\|U_i - U_{i-1}\|^2 + \frac{\gamma}{2}\left[\mathcal{A}_h(U_i, U_i) - \mathcal{A}_h(U_{i-1}, U_{i-1}) + \mathcal{A}_h(U_i - U_{i-1}, U_i - U_{i-1})\right] \\ &\quad + \frac{1}{2}\left[(\nabla U_i, \nabla U_i) - (\nabla U_{i-1}, \nabla U_{i-1}) + (\nabla(U_i - U_{i-1}), \nabla(U_i - U_{i-1}))\right] \\ &\quad + (\Psi(U_i) - \Psi(U_{i-1}), 1) \le \frac{1}{2}\|U_i - U_{i-1}\|^2.\end{aligned}$$

We sum over $i = 1$ to $n$ and find

$$\begin{aligned}&\left(\frac{1}{k} - \frac{1}{2}\right)\sum_{i=1}^{n}\|U_i - U_{i-1}\|^2 + \frac{\gamma}{2}\mathcal{A}_h(U_n, U_n) + \frac{1}{2}(\nabla U_n, \nabla U_n) + (\Psi(U_n), 1) \\ &\quad \le \frac{\gamma}{2}\mathcal{A}_h(U_0, U_0) + \frac{1}{2}(\nabla U_0, \nabla U_0) + (\Psi(U_0), 1).\end{aligned}$$

Now using Lemma 2.5 and the assumption that $k < 1/2$, we deduce the first inequality. To prove the second inequality, let $v = U_i$ in (2.18) with $n = i$ and arrive at

$$(U_i - U_{i-1}, U_i) + \gamma k \mathcal{A}_h(U_i, U_i) + k\|\nabla U_i\|^2 + k(\phi(U_i), U_i) = 0.$$

Since

$$(U_i - U_{i-1}, U_i) \ge \frac{1}{2}\left(\|U_i\|^2 - \|U_{i-1}\|^2\right),$$

and

$$-(\phi(U_i), U_i) \le \|U_i\|^2,$$

we find

$$\frac{1}{2}\left(\|U_i\|^2 - \|U_{i-1}\|^2\right) \le k\|U_i\|^2.$$

By summing over $i = 1$ to $n$, we obtain

$$\frac{1}{2}\|U_n\|^2 \le k\sum_{i=1}^{n}\|U_i\|^2 + \frac{1}{2}\|U_0\|^2.$$

Now discrete Gronwall's lemma completes the proof. □

## 3. Regularity estimate

In this section, we derive the $L_\infty(0, T; H^3(\Omega))$ regularity estimate for the solution $u$ of (2.1)–(2.2) that is explicit in $\gamma$. For this, we assume that the initial data $u_0$ is in $H^4(\Omega)$ and satisfies the boundary conditions (1.2). The resulting regularity estimate will be used in our error analysis to derive *a priori* error estimates that are robust in $\gamma$. To this end, we use the elliptic regularity result for second order problems. The idea of using elliptic regularity results for second order problems in the error analysis of the $C^0$ interior penalty method for a fourth order elliptic singular perturbation problem is exploited in [18].

**Theorem 3.1.** *Let $u_0 \in H^4(\Omega)$ be such that*

$$\gamma \|\Delta^2 u_0\| + \|\Delta u_0\| + \|\phi(u_0)\| \le M_2, \tag{3.1}$$

*for some $M_2 > 0$. Moreover, assume that $u_0$ satisfies the boundary conditions (1.2). Then, the solution $u$ of (2.1)–(2.2) satisfies*

$$\|u\|_{L_\infty\left(0,T;H^2(\Omega)\right)} \le C(T, M_1, M_2) \quad \textit{and} \quad \|u\|_{L_\infty\left(0,T;H^3(\Omega)\right)} \le \frac{C(T, M_1, M_2)}{\sqrt{\gamma}}, \tag{3.2}$$

*for some constant $C(T, M_1, M_2)$ independent of $\gamma$.*

**Proof.** First of all, we need to bound the $L_2$ norm of $u_t$. For this, differentiate (2.1) formally with respect to $t$ and then set $v = u_t$ in the resulting equation to obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \gamma\|\Delta u_t\|^2 + \|\nabla u_t\|^2 + ((3u^2 - 1)u_t, u_t) = 0.$$

Since $(3u^2 u_t, u_t) \ge 0$,

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2 + \gamma\|\Delta u_t\|^2 + \|\nabla u_t\|^2 \le \|u_t\|^2. \tag{3.3}$$

By integrating (3.3) from 0 to $t$ for $t \in (0, T]$, we obtain

$$\frac{1}{2}\|u_t(t)\|^2 + \int_0^t \left(\gamma\|\Delta u_t(s)\|^2 + \|\nabla u_t(s)\|^2\right) ds \le \frac{1}{2}\|u_t(0)\|^2 + \int_0^t \|u_t(s)\|^2 ds.$$

An appeal to Gronwall's lemma yields for $t \in (0, T]$,

$$\frac{1}{2}\|u_t(t)\|^2 + \int_0^t \left(\gamma\|\Delta u_t(s)\|^2 + \|\nabla u_t(s)\|^2\right) ds \le C(T)\frac{1}{2}\|u_t(0)\|^2. \tag{3.4}$$

To find a bound for $\|u_t(0)\|$, we take $t \to 0$ in (2.1) and obtain

$$(u_t(0), v) + \gamma(\Delta u_0, \Delta v) + (\nabla u_0, \nabla v) + (\phi(u_0), v) = 0 \quad \forall v \in V. \tag{3.5}$$

Now set $v = u_t(0)$ in (3.5) and derive

$$\begin{aligned} \|u_t(0)\|^2 &= -\gamma(\Delta u_0, \Delta u_t(0)) - (\nabla u_0, \nabla u_t(0)) - (\phi(u_0), u_t(0)) \\ &= -\gamma(\Delta^2 u_0, u_t(0)) + (\Delta u_0, u_t(0)) - (\phi(u_0), u_t(0)) \\ &\le \left(\gamma\|\Delta^2 u_0\| + \|\Delta u_0\| + \|\phi(u_0)\|\right)\|u_t(0)\|. \end{aligned} \tag{3.6}$$

Use (3.1) in (3.6) to find

$$\|u_t(0)\| \le M_2. \tag{3.7}$$

From (3.4) and (3.7), we find for $t \in (0, T]$

$$\frac{1}{2}\|u_t(t)\|^2 + \int_0^t \left(\gamma\|\Delta u_t(s)\|^2 + \|\nabla u_t(s)\|^2\right) ds \le C(T)M_2^2, \tag{3.8}$$

which implies an $L_\infty(0, T; L_2(\Omega))$ bound for $u_t$. From (2.1), it can easily be seen that

$$\int_\Omega (u_t + \phi(u))\, dx = 0,$$

and hence there is a solution $w \in H^1(\Omega)$ to the following problem

$$(\nabla w, \nabla v) = -(u_t + \phi(u), v) \quad \forall v \in H^1(\Omega) \tag{3.9}$$

satisfying $(w, 1) = 0$. Moreover since $\Omega$ is convex, $w \in V$ and satisfies the following elliptic regularity [19]:

$$\|w\|_{H^2(\Omega)} \le C\|u_t + \phi(u)\|_{L_2(\Omega)}. \tag{3.10}$$

From (2.7) and Sobolev embedding, we note that

$$\begin{aligned} \|\phi(u(t))\|^2_{L_2(\Omega)} &= \|u(t)^3 - u(t)\|^2_{L_2(\Omega)} \\ &= \|u(t)\|^6_{L_6(\Omega)} + \|u(t)\|^2_{L_2(\Omega)} - 2\|u(t)\|^4_{L_4(\Omega)} \le C(T, M_1). \end{aligned} \tag{3.11}$$

Combining (3.8), (3.10) and (3.11), we find

$$\|w\|_{H^2(\Omega)} \leq C(T, M_1, M_2). \tag{3.12}$$

Note from (2.1) and (3.9) that,

$$\begin{aligned}\gamma(\Delta(u-w), \Delta(u-w)) + (\nabla(u-w), \nabla(u-w)) &= -\gamma(\Delta w, \Delta(u-w)) \\ &\leq \gamma\|\Delta w\|\,\|\Delta(u-w)\|.\end{aligned} \tag{3.13}$$

Therefore

$$\|\Delta(u-w)\| \leq \|\Delta w\|,$$

which together with (3.12) implies that $\|\Delta u\| \leq C(T, M_1, M_2)$. Using Lemma 2.1 and (2.7), we deduce

$$\|u\|_{H^2(\Omega)} \leq C(T, M_1, M_2)$$

and complete the proof of the first inequality in (3.2). From (3.13), we also find that

$$\|\nabla(u-w)\| \leq \sqrt{\frac{\gamma}{2}}\|\Delta w\|.$$

We note from (2.1) that $u$ satisfies

$$\begin{aligned}(\Delta u, \Delta v) &= \frac{1}{\gamma}(-u_t - \phi(u) + \Delta u, v) \\ &= \frac{1}{\gamma}(\Delta(u-w), v),\end{aligned}$$

and hence using the regularity theory of fourth order elliptic problems on convex domains [25], we find

$$\|u\|_{H^3(\Omega)} \leq \frac{C}{\gamma}\|\Delta(u-w)\|_{H^{-1}(\Omega)} \leq \frac{C}{\gamma}\|\nabla(u-w)\| \leq \frac{C}{\sqrt{\gamma}}\|w\|_{H^2(\Omega)}$$

for some positive constant $C$ which is independent of $\gamma$. This and (3.12) complete the proof. □

## 4. Error analysis

In this section, we derive *a priori* error analysis of our fully discrete $C^0$ interior penalty method. For this, we recall the following approximation properties of the standard Lagrange nodal interpolation operator $I_h : H^3(\Omega) \longrightarrow V_h$ from [26,27].

**Lemma 4.1.** *Let $w \in H^3(\Omega)$. Then there is a constant $C$ independent of $h$ such that*

$$|w - I_h w|_{H^\ell(\tau)} \leq Ch_\tau^{3-\ell}|w|_{H^3(\tau)} \quad \text{for } 0 \leq \ell \leq 3, \tag{4.1}$$

$$\int_{\partial\tau}\left|\frac{\partial}{\partial n}(w - I_h w)\right|^2 ds \leq Ch_\tau^3|w|^2_{H^3(\tau)}, \tag{4.2}$$

$$\int_{\partial\tau}|\Delta(w - I_h w)|^2\, ds \leq Ch_\tau|w|^2_{H^3(\tau)}, \tag{4.3}$$

*where $\tau \in \mathcal{T}_h$ and $\partial\tau$ is the boundary of $\tau$.*

The estimate (4.1) is standard while the estimates (4.2)–(4.3) follow from (4.1) and the following trace inequality with scaling [19]:

$$\|v\|_{L_2(\partial\tau)} \leq Ch_\tau^{-1/2}\left(\|v\|^2_{L_2(\tau)} + h_\tau\|v\|_{L_2(\tau)}\|\nabla v\|_{L_2(\tau)}\right)^{1/2},$$

where $C$ is independent of $h$ and $\tau \in \mathcal{T}_h$.

Using Lemma 4.1, it is easy to show the following [9].

There is some constant $C$ which is independent of $h$ such that

$$\|w - I_h w\|_h \leq Ch\|w\|_{H^3(\Omega)} \quad \forall w \in H^3(\Omega). \tag{4.4}$$

In the error analysis, we require an elliptic projection $R_h : V \cap H^3(\Omega) \to V_h$ defined by

$$\gamma\mathcal{A}_h(w - R_h w, v_h) + \big(\nabla(w - R_h w), v_h\big) = 0, \quad \forall v_h \in V_h, \tag{4.5}$$

$$(w - R_h w, 1) = 0 \tag{4.6}$$

and for all $w \in V \cap H^3(\Omega)$. Using Lemma 2.5, we deduce that the projection $R_h$ is well-defined.

We now prove error estimates for $R_h$.

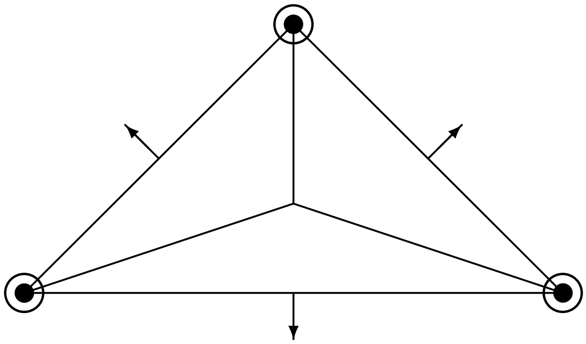

**Fig. 4.1.** Degrees of freedom for the $P_3$-HCT macro element. Thick dots denote the values of the function, circles denote the values of its first order derivatives and arrows denote the values of its normal derivatives.

**Lemma 4.2.** *Let $w \in V \cap H^3(\Omega)$. Then, there exists a constant $C > 0$ which is independent of $h$, $k$ and $\gamma$ such that*

$$\gamma \|w - R_h w\|_h^2 + \|\nabla(w - R_h w)\|^2 \le C\left(\gamma h^2 + h^4\right)\|w\|_{H^3(\Omega)}^2,$$

*and*

$$\|w - R_h w\|^2 \le C\left(\gamma h^4 + h^6 + \gamma^{-1}h^8\right)\|w\|_{H^3(\Omega)}^2.$$

**Proof.** First of all, using the arguments analogous to Cea's Lemma [26] and Lemma 2.5, it is easy to find that

$$\gamma \|w - R_h w\|_h^2 + \|\nabla(w - R_h w)\|^2 \le C \min_{v \in V_h}\left(\gamma \|w - v\|_h^2 + \|\nabla(w - v)\|^2\right).$$

Using Lemma 4.1 and (4.4), we find

$$\gamma \|w - R_h w\|_h^2 + \|\nabla(w - R_h w)\|^2 \le C\left(\gamma h^2 \|w\|_{H^3(\Omega)}^2 + h^4 \|w\|_{H^3(\Omega)}^2\right).$$

To derive the $L_2$ norm error estimate, we consider the following adjoint elliptic problem: find $z \in V$ such that

$$\gamma(\Delta z, \Delta v) + (\nabla z, \nabla v) = (w - R_h w, v), \quad \forall v \in V. \tag{4.7}$$

Since $\Omega$ is convex, we can derive $\|z\|_{H^3(\Omega)} \le C\frac{1}{\sqrt{\gamma}}\|w - R_h w\|$ and $\|z\|_{H^2(\Omega)} \le C\|w - R_h w\|$ using the same technique in Theorem 3.1. Let $z_h = I_h z \in V_h$ be an approximation of $z$ as in Lemma 4.1. Then using (4.7), Lemmas 2.5 and 4.1,

$$\begin{aligned}
\|w - R_h w\|^2 &= \gamma \mathcal{A}_h(z, w - R_h w) + (\nabla z, \nabla(w - R_h w)) \\
&= \gamma \mathcal{A}_h(z - z_h, w - R_h w) + (\nabla(z - z_h), \nabla(w - R_h w)) \\
&\le C\gamma \|w - R_h w\|_h \|z - z_h\|_h + \|\nabla(w - R_h w)\| \, \|\nabla(z - z_h)\| \\
&\le C\gamma h \|w - R_h w\|_h \|z\|_{H^3(\Omega)} + Ch^2 \|\nabla(w - R_h w)\| \, \|z\|_{H^3(\Omega)} \\
&\le C\left(\sqrt{\gamma} h \|w - R_h w\|_h + Ch^2 \frac{1}{\sqrt{\gamma}} \|\nabla(w - R_h w)\|\right)\|w - R_h w\|
\end{aligned}$$

which implies

$$\begin{aligned}
\|w - R_h w\|^2 &\le C\left(\gamma h^2 \|w - R_h w\|_h^2 + Ch^4 \gamma^{-1} \|\nabla(w - R_h w)\|^2\right) \\
&\le C\left(h^2(\gamma h^2 + h^4) + Ch^4 \gamma^{-1}(\gamma h^2 + h^4)\right)\|w\|_{H^3(\Omega)}^2.
\end{aligned}$$

Hence the proof. □

Below, we derive a discrete Sobolev inequality on $V_h$. To prove this without any quasi-uniformity assumption on the mesh $\mathcal{T}_h$, we let $V_c \subset H^2(\Omega)$ be the P3-Hsieh–Clough–Tocher finite element space associated with $\mathcal{T}_h$ [27,26]. In order to construct a P3-HCT finite element, each triangle $\tau \in \mathcal{T}_h$ is subdivided into three triangles as in Fig. 4.1. Then using this mesh for $\tau$, the P3-HCT element is constructed by using $C^1$-piecewise cubic polynomials based on the degrees of freedom given in Fig. 4.1.

From [28,9], note that there is an enriching map $E_h : V_h \to V_c$ such that

$$\sum_{\tau \in \mathcal{T}_h} h_\tau^{-4} \|v - E_h v\|_{L_2(\tau)}^2 \le C \sum_{e \in E_h} \int_e \frac{1}{h_e} [\![\nabla v]\!]^2 \, ds,$$

$$\sum_{\tau\in\mathcal{T}_h} h_\tau^{-2}\|v-E_hv\|_{H^1(\tau)}^2 \le C\sum_{e\in E_h}\int_e \frac{1}{h_e}[\![\nabla v]\!]^2\,ds,$$
$$\sum_{\tau\in\mathcal{T}_h} \|v-E_hv\|_{H^2(\tau)}^2 \le C\sum_{e\in E_h}\int_e \frac{1}{h_e}[\![\nabla v]\!]^2\,ds.$$

Also recall the following local inverse inequality from [27,26]:

$$\|v\|_{L_\infty(\tau)} \le Ch_\tau^{-d/2}\|v\|_{L_2(\tau)}, \tag{4.8}$$

for all $\tau\in\mathcal{T}_h$.

We now prove a lemma on the discrete Sobolev inequality.

**Lemma 4.3.** *There is a constant $C$ which is independent of $h$, $k$ and $\gamma$ such that*

$$\|v\|_{L_\infty(\Omega)} \le C\|v\|_h, \quad \forall v\in V_h.$$

**Proof.** Using the inverse inequality (4.8), Sobolev embedding and Lemma 2.1

$$\begin{aligned}
\|v\|_{L_\infty(\Omega)}^2 &\le \|v-E_hv\|_{L_\infty(\Omega)}^2 + \|E_hv\|_{L_\infty(\Omega)}^2\\
&\le \sum_{\tau\in\mathcal{T}_h}\|v-E_hv\|_{L_\infty(\tau)}^2 + C\|E_hv\|_{H^2(\Omega)}^2\\
&\le C\sum_{\tau\in\mathcal{T}_h} h_\tau^{-d}\|v-E_hv\|_{L_2(\tau)}^2 + C\left(\|\Delta E_hv\|^2 + \|E_hv\|_{H^1(\Omega)}^2\right)\\
&\le C\sum_{e\in E_h}\int_e\frac{1}{h_e}[\![\nabla v]\!]^2\,ds + C\left(\sum_{\tau\in\mathcal{T}_h}\|\Delta(E_hv-v)\|_{L_2(\tau)}^2 + \|E_hv-v\|_{H^1(\Omega)}^2\right)\\
&\quad + C\left(\sum_{\tau\in\mathcal{T}_h}\|\Delta v\|_{L_2(\tau)}^2 + \|v\|_{H^1(\Omega)}^2\right)\\
&\le C\|v\|_h^2.
\end{aligned}$$

Hence the proof. □

Below, we prove some stability estimates for $R_hu$, where $u$ is the solution of (2.1)–(2.2). For this, we recall the following trace inequality [27,26]:

$$\|v\|_{L_2(\partial\tau)} \le Ch_\tau^{-1/2}\|v\|_{L_2(\tau)}, \quad \forall v\in V_h, \tag{4.9}$$

for all $\tau\in\mathcal{T}_h$.

**Lemma 4.4.** *For the solution $u$ of (2.1)–(2.2), it holds that*

$$\|R_hu\|_{H^1(\Omega)} \le C, \tag{4.10}$$

*where $C$ is a constant independent of $h$, $k$ and $\gamma$. Moreover, if $h\le\sqrt{\gamma}$*

$$\|R_hu\|_h \le C, \tag{4.11}$$
$$\|R_hu\|_{L_\infty(\Omega)} \le C. \tag{4.12}$$

**Proof.** First of all, we note from Theorem 3.1 that

$$\|u\|_{H^3(\Omega)} \le \frac{C}{\sqrt{\gamma}} \quad\text{and}\quad \|u\|_{H^2(\Omega)} \le C. \tag{4.13}$$

Using Lemma 2.5, Lemma 4.1 and (4.13), we obtain

$$\begin{aligned}
\gamma\|u-R_hu\|_h^2 + \|\nabla(u-R_hu)\|^2 &\le C\left(\gamma\|u-I_hu\|_h^2 + \|\nabla(u-I_hu)\|^2\right)\\
&\le C\left(\gamma h^2\|u\|_{H^3(\Omega)}^2 + h^2\|u\|_{H^2(\Omega)}^2\right)\\
&\le Ch^2,
\end{aligned}$$

which implies

$$\gamma \|u - R_h u\|_h^2 \le Ch^2,$$
$$\|\nabla(u - R_h u)\|^2 \le Ch^2. \tag{4.14}$$

The proof of (4.10) follows from (4.14) and (4.13). Now since $h \le \sqrt{\gamma}$,

$$\|u - R_h u\|_h^2 \le C\gamma^{-1}h^2 \le C.$$

To prove (4.11), first we note using the trace inequality (4.9) that

$$\begin{aligned}
\|R_h u\|_h^2 &= \left(\sum_{\tau\in\mathcal{T}_h} \|\Delta R_h u\|_{L_2(\tau)}^2 + \sum_{e\in\mathcal{E}_h} \int_e \frac{h_e}{\eta}\{\{\Delta R_h u\}\}^2\, ds + \sum_{e\in\mathcal{E}_h} \int_e \frac{\eta}{h_e}[\![\nabla R_h u\,]\!]^2\, ds\right)\\
&\le C\left(\sum_{\tau\in\mathcal{T}_h} \|\Delta R_h u\|_{L_2(\tau)}^2 + \sum_{e\in\mathcal{E}_h} \int_e \frac{\eta}{h_e}[\![\nabla R_h u\,]\!]^2\, ds\right)\\
&\le C\left(\sum_{\tau\in\mathcal{T}_h} \|\Delta(u - R_h u)\|_{L_2(\tau)}^2 + \sum_{e\in\mathcal{E}_h} \int_e \frac{\eta}{h_e}[\![\nabla (u - R_h u)\,]\!]^2\, ds\right) + C\left(\sum_{\tau\in\mathcal{T}_h} \|\Delta u\|_{L_2(\tau)}^2\right)\\
&\le C,
\end{aligned}$$

which proves (4.11). The proof of (4.12) follows from (4.11) and Lemma 4.3. □

We now derive *a priori* error estimates.

**Theorem 4.5.** *There is a constant $C = C(T)$ which depends on $T$ but is independent of $h$, $k$ and $\gamma$ such that*

$$\begin{aligned}
\|U_n - u(t_n)\|^2 \le\ & \|u(t_n) - R_h u(t_n)\|^2 + C(T)\left(\|\Pi_h u_0 - R_h(u_0)\|^2 + \|u_0 - R_h u_0\|^2\right)\\
& + C(T)\left(\int_0^{t_n} \|(R_h - I)u_t\|^2 ds + k^2 \int_0^{t_n} \|u_{tt}(s)\|^2 ds\right)
\end{aligned}$$

*for all $1 \le n \le N$.*

**Proof.** Let $1 \le i \le n \le N$, $\theta_i = R_h u(t_i) - U_i$ and $\rho_i = R_h u(t_i) - u(t_i)$. Then

$$\begin{aligned}
(\partial\theta_i, v) + \gamma\mathcal{A}_h(\theta_i, v) + (\nabla\theta_i, \nabla v) &= (\phi(U_i) - \phi(u(t_i)), v) + (\partial R_h u(t_i) - u_t(t_i), v)\\
&= (\phi(U_i) - \phi(u(t_i)), v) + (\omega_i, v),
\end{aligned} \tag{4.15}$$

where $\omega_i = \omega_{i,1} + \omega_{i,2}$, $\omega_{i,1} = (R_h - I)\partial u(t_i)$ and $\omega_{i,2} = \partial u(t_i) - u_t(t_i)$.

Set $v = \theta_i$ in (4.15),

$$(\partial\theta_i, \theta_i) + \gamma\mathcal{A}_h(\theta_i, \theta_i) + \|\nabla\theta_i\|^2 = (\phi(U_i) - \phi(u(t_i)), \theta_i) + (\omega_i, \theta_i).$$

Since we can write

$$(\phi(U_i) - \phi(u(t_i)), \theta_i) = (\phi(U_i) - \phi(R_h u(t_i)), \theta_i) + (\phi(R_h u(t_i)) - \phi(u(t_i)), \theta_i),$$

and since the first term

$$(\phi(U_i) - \phi(R_h u(t_i)), \theta_i) = \|\theta_i\|^2 - (U_i^2 + U_i R_h u(t_i) + R_h u(t_i)^2, \theta_i^2) \le \|\theta_i\|^2,$$

we have

$$(\partial\theta_i, \theta_i) + \gamma\mathcal{A}_h(\theta_i, \theta_i) + \|\nabla\theta_i\|^2 \le (\phi(R_h u(t_i)) - \phi(u(t_i)), \theta_i) + (\omega_i, \theta_i) + \|\theta_i\|^2. \tag{4.16}$$

The first term on the right-hand side of (4.16) is estimated using Theorem 2.2 and (4.10) as

$$\begin{aligned}
(\phi(R_h u(t_i)) - \phi(u(t_i)), \theta_i) &= ((u(t_i)^2 + u(t_i)R_h u(t_i) + R_h u(t_i)^2 - 1)\rho_i, \theta_i)\\
&\le \frac{3}{2}(u(t_i)^2|\rho_i|, |\theta_i|) + \frac{3}{2}(R_h u(t_i)^2|\rho_i|, |\theta_i|) + \|\rho_i\|\,\|\theta_i\|\\
&\le \frac{3}{2}\left[\|u(t_i)\|_{L_6(\Omega)}^2 + \|R_h u(t_i)\|_{L_6(\Omega)}^2\right]\|\rho_i\|\,\|\theta_i\|_{L_6(\Omega)} + \|\rho_i\|\,\|\theta_i\|\\
&\le C\|\rho_i\|^2 + \frac{1}{2}\left(\|\nabla\theta_i\|^2 + \|\theta_i\|^2\right).
\end{aligned}$$

The Cauchy–Schwarz inequality implies

$$(\omega_i, \theta_i) \le \frac{1}{2}\|\omega_i\|^2 + \frac{1}{2}\|\theta_i\|^2.$$

From (4.16),

$$(\partial\theta_i, \theta_i) + \gamma\mathcal{A}_h(\theta_i, \theta_i) + \frac{1}{2}\|\nabla\theta_i\|^2 \le C\left(\|\theta_i\|^2 + \|\rho_i\|^2 + \|\omega_i\|^2\right).$$

Since

$$(\partial\theta_i, \theta_i) \ge \frac{1}{2k}\|\theta_i\|^2 - \frac{1}{2k}\|\theta_{i-1}\|^2,$$

we obtain by summing over $i = 1$ to $n$ that

$$\|\theta_n\|^2 \le \|\theta_0\|^2 + Ck\sum_{i=1}^{n}\left(\|\theta_i\|^2 + \|\rho_i\|^2 + \|\omega_i\|^2\right). \tag{4.17}$$

Using the similar techniques in [29, Theorem 1.5], we estimate the third term on the right-hand side of (4.17) as follows: Since $\omega_i$ in (4.15) is $\omega_i = \omega_{i,1} + \omega_{i,2}$, we first take

$$\omega_{i,1} = k^{-1}(R_h - I)\int_{t_{i-1}}^{t_i} u_t(s)ds.$$

We square, integrate over $\Omega$ and use Cauchy–Schwarz for the above,

$$\|\omega_{i,1}\|^2 \le k^{-1}\int_{t_{i-1}}^{t_i}\|(R_h - I)u_t(s)\|^2 ds.$$

By summing over $i = 1$ to $n$, we obtain

$$k\sum_{i=1}^{n}\|\omega_{i,1}\|^2 = \int_0^{t_n}\|\rho_t\|^2 ds.$$

Next we take

$$k\omega_{i,2} = u(t_i) - u(t_{i-1}) - ku_t(t_i) = -\int_{t_{i-1}}^{t_i}(s - t_{i-1})u_{tt}(s)ds.$$

It implies

$$\|\omega_{i,2}\|^2 \le k\int_{t_{i-1}}^{t_i}\|u_{tt}(s)\|^2 ds,$$

and

$$k\sum_{i=1}^{n}\|\omega_{i,2}\|^2 \le k^2\int_0^{t_n}\|u_{tt}(s)\|^2 ds.$$

Now we will bound $\|\rho_i\|$. We can write

$$\rho_i = (R_h - I)u(t_i) = (R_h - I)u(0) + \int_0^{t_i}(R_h - I)u_t(s)ds.$$

Therefore

$$\|\rho_i\|^2 \le 2\left(\|\rho_0\|^2 + t_i\int_0^{t_i}\|\rho_t\|^2 ds\right)$$

and

$$k\sum_{i=1}^{n}\|\rho_i\|^2 \le 2t_n\left(\|\rho_0\|^2 + t_n\int_0^{t_n}\|\rho_t\|^2 ds\right).$$

We obtain

$$\|\theta_n\|^2 \le \left(\|\theta_0\|^2 + Ct_n\|\rho_0\|^2 + C(1 + 2t_n^2)\int_0^{t_n}\|\rho_t\|^2 ds + Ck^2\int_0^{t_n}\|u_{tt}(s)\|^2 ds\right) + Ck\sum_{i=1}^{n}\|\theta_i\|^2.$$

Discrete Gronwall's lemma implies

$$\|U^n - u(t_n)\|^2 \le \|u(t_n) - R_h u(t_n)\|^2 + C(T)\left(\|\Pi_h u_0 - R_h(u_0)\|^2 + \|u_0 - R_h u_0\|^2\right) + C(T)\left(\int_0^{t_n} \|(R_h - I)u_t\|^2 ds + k^2 \int_0^{t_n} \|u_{tt}(s)\|^2 ds\right).$$

This completes the proof. □

From Theorem 4.5 and Lemma 4.2, we deduce the following.

**Corollary 4.6.** *Let $\Pi_h u_0 = I_h u_0$ or $R_h u_0$. Then, there is a constant $C = C(T)$ which depends on $T$ but is independent of $h, k$ and $\gamma$ such that*

$$\|U^n - u(t_n)\|^2 \le C(T)\left(\gamma h^4 + h^6 + \gamma^{-1}h^8\right)\left(\|u(t_n)\|^2_{H^3(\Omega)} + \|u_0\|^2_{H^3(\Omega)} + \int_0^{t_n} \|u_t\|^2_{H^3(\Omega)} ds\right) + C(T)k^2 \int_0^{t_n} \|u_{tt}(s)\|^2 ds,$$

*for all $1 \le n \le N$.*

We next derive a super-convergence result for $\left(U_n - R_h u(t_n)\right)$ in the $H^1$ norm.

**Theorem 4.7.** *Let $h \le \sqrt{\gamma}$ and $\Pi_h u_0 = R_h u_0$. Then,*

$$\gamma \|U^n - R_h u(t_n)\|_h^2 + \|\nabla(U^n - R_h u(t_n))\|^2 \le C(T)\|u_0 - R_h u_0\|^2 + C(T)\left(\int_0^{t_n} \|(R_h - I)u_t\|^2 ds + k^2 \int_0^{t_n} \|u_{tt}(s)\|^2 ds\right),$$

*for all $1 \le n \le N$ and for a constant $C(T)$ which is independent of $h, k$ and $\gamma$ but depending on $T$.*

**Proof.** Let $1 \le i \le n \le N$. Then, set $v = \theta_i - \theta_{i-1}$ in (4.15).

$$(\partial\theta_i, \theta_i - \theta_{i-1}) + \gamma \mathcal{A}_h(\theta_i, \theta_i - \theta_{i-1}) + (\nabla\theta_i, \nabla(\theta_i - \theta_{i-1})) = (\phi(U_i) - \phi(u(t_i)), \theta_i - \theta_{i-1}) + (\omega_i, \theta_i - \theta_{i-1}).$$

First note that

$$\begin{aligned}
&(\partial\theta_i, \theta_i - \theta_{i-1}) + \gamma \mathcal{A}_h(\theta_i, \theta_i - \theta_{i-1}) + (\nabla\theta_i, \nabla(\theta_i - \theta_{i-1})) \\
&\quad \ge \frac{1}{k}\|\theta_i - \theta_{i-1}\|^2 + \frac{\gamma}{2}\left(\mathcal{A}_h(\theta_i, \theta_i) - \mathcal{A}_h(\theta_{i-1}, \theta_{i-1})\right) + \frac{1}{2}\left(\|\nabla\theta_i\|^2 - \|\nabla\theta_{i-1}\|^2\right). \\
&(\phi(U_i) - \phi(u(t_i)), \theta_i - \theta_{i-1}) = (\phi(U_i) - \phi(R_h u(t_i)), \theta_i - \theta_{i-1}) + (\phi(R_h u(t_i)) - \phi(u(t_i)), \theta_i - \theta_{i-1}).
\end{aligned}$$

For the first term, we use Lemma 2.7 and (4.10) to find

$$\begin{aligned}
(\phi(U_i) - \phi(R_h u(t_i)), \theta_i - \theta_{i-1}) &= ((U_i^2 + U_i R_h u(t_i) + R_h u(t_i)^2 - 1)\theta_i, \theta_i - \theta_{i-1}) \\
&\le \frac{3}{2}(U_i^2|\theta_i|, |\theta_i - \theta_{i-1}|) + \frac{3}{2}(R_h u(t_i)^2|\theta_i|, |\theta_i - \theta_{i-1}|) + \|\theta_i\|\,\|\theta_i - \theta_{i-1}\| \\
&\le \frac{3}{2}\left[\|U_i\|^2_{L_6(\Omega)} + \|R_h u(t_i)\|^2_{L_6(\Omega)}\right]\|\theta_i\|_{L_6(\Omega)}\|\theta_i - \theta_{i-1}\| + \|\theta_i\|\,\|\theta_i - \theta_{i-1}\| \\
&\le Ck\|\theta_i\|^2_{H^1(\Omega)} + \frac{1}{4k}\|\theta_i - \theta_{i-1}\|^2.
\end{aligned}$$

Using Theorem 3.1, Sobolev embedding and (4.12), we bound the second term as

$$\begin{aligned}
(\phi(R_h u(t_i)) - \phi(u(t_i)), \theta_i - \theta_{i-1}) &= ((u(t_i)^2 + u(t_i)R_h u(t_i) + R_h u(t_i)^2 - 1)\rho_i, \theta_i - \theta_{i-1}) \\
&\le \frac{3}{2}(u(t_i)^2|\rho_i|, |\theta_i|) + \frac{3}{2}(R_h u(t_i)^2|\rho_i|, |\theta_i - \theta_{i-1}|) + \|\rho_i\|\,\|\theta_i - \theta_{i-1}\| \\
&\le \frac{3}{2}\left[\|u(t_i)\|^2_{L_\infty(\Omega)} + \|R_h u(t_i)\|^2_{L_\infty(\Omega)}\right]\|\rho_i\|\,\|\theta_i - \theta_{i-1}\| + \|\rho_i\|\,\|\theta_i - \theta_{i-1}\| \\
&\le Ck\|\rho_i\|^2 + \frac{1}{4k}\|\theta_i - \theta_{i-1}\|^2.
\end{aligned}$$

We obtain,

$$\frac{1}{4k}\|\theta_i-\theta_{i-1}\|^2+\frac{\gamma}{2}\left(\mathcal{A}_h(\theta_i,\theta_i)-\mathcal{A}_h(\theta_{i-1},\theta_{i-1})\right)+\frac{1}{2}\left(\|\nabla\theta_i\|^2-\|\nabla\theta_{i-1}\|^2\right)$$
$$\leq Ck\left(\|\theta_i\|^2_{H^1(\Omega)}+\|\rho_i\|^2+\|\omega_i\|^2\right).$$

By summing over $i=1$ to $n$, we obtain

$$\begin{aligned}\frac{\gamma}{2}\mathcal{A}_h(\theta_n,\theta_n)+\frac{1}{2}\|\nabla\theta_n\|^2 &\leq \frac{\gamma}{2}\mathcal{A}_h(\theta_0,\theta_0)+\frac{1}{2}\|\nabla\theta_0\|^2+Ck\sum_{i=1}^{n}\left(\|\theta_i\|^2_{H^1(\Omega)}+\|\rho_i\|^2+\|\omega_i\|^2\right)\\ &\leq Ck\sum_{i=1}^{n}\|\nabla\theta_i\|^2+Ck\sum_{i=1}^{n}\|\theta_i\|^2+2T\|\rho_0\|^2+2T\int_0^{t_n}\|\rho_t\|^2ds+2k^2\int_0^{t_n}\|u_{tt}(s)\|^2ds\\ &\leq Ck\sum_{i=1}^{n}\|\nabla\theta_i\|^2+\|\theta_0\|^2+C(T)\left(\|\rho_0\|^2+\int_0^{t_n}\|\rho_t\|^2ds+k^2\int_0^{t_n}\|u_{tt}(s)\|^2ds\right).\end{aligned}$$

Discrete Gronwall's lemma completes the proof. □

We now deduce the following super-convergence result from Theorem 4.7.

**Corollary 4.8.** *Let* $h\leq\sqrt{\gamma}$ *and* $\Pi_h u_0=R_h u_0$. *Then,*

$$\gamma\|U^n-R_hu(t_n)\|_h^2+\|\nabla(U^n-R_hu(t_n))\|^2$$
$$\leq C(T)\left(\gamma h^4+h^6+\gamma^{-1}h^8\right)\left(\|u(t_n)\|^2_{H^3(\Omega)}+\|u_0\|^2_{H^3(\Omega)}+\int_0^{t_n}\|u_t\|^2_{H^3(\Omega)}ds\right)+C(T)k^2\int_0^{t_n}\|u_{tt}(s)\|^2ds,$$

*for all* $1\leq n\leq N$ *and for a constant* $C(T)$ *which is independent of* $h$, $k$ *and* $\gamma$ *but depending on* $T$.

## 5. Conclusions

In this article, we have developed a fully discrete $C^0$ interior penalty finite element method for the Extended Fisher–Kolmogorov (EFK) equation. We have derived robust *a priori* error estimates for the numerical scheme. To this end, we have derived a regularity estimate and some stability bounds for the continuous solution. The results in this article are derived under the assumption that the domain is convex polyhedral and the initial data is in $H^4(\Omega)$. We leave the subject of analysis for general polyhedral domains and more general initial data to the future.


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