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A new general iterative scheme for split variational inclusion and fixed point problems of k-strick pseudo-contraction mappings with convergence analysis

Jitsupa Deepho[1], Phatiphat Thounthong[2,3], Poom Kumam[4,5,6,*] and Supak Phiangsungnoen[7,*]

[1] Faculty of Science, Energy and Environment,
King Mongkut's University of Technology North Bangkok, Rayong Campus (KMUTNB)
19 Moo 11, Tambon Nonglalok, Amphur Bankhai, Rayong 21120, Thailand

[2] Renewable Energy Research Centre, King Mongkut's University of Technology North Bangkok (KMUTNB),
Wongsawang, Bangsue, Bangkok 10800, Thailand

[3] Department of Teacher Training in Electrical Engineering, Faculty of Technical Education,
King Mongkut's University of Technology North Bangkok (KMUTNB), Wongsawang, Bangsue, Bangkok 10800, Thailand

[4]KMUTTFixed Point Research Laboratory, Department of Mathematics, Room SCL 802 Fixed Point Laboratory,
Science Laboratory Building, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT)
126 Pracha Uthit Rd., Bang Mod, Thrung Khru, Bangkok 10140, Thailand

[5]KMUTT-Fixed Point Theory and Applications Research Group (KMUTT-FPTA),
Science Laboratory Building,Theoretical and Computational Science Center (TaCS),
Faculty of Science, King Mongkuts University of Technology Thonburi (KMUTT),
126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand

[6]Department of Medical Research, China Medical University Hospital, China Medical University, Taichung 40402, Taiwan

[7]Department of Mathematics, Faculty of Liberal Arts, Rajamangala University of Technology Rattanakosin (RMUTR)
264 Chakkrawat Rd., Chakkrawat, Samphanthawong, Bangkok 10100, Thailand

Abstract

In this paper, we modify the general iterative method to approximate a common element of the set of solutions of split variational inclusion problem and the set of common fixed points of a finite family of k-strictly pseudo-contractive nonself mapping. Strong convergence theorem is established under some suitable conditions in a real Hilbert space, which also solves some variational inequality problems. Results presented in this paper may be viewed as a refinement and important generalizations of the previously known results announced by many other authors. Finally, some examples to study the rate of convergence and some illustrative numerical examples are presented.

*Corresponding authors: Tel.: +66 02 470 8998; fax: +66 02 428 4025.
E-mail: jitsupa.d@sciee.kmutnb.ac.th (J. Deepho), phtt@kmutnb.ac.th (P.Thounthong), poom.kum@kmutt.ac.th (P. Kumam) and supak.pia@rmutr.ac.th (S.Phiangsungnoen).

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1 Introduction

Let H_1 and H_2 be real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C and Q be nonempty closed convex subsets of H_1 and H_2, respectively. Let $\{x_n\}$ be a sequence in H_1, then $x_n \to x$ will denote strong and $x_n \rightharpoonup x$ denote weak convergence of the sequence $\{x_n\}$ respectively. A mapping $S : C \to C$ is called *nonexpansive* if $\|Sx - Sy\| \leq \|x - y\|, \forall x, y \in C$.

The *fixed point problem* (FPP) for the mapping S is to find $x \in C$ such that

$$Sx = x. \tag{1.1}$$

We denote $F(S) := \{x \in C : Sx = x\}$, the set of solutions of FPP.

Throughout in this paper we assumed that S is a nonexpansive mapping such that $F(S) \neq \emptyset$. Recall that a self-mapping $f : C \to C$ is a *contraction* on C if there exists a constant $\alpha \in (0, 1)$ and $x, y \in C$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$.

Given a nonlinear mapping $B : C \to H_1$. Then the *variational inequality problem* (VIP) is to find $u \in C$ such that

$$\langle Bu, v - u\rangle \geq 0, \quad \forall v \in C. \tag{1.2}$$

The solution of VIP (1.2) is denoted by $VI(C, B)$. It is well known that if B is strongly monotone and Lipschitz continuous mapping on C then VIP (1.2) has a unique solution. There are several different approaches towards solving this problem in finite dimensional and infinite dimensional spaces see [5, 6, 7, 8, 9, 10, 11, 12, 13] and the research in this direction is intensively continued. Then VIP is satisfies the following Lemma;

Lemma 1.1. *For a given $z \in H_1, u \in C$ satisfies the inequality*

$$\langle u - z, v - u\rangle \geq 0, \quad \forall v \in C, \quad \textit{iff} \quad u = P_C z, \tag{1.3}$$

where P_C is the projection of H_1 onto a closed convex set C.

Recall that a nonself mapping $T : C \to H_1$ is called a k-strict pseudo-contraction if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{1.4}$$

A mapping T is said to be pseudo-contractive if $k = 1$, and is also said to be strongly pseudo-contractive if there exists a positive constant $\lambda \in (0, 1)$ such that $T + \lambda I$ is pseudo-contractive. Clearly, the class of k-strict pseudo-contractions falls into the one between classes of nonexpansive mappings and pseudo-contraction mappings. We remark also that the class of strongly pseudo-contractive mappings is independent of the class of k-strict pseudo-contraction mapping (see, e.g., [18, 19]).

Iterative schemes for strict pseudo-contractions are far less developed than those for nonexpansive mappings though Browder and Petryshyn [19] initiated their work in 1967 ; the reason is probably that the second term appearing in the right-hand side of (1.4) impedes the convergence analysis for iterative algorithms used to find a fixed point of the strict pseudo-contraction. On the other hand, strict pseudo-contractions have more powerful applications than nonexpansive mappings do in solving inverse problems; see, e.g., [20, 21, 22, 23, 24, 25, 26] and the references therein.

In 2006, Marino and Xu [22] introduced a general iterative method and proved that for a given $x_0 \in H_1$, the sequence $\{x_n\}$ generated by

$$x_{n+1} = \alpha_n \gamma_n f(x_n) + (I - \alpha_n D)Tx_n, \quad \forall n \in \mathbf{N},$$

where T is a self-nonexpansive mapping on H_1, f is a contraction on H_1 into itself and $\{\alpha_n\} \subseteq (0,1)$ satisfies certain conditions, D is a strongly positive bounded linear operator on H_1, converges strongly to $x^* \in F(T)$, which is the unique solution of the following variational inequality:

$$\langle (D - \gamma f)x^*, x^* - x\rangle \leq 0, \quad \forall x \in F(T),$$

and is also the optimality condition for some minimization problem.

Recall also that a multi-valued mapping $M : H_1 \to 2^{H_1}$ is called monotone if, for all $x, y \in H_1, u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v\rangle \geq 0.$$

A monotone mapping M is maximal if the Graph(M) is not properly contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if for $(x, u) \in H_1 \times H_1, \langle x - y, u - v\rangle \geq 0$ for every $(y, v) \in$ Graph(M) implies that $u \in Mx$.

Let $M : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. Then the resolvent mapping $J_\lambda^M : H_1 \to H_1$ associated with M is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \quad \forall x \in H_1,$$

for some $\lambda > 0$, where I stands for the identity operator on H_1. Note that for all $\lambda > 0$ the resolvent operator J_λ^M is single-valued, nonexpansive, and firmly nonexpansive.

In 2011, Moudafi [33] introduced the following split monotone variational inclusion problem: Find $x^* \in H_1$ such that

$$\begin{cases} 0 \in f_1(x^*) + B_1(x^*), \\ y^* = Ax^* \in H_2 : \quad 0 \in f_2(y^*) + B_2(y^*), \end{cases} \tag{1.5}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings.

The split monotone variational inclusion problem (1.5) includes as special cases: the split common fixed point problem, the split variational inequality problem, the split zero problem, and the split feasibility problem, which have already been studied and used in practice as a model in intensity-modulated radiation therapy treatment planning, see e.g. [14, 27, 28]. This formalism is also at the core of the modeling of many inverse problems arising for phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see [29, 30] and the references therein.

If $f_1 \equiv 0$ and $f_2 \equiv 0$, the problem (1.5) reduces to the following split variational inclusion problem: Find $x^* \in H_1$ such that

$$\begin{cases} 0 \in B_1(x^*), \\ y^* = Ax^* \in H_2 : \quad 0 \in B_2(y^*), \end{cases} \tag{1.6}$$

which constitutes a pair of variational inclusion problems connected with a bounded linear operator A in two different Hilbert spaces H_1 and H_2. The solution set of problem (1.6) is denoted by $\bar{\Gamma} = \{x^* \in H_1 : 0 \in B_1(x^*), y^* = Ax^* \in H_2 : 0 \in B_2(y^*)\}$.

Very recently, Byrne et al. [31] studied the weak and strong convergence of the following iterative method for problem (1.6): For given $x_0 \in H_1$ and $\lambda > 0$, compute iterative sequence $\{x_n\}$ generated by the following scheme:

$$x_{n+1} = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]. \tag{1.7}$$

In 2013, Kazmi and Rivi [32] modified scheme (1.6) to the case of a split variational inclusion and the fixed point problem of a nonexpansive mapping. To be more precise, they proved the following strong convergence theorem.

Theorem KR Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\rho \in (0, 1)$ and $T : H_1 \to H_1$ be a nonexpansive mapping such that $\Omega = Fix(T) \cap \bar{\Gamma} \neq \emptyset$. For a given $x_0 \in H_1$ arbitrarily, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n], \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tu_n, \end{cases} \tag{1.8}$$

where $\lambda > 0$ and $\epsilon \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A, and A^* is the adjoint of $A; \{\alpha_n\}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$. Then the sequence $\{u_n\}$ and $\{x_n\}$ both convergence strongly to $z \in \Omega$, where $z = P_\Omega f(z)$.

Inspiration and motivation by research going on in this area, a modified general iterative method for a split variational inclusion and a finite family of k-strictly pseudo-contractive nonself mapping, which is defined in the following way:

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \beta_n u_n + (1 - \beta_n) \sum_{i=1}^{\mathbf{N}} \eta_{i=1}^{(n)} T_i u_n, \\ x_{n+1} = \alpha_n \tau f(x_n) + (I - \alpha_n D)y_n, \ n \geq 1, \end{cases} \tag{1.9}$$

where $\lambda > 0$, $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A, $\tau > 0$, f is a contraction and D is operator, $\{T_i\}_{i=1}^{\mathbf{N}} : C \to H_1$ is a finite family of k_i-strict pseudo-contractions, $\{\eta_i^{(n)}\}_{i=1}^{\mathbf{N}}$ is a finite sequence of positive numbers, $\{\alpha_n\}$ and $\{\beta_n\}$ are some sequences with certain conditions.

Our purpose is not only to modify the general iterative method to the case of a finite family of k_i-strictly pseudo-contractive nonself mappings, but also to establish strong convergence theorems for split variational inclusion problem and k_i-strict pseudo-contractions in a real Hilbert space, which also solves some variational inequality problems.

2 Preliminaries

Let H_1 be a real Hilbert space. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \tag{2.1}$$

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle, \tag{2.2}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2.3}$$

for all $x, y \in H_1$ and $y \in [0,1]$.

We recall some concepts and results which are needed in sequel. A mapping P_C is said to be *metric projection* of H_1 onto C if for every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \forall y \in C. \tag{2.4}$$

It is well known that P_C is a nonexpansive mapping and is characterized by the following property:

$$\|P_C x - P_C y\|^2 \leq \langle x - y, P_C x - P_C y\rangle, \quad \forall x, y \in H_1. \tag{2.5}$$

Moreover, $P_C x$ is characterized by the following properties:

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.6}$$

$$\|x-y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \quad \forall x \in H_1, y \in C, \tag{2.7}$$

and

$$\|(x-y) - (P_C x - P_C y)\|^2 \geq \|x-y\|^2 - \|P_C x - P_C y\|^2, \quad \forall x, y \in H_1. \tag{2.8}$$

It is known that every nonexpansive operator $S : H_1 \to H_1$ satisfies, for all $(x,y) \in H_1 \times H_1$, the inequality

$$\langle (x - S(x)) - (y - S(y)), S(y) - S(x)\rangle \leq \frac{1}{2}\|(S(x) - x) - (S(y) - y)\|^2, \tag{2.9}$$

and therefore, we get, for all $(x,y) \in H_1 \times F(S)$,

$$\langle x - S(x), y - S(x)\rangle \leq \frac{1}{2}\|S(x) - x\|^2, \tag{2.10}$$

(see, e.g., Theorem 3 in [1] and Theorem 1 in [2]).

Lemma 2.1. *A point $x^* \in C$ is a solution of the variational inequality if and only if $x^* \in C$ satisfies the ralation*

$$x^* = P_C(x^* - \lambda B x^*), \tag{2.11}$$

where P_C is the projection of H_1 onto a closed convex set C and $\lambda > 0$ is a constant.

Lemma 2.2. *[22] Assume that D is a strongly positive linear operator on the Hilbert space H_1 with a coefficient $\bar{\tau} > 0$ and $0 < \varrho < \|D\|^{-1}$. Then $\|I - \varrho D\| \leq 1 - \varrho\bar{\tau}$.*

Lemma 2.3. *[24] If $T : C \to H_1$ is a k-strict pseudo-contraction, then the fixed point set $F(T)$ is closed convex so that the projection $P_{F(T)}$ is well defined.*

Lemma 2.4. *[24] Let $T : C \to H_1$ be a k-strict pseudo-contraction. For $\lambda \in [k, 1)$, define $S : C \to H_1$ by $Sx = \lambda x + (1-\lambda)Tx$ for each $x \in C$. Then S is a nonexpansive mapping such that $F(S) = F(T)$.*

Proposition 2.5. *[16] Let C be a nonempty closed convex subset of the Hilbert space H_1. Given an integer $\mathbf{N} \geq 1$, assume that $\{T_i\}_{i=1}^{\mathbf{N}} : C \to H_1$ is a finite family of k_i-strict pseudo-contractions. Suppose that $\{\eta_i\}_{i=1}^{\mathbf{N}}$ is a positive sequence such that $\sum\limits_{i=1}^{\mathbf{N}} \eta_i = 1$. Then $\sum\limits_{i=1}^{\mathbf{N}} \eta_i T_i$ is a k-strict pseudo-contraction with $k = \max\{k_i : 1 \leq i \leq \mathbf{N}\}$.*

Proposition 2.6. *[16] Let $\{T_i\}_{i=1}^{\mathbf{N}}$ and $\{\eta_i\}_{i=1}^{\mathbf{N}}$ be given as in Proposition 2.5 above. Then $F(\sum\limits_{i=1}^{\mathbf{N}} \eta_i T_i) = \bigcap\limits_{i=1}^{\mathbf{N}} F(T_i)$.*

Lemma 2.7. *[3] Let E be an inner product space. Then, for any $x, y, z \in E$ and $\alpha, \beta, \gamma \in [0, 1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2.$$

Lemma 2.8. *[4] Each Hilbert space H_1 satisfies the Opial condition that is, for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality $\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|$, holds for every $y \in H$ with $y \neq x$.*

Lemma 2.9. *[15] Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \; n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a real sequence such that a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$,

(ii) $\limsup_{n\longrightarrow\infty} \delta_n \leq 0$ or $\sum\limits_{n=1}^{\infty} |\gamma_n\delta_n| < \infty$.

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 2.10. *[17] Assume that T is nonexpansive self mapping of a closed convex subset C of a Hilbert space H_1. If T has a fixed point, then $I - T$ is demiclosed, i.e., whenever $\{x_n\}$ converges strongly to some y, it follows that $(I - T)x = y$. Here I is the identity mapping on H_1.*

Lemma 2.11. *[22] Let C be a nonempty, closed and convex subset of a Hilbert space H_1. Assume that $f : C \to C$ is a contraction with a coefficient $\rho \in (0, 1)$ and D is a strongly positive linear bounded operator with a coefficient $\bar{\gamma} > 0$. Then, for $0 < \gamma < \frac{\bar{\gamma}}{\rho}$,*

$$\langle x - y, (D - \gamma f)x - (D - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\rho)\|x - y\|^2, \quad \forall x, y \in H_1.$$

That is, $D - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\rho$.

3 Main Result

Theorem 3.1. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Let D be a strongly positive bounded linear* *operator on H_1 with a coefficient $\bar{\tau} > 0$. Assume that $\{T\}_{i=1}^{\mathbf{N}} : C \to H_1$ be a finite family of k_i-strict pseudo-contraction mappings. such that $\mathcal{F} := \cap_i^{\mathbf{N}} F(T_i) \cap \bar{\Gamma} \neq \emptyset$. Suppose that $f \in \prod_C$ with a coefficient $\rho \in (0,1)$ and $\{\eta_i^{(n)}\}_{i=1}^{\mathbf{N}}$ are finite sequences of positive numbers such that $\sum_{i=1}^{\mathbf{N}} \eta_i^{(n)} = 1$ for all $n \geq 0$, for a given point $x_0 \in C, \alpha_n, \beta_n \in (0,1)$ and $0 < \tau < \frac{\bar{\tau}}{\rho}$. Let $\{x_n\}$ be a sequence generated in the following;*

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \beta_n u_n + (1-\beta_n)\sum_{i=1}^{\mathbf{N}} \eta_{i=1}^{(n)} T_i u_n, \\ x_{n+1} = \alpha_n \tau f(x_n) + (I - \alpha_n D)y_n, \ n \geq 1, \end{cases} \tag{3.1}$$

where $\lambda > 0$ and $\gamma \in (0, \frac{1}{L}), L$ is the spectral radius of the operator A^*A, and A^* is the adjoint of A. The following control conditions are satisfied:

(C1) $\lim\limits_{n\to\infty} \alpha_n = 0, \sum\limits_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum\limits_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(C2) $k_i \leq \beta_n \leq \ell < 1, \lim\limits_{n\to\infty} \beta_n = \ell$ *and* $\sum\limits_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$;

(C3) $\sum\limits_{n=1}^{\infty}\sum\limits_{i=1}^{\mathbf{N}} |\eta_i^{(n)} - \eta_i^{(n-1)}| < \infty$.

Then the sequence $\{x_n\}$ generated by (3.1) converges strongly to $q \in \mathcal{F}$ which solves the variational inequality

$$\langle (D - \tau f)q, q - p\rangle \leq 0, \ \forall p \in \mathcal{F}.$$

Proof. **Step 1.** First we will prove that $\{x_n\}$ is bounded.

Putting $W_n = \sum\limits_{i=1}^{\mathbf{N}} \eta_i^{(n)} T_i$ we have $W_n : C \to H_1$ is a k-strict pseudo-contraction and $F(W_n) = \bigcap\limits_{i=1}^{\mathbf{N}} F(T_i)$ by Proposition 2.5 and 2.6, where $k = \max\{k_i : 1 \leq i \leq \mathbf{N}\}$.

First, we show that the mapping $I - r_n D$ is nonexpansive. Indeed, for each $x, y \in C$, we have

$$\begin{aligned} \|(I - r_nD)x - (I - r_nD)y\|^2 &= \|x-y\|^2 - 2r_n\langle x-y, Dx - Dy\rangle + r_n^2\|Dx - Dy\|^2 \\ &\leq \|x-y\|^2 - 2\alpha r_n\|Dx-Dy\|^2 + r_n^2\|Dx-Dy\|^2 \\ &= \|x-y\|^2 - r_n(2\alpha - r_n)\|Dx-Dy\|^2. \end{aligned}$$

It follows from the condition $r_n \in (0, 2\alpha)$ that the mapping $I - r_nD$ is nonexpansive.

Let $p \in \mathcal{F}$, we have $p = J_\lambda^{B_1}p, Ap = J_\lambda^{B_2}(Ap)$ and $W_np = p$. We estimate

$$\begin{aligned} \|u_n - p\|^2 &= \|J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) - p\|^2 \\ &= \|J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n) - J_\lambda^{B_1}p\|^2 \\ &\leq \|x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \\ &\leq \|x_n - p\|^2 + \gamma^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 + 2\gamma\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle. \end{aligned} \tag{3.2}$$

Thus, we have

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + \gamma^2\langle (J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle + 2\gamma\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle. \tag{3.3}$$

Now, we have

$$\begin{aligned}\gamma^2\langle (J_\lambda^{B_2}-I)Ax_n, AA^*(J_\lambda^{B_2}-I)Ax_n\rangle &\le L\gamma^2\langle (J_\lambda^{B_2}-I)Ax_n, (J_\lambda^{B_2}-I)Ax_n\rangle \\ &= L\gamma^2\|(J_\lambda^{B_2}-I)Ax_n\|^2. \end{aligned}\tag{3.4}$$

Setting $\Lambda := 2\gamma\langle x_n-p, A^*(J_\lambda^{B_2}-I)Ax_n\rangle$ and using (2.10), we have

$$\begin{aligned}\Lambda &= 2\gamma\langle x_n-p, A^*(J_\lambda^{B_2}-I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n-p), (J_\lambda^{B_2}-I)Ax_n\rangle \\ &= 2\gamma\langle A(x_n-p)+(J_\lambda^{B_2}-I)Ax_n-(J_\lambda^{B_2}-I)Ax_n, (J_\lambda^{B_2}-I)Ax_n\rangle \\ &= 2\gamma\left\{\langle J_\lambda^{B_2}Ax_n-Ap, (J_\lambda^{B_2}-I)Ax_n\rangle-\|(J_\lambda^{B_2}-I)Ax_n\|^2\right\} \\ &\le 2\gamma\left\{\frac{1}{2}\|(J_\lambda^{B_2}-I)Ax_n\|^2-\|(J_\lambda^{B_2}-I)Ax_n\|^2\right\} \\ &\le -\gamma\|(J_\lambda^{B_2}-I)Ax_n\|^2. \end{aligned}\tag{3.5}$$

Using (3.3), (3.4) and (3.5), we obtain

$$\|u_n-p\|^2 \le \|x_n-p\|^2+\gamma(L\gamma-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2. \tag{3.6}$$

Since $\gamma\in(0,\frac{1}{L})$, we obtain

$$\|u_n-p\|^2\le\|x_n-p\|^2. \tag{3.7}$$

From (3.1), condition (C2), (2.3) and (3.7), we have

$$\begin{aligned}\|y_n-p\|^2 &= \|\beta_n(u_n-p)+(1-\beta_n)(W_nu_n-p)\|^2 \\ &= \beta_n\|u_n-p\|^2+(1-\beta_n)\|W_nu_n-p\|^2-\beta_n(1-\beta_n)\|u_n-W_nu_n\|^2 \\ &\le \beta_n\|u_n-p\|^2+(1-\beta_n)[\|u_n-p\|^2+k\|u_n-W_nu_n\|^2]-\beta_n(1-\beta_n)\|u_n-W_nu_n\|^2 \\ &= \|u_n-p\|^2-(1-\beta_n)(\beta_n-k)\|u_n-W_nu_n\|^2 \\ &\le \|u_n-p\|^2. \end{aligned}\tag{3.8}$$

This together with (3.7) and (3.8), we obtain

$$\|y_n-p\|\le\|u_n-p\|\le\|x_n-p\|. \tag{3.9}$$

Furthermore, by Lemma 2.2, we have

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n[\tau f(x_n)-Dp]+(I-\alpha_nD)(y_n-p)\| \\ &\le (1-\alpha_n\bar{\tau})\|y_n-p\|+\alpha_n\|\tau f(x_n)-Dp\| \\ &\le (1-\alpha_n\bar{\tau})\|y_n-p\|+\alpha_n[\|\tau f(x_n)-\tau f(p)\|+\|\tau f(p)-Dp\|] \\ &\le [1-(\bar{\tau}-\tau\rho)\alpha_n]\|x_n-p\|+\alpha_n\|\tau f(p)-Dp\|. \end{aligned}$$

It follows from induction that

$$\|x_n-p\|\le\max\left\{\|x_0-p\|, \frac{1}{\bar{\tau}-\tau\rho}\|\tau f(p)-Dp\|\right\}, \quad n\ge 1, \tag{3.10}$$

which gives that sequence $\{x_n\}$ is bounded, and so are $\{u_n\}$ and $\{y_n\}$.

Step 2. We will prove that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.

Define a mapping $S_n x := \beta_n x + (1-\beta_n)W_n x$ for each $x \in C$. Then $S_n : C \to H_1$ is nonexpansive. Indeed, by using (1.4), (2.3) and condition (C2), we have for all $x, y \in C$ that

$$\begin{aligned}
\|S_n x - S_n y\|^2 &= \|\beta_n(x-y) + (1-\beta_n)(W_n x - W_n y)\|^2 \\
&= \beta_n\|x-y\|^2 + (1-\beta_n)\|W_n x - W_n y\|^2 \\
&\quad -\beta_n(1-\beta_n)\|x - W_n x - (y - W_n y)\|^2 \\
&\leq \beta_n\|x-y\|^2 + (1-\beta_n)[\|x-y\|^2 + k\|x - W_n x - (y - W_n y)\|^2] \\
&\quad -\beta_n(1-\beta_n)\|x - W_n x - (y - W_n y)\|^2 \\
&= \|x-y\|^2 - (1-\beta_n)(\beta_n - k)\|x - W_n x - (y - W_n y)\|^2 \\
&\leq \|x-y\|^2,
\end{aligned}$$

which shows that $S_n : C \to H_1$ is nonexpansive.
Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. From (3.1) and Lemma 2.2, we have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n \tau f(x_n) + (I - \alpha_n D)y_n - [\alpha_{n-1}\tau f(x_{n-1}) + (I - \alpha_{n-1}D)y_{n-1}] \\
&= \|\alpha_n \tau f(x_n) + (I - \alpha_n D)y_n - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) \\
&\quad -\alpha_{n-1}\tau f(x_{n-1}) + (I - \alpha_{n-1}D)y_{n-1}\| \\
&\leq \alpha_n \tau\|f(x_n) - f(x_{n-1})\| + |\alpha_n - \alpha_{n-1}|[\|\tau f(x_{n-1})\| + \|Dy_{n-1}\|] \\
&\quad +\|(I - \alpha_n D)(y_n - y_{n-1})\| \\
&\leq \alpha_n \tau\rho\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|M_1 + (1 - \alpha_n\bar{\tau})\|y_n - y_{n-1}\|,
\end{aligned} \tag{3.11}$$

where $M_1 = sup_{n\geq 1}\{\tau\|f(x_n)\| + \|Dy_n\|\} < \infty$. Moreover, we note that $y_n = S_n u_n$ and

$$\begin{aligned}
\|y_n - y_{n-1}\| &\leq \|S_n u_n - S_n u_{n-1}\| + \|S_n u_{n-1} - S_{n-1}u_{n-1}\| \\
&\leq \|u_n - u_{n-1}\| + \|\beta_n u_{n-1} + (1-\beta_n)W_n u_{n-1} \\
&\quad -[\beta_{n-1}u_{n-1} + (1-\beta_{n-1})W_{n-1}u_{n-1}]\| \\
&\leq \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}|\|u_{n-1} - W_{n-1}u_{n-1}\| \\
&\quad +(1-\beta_n)\|W_n u_{n-1} - W_{n-1}u_{n-1}\| \\
&\leq \|u_n - u_{n-1}\| + |\beta_n - \beta_{n-1}|M_2 \\
&\quad +(1-\beta_n)\sum_{i=1}^{\mathbf{N}}|\eta_i^{(n)} - \eta_i^{(n-1)}|\|T_i u_{n-1}\|,
\end{aligned} \tag{3.12}$$

where $M_2 = \sup_{n\geq 1}\{\|u_{n-1} - W_{n-1}u_{n-1}\|\}$. Since, for $\gamma \in (0, \frac{1}{L})$, the mapping $J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)$ is averaged and hence nonexpansive, then we obtain

$$\begin{aligned}
\|u_n - u_{n-1}\| &= \|J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n - J_\lambda^{B_1}(x_{n-1} + \gamma A^*(J_\lambda^{B_2} - I)Ax_{n-1}\| \\
&\leq \|J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)x_n - J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)x_{n-1}\| \\
&\leq \|x_n - x_{n-1}\|.
\end{aligned} \tag{3.13}$$

Substitution (3.13) into (3.12), we get

$$\|y_n - y_{n-1}\| \leq \|x_n - x_{n-1}\| + |\beta_n - \beta_{n-1}|M_2 + (1-\beta_n)\sum_{i=1}^{\mathbf{N}}|\eta_i^{(n)} - \eta_i^{(n-1)}|\|T_i u_{n-1}\|. \tag{3.14}$$

Combining (3.11), (3.12), (3.13) and (3.14), we have

$$\begin{aligned}\|x_{n+1}-x_n\| &\leq \alpha_n\tau\rho\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_1+(1-\alpha_n\bar{\tau})[\|x_n-x_{n-1}\|\\ &\quad +|\beta_n-\beta_{n-1}|M_2+(1-\beta_n)\sum_{i=1}^{\mathbf{N}}|\eta_i^{(n)}-\eta_i^{(n-1)}|\|T_iu_{n-1}\|]\\ &\leq [1-(\bar{\tau}-\tau\rho)\alpha_n]\|x_n-x_{n-1}\|+|\alpha_n-\alpha_{n-1}|M_1+|\beta_n-\beta_{n-1}|M_2\\ &\quad +\sum_{i=1}^{\mathbf{N}}|\eta_i^{(n)}-\eta_i^{(n-1)}|\|T_iu_{n-1}\|. \end{aligned}\tag{3.15}$$

It follows from $0<\tau<\frac{\bar{\tau}}{\rho}$ and Lemma 2.9 that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.16}$$

Moreover, we observe that

$$\begin{aligned}\|x_n-y_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-y_n\|\\ &= \|x_n-x_{n+1}\|+\|\alpha_n\tau f(x_n)+(I-\alpha_nD)y_n-y_n\|\\ &\leq \|x_n-x_{n+1}\|+\alpha_n\|\tau f(x_n)-Dy_n\|.\end{aligned}$$

It follows from (3.16) and $\lim_{n\to\infty}\alpha_n=0$ that

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \tag{3.17}$$

Step 3. We will prove that $\lim_{n\to\infty}\|x_n-W_nx_n\|=0$.

It follows from (3.1) and (3.6) that

$$\begin{aligned}\|x_{n+1}-p\|^2 &= \|\alpha_n[\tau f(x_n)-Dp]+(I-\alpha_nD)(y_n-p)\|^2\\ &\leq (1-\alpha_n\bar{\tau})^2\|y_n-p\|^2+\alpha_n^2\|\tau f(x_n)-Dp\|^2\\ &\quad +2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|\\ &\leq (1-\alpha_n\bar{\tau})^2\|u_n-p\|^2+\alpha_n^2\|\tau f(x_n)-Dp\|^2\\ &\quad +2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|\\ &\leq (1-\alpha_n\bar{\tau})^2[\|x_n-p\|^2+\gamma(L\gamma-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2]\\ &\quad +\alpha_n^2\|\tau f(x_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|\\ &= (1-2\alpha_n\bar{\tau}+(\alpha_n\bar{\tau})^2)\|x_n-p\|^2+(1-\alpha_n\bar{\tau})^2\gamma(L\gamma-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2\\ &\quad +\alpha_n^2\|\tau f(x_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|\\ &\leq \|x_n-p\|^2+(\alpha_n\bar{\tau})^2\|x_n-p\|^2+(1-\alpha_n\bar{\tau})^2\gamma(L\gamma-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2\\ &\quad +\alpha_n^2\|\tau f(x_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|\\ &= \|x_n-p\|^2+(\alpha_n\bar{\tau})^2\|x_n-p\|^2-(1-\alpha_n\bar{\tau})^2\gamma(1-L\gamma)\|(J_\lambda^{B_2}-I)Ax_n\|^2\\ &\quad +\alpha_n^2\|\tau f(x_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|, \end{aligned}\tag{3.18}$$

and hence

$$\begin{aligned}
&(1-\alpha_n\bar{\tau})^2\gamma(1-L\gamma)\|(J_\lambda^{B_2}-I)Ax_n\|^2\\
&\quad\leq (\alpha_n\bar{\tau})^2\|x_n-p\|^2+\alpha_n^2\|\tau f(x_n)-Dp\|^2\\
&\qquad+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|\\
&\qquad+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\
&\quad= (\alpha_n\bar{\tau})^2\|x_n-p\|^2+\alpha_n^2\|\tau f(x_n)-Dp\|^2\\
&\qquad+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|\\
&\qquad+\|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|).
\end{aligned}$$

Since $\gamma(1-L\gamma)>0, \alpha_n\to 0, \{x_n\}$ is bounded and $\|x_n-x_{n+1}\|\to 0$ as $n\to\infty$, we have

$$\lim_{n\to\infty}\|(J_\lambda^{B_2}-I)Ax_n\|=0. \tag{3.19}$$

Furthermore, using (3.2), (3.6) and $\gamma\in(0,\frac{1}{L})$, we observe that

$$\begin{aligned}
\|u_n-p\|^2 &= \|J_\lambda^{B_1}(x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n)-p\|^2\\
&= \|J_\lambda^{B_1}(x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n)-J_\lambda^{B_1}p\|^2\\
&\leq \langle u_n-p, x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n-p\rangle\\
&= \frac{1}{2}\{\|u_n-p\|^2+\|x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n-p\|^2\\
&\quad-\|(u_n-p)-[x_n+\gamma A^*(J_\lambda^{B_2}-I)Ax_n-p]\|^2\}\\
&= \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2+\gamma(L\gamma-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2\\
&\quad-\|u_n-x_n-\gamma A^*(J_\lambda^{B_2}-I)Ax_n\|^2\}\\
&\leq \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2-[\|u_n-x_n\|^2+\gamma^2\|A^*(J_\lambda^{B_2}-I)Ax_n\|^2\\
&\quad-2\gamma\langle u_n-x_n, A^*(J_\lambda^{B_2}-I)Ax_n\rangle]\}\\
&\leq \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\}.
\end{aligned}$$

Hence, we obtain

$$\|u_n-p\|^2\leq\|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|. \tag{3.20}$$

From (3.18) and (3.20), we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq (1-\alpha_n\bar{\tau})^2\|u_n-p\|^2+\alpha_n^2\|\tau f(x_n)-Dp\|^2\\
&\quad+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|\\
&\leq (1-\alpha_n\bar{\tau})^2[\|x_n-p\|^2-\|u_n-x_n\|^2+2\gamma\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|]\\
&\quad+\alpha_n^2\|\tau f(x_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|\\
&= (1-2\alpha_n\bar{\tau}+(\alpha_n\bar{\tau})^2)\|x_n-p\|^2-(1-\alpha_n\bar{\tau})\|u_n-x_n\|^2\\
&\quad+2\gamma(1-\alpha_n\bar{\tau})^2\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\\
&\quad+\alpha_n^2\|\tau f(x_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|\\
&\leq \|x_n-p\|^2+(\alpha_n\bar{\tau})^2\|x_n-p\|^2-(1-\alpha_n\bar{\tau})^2\|u_n-x_n\|^2\\
&\quad+2\gamma(1-\alpha_n\bar{\tau})^2\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\\
&\quad+\alpha_n^2\|\tau f(x_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|,
\end{aligned}$$

we get

$$
\begin{aligned}
&(1-\alpha_n\bar{\tau})^2\|u_n-x_n\|^2 \\
&\quad \leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+(\alpha_n\bar{\tau})^2+2\gamma(1-\alpha_n\bar{\tau})^2\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\| \\
&\quad\quad +\alpha_n^2\|\tau f(x_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\| \\
&\quad \leq (\|x_n-p\|+\|x_{n+1}-p\|)\|x_n-x_{n+1}\|+(\alpha_n\bar{\tau})^2\|x_n-p\|^2 \\
&\quad\quad +2\gamma(1-\alpha_n\bar{\tau})^2\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\| \\
&\quad\quad +\alpha_n^2\|\tau f(x_n)-Dp\|^2+2\alpha_n(1-\alpha_n\bar{\tau})\|\tau f(x_n)-Dp\|\|y_n-p\|.
\end{aligned} \tag{3.21}
$$

Since from condition (C1), (3.16) and (3.19), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{3.22}$$

By the nonexpansion of S_n, we have

$$
\begin{aligned}
\|x_n-S_nx_n\| &\leq \|x_n-x_{n+1}\|+\|x_{n+1}-S_nx_n\| \\
&\leq \|x_n-x_{n+1}\|+\|\alpha_n\tau f(x_n)+(I-\alpha_nD)y_n-S_nx_n\| \\
&\leq \|x_n-x_{n+1}\|+\alpha_n[\|\tau f(x_n)\|+\|Dy_n\|]+\|S_nu_n-S_nx_n\| \\
&\leq \|x_n-x_{n+1}\|+\alpha_n[\|\tau f(x_n)\|+\|Dy_n\|]+\|u_n-x_n\|.
\end{aligned}
$$

This together (3.16), (3.22) and condition (C1), we obtain

$$\lim_{n\to\infty}\|x_n-S_nx_n\|=0. \tag{3.23}$$

Furthermore, we note that

$$
\begin{aligned}
\|x_n-S_nx_n\| &= \|\beta_nx_n+(1-\beta_n)W_nx_n-x_n\| \\
&= \|\beta_nx_n+(1-\beta_n)W_nx_n-(\beta_n+(1-\beta_n))x_n\| \\
&= \|\beta_nx_n+(1-\beta_n)W_nx_n-\beta_nx_n-(1-\beta_n)x_n\| \\
&= (1-\beta_n)\|x_n-W_nx_n\|.
\end{aligned}
$$

It follows from condition (C2),

$$\lim_{n\to\infty}\|x_n-W_nx_n\|=0. \tag{3.24}$$

On the other hand, by condition (C3), we may assume that $\eta_i^{(n)}\to\eta_i$ as $n\to\infty$ for every $1\leq i\leq \mathbf{N}$. It is easy to see that each $\eta_i>0$ and $\sum_{i=1}^{\mathbf{N}}\eta_i=1$. Define $W=\sum_{i=1}^{\mathbf{N}}\eta_iT_i$, then $W:C\to H_1$ is a k-strict pseudo-contraction such that $F(W)=F(W_n)=\bigcap_{i=1}^{\mathbf{N}}F(T_i)$ by Proposition 2.5 and 2.6. Consequently,

$$
\begin{aligned}
\|x_n-Wx_n\| &\leq \|x_n-W_nx_n\|+\|W_nx_n-Wx_n\| \\
&\leq \|x_n-W_nx_n\|+\sum_{i=1}^{\mathbf{N}}|\eta_i^{(n)}-\eta_i|\|T_ix_n\|,
\end{aligned}
$$

which implies that

$$\lim_{n\to\infty}\|x_n-Wx_n\|=0. \tag{3.25}$$

Observe that

$$\|W_n x_n - W x_n\| = \|W_n x_n - x_n + x_n - W x_n\| \leq \|W_n x_n - x_n\| + \|x_n - W x_n\|.$$

From (3.24) and (3.25), we obtain that

$$\lim_{n\to\infty} \|W_n x_n - W x_n\| = 0. \tag{3.26}$$

Define $S : C \to H_1$ by $Sx = \lambda x + (1-\lambda)Wx$. Again by condition (C2) again, we have $\lim_{n\to\infty} \beta_n = \ell \in [k, 1)$. Then, S is nonexpansive with $F(S) = F(W)$ by Lemma 2.4. Notice that

$$\begin{aligned}
\|x_n - Sx_n\| &\leq \|x_n - S_n x_n\| + \|S_n x_n - Sx_n\| \\
&= \|x_n - S_n x_n\| + \|\beta_n x_n + (1-\beta_n)W_n x_n - \ell x_n - (1-\ell)W x_n\| \\
&\leq \|x_n - S_n x_n\| + |\beta_n - \ell|\|x_n - W x_n\| + (1-\beta_n)\|W_n x_n - W x_n\|.
\end{aligned}$$

It follows from (3.23), (3.25) and (3.26), we get

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.27}$$

Step 4. We will prove that $q \in \mathcal{F}$.

We next show that $q \in F(W) = F(W_n) = \bigcap_{n=1}^{\mathbf{N}} F(T_i)$. Assume that $q \neq F(W)$. Since $x_{n_k} \rightharpoonup q$ and $q \neq Wq$, from the Opial condition we have

$$\begin{aligned}
\liminf_{k\to\infty} \|x_{n_k} - q\| &< \liminf_{k\to\infty} \|x_{n_k} - Wq\| \\
&\leq \liminf_{k\to\infty} \{\|x_{n_k} - W x_{n_k}\| + \|W x_{n_k} - Wq\|\} \\
&\leq \liminf_{k\to\infty} \|x_{n_k} - q\|,
\end{aligned}$$

which is a contradiction. Thus, we get $q \in F(W) = F(W_n) = \bigcap_{i=1}^{\mathbf{N}} F(T_i)$.

On the other hand, $u_{n_k} = J_\lambda^{B_1}(x_{n_k} + \gamma A^*(J_\lambda^{B_2} - I)A x_{n_k})$ can be rewritten as

$$\frac{(x_{n_k} - u_{n_k}) + \gamma A^*(J_\lambda^{B_2} - I)A x_{n_k}}{\lambda} \in B_1 u_{n_k}. \tag{3.28}$$

By passing to limit $k \to \infty$ in (3.28) and by taking into account (3.19) and (3.22) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(q)$, i.e., $q \in \mathrm{SOLVIP}(B_1)$. Furthermore, since $\{x_n\}$ and $\{u_n\}$ have the same asymptotical behavior, $\{A x_{n_k}\}$ weakly converges to Aq. Again, by (3.19) and the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive and Lemma 2.10, we obtain that $Aq \in B_2(Aq)$, i.e., $Aq \in \mathrm{SOLVIP}(B_2)$. Thus, $q \in \mathcal{F}$.

Step 5. We will prove that $\limsup_{k\to\infty} \langle (D - \tau f)q, q - x_n\rangle \leq 0$, where $q = \lim_{t\to 0} x_t$ with x_t being the fixed point of the contraction Ψ_n on H_1 defined by

$$\Psi_n x = t\tau f(x) + (I - \tau D)S_n J_\lambda^{B_1}(I + \gamma A^*(J_\lambda^{B_2} - I)A)x, \quad \forall x \in H_1,$$

where $t \in (0,1)$. Indeed, by Lemma 2.2, we have

$$
\begin{aligned}
&\|\Psi_n x - \Psi_n y\| \\
&\quad \leq \|t\tau f(x) + (I-\tau D)S_n J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)x - [t\tau f(y) + (I-\tau D)S_n J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)y]\| \\
&\quad \leq t\tau\|f(x)-f(y)\| + (1-t\bar{\tau})\|S_n J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)x - S_n J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)y)\| \\
&\quad \leq t\tau\|f(x)-f(y)\| + (1-t\bar{\tau})\|S_n x - S_n y\| \\
&\quad \leq t\tau\rho\|x-y\| + (1-t\bar{\tau})\|x-y\| \\
&\quad \leq (1-t(\bar{\tau}-\tau\rho))\|x-y\|,
\end{aligned}
$$

for all $x,y \in H_1$. Since $0 < 1-t(\bar{\tau}-\tau\rho) < 1$, it follows that Ψ_n is a contraction. Therefore, by the Banach contraction principle, Ψ_n has a unique fixed point $x_t \in H_1$ such that

$$x_t = t\tau f(x_t) + (I-\tau D)S_n J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)x_t.$$

By (2.3) and (3.17), we have

$$
\begin{aligned}
\|x_t - x_n\|^2 &= \|(I-tD)[S_n J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)x_t - x_n] + t[\tau f(x_t) - Dx_t]\|^2 \\
&\leq (1-\bar{\tau}t)^2\|S_n J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)x_t - x_n\|^2 + 2t\langle \tau f(x_t) - Dx_t, x_t - x_n\rangle \\
&\leq (1-\bar{\tau}t)^2\|S_n J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)x_t - S_n J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)x_n \\
&\quad + S_n J_\lambda^{B_1}(I+\gamma A^*(J_\lambda^{B_2}-I)A)x_n\|^2 + 2t\langle \tau f(x_t) - Dx_n, x_t - x_n\rangle \\
&\leq (1-\bar{\tau}t)^2[\|x_t - x_n\| + \|y_n - x_n\|]^2 + 2t\langle \tau f(x_t) - Dx_n, x_t - x_n\rangle \\
&\leq (1-\bar{\tau}t)^2\|x_t - x_n\|^2 + 2(1-\bar{\tau}t)^2\|x_t - x_n\|\|y_n - x_n\| + (1-\bar{\tau}t)^2\|y_n - x_n\|^2 \\
&\quad + 2t\langle \tau f(x_t) - Dx_n + Dx_t - Dx_t, x_t - x_n\rangle \\
&\leq (1-\bar{\tau}t)^2\|x_t - x_n\|^2 + \Psi_n(t) + 2t\langle \tau f(x_t) - Dx_t, x_t - x_n\rangle \\
&\quad + 2t\langle Dx_t - Dx_n, x_t - x_n\rangle, \qquad (3.29)
\end{aligned}
$$

where $\Psi_n(t) = (1-\bar{\tau}t)^2(2\|x_t - x_n\| + \|y_n - x_n\|)\|y_n - x_n\| \to 0$ as $n \to \infty$. Observe D is strongly positive, we obtain

$$\langle Dx_t - Dx_n, x_t - x_n\rangle = \langle D(x_t - x_n), x_t - x_n\rangle \geq \bar{\tau}\|x_t - x_n\|^2. \qquad (3.30)$$

Combining (3.29) and (3.30), we have

$$
\begin{aligned}
\|x_t - x_n\|^2 &\leq (1-\bar{\tau}t)^2\|x_t - x_n\|^2 + \Psi_n(t) + 2t\langle \tau f(x_t) - Dx_t, x_t - x_n\rangle + 2t\langle Dx_t - Dx_n, x_t - x_n\rangle \\
&= (1-2\bar{\tau}t + (\bar{\tau}t)^2)\|x_t - x_n\|^2 + \Psi_n(t) + 2t\langle \tau f(x_t) - Dx_t, x_t - x_n\rangle + 2t\langle Dx_t - Dx_n, x_t - x_n\rangle,
\end{aligned}
$$

so,

$$
\begin{aligned}
2t\langle Dx_t - \tau f(x_t), x_t - x_n\rangle &\leq (\bar{\tau}^2 t^2 - 2\bar{\tau}t)\|x_t - x_n\|^2 + \Psi_n(t) + 2t\langle Dx_t - Dx_n, x_t - x_n\rangle \\
&\leq (\bar{\tau}t^2 - 2t)\langle D(x_t - x_n), x_t - x_n\rangle + \Psi_n(t) + 2t\langle Dx_t - Dx_n, x_t - x_n\rangle \\
&= \bar{\tau}t^2\langle Dx_t - Dx_n, x_t - x_n\rangle + \Psi_n(t).
\end{aligned}
$$

It follows that

$$\langle Dx_t - \tau f(x_t), x_t - x_n\rangle \leq \frac{\bar{\tau}t}{2}\langle Dx_t - Dx_n, x_t - x_n\rangle + \frac{1}{2t}\Psi_n(t). \qquad (3.31)$$

Let $n \to \infty$ in (3.31) and note that $\Psi_n(t) \to 0$ as $n \to \infty$ yields,

$$\limsup_{n\to\infty}\langle Dx_t - \tau f(x_t), x_t - x_n\rangle \leq \frac{t}{2}M_4, \qquad (3.32)$$

where M_4 is an approximate positive constant such that $M_4 \geq \bar{\tau}\langle Dx_t - Dx_n, x_t - x_n\rangle$ for all $t \in (0,1)$ and $n \geq 1$. Taking $t \to 0$ from (3.32), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle Dx_t - \tau f(x_t), x_t - x_n\rangle \leq 0. \tag{3.33}$$

On the other hand, we have

$$\begin{aligned}\langle \tau f(q) - Dq, x_n - q\rangle &= \langle \tau f(q) - Dq, x_n - q\rangle - \langle \tau f(q) - Dq, x_n - x_t\rangle \\ &\quad + \langle \tau f(q) - Dq, x_n - x_t\rangle - \langle \tau f(q) - Dx_t, x_n - x_t\rangle \\ &\quad + \langle \tau f(q) - Dx_t, x_n - x_t\rangle - \langle \tau f(x_t) - Dx_t, x_n - x_t\rangle \\ &\quad + \langle \tau f(x_t) - Dx_t, x_n - x_t\rangle.\end{aligned}$$

It follows that

$$\begin{aligned}&\limsup_{n\to\infty}\langle \tau f(q) - Dq, x_n - q\rangle \\ &\quad\leq \|\tau f(q) - Dq\|\|x_t - q\| + \|D\|\|x_t - q\|\lim_{n\to\infty}\|x_n - x_t\| \\ &\qquad + \tau\rho\|x_t - q\|\lim_{n\to\infty}\|x_n - x_t\| + \limsup_{n\to\infty}\langle \tau f(x_t) - Dx_t, x_n - x_t\rangle.\end{aligned}$$

Therefore, from (3.33) and $\lim_{t\to 0} x_t = q$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle \tau f(q) - Dq, x_n - q\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle \tau f(q) - Dq, x_n - q\rangle \\ &\leq \limsup_{t\to 0}\|\tau f(q) - Dq\|\|x_t - q\| \\ &\quad + \limsup_{t\to 0}\|D\|\|x_t - q\|\lim_{n\to\infty}\|x_n - x_t\| \\ &\quad + \limsup_{t\to 0}\tau\rho\|x_t - q\|\lim_{n\to\infty}\|x_n - x_t\| \\ &\quad + \limsup_{t\to 0}\limsup_{n\to\infty}\langle \tau f(x_t) - Dx_t, x_n - x_t\rangle \\ &\leq 0.\end{aligned} \tag{3.34}$$

On the other hand, we shall show that the uniqueness of a solution of the variational inequality

$$\langle (D - \tau f)x, x - q\rangle \leq 0, \quad q \in \mathcal{F}. \tag{3.35}$$

Suppose that $q \in \mathcal{F}$ and $\hat{q} \in \mathcal{F}$ both are solutions to (3.35), then

$$\langle (D - \tau f)q, q - \hat{q}\rangle \leq 0 \tag{3.36}$$

and

$$\langle (D - \tau f)\hat{q}, \hat{q} - q\rangle \leq 0. \tag{3.37}$$

Adding up (3.36) and (3.37) one gets

$$\langle (D - \tau f)q - (D - \tau f)\hat{q}, q - \hat{q}\rangle \leq 0. \tag{3.38}$$

By Lemma 2.11, the strong monotonicity of $D - \gamma f$, we obtain $q = \hat{q}$ and the uniqueness is proved.

Finally, we prove that $x_n \to q$ as $n \to \infty$. From (3.1) and (3.9) again, we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \langle \alpha_n \tau f(x_n) + (I - \alpha_n D) y_n - q, x_{n+1} - q\rangle \\
&= \alpha_n \langle \tau f(x_n) - Dq, x_{n+1} - q\rangle + \langle (I - \alpha_n D)(y_n - q), x_{n+1} - q\rangle \\
&\leq \alpha_n \tau \langle f(x_n) - f(q), x_{n+1} - q\rangle + \alpha_n \langle \tau f(q) - Dq, x_{n+1} - q\rangle \\
&\quad + (1 - \alpha_n \bar{\tau}) \|y_n - q\| \|x_{n+1} - q\| \\
&\leq \alpha_n \tau \rho \|x_n - q\| \|x_{n+1} - q\| + \alpha_n \langle \tau f(q) - Dq, x_{n+1} - q\rangle \\
&\quad + (1 - \alpha_n \bar{\tau}) \|x_n - q\| \|x_{n+1} - q\| \\
&= [1 - (\bar{\tau} - \tau\rho)\alpha_n] \|x_n - q\| \|x_{n+1} - q\| + \alpha_n \langle \tau f(q) - Dq, x_{n+1} - q\rangle \\
&\leq \frac{1 - (\bar{\tau} - \tau\rho)\alpha_n}{2} (\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + \alpha_n \langle \tau f(q) - Dq, x_{n+1} - q\rangle \\
&\leq \frac{1 - (\bar{\tau} - \tau\rho)\alpha_n}{2} \|x_n - q\|^2 + \frac{1}{2}\|x_{n+1} - q\|^2 + \alpha_n \langle \tau f(q) - Dq, x_{n+1} - q\rangle.
\end{aligned}$$

It follows that

$$\|x_{n+1} - q\|^2 \leq [1 - (\bar{\tau} - \tau\rho)\alpha_n] \|x_n - q\|^2 + 2\alpha_n \langle \tau f(q) - Dq, x_{n+1} - q\rangle. \tag{3.39}$$

From $0 < \tau < \frac{\bar{\tau}}{\rho}$, condition (C1) and (3.34), we can arrive at the desired conclusion $\lim_{n\to\infty} \|x_n - q\| = 0$ by Lemma 2.9. This complete the proof. □

4 Consequently results

Corollary 4.1. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subseteq H_1$ and $Q \subseteq H_2$ be nonempty closed convex subsets. Let $A : H_1 \to H_2$ be a bounded linear operator. Let D be a strongly positive bounded linear operator on H_1 with $\bar{\tau} > 0$ and $0 < \tau < \frac{\bar{\tau}}{\rho}$. Let $T : C \to H_1$ be a k-strict pseudo-contractions such that $\mathcal{F} := F(T) \cap \bar{\Gamma} \neq \emptyset$. Let $\{x_n\}$ be a sequence generated in the following;*

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \beta_n u_n + (1 - \beta_n) T u_n, \\ x_{n+1} = \alpha_n \tau f(x_n) + (I - \alpha_n D) y_n, \ n \geq 1, \end{cases} \tag{4.1}$$

where $\lambda > 0$, $\gamma \in (0, \frac{1}{L})$, L is the spectral radius of the operator A^*A and A^* is the adjoint of A. The following control conditions are satisfied:

(C1) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n-1}| < \infty$;

(C2) $k \leq \beta_n \leq \ell < 1$, $\lim_{n\to\infty} \beta_n = \ell$ and $\sum_{n=1}^{\infty} |\beta_n - \beta_{n-1}| < \infty$.

Then the sequence $\{x_n\}$ generated by (4.1) converges strongly to $q \in \mathcal{F}$ which solves the variational inequality

$$\langle (D - \tau f)q, q - p\rangle \leq 0, \ \forall p \in \mathcal{F}.$$

Proof. Putting $N = 1$ and $W_n = T$, the desired conclusion follows immediately from Theorem 3.1. This completes the proof. □

5 Numerical examples

In this section, we give an example and numerical results to illustrate our algorithm and the main result of this paper.

Example 5.1. *Let $H_1 = H_2 = \mathbb{R}^3$. Let two operators of matrix multiplication $B_1, B_2 : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by*

$$B_1 = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 18 & 0 \\ 0 & 0 & 27 \end{bmatrix} \quad \text{and} \quad B_2 = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 19 & 0 \\ 0 & 0 & 14 \end{bmatrix}.$$

So, we can define the resolvent mappings $J_\lambda^{B_1}(x) := (I + \lambda B_1)^{-1}(x)$ and $J_\lambda^{B_2} := (I + \lambda B_2)^{-1}(x)$ on $\mathbb{R}^3$ associated with B_1 and B_2 where $\lambda > 0$.

Let $A \in \mathbb{R}^{3\times 3}$ be a singular matrix operator in which elements are random and A^* be an adjoint of A. The mappings $T_i : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T_1 = \frac{x}{20(1+x)}$, $T_2 = \frac{\sin x}{30(1+x)}$ and $T_3 = \frac{x}{40+x}$ are k_i-strict pseudo-contractive for $i = 1, 2, 3$ (see [34]). Then we present the following algorithm.

Algorithm 5.2. (Iteration algorithm for split variational inclusion)

Step 0. Choose the initial point $x_0 \in \mathbb{R}^3$. Put $\lambda = \frac{1}{2}$, $\gamma = \frac{1}{10}$, $\tau = 10$, $\alpha_n = \frac{1}{2}$, $\beta_n = \frac{1}{10}$, $\eta_1 = \eta_2 = \eta_3 = \frac{1}{3}$, $D = I$ and let $n = 1$.

Step 1. Given $x_n \in \mathbb{R}^3$ and compute $x_{n+1} \in \mathbb{R}^3$ as follows.

$$\begin{cases} u_n = J_\lambda^{B_1}(x_n + \gamma A^*(J_\lambda^{B_2} - I)Ax_n), \\ y_n = \beta_n u_n + (1 - \beta_n) \sum_{i=1}^{\mathbf{N}} \eta_{i=1}^{(n)} T_i u_n, \\ x_{n+1} = \alpha_n \tau f(x_n) + (I - \alpha_n D) y_n, \ n \geq 1, \end{cases}$$

Step 2. Put $n := n + 1$ and go to step 1.

Setting $\|x_n - x_{n+1}\| \leq 10^{-10}$ as stop criterion, then we obtain the numerical results of iteration scheme for Algorithm 5.2.

Figure 1 shows the convergent of Algorithm 5.2 with different initial points where $x_1 = 2, 4, -4$.

Figure 2 shows the convergent of Algorithm 5.2 with different contraction mappings $f(x) = 0.1x$, $f(x) = 0.02x$ and $f(x) = 0.05x$.

We can see that both of Figure 1 and Figure 2, we can see that the sequence x_n coverges to 0, that is, 0 is the solution in Example 5.1.

$x_1 = 2$		$x_1 = 4$		$x_1 = -4$	
n	$\lvert x_n - x_{n+1}\rvert$	n	$\lvert x_n - x_{n+1}\rvert$	n	$\lvert x_n - x_{n+1}\rvert$
1	8.563025	1	6.21	1	1.03
2	2.597529×10^{-01}	2	1.87×10^{-01}	2	4.26×10^{-01}
3	9.873640×10^{-03}	3	7.20×10^{-03}	3	1.72×10^{-02}
4	3.855163×10^{-04}	4	2.88×10^{-04}	4	6.91×10^{-04}
5	1.516538×10^{-05}	5	1.17×10^{-05}	5	2.78×10^{-05}
6	6.005162×10^{-07}	6	4.83×10^{-07}	6	1.13×10^{-06}
7	2.393081×10^{-08}	7	2.00×10^{-08}	7	4.58×10^{-08}
8	9.594050×10^{-10}	8	8.33×10^{-10}	8	1.87×10^{-09}
9	3.867765×10^{-11}	9	3.47×10^{-11}	9	7.67×10^{-11}

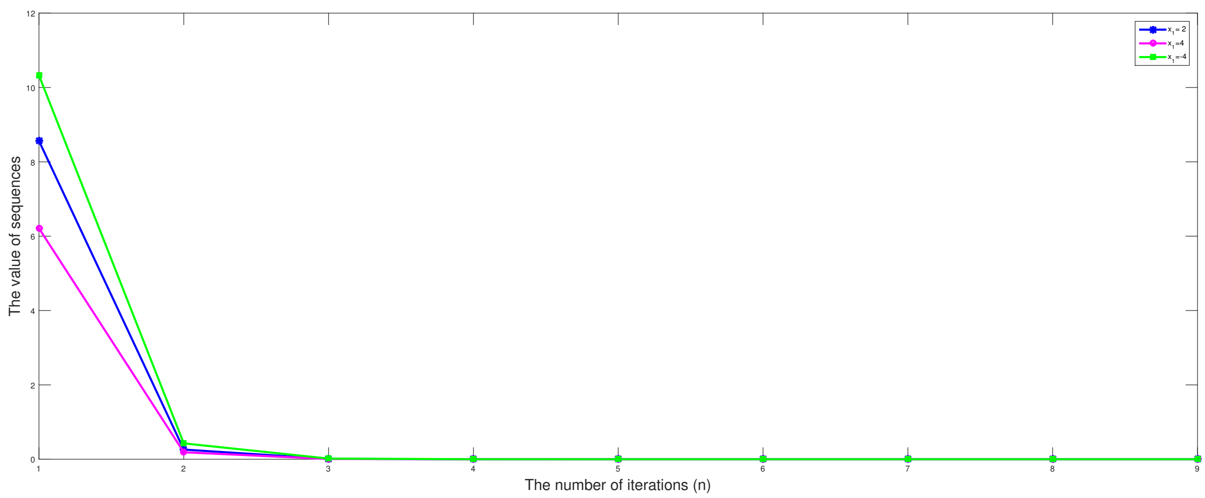

Figure 1: The iteration chart with initial pointst $x_1 = 2, 4, -4$.

$f(x)=0.1x$		$f(x)=0.02x$		$f(x)=0.05x$	
n	$\|x_n - x_{n+1}\|$	n	$\|x_n - x_{n+1}\|$	n	$\|x_n - x_{n+1}\|$
1	75.39948	1	131.9665	1	110.7537
5	34.90713	3	7.11×10^{-01}	3	5.324926
10	9.78×10^{-02}	5	5.41×10^{-03}	6	6.86×10^{-02}
15	2.68×10^{-03}	7	4.27×10^{-05}	9	9.17×10^{-04}
20	7.40×10^{-05}	9	3.44×10^{-07}	12	1.24×10^{-05}
25	2.05×10^{-06}	10	3.11×10^{-08}	14	7.04×10^{-07}
30	5.72×10^{-08}	11	2.81×10^{-09}	16	4.02×10^{-08}
35	1.60×10^{-09}	12	2.56×10^{-10}	18	2.30×10^{-09}
39	9.19×10^{-11}	13	2.33×10^{-11}	21	3.16×10^{-11}

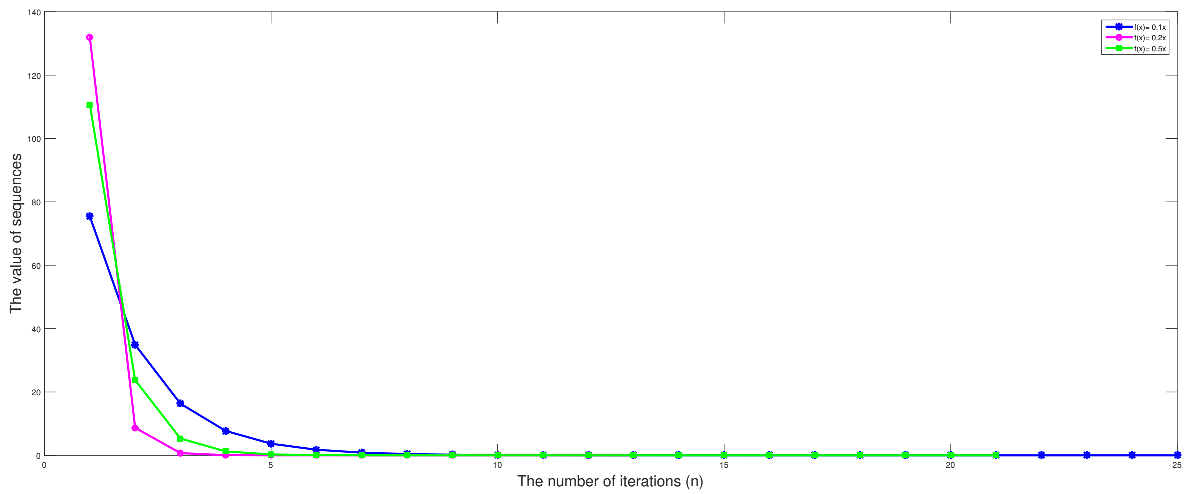

Figure 2 The iteration chart with different contraction mappings $f(x) = 0.1x, 0.02x, 0.05x$

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References

[1] T. Suzuki, Strong convergence theorems for an infinite family of nonexpansive mappings in general Banach spaces, Fixed Point Theory Appl., Vol. 1, 103–123 (2005).

[2] K. Shimoji, W. Takahashi, Strong convergence to common fixed points of infinite nonexpansive mappings and applications, Taiwanese J. Math., Vol. 5 (2), 87–404 (2001).

[3] M. O., Osilike and D. I., Igbokwe,Weak and Strong Convergence Theorems for Fixed Points of Pseudocontractions and Solutions of Monotone Type Operator Equations, Computers & Mathematics with Applications, Vol. 40, 559–567, (2000).

[4] Z., Opial, Weak Convergence of Successive Approximations for Nonexpansive Mappings, Bulletin of the American Mathematical Society, Vol. 73, 591–597, (1967).

[5] L. C. Ceng, J. C. Yao, An extragradient like approximation method for variational inequality problems and fixed point problems, Appl. Math. Comput. 190 (2007), 205–215.

[6] L.C. Ceng, J.C. Yao, On the convergence analysis of inexact hybrid extragradient proximal point algorithms for maximal monotone operators, J. Comput. Appl. Math. 217 (2008), 326–338.

[7] L.C. Ceng, J.C. Yao, Approximate proximal algorithms for generalized variational inequalities with pseudomonotone multifunctions, J. Comput. Appl. Math. 213 (2008), 423–438.

[8] R. Glowinski, Numerical Methods for Nonlinear Variational Problems, Springer-Verlag, New York, 1984.

[9] C. Jaiboon, P. Kumam, H.W. Humphries, Weak convergence theorem by extragradient method for variational inequality, equilibrium problems and xed point problems, Bull. Malaysian Math. Sci. Soc. 2 (32) (2009), 173–185.

[10] F. Liu, M.Z. Nasheed, Regularization of nonlinear ill-posed variational inequalities and convergence rates, Set Valued Anal. 6 (1998), 313–344.

[11] G. Stampacchia, Formes bilineaires coercitivies sur les ensembles convexes, C.R. Acad. Sci. Paris, 258 (1964), 4413–4416.

[12] R. Wangkeeree, R. Wangkeeree, A general iterative methods for variational inequality problems and mixed equilibrium problems and fixed point problems of strictly pseudocontractive mappings in Hilbert spaces, Fixed Point Theory Appl., (2007) 32, (Article ID 519065).

[13] S.S. Zeng, N.C. Wong, J.C. Yao, Convergence of hybrid steepest-descent methods for generalized variational inequalities, Acta Math. Sin. English Ser., 22 (1) (2006), 1–12.

[14] Y. Censor and T. Elfving, A multiprojection algorithm using Bregman projections in a product space, Numerical Algorithms, Vol. 8, No. 2–4, 221–239, (1994).

[15] Xu, H.K., An iterative approach to quadratic optimization, J. Optim. Theory Appl., 116, 659–678 (2003).

[16] Acedo, G.L., Xu, H.K., Iteration methods for strict pseudo-contractions in Hilbert spaces, Nonlinear Anal., 67, 2258–2271 (2007).

[17] K. Geobel, W. A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Mathematics, Vol. 28, cambridge University Press, 1990.

[18] Browder, FE: Convergence of approximants to fixed points of nonexpansive nonlinear mappings in Banach spaces. Arch. Ration. Mech. Anal. 24, 82–90 (1967).

[19] Browder, FE, Petryshyn, WV: Construction of fixed points of nonlinear mappings in Hilbert space. J. Math. Anal. Appl. 20, 197–228 (1967).

[20] Scherzer, O: Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems. J. Math. Anal. Appl. 194, 911–933 (1991).

[21] Plubtieng, S, Punpaeng, R: A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces. J. Math. Anal. Appl. 336, 455–469 (2007).

[22] Marino, G, Xu, HK: A general iterative method for nonexpansive mappings in Hilbert spaces. J. Math. Anal. Appl. 318, 43–52 (2006).

[23] Ceng, LC, Yao, JC: Hybrid viscosity approximation schemes for equilibrium problems and fixed point problems of infinitely many nonexpansive mappings. Appl. Math. Comput. 198, 729–741 (2008).

[24] Zhou, H: Convergence theorems of fixed points for k-strict pseudo-contractions in Hilbert spaces. Nonlinear Anal. 69, 456–462 (2008).

[25] Marino, G, Xu, HK: Weak and strong convergence theorems for k-strict pseudo-contractions in Hilbert spaces. J. Math. Anal. Appl. 329, 336–349 (2007)

[26] Takahashi, S, Takahashi, W: Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces. J. Math. Anal. Appl. 331, 506–515 (2007).

[27] Censor, Y, Bortfeld, T, Martin, B, Trofimov, A: A unified approach for inversion problems in intensity modulated radiation therapy. Phys. Med. Biol. 51, 2353–2365 (2006).

[28] Censor, Y, Gibali, A, Reich, S: Algorithms for the split variational inequality problem. Numer. Algorithms 59, 301–323 (2012).

[29] Combettes, PL: The convex feasibility problem in image recovery. Adv. Imaging Electron Phys. 95, 155–453 (1996).

[30] Byrne, C: Iterative oblique projection onto convex sets and the split feasibility problem. Inverse Probl. 18, 441–453 (2002).

[31] Byrne, C, Censor, Y, Gibali, A, Reich, S: Weak and strong convergence of algorithms for the split common null point problem. J. Nonlinear Convex Anal. 13, 759–775 (2012).

[32] Kazmi, KR, Rizvi, SH: An iterative method for split variational inclusion problem and fixed point problem for a nonexpansive mapping. Optim. Lett., 2014, Volume 8, Issue 3, pp 1113–1124.

[33] Moudafi, A: Split monotone variational inclusions. J. Optim. Theory Appl. 150, 275-283 (2011).

[34] A. Razani, R. Moradi: On the iterates of asymptotically k-strict pseudocontractive mappings in Hilbert spaces with convergence analysis. Fixed Point Theory and Applications, 2015:21 (2015).