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Weak Galerkin finite element method for Biot's consolidation problem ^{* †‡§}

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Abstract

In this paper, a fully discrete weak Galerkin (WG) finite element method is proposed to solve Biot's consolidation problem, where weakly defined gradient and divergence operators over discontinuous functions are introduced. $\mathbb{P}_l$ - $\mathbb{P}_l$ ($l \geq 1$) finite element combination is used for the displacement and pressure approximations in the interior of the elements, and $\mathbb{P}_l$ - $\mathbb{P}_{l-1}$ combination for the corresponding trace approximations on the interfaces of the finite element partition. The existence and uniqueness of the discrete linear system at each time step is derived, and error estimates for the approximation of displacement and pressure are obtained. Numerical experiments confirm the theoretical results and show that the proposed WG method is capable of overcoming pressure oscillations.

1 Introduction

Let Ω be a convex polyhedral domain in $\mathbb{R}^d$ ($d = 2, 3$) with boundary $\Gamma = \Gamma_D \cup \Gamma_N = \Gamma_{\bar{D}} \cup \Gamma_{\bar{N}}$, where $\text{meas}(\Gamma_D) > 0$ and $\text{meas}(\Gamma_{\bar{D}}) > 0$. For $t \in (0, T]$, we consider Biot's consolidation problem [5]

$$-\nabla \cdot (2\mu\varepsilon(\mathbf{u}) + \lambda\nabla \cdot \mathbf{u}\mathbf{I}) + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega \times (0, T], \quad (1.1)$$

$$\nabla \cdot D_t \mathbf{u} - \frac{\kappa}{\eta_f} \Delta p = g \quad \text{in} \quad \Omega \times (0, T], \quad (1.2)$$

with the boundary conditions

$$\mathbf{u} = \mathbf{0} \text{ on } \Gamma_D, \quad (2\mu\varepsilon(\mathbf{u}) + \lambda\nabla \cdot \mathbf{u}\mathbf{I} - p\mathbf{I})\mathbf{n} = \boldsymbol{\beta} \text{ on } \Gamma_N, \quad (1.3)$$

$$p = 0 \text{ on } \Gamma_{\bar{D}}, \quad \frac{\kappa}{\eta_f} \nabla p \cdot \mathbf{n} = \gamma \text{ on } \Gamma_{\bar{N}} \quad (1.4)$$

and the initial condition

$$\nabla \cdot \mathbf{u}(0) = 0 \quad \text{in} \quad \Omega. \quad (1.5)$$

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Here $\mathbf{u}(t) : \Omega \rightarrow \mathbb{R}^d$ is the displacement vector of the porous medium, $p(t) : \Omega \rightarrow \mathbb{R}$ is the pore pressure of the interstitial fluid, $\mathbf{f}(t) : \Omega \rightarrow \mathbb{R}^d$ is the body force, $g(t) : \Omega \rightarrow \mathbb{R}$ is the source/sink density function of the fluid, $\boldsymbol{\beta}(t) : \Omega \rightarrow \mathbb{R}^d$ is the prescribed surface traction, and $\gamma(t) : \Omega \rightarrow \mathbb{R}$ is the prescribed discharge on the boundary. The symbols Δ , ∇ and $\nabla \cdot$ denote the Laplacian, gradient and divergence operators, respectively. $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ is the strain tensor, λ , μ are the Lamé constants, κ is the permeability of the porous skeleton, and η_f denotes the viscosity of the pore fluid. $\mathbf{n}$ is an outward unit normal vector, D_t represents the time derivative. We note that the initial condition (1.5) characterizes the incompressible response of the solid-fluid aggregate in the beginning of the salivation process.

Biot's consolidation model describes the deformation of an elastic porous medium and the viscous fluid flow inside it when the medium is saturated by the fluid. This model has a wide range of applications including petroleum engineering, soil mechanics, environment engineering, and biomedical engineering.

In recent two decades, extensive research has been done for numerical treatment of Biot's consolidation model. The most commonly used numerical methods are finite element methods; see, for instance, [11] [12]-[15] [21]-[23] [25]-[27] [34]-[37]. Standard Galerkin finite element methods suffer from locking in poroelasticity [6], i.e. nonphysical oscillations of the pressure approximation may occur when the permeability is low or the time step size is small. In [27] Phillips and Wheeler examined heuristically the reason for the locking in poroelasticity, and in [26] they approximated the flow variables by a mixed finite-element space and the displacement by a family of discontinuous Galerkin methods so as to overcome the nonphysical pressure oscillations. Yi [36] developed a nonconforming mixed finite element method to solve a three-field Biot's consolidation model, where the fluid flux was introduced as an independent unknown. The proposed method was shown to be effective to overcome the locking. A stabilized finite element scheme for the poroelasticity equations was proposed in [2], which could obtain solutions without oscillations independently of the chosen discretization parameters. Hu et al. [12] proposed a stable nonconforming finite element scheme for the three-field Biot's model to avoid the pressure oscillations. Based on MINI and stabilized P_1 - P_1 elements, Rodrigo et al. [28] introduced a stabilization term which removed the oscillations.

This paper is to develop a weak Galerkin (WG) finite element method for Biot's consolidation problem (1.1)-(1.5). The WG method was first introduced and analyzed by Wang and Ye in [32] for second order elliptic equations. Later it was extended to different models, for example Stokes equations [33, 7], biharmonic equations [19, 30], parabolic equations [17], convection-diffusion equations [8], Maxwell equations [20], and linear elasticity problems [9, 31]. The method is designed by using a weakly defined gradient operator over functions with discontinuity, and allows the use of totally discontinuous functions in the finite element procedure. We refer to [16] for a multigrid WG algorithm, and to [10] for an auxiliary space multigrid preconditioner for the WG discretization of diffusion equations. We mention that in [9, 31] the proposed arbitrary order WG finite element methods are locking-free from Poisson locking.

In this paper, we shall propose a fully discrete WG method for the model (1.1)-(1.5). We use the $\mathbb{P}_l$ - $\mathbb{P}_l$ ($l \geq 1$) finite element combination for the displacement and pressure approximations in the interior of the elements, and the $\mathbb{P}_l$ - $\mathbb{P}_{l-1}$ combination for the corresponding trace approximations on the interfaces

of the finite element partition. We will derive error estimates for the displacement and pressure approximations. Numerical experiments will be provided to show the performance of the proposed WG method.

The rest of this paper is organized as follows. In Section 2 we introduce notation and weak formulations for the model problem. Section 3 describes semi-discrete and fully discrete WG finite element schemes. Section 4 is devoted to the error estimation. Finally, Section 5 provides some numerical experiments.

2 Notation and weak formulations

Let $L^2(\Omega)$ be the set of square-integrable real-valued functions on Ω , with inner product and norm denoted by $(\cdot, \cdot)$ and $\|\cdot\|_0$, respectively. For any nonnegative integer s , $H^s(\Omega)$ denotes the standard Sobolev spaces of real-valued functions and the corresponding norm is denoted by $\|\cdot\|_s$. For $s \geq 1$, we use $H_0^s(\Omega) := \{\mathbf{v} \in H^s(\Omega) : \mathbf{v}|_{\partial\Omega} = \mathbf{0}\}$ to denote the subspace of $H^s(\Omega)$ with vanishing trace on $\partial\Omega$. We also need the Sobolev spaces

$$\mathbf{H}_{0,\Gamma_D}^1(\Omega) := \{\mathbf{v} \in [H^1(\Omega)]^d, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\},$$

$$H_{0,\Gamma_D}^1(\Omega) := \{q \in H^1(\Omega), q = 0 \text{ on } \Gamma_D\}.$$

Given any Banach space U with norm $\|\cdot\|_U$, for $0 < t_1 < t_2$ and $1 \leq q \leq \infty$, we introduce the space

$$L^q(t_1, t_2; U) := \{v : v(t) \in U \text{ for } t_1 \leq t \leq t_2, \text{ and } \|v(t)\|_U \in L^q([t_1, t_2])\}$$

with norm $\|\cdot\|_{L^q(t_1, t_2; U)}$ defined as follows: for any $v \in L^q(t_1, t_2; U)$,

$$\|v\|_{L^q(t_1, t_2; U)} := \left(\int_{t_1}^{t_2} \|v(t)\|_U^q dt \right)^{1/q} \text{ when } 1 \leq q < \infty,$$

$$\|v\|_{L^\infty(t_1, t_2; U)} := \operatorname{esssup}_{t_1 \leq t \leq t_2} \|v(t)\|_U \text{ when } q = \infty.$$

Let $\mathfrak{S}_h = \bigcup \{K\}$ be a shape-regular decomposition of the domain $\Omega \in \mathbb{R}^d$ ($d = 2, 3$) consisting of polygons or polyhedrons. For any $K \in \mathfrak{S}_h$ with boundary ∂K , let h_K be its diameter. Denote $h := \max_{K \in \mathfrak{S}_h} h_K$. Let $\mathcal{E}_h$ be the union of all edges (faces) of $K \in \mathfrak{S}_h$.

Throughout the rest of the paper, vector valued functions are written in boldface. We use $a \lesssim b$ ($a \gtrsim b$) to represent that there exists a positive constant c , independent of the mesh parameters h and Δt , such that $a \leq cb$ ($a \geq cb$).

The variational formulation of the evolutionary consolidation problem (1.1)-(1.5) is as follows.

Problem $\mathbf{M}_t$. For any $t \in (0, T]$, find $(\mathbf{u}(t), p(t)) \in \mathbf{H}_{0,\Gamma_D}^1(\Omega) \times H_{0,\Gamma_D}^1(\Omega)$ such that

$$2\mu(\varepsilon(\mathbf{u}), \varepsilon(\mathbf{v})) + \lambda(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \langle \beta, \mathbf{v} \rangle_{\Gamma_N}, \forall \mathbf{v} \in \mathbf{H}_{0,\Gamma_D}^1(\Omega), \quad (2.1)$$

$$(\nabla \cdot D_t \mathbf{u}, q) + \frac{\kappa}{\eta_f} (\nabla p, \nabla q) = (g, q) + \langle \gamma, q \rangle_{\Gamma_N}, \quad \forall q \in H_{0,\Gamma_D}^1(\Omega), \quad (2.2)$$

with the initial condition

$$(\nabla \cdot \mathbf{u}(0), q) = 0, \quad \forall q \in L^2(\Omega). \quad (2.3)$$

3 Weak Galerkin finite element method

3.1 Discrete Weak Operators

In order to design a WG finite element scheme for Biot's problem (1.1)-(1.5), we first follow [32] to introduce the discrete weak gradient and divergence operators for the displacement approximation and the discrete weak gradient operator for the pressure approximation.

Denote by $\nu(K)$ the space of weak vector-valued functions in K with

$$\nu(K) := \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \mid \mathbf{v}_0 \in [L^2(K)]^d, \mathbf{v}_b \in [H^{\frac{1}{2}}(\partial K)]^d\}.$$

For any non-negative integer r , denote respectively by $\mathbb{P}_r(K)$ and $\mathbb{P}_r(e)$ the sets of polynomials on K and e with degree no more than r .

Definition 3.1. *The discrete weak gradient operator of $\mathbf{v} \in \nu(K)$, denoted by $\nabla_{w,r,K}\mathbf{v}$, is defined as the unique polynomial $\nabla_{w,r,K}\mathbf{v} \in [\mathbb{P}_r(K)]^{d \times d}$ satisfying the following equation,*

$$(\nabla_{w,r,K}\mathbf{v}, \boldsymbol{\tau})_K := -(\mathbf{v}_0, \nabla \cdot \boldsymbol{\tau})_K + \langle \mathbf{v}_b, \boldsymbol{\tau} \cdot \mathbf{n} \rangle_{\partial K}, \quad \forall \boldsymbol{\tau} \in [\mathbb{P}_r(K)]^{d \times d}. \quad (3.1)$$

Denote by $W(K)$ the space of weak functions on K with

$$W(K) := \{\mathbf{v} = \{\mathbf{v}_0, \mathbf{v}_b\} \mid \mathbf{v}_0 \in [L^2(K)]^d, \mathbf{v}_b \cdot \mathbf{n} \in H^{-\frac{1}{2}}(\partial K)\}.$$

Definition 3.2. *The discrete weak divergence operator of $\mathbf{v} \in W(K)$, denoted by $\nabla_{w,r,K} \cdot \mathbf{v}$, is defined as the unique polynomial $\nabla_{w,r,K} \cdot \mathbf{v} \in \mathbb{P}_r(K)$ satisfying the following equation,*

$$(\nabla_{w,r,K} \cdot \mathbf{v}, q)_K := -(\mathbf{v}_0, \nabla q)_K + \langle \mathbf{v}_b \cdot \mathbf{n}, q \rangle_{\partial K}, \quad \forall q \in \mathbb{P}_r(K). \quad (3.2)$$

In particular, we write $\nabla_{w,r,K} \cdot \mathbf{v}_0 := \nabla_{w,r,K} \cdot \{\mathbf{v}_0, \mathbf{0}\}$ for simplicity.

Denote by $M(K)$ the space of weak functions on K with

$$M(K) := \{q = \{q_0, q_b\} \mid q_0 \in L^2(K), q_b \in H^{\frac{1}{2}}(\partial K)\}.$$

Definition 3.3. *The discrete weak gradient operator of $q \in M(K)$, denoted by $\tilde{\nabla}_{w,r,K} q$, is defined as the unique polynomial $\tilde{\nabla}_{w,r,K} q \in [\mathbb{P}_r(K)]^d$ satisfying the following equation,*

$$(\tilde{\nabla}_{w,r,K} q, \boldsymbol{\varphi})_K := -(q_0, \nabla \cdot \boldsymbol{\varphi})_K + \langle q_b, \boldsymbol{\varphi} \cdot \mathbf{n} \rangle_{\partial K}, \quad \forall \boldsymbol{\varphi} \in [\mathbb{P}_r(K)]^d. \quad (3.3)$$

3.2 WG discretization

For any integer $l \geq 1$, we define the following weak Galerkin finite element spaces respectively for the velocity and pressure variables:

$$V^h := \{\mathbf{v}_h = \{\mathbf{v}_{h0}, \mathbf{v}_{hb}\} \mid \mathbf{v}_{h0}|_K \in [\mathbb{P}_l(K)]^d, \mathbf{v}_{hb}|_e \in [\mathbb{P}_l(e)]^d, \forall K \in \mathfrak{S}_h, \forall e \in \mathcal{E}_h\},$$

$$\Theta^h := \{q_h = \{q_{h0}, q_{hb}\} \mid q_{h0}|_K \in \mathbb{P}_l(K), q_{hb}|_e \in \mathbb{P}_{l-1}(e), \forall K \in \mathfrak{S}_h, \forall e \in \mathcal{E}_h\}.$$

Denote

$$V_{0,D}^h := \{\mathbf{v}_h = \{\mathbf{v}_{h0}, \mathbf{v}_{hb}\} \in V^h, \mathbf{v}_{hb} = 0 \text{ on } \Gamma_D\},$$

$$\Theta_{0,\tilde{D}}^h := \{q_h = \{q_{h0}, q_{hb}\} \in \Theta^h, q_{hb} = 0 \text{ on } \Gamma_{\tilde{D}}\}.$$

For any $\mathbf{v} \in V^h$ and $q \in \Theta^h$, the discrete weak gradients $\nabla_{w,l}\mathbf{v}$, $\tilde{\nabla}_{w,l-1}q$, and the discrete weak divergence $\nabla_{w,l} \cdot \mathbf{v}$ are computed by using (3.1), (3.3) and (3.2) on each element K , respectively. More precisely, we have

$$\begin{aligned} (\nabla_{w,l}\mathbf{v})|_K &= \nabla_{w,l,K}(\mathbf{v}|_K), \quad (\nabla_{w,l} \cdot \mathbf{v})|_K = \nabla_{w,l,K} \cdot (\mathbf{v}|_K), \\ (\tilde{\nabla}_{w,l-1}q)|_K &= \tilde{\nabla}_{w,l-1,K}(q|_K) \end{aligned}$$

and we define the weak strain tensor $\varepsilon_{w,l}(\mathbf{v})$ by

$$\varepsilon_{w,l}(\mathbf{v}) := \frac{1}{2}(\nabla_{w,l}\mathbf{v} + (\nabla_{w,l}\mathbf{v})^T).$$

For simplicity of notation, from now on we shall write

$$\nabla_w := \nabla_{w,l}, \quad \varepsilon_w(\cdot) := \varepsilon_{w,l}(\cdot), \quad \tilde{\nabla}_w := \tilde{\nabla}_{w,l-1}.$$

3.2.1 Semi-discrete WG scheme

For each edge/face $e \in \mathcal{E}_h$, let $Q_b : L^2(e) \rightarrow \mathbb{P}_{l-1}(e)$ be L^2 -projection operator. The semi-discrete or time-continuous WG finite element scheme for Problem $\mathbf{M}_t$, i.e. (2.1)-(2.3), is as follows.

Problem $\mathbf{CM}_t^h$. For any $t \in (0, T]$, find $\mathbf{u}_h(\cdot, t) = \{\mathbf{u}_{h0}(\cdot, t), \mathbf{u}_{hb}(\cdot, t)\} \in V_{0,D}^h, p_h(\cdot, t) = \{p_{h0}(\cdot, t), p_{hb}(\cdot, t)\} \in \Theta_{0,\tilde{D}}^h$ such that

$$a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_{h0}) = (\mathbf{f}, \mathbf{v}_{h0}) + \langle \boldsymbol{\beta}, \mathbf{v}_{hb} \rangle_{\Gamma_N}, \quad \forall \mathbf{v}_h = \{\mathbf{v}_{h0}, \mathbf{v}_{hb}\} \in V_{0,D}^h, \quad (3.4)$$

$$-b(D_t \mathbf{u}_h, q_{h0}) + d(p_h, q_h) = (g, q_{h0}) + \langle \gamma, q_{hb} \rangle_{\Gamma_{\tilde{N}}}, \quad \forall q_h = \{q_{h0}, q_{hb}\} \in \Theta_{0,\tilde{D}}^h, \quad (3.5)$$

with

$$b(\mathbf{u}_h(0), q_{h0}) = 0, \quad \forall q_h = \{q_{h0}, q_{hb}\} \in \Theta_{0,\tilde{D}}^h, \quad (3.6)$$

where

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}_h) &= 2\mu \sum_{K \in \mathfrak{S}_h} (\varepsilon_w(\mathbf{u}_h), \varepsilon_w(\mathbf{v}_h))_K + \lambda \sum_{K \in \mathfrak{S}_h} (\nabla_w \cdot \mathbf{u}_h, \nabla_w \cdot \mathbf{v}_h)_K \\ &\quad + \mu \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle \mathbf{u}_{h0} - \mathbf{u}_{hb}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K}, \\ b(\mathbf{v}_h, p_{h0}) &= - \sum_{K \in \mathfrak{S}_h} (\nabla_w \cdot \mathbf{v}_h, p_{h0})_K, \\ d(p_h, q_h) &= \frac{\kappa}{\eta_{\mathbf{f}}} \sum_{K \in \mathfrak{S}_h} (\tilde{\nabla}_w p_h, \tilde{\nabla}_w q_h)_K \\ &\quad + \frac{\kappa}{\eta_{\mathbf{f}}} \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle Q_b p_{h0} - p_{hb}, Q_b q_{h0} - q_{hb} \rangle_{\partial K}. \end{aligned}$$

3.2.2 Fully discrete WG scheme

We consider a first-order backward Euler fully discrete approximation based on the semi-discrete WG scheme (3.4)-(3.6) for Problem $\mathbf{M}_t$.

Let N be a positive integer with the time step $\Delta t := \frac{T}{N}$. Let the pair $\{\mathbf{u}_h^m, p_h^m\}$ denote the approximation of $\{\mathbf{u}^m, p^m\} := \{\mathbf{u}(t_m), p(t_m)\}$ with the discrete time $t_m = m\Delta t$ for $m = 1, 2, \dots, N$. By replacing the time derivative $D_t \mathbf{u}_h$ at time t_m by the backward difference quotient

$$\tilde{\partial}_t \mathbf{u}_h^m := \frac{\mathbf{u}_h^m - \mathbf{u}_h^{m-1}}{\Delta t}$$

in (3.5), the backward Euler WG finite element scheme for Biot's consolidation problem is as follows.

Problem $\mathbf{DM}_t^h$. Find $(\mathbf{u}_h^m, p_h^m) = (\{\mathbf{u}_{h0}^m, \mathbf{u}_{hb}^m\}, \{p_{h0}^m, p_{hb}^m\}) \in V_{0,D}^h \times \Theta_{0,\tilde{D}}^h$ such that, for $m = 1, 2, \dots, N$,

$$a(\mathbf{u}_h^m, \mathbf{v}_h) + b(\mathbf{v}_h, p_{h0}^m) = (\mathbf{f}^m, \mathbf{v}_{h0}) + \langle \boldsymbol{\beta}^m, \mathbf{v}_{hb} \rangle_{\Gamma_N}, \quad \forall \mathbf{v}_h = \{\mathbf{v}_{h0}, \mathbf{v}_{hb}\} \in V_{0,D}^h, \quad (3.7)$$

$$-b(\tilde{\partial}_t \mathbf{u}_h^m, q_{h0}) + d(p_h^m, q_h) = (g^m, q_{h0}) + \langle \gamma^m, q_{hb} \rangle_{\Gamma_{\tilde{N}}}, \quad \forall q_h = \{q_{h0}, q_{hb}\} \in \Theta_{0,\tilde{D}}^h, \quad (3.8)$$

with

$$\nabla_w \cdot \mathbf{u}_h^0 = 0. \quad (3.9)$$

We note that this initial condition means $b(\mathbf{u}_h^0, q_{h0}) = 0$ for any q_{h0} .

To show this linear system admits a unique solution at each step m , we rewrite the equations (3.7)-(3.8) as follows:

$$a(\mathbf{u}_h^m, \mathbf{v}_h) + b(\mathbf{v}_h, p_{h0}^m) = (\mathbf{f}^m, \mathbf{v}_{h0}) + \langle \boldsymbol{\beta}^m, \mathbf{v}_{hb} \rangle_{\Gamma_N}, \quad (3.10)$$

$$\begin{aligned} -b(\mathbf{u}_h^m, q_{h0}) + \Delta t d(p_h^m, q_h) &= \Delta t (g^m, q_{h0}) + \Delta t \langle \gamma^m, q_{hb} \rangle_{\Gamma_{\tilde{N}}} \\ &\quad -b(\mathbf{u}_h^{m-1}, q_{h0}). \end{aligned} \quad (3.11)$$

We introduce the following triple-bar semi-norms:

$$|||\mathbf{v}|||_V^2 := 2\mu \sum_{K \in \mathfrak{S}_h} \|\varepsilon_w(\mathbf{v})\|_{0,K}^2 + \lambda \sum_{K \in \mathfrak{S}_h} \|\nabla_w \cdot \mathbf{v}\|_{0,K}^2 + \mu \sum_{K \in \mathfrak{S}_h} h_K^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial K}^2, \quad (3.12)$$

$$|||q|||_\Theta^2 := \frac{\kappa}{\eta_f} \sum_{K \in \mathfrak{S}_h} \|\tilde{\nabla}_w q\|_{0,K}^2 + \frac{\kappa}{\eta_f} \sum_{K \in \mathfrak{S}_h} h_K^{-1} \|Q_b q_0 - q_b\|_{\partial K}^2, \quad (3.13)$$

Similar to [30, 32], we have the following conclusion.

Lemma 3.1. *The semi-norms $|||\cdot|||_V$ and $|||\cdot|||_\Theta$ are norms on $V_{0,D}^h$ and $\Theta_{0,\tilde{D}}^h$, respectively.*

Proof. Since the proof for $|||\cdot|||_\Theta$ on $\Theta_{0,\tilde{D}}^h$ follows similarly, we only show $|||\cdot|||_V$ defined in (3.12) is a norm on $V_{0,D}^h$. It suffices to verify the positivity property of $|||\cdot|||_V$. To this end, assume that $|||\mathbf{v}|||_V = 0$ for some $\mathbf{v} \in V_{0,D}^h$. Then, from (3.12) it follows $\varepsilon_w(\mathbf{v}) = 0$, $\nabla_w \cdot \mathbf{v} = 0$ on each element $K \in \mathfrak{S}_h$, and $\mathbf{v}_0 = \mathbf{v}_b$ on ∂K . Using the definition of the weak gradient operator, we have

$$\begin{aligned} (\nabla_w \mathbf{v}, \boldsymbol{\tau})_K &= -(\mathbf{v}_0, \nabla \cdot \boldsymbol{\tau})_K + \langle \mathbf{v}_b, \boldsymbol{\tau} \mathbf{n} \rangle_{\partial K} \\ &= (\nabla \mathbf{v}_0, \boldsymbol{\tau})_K - \langle \mathbf{v}_0 - \mathbf{v}_b, \boldsymbol{\tau} \mathbf{n} \rangle_{\partial K} \\ &= (\nabla \mathbf{v}_0, \boldsymbol{\tau})_K, \quad \forall \boldsymbol{\tau} \in [\mathbb{P}_l(K)]^{d \times d}, \end{aligned}$$

and

$$\begin{aligned} ((\nabla_w \mathbf{v})^T, \boldsymbol{\tau})_K &= (\nabla_w \mathbf{v}, \boldsymbol{\tau}^T)_K \\ &= (\nabla \mathbf{v}_0, \boldsymbol{\tau}^T)_K = ((\nabla \mathbf{v}_0)^T, \boldsymbol{\tau})_K, \quad \forall \boldsymbol{\tau} \in [\mathbb{P}_l(K)]^{d \times d}. \end{aligned}$$

The above two equations yield $\varepsilon(\mathbf{v}_0) = 0$ on each element $K \in \mathfrak{S}_h$. Thus, $\mathbf{v}_0 \in RM(K)$ satisfies $\mathbf{v}_0|_e = \mathbf{v}_b$, where

$$RM(K) := \{\mathbf{a} + \eta \mathbf{x} : \mathbf{a} \in \mathbb{R}^d, \eta \in so(d)\}$$

with $\mathbf{x}$ denoting the position vector on K and $so(d)$ the space of skew-symmetric $d \times d$ matrices. This implies the continuity of $\mathbf{v}_0$ in the whole domain Ω . The boundary condition $\mathbf{v}_b = 0$ on Γ_D implies that $\mathbf{v}_0 = 0$ on Γ_D . From the second Korn's inequality, we have $\mathbf{v}_0 = 0$ in Ω . The fact that $\mathbf{v}_0 = \mathbf{v}_b$ on ∂K gives $\mathbf{v}_b = 0$ in Ω . As a result, we obtain $\mathbf{v} = \mathbf{0}$. This completes the proof. $\square$

From the definitions of the norms we easily get the following stability results for $a(\cdot, \cdot)$ and $d(\cdot, \cdot)$.

Lemma 3.2. *For any $\mathbf{u}, \mathbf{v} \in V_{0,D}^h$ and $p, q \in \Theta_{0,\bar{D}}^h$, we have*

$$\begin{aligned} |a(\mathbf{u}, \mathbf{v})| &\leq |||\mathbf{u}|||_V |||\mathbf{v}|||_V, \quad a(\mathbf{v}, \mathbf{v}) = |||\mathbf{v}|||_V^2, \\ |d(p, q)| &\leq |||p|||_\Theta |||q|||_\Theta, \quad d(q, q) = |||q|||_\Theta^2. \end{aligned}$$

In light of Lemmas 3.1-3.2, we can obtain the existence and uniqueness of the solution to the discrete system (3.10)-(3.11).

Theorem 3.1. *The fully discrete WG finite element scheme (3.10)-(3.11) admits a unique solution $(\mathbf{u}_h^m, p_h^m) \in V_{0,D}^h \times \Theta_{0,\bar{D}}^h$ for $m = 1, 2, \dots, N$.*

Proof. The scheme (3.10)-(3.11) is a linear system with the same number of equations and unknowns. So we only need to show that the conditions $\mathbf{u}_h^{m-1} = \mathbf{f}^m = \beta^m = \mathbf{0}$, $g^m = \gamma^m = 0$ indicate $\mathbf{u}_h^m = \mathbf{0}$, $p_h^m = 0$. Taking $\mathbf{v}_h = \mathbf{u}_h^m$, $q_h = p_h^m$ and adding the two equations (3.10) and (3.11), then we have

$$a(\mathbf{u}_h^m, \mathbf{u}_h^m) + \triangle t d(p_h^m, p_h^m) = 0,$$

which, together with Lemmas 3.1-3.2, implies $\mathbf{u}_h^m = \mathbf{0}$, $p_h^m = 0$. $\square$

4 Error estimates

To estimate the errors $\mathbf{u}^m - \mathbf{u}_h^m$ and $p^m - p_h^m$ of the proposed fully discrete WG scheme (3.10)-(3.11) for $m = 1, 2, \dots, N$, we need to introduce the following L^2 projection operators: for each element $K \in \mathfrak{S}_h$ and each edge/face $e \in \mathcal{E}_h$,

$$\begin{aligned} \mathbf{Q}_0 : \quad & [L^2(K)]^d \rightarrow [\mathbb{P}_l(K)]^d, \\ \mathbf{Q}_b : \quad & [L^2(e)]^d \rightarrow [\mathbb{P}_l(e)]^d, \\ Q_0 : \quad & L^2(K) \rightarrow \mathbb{P}_l(K), \\ \mathbb{Q}_0 : \quad & [L^2(K)]^{d \times d} \rightarrow [\mathbb{P}_l(e)]^{d \times d}, \\ \mathcal{Q}_0 : \quad & [L^2(K)]^d \rightarrow [\mathbb{P}_{l-1}(K)]^d. \end{aligned}$$

Recalling that Q_b is the projection from $L^2(e)$ onto $\mathbb{P}_{l-1}(e)$, we define

$$\mathbf{P}_w := \{\mathbf{Q}_0, \mathbf{Q}_b\}, \quad P_w := \{Q_0, Q_b\}.$$

The following commutative properties hold for the projection operators $\mathbf{P}_w$, P_w , $\mathbf{Q}_0$, Q_0 and $\mathbf{Q}_b$.

Lemma 4.1. *For any $\mathbf{v} \in [H^1(\Omega)]^d$ and $q \in H^1(\Omega)$, we have*

$$\nabla_w(\mathbf{P}_w \mathbf{v}) = \mathbf{Q}_0(\nabla \mathbf{v}), \quad (4.1)$$

$$\nabla_w \cdot (\mathbf{P}_w \mathbf{v}) = Q_0(\nabla \cdot \mathbf{v}), \quad (4.2)$$

$$\varepsilon_w(\mathbf{P}_w \mathbf{v}) = \mathbf{Q}_0(\varepsilon(\mathbf{v})), \quad (4.3)$$

$$\tilde{\nabla}_w(P_w q) = \mathbf{Q}_0(\nabla q). \quad (4.4)$$

Proof. The proofs of (4.1)-(4.3) can be found in [30]. We only prove (4.4). For any $q \in H^1(\Omega)$, $K \in \mathfrak{S}_h$, and $\varphi \in [\mathbb{P}_{l-1}(K)]^d$, by (3.3) and Green's formula we have

$$\begin{aligned} (\tilde{\nabla}_w(P_w q), \varphi)_K &= -(Q_0 q, \nabla \cdot \varphi)_K + \langle Q_b q, \varphi \cdot \mathbf{n} \rangle_{\partial K} \\ &= -(q, \nabla \cdot \varphi)_K + \langle q, \varphi \cdot \mathbf{n} \rangle_{\partial K} \\ &= (\nabla q, \varphi)_K = (\mathbf{Q}_0(\nabla q), \varphi)_K. \end{aligned}$$

Since $\tilde{\nabla}_w(P_w q), \mathbf{Q}_0(\nabla q) \in [\mathbb{P}_{l-1}(K)]^d$, it follows

$$\tilde{\nabla}_w(P_w q)|_K = \mathbf{Q}_0(\nabla q)|_K,$$

which implies (4.4). $\square$

Lemma 4.2. *Let $(\mathbf{u}(t), p(t)) \in [H^{1+s}(\Omega)]^d \times H^{1+s}(\Omega)$ satisfy (1.1)-(1.4) with $s > 0$. Then, for any $(\mathbf{v}_h, q_h) \in V_{0,D}^h \times \Theta_{0,\bar{D}}^h$, we have*

$$a(\mathbf{P}_w \mathbf{u}, \mathbf{v}_h) + b(\mathbf{v}_h, Q_0 p) = (\mathbf{f}, \mathbf{v}_{h0}) + \langle \boldsymbol{\beta}, \mathbf{v}_{hb} \rangle_{\Gamma_N} + E_{\mathbf{v}}(\mathbf{u}, \mathbf{v}_h), \quad (4.5)$$

$$-b(D_t \mathbf{P}_w \mathbf{u}, q_{h0}) + d(P_w p, q_h) = (g, q_{h0}) + \langle \gamma, q_{hb} \rangle_{\Gamma_{\bar{N}}} + E_q(p, q_h), \quad (4.6)$$

where

$$\begin{aligned} E_{\mathbf{v}}(\mathbf{u}, \mathbf{v}_h) &:= 2\mu \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathbf{Q}_0)(\varepsilon(\mathbf{u}))\mathbf{n}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K} \\ &\quad + \lambda \sum_{K \in \mathfrak{S}_h} \langle (I - Q_0)(\nabla \cdot \mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K} \\ &\quad + \mu \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle \mathbf{Q}_0 \mathbf{u} - \mathbf{Q}_b \mathbf{u}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K}, \\ &\quad - \sum_{K \in \mathfrak{S}_h} \langle (I - Q_0)p, (\mathbf{v}_{h0} - \mathbf{v}_{hb}) \cdot \mathbf{n} \rangle_{\partial K} \\ &:= \sum_{i=1}^4 E_i \end{aligned}$$

and

$$\begin{aligned}
 E_q(p, q_h) &:= \frac{\kappa}{\eta_{\mathbf{f}}} \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathcal{Q}_0) \nabla p \cdot \mathbf{n}, q_{h0} - q_{hb} \rangle_{\partial K} \\
 &\quad + \frac{\kappa}{\eta_{\mathbf{f}}} \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle Q_b Q_0 p - Q_b p, Q_b q_{h0} - q_{hb} \rangle_{\partial K} \\
 &:= \sum_{i=5}^6 E_i.
 \end{aligned}$$

Proof. For any $\mathbf{v}_h = \{\mathbf{v}_{h0}, \mathbf{v}_{hb}\} \in V_{0,D}^h$, by the definitions of $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ we have

$$\begin{aligned}
 &a(\mathbf{P}_w \mathbf{u}, \mathbf{v}_h) + b(\mathbf{v}_h, Q_0 p) \\
 &= 2\mu \sum_{K \in \mathfrak{S}_h} (\varepsilon_w(\mathbf{P}_w \mathbf{u}), \varepsilon_w(\mathbf{v}_h))_K + \lambda \sum_{K \in \mathfrak{S}_h} (\nabla_w \cdot \mathbf{P}_w \mathbf{u}, \nabla_w \cdot \mathbf{v}_h)_K \\
 &\quad + \mu \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle \mathbf{Q}_0 \mathbf{u} - \mathbf{Q}_b \mathbf{u}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K} \\
 &\quad - \sum_{K \in \mathfrak{S}_h} (\nabla_w \cdot \mathbf{v}_h, Q_0 p)_K. \tag{4.7}
 \end{aligned}$$

From (3.1), (4.3), the definition of projection $\mathbb{Q}_0$ and Green's formula, it follows

$$\begin{aligned}
 &\sum_{K \in \mathfrak{S}_h} (\varepsilon_w(\mathbf{P}_w \mathbf{u}), \varepsilon_w(\mathbf{v}_h))_K = \sum_{K \in \mathfrak{S}_h} (\mathbb{Q}_0(\varepsilon(\mathbf{u})), \varepsilon_w(\mathbf{v}_h))_K \\
 &= - \sum_{K \in \mathfrak{S}_h} (\nabla \cdot \mathbb{Q}_0(\varepsilon(\mathbf{u})), \mathbf{v}_{h0})_K + \sum_{K \in \mathfrak{S}_h} \langle \mathbb{Q}_0(\varepsilon(\mathbf{u})) \mathbf{n}, \mathbf{v}_{hb} \rangle_{\partial K} \\
 &= \sum_{K \in \mathfrak{S}_h} (\mathbb{Q}_0(\varepsilon(\mathbf{u})), \nabla \mathbf{v}_{h0})_K - \sum_{K \in \mathfrak{S}_h} \langle \mathbb{Q}_0(\varepsilon(\mathbf{u})) \mathbf{n}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K} \\
 &= \sum_{K \in \mathfrak{S}_h} (\varepsilon(\mathbf{u}), \nabla \mathbf{v}_{h0})_K - \sum_{K \in \mathfrak{S}_h} \langle \mathbb{Q}_0(\varepsilon(\mathbf{u})) \mathbf{n}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K}. \tag{4.8}
 \end{aligned}$$

Similarly, by (3.2) and (4.2) we obtain

$$\begin{aligned}
 &\sum_{K \in \mathfrak{S}_h} (\nabla_w \cdot \mathbf{P}_w \mathbf{u}, \nabla_w \cdot \mathbf{v}_h)_K = \sum_{K \in \mathfrak{S}_h} (Q_0(\nabla \cdot \mathbf{u}), \nabla_w \cdot \mathbf{v}_h)_K \\
 &= - \sum_{K \in \mathfrak{S}_h} (\nabla Q_0(\nabla \cdot \mathbf{u}), \mathbf{v}_{h0})_K + \sum_{K \in \mathfrak{S}_h} \langle Q_0(\nabla \cdot \mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_{hb} \rangle_{\partial K} \\
 &= \sum_{K \in \mathfrak{S}_h} (Q_0(\nabla \cdot \mathbf{u}), \nabla \cdot \mathbf{v}_{h0})_K - \sum_{K \in \mathfrak{S}_h} \langle Q_0(\nabla \cdot \mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K} \\
 &= \sum_{K \in \mathfrak{S}_h} (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}_{h0})_K - \sum_{K \in \mathfrak{S}_h} \langle Q_0(\nabla \cdot \mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K}. \tag{4.9}
 \end{aligned}$$

We also have

$$\begin{aligned}
 & - \sum_{K \in \mathfrak{S}_h} (\nabla_w \cdot \mathbf{v}_h, Q_0 p)_K \\
 &= \sum_{K \in \mathfrak{S}_h} (\mathbf{v}_{h0}, \nabla Q_0 p)_K - \sum_{K \in \mathfrak{S}_h} \langle \mathbf{v}_{hb} \cdot \mathbf{n}, Q_0 p \rangle_{\partial K} \\
 &= - \sum_{K \in \mathfrak{S}_h} (Q_0 p, \nabla \cdot \mathbf{v}_{h0})_K + \sum_{K \in \mathfrak{S}_h} \langle Q_0 p, (\mathbf{v}_{h0} - \mathbf{v}_{hb}) \cdot \mathbf{n} \rangle_{\partial K} \\
 &= - \sum_{K \in \mathfrak{S}_h} (p, \nabla \cdot \mathbf{v}_{h0})_K + \sum_{K \in \mathfrak{S}_h} \langle Q_0 p, (\mathbf{v}_{h0} - \mathbf{v}_{hb}) \cdot \mathbf{n} \rangle_{\partial K}. \quad (4.10)
 \end{aligned}$$

Testing the equation (4.1) with the component $\mathbf{v}_{h0}$ of $\mathbf{v}_h = \{\mathbf{v}_{h0}, \mathbf{v}_{hb}\} \in V_{0,D}^h$ yields

$$\begin{aligned}
 & 2\mu \sum_{K \in \mathfrak{S}_h} (\varepsilon(\mathbf{u}), \nabla \mathbf{v}_{h0})_K - 2\mu \sum_{K \in \mathfrak{S}_h} \langle \varepsilon(\mathbf{u}) \mathbf{n}, \mathbf{v}_{h0} \rangle_{\partial K} \\
 & + \lambda \sum_{K \in \mathfrak{S}_h} (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}_{h0})_K - \lambda \sum_{K \in \mathfrak{S}_h} \langle (\nabla \cdot \mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_{h0} \rangle_{\partial K} \\
 & - \sum_{K \in \mathfrak{S}_h} (p, \nabla \cdot \mathbf{v}_{h0})_K + \sum_{K \in \mathfrak{S}_h} \langle p, \mathbf{v}_{h0} \cdot \mathbf{n} \rangle_{\partial K} \\
 &= (\mathbf{f}, \mathbf{v}_{h0}).
 \end{aligned}$$

Thus, by the boundary condition (4.3) we have

$$\begin{aligned}
 & 2\mu \sum_{K \in \mathfrak{S}_h} (\varepsilon(\mathbf{u}), \nabla \mathbf{v}_{h0})_K + \lambda \sum_{K \in \mathfrak{S}_h} (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}_{h0})_K - \sum_{K \in \mathfrak{S}_h} (p, \nabla \cdot \mathbf{v}_{h0})_K \\
 &= (\mathbf{f}, \mathbf{v}_{h0}) + \langle \beta, \mathbf{v}_{hb} \rangle_{\Gamma_N} + 2\mu \sum_{K \in \mathfrak{S}_h} \langle \varepsilon(\mathbf{u}) \mathbf{n}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K} \\
 & + \lambda \sum_{K \in \mathfrak{S}_h} \langle (\nabla \cdot \mathbf{u}) \cdot \mathbf{n}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K} - \sum_{K \in \mathfrak{S}_h} \langle p, (\mathbf{v}_{h0} - \mathbf{v}_{hb}) \cdot \mathbf{n} \rangle_{\partial K}. \quad (4.11)
 \end{aligned}$$

As a result, substituting (4.8)-(4.10) into (4.7) and using (4.11), we arrive at the desired relation (4.5).

Now we turn to show (4.6). From the definitions of $b(\cdot, \cdot)$ and $d(\cdot, \cdot)$ it follows

$$\begin{aligned}
 & -b(D_t P_w \mathbf{u}, q_{h0}) + d(P_w p, q_h) \\
 &= -b(D_t \mathbf{P}_w \mathbf{u}, q_{h0}) + \frac{\kappa}{\eta_{\mathbf{f}}} (\tilde{\nabla}_w P_w p, \tilde{\nabla}_w q_h) \\
 & + \frac{\kappa}{\eta_{\mathbf{f}}} \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle Q_b Q_0 p - Q_b p, Q_b q_{h0} - q_{hb} \rangle_{\partial K}. \quad (4.12)
 \end{aligned}$$

Using the commutative property (4.2), we get

$$\begin{aligned}
 & -b(D_t \mathbf{P}_w \mathbf{u}, q_{h0}) = \sum_{K \in \mathfrak{S}_h} (\nabla_w \cdot D_t \mathbf{P}_w \mathbf{u}, q_{h0})_K \\
 &= \sum_{K \in \mathfrak{S}_h} (Q_0 (\nabla \cdot D_t \mathbf{u}), q_{h0})_K = \sum_{K \in \mathfrak{S}_h} (\nabla \cdot D_t \mathbf{u}, q_{h0})_K. \quad (4.13)
 \end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
 (\tilde{\nabla}_w P_w p, \tilde{\nabla}_w q_h) &= \sum_{K \in \mathfrak{S}_h} (\mathcal{Q}_0(\nabla p), \tilde{\nabla}_w q_h)_K \\
 &= - \sum_{K \in \mathfrak{S}_h} (\nabla \cdot (\mathcal{Q}_0(\nabla p)), q_{h0})_K + \sum_{K \in \mathfrak{S}_h} \langle \mathcal{Q}_0(\nabla p) \cdot \mathbf{n}, q_{hb} \rangle_{\partial K} \\
 &= \sum_{K \in \mathfrak{S}_h} (\mathcal{Q}_0(\nabla p), \nabla q_{h0})_K - \sum_{K \in \mathfrak{S}_h} \langle \mathcal{Q}_0(\nabla p) \cdot \mathbf{n}, q_{h0} - q_{hb} \rangle_{\partial K} \\
 &= \sum_{K \in \mathfrak{S}_h} (\nabla p, \nabla q_{h0})_K - \sum_{K \in \mathfrak{S}_h} \langle \mathcal{Q}_0(\nabla p) \cdot \mathbf{n}, q_{h0} - q_{hb} \rangle_{\partial K}. \quad (4.14)
 \end{aligned}$$

By testing the equation (1.2) with the component q_{h0} of $q_h = \{q_{h0}, q_{hb}\} \in \Theta_{0,\bar{D}}^h$ and using the boundary condition (1.4), we get

$$\begin{aligned}
 &\sum_{K \in \mathfrak{S}_h} (\nabla \cdot D_t \mathbf{u}, q_{h0})_K + \frac{\kappa}{\eta_{\mathbf{f}}} \sum_{K \in \mathfrak{S}_h} (\nabla p, \nabla q_{h0})_K \\
 &= (g, q_{h0}) + \langle \gamma, q_{hb} \rangle_{\Gamma_{\bar{N}}} + \sum_{K \in \mathfrak{S}_h} \langle \nabla p \cdot \mathbf{n}, q_{h0} - q_{hb} \rangle_{\partial K}.
 \end{aligned}$$

Substituting (4.13)-(4.14) into (4.12) and using the above inequality, we finally obtain the desired relation (4.6). $\square$

We have an inf-sup stability result for the bilinear form $b(\cdot, \cdot)$:

Lemma 4.3. *For any $q = \{q_0, q_b\} \in \Theta_{0,\bar{D}}^h$, it holds*

$$\sup_{\mathbf{0} \neq \mathbf{v} \in V_{0,\bar{D}}^h} \frac{-b(\mathbf{v}, q_0)}{\|\mathbf{v}\|_V} \gtrsim \|q_0\|_0. \quad (4.15)$$

Proof. For any $q = \{q_0, q_b\} \in \Theta_{0,\bar{D}}^h$, owing to a result by Nečas [24], there is $\tilde{\mathbf{v}} \in H_{0,\bar{D}}^1$, such that

$$\nabla \cdot \tilde{\mathbf{v}} = q_0, \quad \|\tilde{\mathbf{v}}\|_1 \lesssim \|q_0\|_0.$$

Taking $\mathbf{v} = \mathbf{P}_w \tilde{\mathbf{v}}$, from the definition of the discrete weak divergence (3.2) and the commutative property (4.2), we have

$$\begin{aligned}
 -b(\mathbf{v}, q_0) &= \sum_{K \in \mathfrak{S}_h} (\nabla_w \cdot (\mathbf{P}_w \tilde{\mathbf{v}}), q_0)_K = \sum_{K \in \mathfrak{S}_h} (\mathcal{Q}_0(\nabla \cdot \tilde{\mathbf{v}}), q_0)_K \\
 &= \sum_{K \in \mathfrak{S}_h} (\nabla \cdot \tilde{\mathbf{v}}, q_0)_K = \sum_{K \in \mathfrak{S}_h} \|q_0\|_{0,K}^2 = \|q_0\|_0^2. \quad (4.16)
 \end{aligned}$$

Next we show $\|\mathbf{v}\|_V \lesssim \|\tilde{\mathbf{v}}\|_1$. By the commutative properties (4.2)-(4.3) and the projection property, we have

$$\begin{aligned}
 \sum_{K \in \mathfrak{S}_h} \|\varepsilon_w(\mathbf{v})\|_{0,K}^2 &= \sum_{K \in \mathfrak{S}_h} \|\varepsilon_w(\mathbf{P}_w \tilde{\mathbf{v}})\|_{0,K}^2 = \sum_{K \in \mathfrak{S}_h} \|\mathbb{Q}_0(\varepsilon(\tilde{\mathbf{v}}))\|_{0,K}^2 \\
 &\lesssim \sum_{K \in \mathfrak{S}_h} \|\varepsilon(\tilde{\mathbf{v}})\|_{0,K}^2 \lesssim \sum_{K \in \mathfrak{S}_h} \|\nabla(\tilde{\mathbf{v}})\|_{0,K}^2,
 \end{aligned}$$

$$\begin{aligned} \sum_{K \in \mathfrak{S}_h} \|\nabla_w \cdot \mathbf{v}\|_{0,K}^2 &= \sum_{K \in \mathfrak{S}_h} \|\nabla_w \cdot (\mathbf{P}_w \tilde{\mathbf{v}})\|_{0,K}^2 = \sum_{K \in \mathfrak{S}_h} \|Q_0(\nabla \cdot \tilde{\mathbf{v}})\|_{0,K}^2 \\ &\lesssim \sum_{K \in \mathfrak{S}_h} \|\nabla \cdot \tilde{\mathbf{v}}\|_{0,K}^2 \lesssim \sum_{K \in \mathfrak{S}_h} \|\nabla \tilde{\mathbf{v}}\|_{0,K}^2. \end{aligned}$$

From the projection properties and the trace inequality (see for example in [3]), it follows

$$\begin{aligned} \sum_{K \in \mathfrak{S}_h} h_K^{-1} \|\mathbf{v}_0 - \mathbf{v}_b\|_{\partial K}^2 &= \sum_{K \in \mathfrak{S}_h} h_K^{-1} \|\mathbf{Q}_b \mathbf{Q}_0 \tilde{\mathbf{v}} - \mathbf{Q}_b \tilde{\mathbf{v}}\|_{\partial K}^2 \\ &\lesssim \sum_{K \in \mathfrak{S}_h} h_K^{-1} \|\mathbf{Q}_0 \tilde{\mathbf{v}} - \tilde{\mathbf{v}}\|_{\partial K}^2 \\ &\lesssim \sum_{K \in \mathfrak{S}_h} h_K^{-1} (h_K^{-1} \|\mathbf{Q}_0 \tilde{\mathbf{v}} - \tilde{\mathbf{v}}\|_{0,K}^2 + h_K \|\nabla(\mathbf{Q}_0 \tilde{\mathbf{v}} - \tilde{\mathbf{v}})\|_{0,K}^2) \\ &\lesssim \sum_{K \in \mathfrak{S}_h} \|\nabla \tilde{\mathbf{v}}\|_{0,K}^2. \end{aligned}$$

By the above three inequalities and $\|\tilde{\mathbf{v}}\|_1 \lesssim \|q_0\|_0$, we get

$$\|\mathbf{v}\|_V^2 \lesssim \sum_{K \in \mathfrak{S}_h} \|\nabla \tilde{\mathbf{v}}\|_{0,K}^2 \lesssim \|q_0\|_0^2. \quad (4.17)$$

Combining (4.16) and (4.17) leads to the desired inequality (4.15). This completes the proof. $\square$

Lemma 4.4. *For any $q = \{q_0, q_b\} \in \Theta^h$, we have*

$$\sum_{K \in \mathfrak{S}_h} \|\nabla q_0\|_{0,K}^2 \lesssim \|q\|_{\Theta}^2. \quad (4.18)$$

Proof. For any $q = \{q_0, q_b\} \in \Theta^h$, from the integration by parts and the definition of weak gradient and Q_b , it follows

$$\begin{aligned} (\nabla q_0, \nabla q_0)_K &= -(q_0, \nabla \cdot \nabla q_0)_K + \langle q_0, \nabla q_0 \cdot \mathbf{n} \rangle_{\partial K} \\ &= -(q_0, \nabla \cdot \nabla q_0)_K + \langle q_b, \nabla q_0 \cdot \mathbf{n} \rangle_{\partial K} + \langle q_0 - q_b, \nabla q_0 \cdot \mathbf{n} \rangle_{\partial K} \\ &= (\tilde{\nabla}_w q, \nabla q_0)_K + \langle Q_b q_0 - q_b, \nabla q_0 \cdot \mathbf{n} \rangle_{\partial K}, \end{aligned}$$

which, together with the trace inequality and the inverse inequality, yields

$$\|\nabla q_0\|_{0,K}^2 \lesssim \|\tilde{\nabla}_w q\|_{0,K} \|\nabla q_0\|_{0,K} + h_K^{-\frac{1}{2}} \|Q_b q_0 - q_b\|_{0,\partial K} \|\nabla q_0\|_{0,K}.$$

Thus,

$$\|\nabla q_0\|_{0,K} \lesssim \|\tilde{\nabla}_w q\|_{0,K} + h_K^{-\frac{1}{2}} \|Q_b q_0 - q_b\|_{0,\partial K}.$$

Therefore, we have

$$\|\nabla q_0\|_{0,K}^2 \lesssim \|\tilde{\nabla}_w q\|_{0,K}^2 + h_K^{-1} \|Q_b q_0 - q_b\|_{0,\partial K}^2,$$

which gives rise to (4.18) after a summation over all $K \in \mathfrak{S}_h$. $\square$

We introduce a space Θ^{h0} for the first components of $q_h = \{q_{h0}, q_{hb}\} \in \Theta_{0,\bar{D}}^h$, i.e.

$$\Theta^{h0} := \{q_{h0} \mid q_{h0}|_K \in \mathbb{P}_l(K), \forall K \in \mathfrak{S}_h\}.$$

From [4], the following result holds:

Lemma 4.5. *There exists an interpolation $I_h^2 : \Theta^{h0} \rightarrow \Theta^{h0} \cap H_{0,\Gamma_{\bar{D}}}^1(\Omega)$ such that, for any $q_{h0} \in \Theta^{h0}$, $I_h^2 q_{h0} \in \Theta^{h0} \cap H_{0,\Gamma_{\bar{D}}}^1(\Omega)$ and*

$$\|q_{h0} - I_h^2 q_{h0}\|_0^2 \lesssim \sum_{e \in \mathcal{E}_h} h_e \|[q_{h0}]\|_{0,e}^2. \quad (4.19)$$

Lemma 4.6. *For any $q_h = \{q_{h0}, q_{hb}\} \in \Theta_{0,\bar{D}}^h$, we have*

$$\|q_{h0}\|_0 \lesssim \|q_h\|_{\Theta}. \quad (4.20)$$

Proof. Using (4.18), the projection property, the trace inequality, and the inverse inequality, we easily obtain, for any $q_h = \{q_{h0}, q_{hb}\} \in \Theta_{0,\bar{D}}^h$,

$$\begin{aligned} & \sum_{K \in \mathfrak{S}_h} \|q_{h0} - q_{hb}\|_{0,\partial K}^2 \\ & \lesssim \sum_{K \in \mathfrak{S}_h} (\|q_{h0} - Q_b q_{h0}\|_{0,\partial K}^2 + \|Q_b q_{h0} - q_{hb}\|_{0,\partial K}^2) \\ & \lesssim \sum_{K \in \mathfrak{S}_h} h_K (\|\nabla q_{h0}\|_{0,K}^2 + h_K^{-1} \|Q_b q_{h0} - q_{hb}\|_{0,\partial K}^2) \\ & \lesssim h \|q_h\|_{\Theta}^2. \end{aligned}$$

This inequality gives

$$\begin{aligned} \sum_{e \in \mathcal{E}_h} h_e \|[q_{h0}]\|_{0,e}^2 &= \sum_{e \in \mathcal{E}_h} h_e \|q_{h0} - q_{hb}\|_{0,e}^2 \\ &\lesssim h^2 \|q_h\|_{\Theta}^2. \end{aligned} \quad (4.21)$$

Since $I_h^2 q_{h0} \in \Theta^{h0} \cap H_{0,\bar{D}}^1(\Omega)$, from (4.18) and Poincaré's inequality it follows

$$\begin{aligned} \|I_h^2 q_{h0}\|_0^2 &\lesssim \|\nabla I_h^2 q_{h0}\|_0^2 \\ &\lesssim \sum_{K \in \mathfrak{S}_h} \|\nabla(q_{h0} - I_h^2 q_{h0})\|_{0,K}^2 + \sum_{K \in \mathfrak{S}_h} \|\nabla q_{h0}\|_{0,K}^2 \\ &\lesssim \|q_h\|_{\Theta}^2. \end{aligned}$$

Then the above inequality, (4.19) and (4.21) yield

$$\|q_{h0}\|_0^2 \lesssim \|q_{h0} - I_h^2 q_{h0}\|_0^2 + \|I_h^2 q_{h0}\|_0^2 \lesssim \|q_h\|_{\Theta}^2.$$

This completes the proof. $\square$

Lemma 4.7. *Assume that $\mathbf{u} \in [H^{l+1}(\Omega)]^d$ and $p \in H^{l+1}(\Omega)$. Then for any $(\mathbf{v}_h, q_h) \in V_{0,D}^h \times \Theta_{0,\bar{D}}^h$ and $m = 1, 2, \dots, N$, we have the estimate*

$$\begin{aligned} & |E_{\mathbf{v}}(\mathbf{u}^m, \mathbf{v}_h) + E_q(p^m, q_h)| \\ & \lesssim h^l (\|\mathbf{u}^m\|_{l+1} \|\mathbf{v}_h\|_V + \|p^m\|_{l+1} \|q_h\|_{\Theta} + \|p^m\|_{l+1} \|\mathbf{v}_h\|_V). \end{aligned} \quad (4.22)$$

Proof. Recall that

$$E_{\mathbf{v}}(\mathbf{u}^m, \mathbf{v}_h) := \sum_{i=1}^4 E_i, \quad E_q(p^m, q_h) := \sum_{i=5}^6 E_i.$$

Using the trace inequality and the project property, we get

$$\begin{aligned} |E_1| &= |2\mu \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathbb{Q}_0)(\varepsilon(\mathbf{u}^m))\mathbf{n}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K}| \\ &\lesssim \left(\sum_{K \in \mathfrak{S}_h} h_K \|(\mathbf{I} - \mathbb{Q}_0)(\varepsilon(\mathbf{u}^m))\mathbf{n}\|_{0,\partial K}^2 \right)^{1/2} \\ &\quad \left(\sum_{K \in \mathfrak{S}_h} h_K^{-1} \|\mathbf{v}_{h0} - \mathbf{v}_{hb}\|_{0,\partial K}^2 \right)^{1/2} \\ &\lesssim h^l \|\mathbf{u}^m\|_{l+1} \|\mathbf{v}_h\|_V \end{aligned} \quad (4.23)$$

and

$$\begin{aligned} |E_2| &= |\lambda \sum_{K \in \mathfrak{S}_h} \langle (I - Q_0)(\nabla \cdot \mathbf{u}^m) \cdot \mathbf{n}, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K}| \\ &\lesssim \left(\sum_{K \in \mathfrak{S}_h} h_K \|(I - Q_0)(\nabla \cdot \mathbf{u}^m) \cdot \mathbf{n}\|_{0,\partial K}^2 \right)^{1/2} \\ &\quad \left(\sum_{K \in \mathfrak{S}_h} h_K^{-1} \|\mathbf{v}_{h0} - \mathbf{v}_{hb}\|_{0,\partial K}^2 \right)^{1/2} \\ &\lesssim h^l \|\mathbf{u}^m\|_{l+1} \|\mathbf{v}_h\|_V. \end{aligned} \quad (4.24)$$

We also have

$$\begin{aligned} |E_3| &= |\mu \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle \mathbf{Q}_0 \mathbf{u}^m - \mathbf{Q}_b \mathbf{u}^m, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K}| \\ &= |\mu \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle \mathbf{Q}_b(\mathbf{Q}_0 \mathbf{u}^m - \mathbf{u}^m), \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K}| \\ &= |\mu \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle \mathbf{Q}_0 \mathbf{u}^m - \mathbf{u}^m, \mathbf{v}_{h0} - \mathbf{v}_{hb} \rangle_{\partial K}| \\ &\lesssim \left(\sum_{K \in \mathfrak{S}_h} (h_K^{-2} \|\mathbf{Q}_0 \mathbf{u}^m - \mathbf{u}^m\|_{0,K}^2 + \|\nabla(\mathbf{Q}_0 \mathbf{u}^m - \mathbf{u}^m)\|_{0,K}^2) \right)^{1/2} \\ &\quad \left(\sum_{K \in \mathfrak{S}_h} h_K^{-1} \|\mathbf{v}_{h0} - \mathbf{v}_{hb}\|_{0,\partial K}^2 \right)^{1/2} \\ &\lesssim h^l \|\mathbf{u}^m\|_{l+1} \|\mathbf{v}_h\|_V, \end{aligned} \quad (4.25)$$

and

$$\begin{aligned}
 |E_4| &= \left| - \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{v}_{h0} - \mathbf{v}_{hb}) \cdot \mathbf{n}, (I - Q_0)p^m \rangle_{\partial K} \right| \\
 &\lesssim \left(\sum_{K \in \mathfrak{S}_h} h_K \|(I - Q_0)p^m\|_{0,\partial K}^2 \right)^{1/2} \\
 &\quad \left(\sum_{K \in \mathfrak{S}_h} h_K^{-1} \|\mathbf{v}_{h0} - \mathbf{v}_{hb}\|_{0,\partial K}^2 \right)^{1/2} \\
 &\lesssim h^l \|p^m\|_{l+1} \|\mathbf{v}_h\|_V.
 \end{aligned} \tag{4.26}$$

For the term E_5 , we have

$$\begin{aligned}
 |E_5| &= \left| \frac{\kappa}{\eta_f} \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}, q_{h0} - q_{hb} \rangle_{\partial K} \right| \\
 &\lesssim \left| \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}, q_{h0} - Q_b q_{h0} \rangle_{\partial K} \right| \\
 &\quad + \left| \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}, Q_b q_{h0} - q_{hb} \rangle_{\partial K} \right|.
 \end{aligned}$$

For the first term on the right-hand side of the above inequality, we use the trace inequality, (4.18), and the projection property to obtain

$$\begin{aligned}
 &\left| \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}, q_{h0} - Q_b q_{h0} \rangle_{\partial K} \right| \\
 &\lesssim \sum_{K \in \mathfrak{S}_h} h_K \|(\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}\|_{0,\partial K} \|\nabla q_{h0}\|_{0,\partial K} \\
 &\lesssim \left(\sum_{K \in \mathfrak{S}_h} h_K \|(\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}\|_{0,\partial K}^2 \right)^{1/2} \left(\sum_{K \in \mathfrak{S}_h} h_K \|\nabla q_{h0}\|_{0,\partial K}^2 \right)^{1/2} \\
 &\lesssim \left(\sum_{K \in \mathfrak{S}_h} h_K \|(\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}\|_{0,\partial K}^2 \right)^{1/2} \left(\sum_{K \in \mathfrak{S}_h} \|\nabla q_{h0}\|_{0,K}^2 \right)^{1/2} \\
 &\lesssim h^l \|p^m\|_{l+1} \|q_h\|_{\Theta}.
 \end{aligned}$$

Similarly, for the second term we have

$$\begin{aligned}
 &\left| \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}, Q_b q_{h0} - q_{hb} \rangle_{\partial K} \right| \\
 &\lesssim \left(\sum_{K \in \mathfrak{S}_h} h_K \|(\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}\|_{0,\partial K}^2 \right)^{1/2} \left(\sum_{K \in \mathfrak{S}_h} h_K^{-1} \|Q_b q_{h0} - q_{hb}\|_{0,\partial K}^2 \right)^{1/2} \\
 &\lesssim h^l \|p^m\|_{l+1} \|q_h\|_{\Theta}.
 \end{aligned}$$

Combining the above three estimates, we get

$$|E_5| \lesssim h^l \|p^m\|_{l+1} \|q_h\|_{\Theta}. \tag{4.27}$$

For the term E_6 , from the trace inequality, the inverse inequality, and the project

property, it follows

$$\begin{aligned}
 |E_6| &= \left| \frac{\kappa}{\eta_{\mathbf{f}}} \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle Q_b Q_0 p^m - Q_b p^m, Q_b q_{h0} - q_{hb} \rangle_{\partial K} \right| \\
 &\lesssim \left(\sum_{K \in \mathfrak{S}_h} h_K^{-1} \|Q_b Q_0 p^m - Q_b p^m\|_{0, \partial K}^2 \right)^{1/2} \\
 &\quad \left(\sum_{K \in \mathfrak{S}_h} h_K^{-1} \|Q_b q_{h0} - q_{hb}\|_{0, \partial K}^2 \right)^{1/2} \\
 &\lesssim h^l \|p^m\|_{l+1} \|q_h\|_{\Theta}.
 \end{aligned} \tag{4.28}$$

The inequalities (4.23)-(4.28) imply the desired result (4.22). $\square$

To derive the error estimation of the fully discrete scheme (3.10)-(3.11), following the standard technique by Thomée [29] for the error analysis of time-dependent problems, we introduce the elliptic projections $\bar{\mathbf{u}}_h \in V_{0,D}^h$ and $\bar{p}_h \in \Theta_{0,\bar{D}}^h$ defined as follows:

$$a(\bar{\mathbf{u}}_h, \mathbf{v}_h) + b(\mathbf{v}_h, \bar{p}_{h0}) = a(\mathbf{P}_w \mathbf{u}, \mathbf{v}_h) + b(\mathbf{v}_h, P_w p) - E_{\mathbf{v}}(\mathbf{u}, \mathbf{v}_h), \tag{4.29}$$

$$d(\bar{p}_h, q_h) = d(P_w p, q_h) - E_q(p, q_h), \tag{4.30}$$

where the terms $E_{\mathbf{v}}(\mathbf{u}, \mathbf{v}_h)$ and $E_q(p, q_h)$ are defined in Lemma 4.2.

Note that the above projections are actually decoupled. In fact, $\bar{p}_h$ is determined by (4.30), and then $\bar{\mathbf{u}}_h$ is determined by $\bar{p}_h$ and (4.29).

We split the errors between the L^2 projections $\mathbf{P}_w \mathbf{u}, P_w p$ and the WG finite element solutions $\mathbf{u}_h^m, p_h^m$ at $t = t_m$ as

$$\mathbf{P}_w \mathbf{u}^m - \mathbf{u}_h^m = (\mathbf{P}_w \mathbf{u}^m - \bar{\mathbf{u}}_h^m) + (\bar{\mathbf{u}}_h^m - \mathbf{u}_h^m) =: \rho_{\mathbf{u}}^m + \xi_{\mathbf{u}}^m, \tag{4.31}$$

$$P_w p^m - p_h^m = (P_w p^m - \bar{p}_h^m) + (\bar{p}_h^m - p_h^m) =: \rho_p^m + \xi_p^m. \tag{4.32}$$

Lemmas 4.8-4.9 are to estimate the four terms $\rho_{\mathbf{u}}^m, \rho_p^m, \xi_{\mathbf{u}}^m, \xi_p^m$.

Lemma 4.8. For $\mathbf{u} \in [H^{l+1}(\Omega)]^d \cap \mathbf{H}_{0,\Gamma_D}^1(\Omega)$ and $p \in H^{l+1}(\Omega) \cap H_{0,\Gamma_{\bar{D}}}^1(\Omega)$ and $m = 1, 2, \dots, N$, we have

$$\|\rho_{\mathbf{u}}^m\|_V \lesssim h^l (\|\mathbf{u}^m\|_{l+1} + \|p^m\|_{l+1}), \tag{4.33}$$

$$\|\rho_p^m\|_{\Theta} \lesssim h^l \|p^m\|_{l+1}, \tag{4.34}$$

$$\|D_t \rho_{\mathbf{u}}^m\|_V \lesssim h^l (\|D_t \mathbf{u}^m\|_{l+1} + \|D_t p^m\|_{l+1}). \tag{4.35}$$

Proof. Using (4.27), (4.28) and (4.30), we get

$$|d(\rho_p^m, q_h)| = |d(P_w p^m - \bar{p}_h^m, q_h)| = |E_5 + E_6| \lesssim h^l \|p^m\|_{l+1} \|q_h\|_{\Theta},$$

which gives the desired result (4.34) by letting $q_h = \rho_p^m$. The above inequality, together with (4.23)-(4.26), (4.29) and (4.20), yields

$$\begin{aligned}
 |a(\rho_{\mathbf{u}}^m, \mathbf{v}_h)| &= |-b(\mathbf{v}_h, \rho_p^m) + E_{\mathbf{v}}(\mathbf{u}^m, \mathbf{v}_h)| \\
 &\lesssim \|\mathbf{v}_h\|_V \|\rho_p^m\|_0 + h^l (\|\mathbf{u}^m\|_{l+1} + \|p^m\|_{l+1}) \|\mathbf{v}_h\|_V \\
 &\lesssim \|\mathbf{v}_h\|_V \|\rho_p^m\|_{\Theta} + h^l (\|\mathbf{u}^m\|_{l+1} + \|p^m\|_{l+1}) \|\mathbf{v}_h\|_V.
 \end{aligned}$$

This implies the desired result (4.33) by letting $\mathbf{v}_h = \rho_{\mathbf{u}}^m$.

Similarly, we easily obtain the estimate (4.35) by differentiating the equations (4.29)-(4.30) with respect to t . This completes the proof. $\square$

We follow the idea in [28] to estimate the terms $\xi_{\mathbf{u}}^m$ and $\xi_p^m := \{\xi_{p0}^m, \xi_{pb}^m\}$.

Lemma 4.9. *For $m = 1, 2, \dots, N$, we have*

$$|||\xi_{\mathbf{u}}^m|||_V + \sqrt{\Delta t} |||\xi_p^m|||_{\Theta} + |||\xi_{p0}^m|||_0 \lesssim |||\xi_{\mathbf{u}}^0|||_V + \Delta t \sum_{j=1}^m |||R_{\mathbf{u}}^j|||_V, \quad (4.36)$$

where $R_{\mathbf{u}}^j := D_t \mathbf{P}_w \mathbf{u}^m - \tilde{\partial}_t \tilde{\mathbf{u}}_h^j$.

Proof. Subtracting (4.29) and (4.30) from (3.7) and (3.8), respectively, using (4.5) and (4.6) we have,

$$a(\xi_{\mathbf{u}}^m, \mathbf{v}_h) + b(\mathbf{v}_h, \xi_{p0}^m) = 0, \quad (4.37)$$

$$-b(\tilde{\partial}_t \xi_{\mathbf{u}}^m, q_h) + d(\xi_p^m, q_h) = -b(R_{\mathbf{u}}^m, q_h). \quad (4.38)$$

Taking $\mathbf{v}_h = \tilde{\partial}_t \xi_{\mathbf{u}}^m$, $q_h = \xi_p^m$ in (4.37) and (4.38), respectively, we obtain

$$\begin{aligned} |||\xi_{\mathbf{u}}^m|||_V^2 + \Delta t |||\xi_p^m|||_{\Theta}^2 &= a(\xi_{\mathbf{u}}^m, \xi_{\mathbf{u}}^{m-1}) + \Delta t b(R_{\mathbf{u}}^m, \xi_{p0}^m) \\ &\lesssim |||\xi_{\mathbf{u}}^m|||_V |||\xi_{\mathbf{u}}^{m-1}|||_V + \Delta t |||R_{\mathbf{u}}^m|||_V |||\xi_{p0}^m|||_0. \end{aligned}$$

Thanks to Lemma 4.3 and (4.37), we get

$$|||\xi_{p0}^m|||_0 \lesssim \sup_{\mathbf{v}_h \in V_{0,D}^h, \mathbf{v}_h \neq \mathbf{0}} \frac{b(\mathbf{v}_h, \xi_{p0}^m)}{|||\mathbf{v}_h|||_V} \lesssim \sup_{\mathbf{v}_h \in V_{0,D}^h, \mathbf{v}_h \neq \mathbf{0}} \frac{a(\xi_{\mathbf{u}}^m, \mathbf{v}_h)}{|||\mathbf{v}_h|||_V} \lesssim |||\xi_{\mathbf{u}}^m|||_V.$$

The above two inequalities imply

$$|||\xi_{\mathbf{u}}^m|||_V + \sqrt{\Delta t} |||\xi_p^m|||_{\Theta} \lesssim |||\xi_{\mathbf{u}}^{m-1}|||_V + \Delta t |||R_{\mathbf{u}}^m|||_V, \quad (4.39)$$

which yields the desired result (4.36). $\square$

Next lemma gives the estimation of $|||R_{\mathbf{u}}^j|||_V$.

Lemma 4.10. *Assume that $D_{tt} \mathbf{u}(t) \in L^1(0, T; \mathbf{H}_{0,\Gamma_D}^1(\Omega))$, we have*

$$\sum_{j=1}^m |||R_{\mathbf{u}}^j|||_V \lesssim \int_0^{t_m} \|D_{tt} \mathbf{u}\|_1 dt + \frac{1}{\Delta t} \int_0^{t_m} \|D_t \rho_{\mathbf{u}}\|_V dt. \quad (4.40)$$

Proof. Based on the definition of $R_{\mathbf{u}}^m$, we have

$$R_{\mathbf{u}}^m = (D_t \mathbf{P}_w \mathbf{u}^m - \tilde{\partial}_t \mathbf{P}_w \mathbf{u}^m) + (\tilde{\partial}_t \mathbf{P}_w \mathbf{u}^m - \tilde{\partial}_t \tilde{\mathbf{u}}_h^m).$$

From Taylor series expansion with the remainder in terms of the integral form

$$f(b) = f(a) + f'(a)(b-a) + \frac{1}{2} \int_a^b f''(x)(b-x) dx,$$

it follows

$$D_t \mathbf{P}_w \mathbf{u}^m - \tilde{\partial}_t \mathbf{P}_w \mathbf{u}^m = \frac{1}{2\Delta t} \int_{t_{m-1}}^{t_m} D_{tt} \mathbf{P}_w \mathbf{u}(\cdot, t)(t_{m-1} - t) dt.$$

Therefore, it holds

$$\begin{aligned}
 & \|D_t \mathbf{P}_w \mathbf{u}^m - \tilde{\partial}_t \mathbf{P}_w \mathbf{u}^m\|_V = \frac{1}{2\Delta t} \left\| \int_{t_{m-1}}^{t_m} D_{tt} \mathbf{P}_w \mathbf{u}(\cdot, t)(t_{m-1} - t) dt \right\|_V \\
 & \lesssim \frac{1}{\Delta t} \int_{t_{m-1}}^{t_m} \|D_{tt} \mathbf{P}_w \mathbf{u}(\cdot, t)(t_{m-1} - t)\|_V dt \\
 & \lesssim \int_{t_{m-1}}^{t_m} \|D_{tt} \mathbf{P}_w \mathbf{u}\|_V dt \lesssim \int_{t_{m-1}}^{t_m} \|D_{tt} \mathbf{u}\|_1 dt.
 \end{aligned} \tag{4.41}$$

Recalling that $\rho_{\mathbf{u}}^m = \mathbf{P}_w \mathbf{u}^m - \bar{\mathbf{u}}_h^m$, we obtain

$$\begin{aligned}
 & \|\tilde{\partial}_t \mathbf{P}_w \mathbf{u}^m - \tilde{\partial}_t \bar{\mathbf{u}}^m\|_V = \left\| \frac{\rho_{\mathbf{u}}^m - \rho_{\mathbf{u}}^{m-1}}{\Delta t} \right\|_V \\
 & = \frac{1}{\Delta t} \left\| \int_{t_{m-1}}^{t_m} D_t \rho_{\mathbf{u}}(t) dt \right\|_V \lesssim \frac{1}{\Delta t} \int_{t_{m-1}}^{t_m} \|D_t \rho_{\mathbf{u}}\|_V dt.
 \end{aligned} \tag{4.42}$$

Finally, combining (4.41) and (4.42), we have

$$\|R_{\mathbf{u}}^m\|_V \lesssim \int_{t_{m-1}}^{t_m} \|D_{tt} \mathbf{u}\|_1 dt + \frac{1}{\Delta t} \int_{t_{m-1}}^{t_m} \|D_t \rho_{\mathbf{u}}\|_V dt.$$

This yields (4.40) immediately. $\square$

In view of Lemmas 4.9-4.10, we can obtain the following main conclusion.

Theorem 4.1. *Assume that*

$$\begin{aligned}
 & \mathbf{u}(t) \in L^\infty(0, T; [H^{l+1}(\Omega)]^d \cap \mathbf{H}_{0, \Gamma_D}^1(\Omega)), \\
 & D_t \mathbf{u}(t) \in L^1(0, T; [H^{l+1}(\Omega)]^d), \quad D_{tt} \mathbf{u}(t) \in L^1(0, T; \mathbf{H}_{0, \Gamma_D}^1(\Omega)), \\
 & p(t) \in L^\infty(0, T; H^{l+1}(\Omega) \cap H_{0, \Gamma_D}^1(\Omega)), \\
 & D_t p(t) \in L^1(0, T; H^{l+1}(\Omega)).
 \end{aligned}$$

Then, for $m = 1, 2, \dots, N$, we have the error estimate

$$\begin{aligned}
 & \|\mathbf{u}^m - \mathbf{u}_h^m\|_V + \sqrt{\Delta t} \|p^m - p_h^m\|_\Theta \\
 & \lesssim \|\xi_{\mathbf{u}}^0\|_V + \Delta t \int_0^{t_m} \|D_{tt} \mathbf{u}\|_1 dt \\
 & \quad + h^l \left(\|\mathbf{u}^m\|_{l+1} + \|p^m\|_{l+1} + \int_0^{t_m} (\|D_t \mathbf{u}^m\|_{l+1} + \|D_t p^m\|_{l+1}) dt \right)
 \end{aligned} \tag{4.43}$$

Moreover, for the pore pressure, we have

$$\begin{aligned}
 & \|p^m - p_{h0}^m\|_0 \\
 & \lesssim \|\xi_{\mathbf{u}}^0\|_V + \Delta t \int_0^{t_m} \|D_{tt} \mathbf{u}\|_1 dt \\
 & \quad + h^{l+1} \|p^m\|_{l+1} + h^l \left[\int_0^{t_m} (\|D_t \mathbf{u}^m\|_{l+1} + \|D_t p^m\|_{l+1}) dt \right].
 \end{aligned} \tag{4.44}$$

Proof. Using (4.31), (4.33)-(4.36), (4.40) and the triangle inequality, we have

$$\begin{aligned} & |||\mathbf{P}_w \mathbf{u}^m - \mathbf{u}_h^m|||_V + \sqrt{\Delta t} |||\mathbf{P}_w p^m - p_h^m|||_\Theta \\ & \lesssim |||\xi_{\mathbf{u}}^0|||_V + \Delta t \int_0^{t_m} \|D_{tt} \mathbf{u}\|_1 dt \\ & \quad + h^l \left(\|\mathbf{u}^m\|_{l+1} + \|p^m\|_{l+1} + \sqrt{\Delta t} \|p^m\|_{l+1} \right. \\ & \quad \left. + \int_0^{t_m} (\|D_t \mathbf{u}^m\|_{l+1} + \|D_t p^m\|_{l+1} dt) \right), \end{aligned}$$

which, together with standard estimates for the L^2 projections, implies the result (4.43).

To derive the L^2 error estimate (4.44), we shall perform a duality argument. Recalling that $\rho_p^m = P_w p^m - \bar{p}_h^m =: \{\rho_{p0}^m, \rho_{pb}^m\}$, we consider the following auxiliary problem: find $\Phi \in H_0^1(\Omega)$ such that

$$-\frac{\kappa}{\eta_f} \Delta \Phi = \rho_{p0}^m \quad \text{in } \Omega. \quad (4.45)$$

Since Ω is a convex polyhedral domain, we have

$$\|\Phi\|_2 \lesssim \|\rho_{p0}^m\|_0. \quad (4.46)$$

From (4.45), (1.2), (4.30), and the definition of $E_q(p^m, q_h)$, it follows

$$\begin{aligned} \|\rho_{p0}^m\|_0^2 &= d(P_w \Phi, \rho_p^m) - E_q(\Phi, \rho_p^m) \\ &= E_q(p^m, P_w \Phi) - E_q(\Phi, \rho_p^m) \\ &= \frac{\kappa}{\eta_f} \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}, Q_0 \Phi - Q_b \Phi \rangle_{\partial K} \\ & \quad + \frac{\kappa}{\eta_f} \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle Q_b Q_0 p^m - Q_b p^m, Q_b Q_0 \Phi - Q_b \Phi \rangle_{\partial K} \\ & \quad - \frac{\kappa}{\eta_f} \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle Q_b \rho_{p0}^m - \rho_{pb}^m, Q_b Q_0 \Phi - Q_b \Phi \rangle_{\partial K} \\ & \quad - \frac{\kappa}{\eta_f} \sum_{K \in \mathfrak{S}_h} \langle (Q_0(\nabla \Phi) - \nabla \Phi) \cdot \mathbf{n}, \rho_{p0}^m - \rho_{pb}^m \rangle_{\partial K} \\ &=: \tilde{E}_1 + \tilde{E}_2 + \tilde{E}_3 + \tilde{E}_4. \end{aligned} \quad (4.47)$$

Next Let us bound the four terms $\tilde{E}_i$ ($i = 1, 2, \dots, 4$). In view of the trace inequality and the projection properties, it is easy to see

$$\begin{aligned} |\tilde{E}_1| &= \left| \frac{\kappa}{\eta_f} \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}, Q_0 \Phi - Q_b \Phi \rangle_{\partial K} \right| \\ &\lesssim \left| \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}, Q_0 \Phi - \Phi \rangle_{\partial K} \right| \\ & \quad + \left| \sum_{K \in \mathfrak{S}_h} \langle (\mathbf{I} - \mathcal{Q}_0) \nabla p^m \cdot \mathbf{n}, \Phi - Q_b \Phi \rangle_{\partial K} \right| \\ &\lesssim h^{l+1} \|p^m\|_{l+1} \|\Phi\|_2. \end{aligned} \quad (4.48)$$

Similarly, we can obtain

$$\begin{aligned} |\tilde{E}_2| &= \left| \frac{\kappa}{\eta_{\mathbf{f}}} \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle Q_b Q_0 p^m - Q_b p^m, Q_b Q_0 \Phi - Q_b \Phi \rangle_{\partial K} \right| \\ &\lesssim h^{l+1} \|p^m\|_{l+1} \|\Phi\|_2, \end{aligned} \quad (4.49)$$

$$\begin{aligned} |\tilde{E}_4| &= \left| -\frac{\kappa}{\eta_{\mathbf{f}}} \sum_{K \in \mathfrak{S}_h} \langle (Q_0(\nabla \Phi) - \nabla \Phi) \cdot \mathbf{n}, \rho_{p0}^m - \rho_{pb}^m \rangle_{\partial K} \right| \\ &\lesssim \left| \sum_{K \in \mathfrak{S}_h} \langle (Q_0(\nabla \Phi) - \nabla \Phi) \cdot \mathbf{n}, \rho_{p0}^m - Q_0 \rho_{p0}^m \rangle_{\partial K} \right| \\ &\quad + \left| \sum_{K \in \mathfrak{S}_h} \langle (Q_0(\nabla \Phi) - \nabla \Phi) \cdot \mathbf{n}, Q_0 \rho_{p0}^m - \rho_{pb}^m \rangle_{\partial K} \right| \\ &\lesssim h^{l+1} \|p^m\|_{l+1} \|\Phi\|_2. \end{aligned} \quad (4.50)$$

For the third term $\tilde{E}_3$, from (4.34) and the project property it follows

$$\begin{aligned} |\tilde{E}_3| &= \left| -\frac{\kappa}{\eta_{\mathbf{f}}} \sum_{K \in \mathfrak{S}_h} h_K^{-1} \langle Q_b \rho_{p0}^m - \rho_{pb}^m, Q_b Q_0 \Phi - Q_b \Phi \rangle_{\partial K} \right| \\ &\lesssim h^{l+1} \|p^m\|_{l+1} \|\Phi\|_2. \end{aligned} \quad (4.51)$$

Substituting (4.48)-(4.51) into (4.47), we have

$$\|\rho_{p0}^m\|_0 \lesssim h^{l+1} \|p^m\|_{l+1},$$

which, together with (4.36), (4.40), (4.35), and the projection property, yields the desired estimate (4.44). $\square$

5 Numerical experiments

In this section, we shall report some numerical results for the proposed fully discrete weak Galerkin finite element scheme (3.7)-(3.9) in two-dimensional cases with $l = 1, 2$.

5.1 Model problem with smooth solution

Consider the Biot's consolidation problem in the square domain $\Omega = (0, 1) \times (0, 1)$ with uniform triangular meshes (Figure 1). We set $T = 1$, $\mu = \lambda = \kappa = \eta_{\mathbf{f}} = 1$. The forcing terms $\mathbf{f}$ and g are chosen such that the analytical solution, $(\mathbf{u}, p)$, to the system (1.1)-(1.2) with Dirichlet boundary conditions, is given by

$$\begin{cases} u_1 = 10x^2(1-x)^2y(1-y)(1-2y\exp(-t)), \\ u_2 = -10x(1-x)(1-2x)y^2(1-y)^2\exp(-2t), \\ p = 10x^2(1-x)^2y(1-y)(1-2y)\exp(-3t). \end{cases} \quad (5.1)$$

Numerical results are presented in Tables 1-2 for $l = 1$ and $l = 2$, respectively, where the time step $\Delta t = h^l$, and

$$\begin{aligned} e_{\mathbf{u},0} &:= \max_{1 \leq m \leq N} \|\mathbf{u}^m - \mathbf{u}_{h0}^m\|_0, & e_{\mathbf{u},1} &:= \max_{1 \leq m \leq N} \|\mathbf{u}^m - \mathbf{u}_h^m\|_V, \\ e_{p,0} &:= \max_{1 \leq m \leq N} \|p^m - p_{h0}^m\|_0, & e_{p,1} &:= \max_{1 \leq m \leq N} \|p^m - p_h^m\|_{\Theta}. \end{aligned}$$

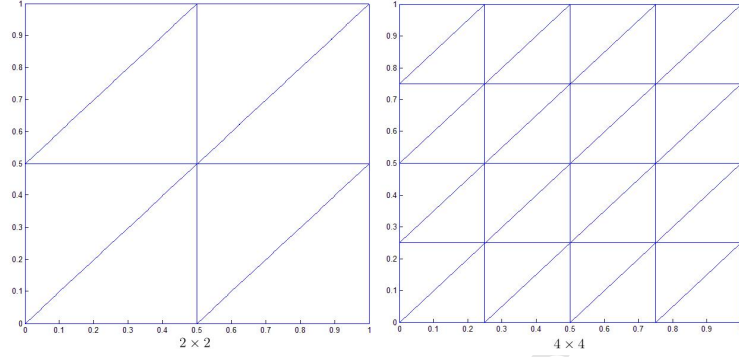


Figure 1: Mesh type

For the displacement error $e_{\mathbf{u},1}$, we can see that the convergence rates of our methods are of first and second orders for $l = 1$ and $l = 2$, respectively. This is conformable with the theoretical results in Theorem 4.1. We note that the convergence rates for $e_{\mathbf{u},0}$ are of second and third order, respectively. For the pressure errors $e_{p,0}$ and $e_{p,1}$, the numerical results are much better than the corresponding theoretical results, $O(\Delta t + h^l) = O(h^l)$ and $O(\sqrt{\Delta t} + h^l/\sqrt{\Delta t}) = O(h^{l/2})$, respectively.

Table 1: Convergence history with $l = 1$ and $\Delta t = h$

h^{-1}	$e_{\mathbf{u},0}$		$e_{\mathbf{u},1}$		$e_{p,0}$		$e_{p,1}$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
4	5.2401E-03		7.0694E-02		1.2539E-02		1.0969E-01	
8	1.2398E-03	2.08	2.9332E-02	1.27	4.6349E-03	1.44	8.1493E-02	0.43
16	3.1002E-04	2.00	1.3186E-02	1.15	1.3828E-03	1.74	4.9533E-02	0.72
32	7.8695E-05	1.98	6.4165E-03	1.04	3.6738E-04	1.91	2.7284E-02	0.86
64	1.9852E-05	1.99	3.2132E-03	1.00	1.0031E-04	1.87	1.4315E-02	0.93

Table 2: Convergence history with $l = 2$ and $\Delta t = h^2$

h^{-1}	$e_{\mathbf{u},0}$		$e_{\mathbf{u},1}$		$e_{p,0}$		$e_{p,1}$	
	Error	Rate	Error	Rate	Error	Rate	Error	Rate
4	6.4331E-04		1.6113E-02		3.3113E-03		4.6684E-02	
8	5.4591E-05	3.56	3.2632E-03	2.30	4.8197E-04	2.78	1.3956E-02	1.74
16	4.9295E-06	3.47	7.1749E-04	2.19	6.4149E-05	2.91	3.6992E-03	1.92
32	5.2521E-07	3.23	1.7136E-04	2.07	8.1734E-06	2.97	9.4485E-04	1.97

5.2 Mandel's problem

Mandel's problem [18] is a benchmark test to verify the accuracy of numerical methods for poroelasticity equations. One important issue in linear poroelasticity is to eliminate numerical oscillations (locking) in the pressure approximation.

The domain is set to be $\Omega = (-a, a) \times (-b, b)$, and, as pointed out in [25], we only need to solve the problem in the upper-right quadrant of Ω , i.e. $(0, a) \times (0, b)$, since the problem is symmetric about the x and y axes. The corresponding boundary conditions and initial conditions are as follows:

$$\begin{aligned} \mathbf{u}_1|_{x=0} &= \mathbf{0}, \quad \mathbf{u}_2|_{y=0} = \mathbf{0}, \quad \mathbf{u}_2|_{y=b} = c_0(\text{Rigid Plate Constraint}), \\ (2\mu\epsilon(\mathbf{u}) + \lambda\nabla \cdot \mathbf{u}\mathbf{I} - p\mathbf{I})\mathbf{n} &= \begin{cases} -2\mathbf{F} & \text{when } y = b, \\ \mathbf{0} & \text{otherwise,} \end{cases} \\ p|_{x=a} &= 0, \\ \frac{\kappa}{\eta_f} \nabla p \cdot \mathbf{n}|_{x=0, \text{ or } y=0, \text{ or } y=b} &= 0, \\ \mathbf{u}|_{t=0} &= \mathbf{0}, \quad p|_{t=0} = 0. \end{aligned}$$

We refer to [1, 25] for the analytical series solutions for the displacement $\mathbf{u}$ and the pressure p . We set $\mu = \frac{10^{-5}}{2}$, $\lambda = 0$, $\frac{\kappa}{\eta_f} = 10^{-6}$, $\mathbf{F} = (0, 1)^T$, $a = b = 1$, and $c_0 = 10^{-4}$. The computational meshes as same as in Figure 1. We choose the mesh size $h = 1/20$ when $l = 1, 2$. To minimize the effects of the error produced by the time discretization, we use a small time step of $\Delta t = 10^{-6}$.

The numerical pressure and real pressure at $t = 10^{-6}$ are presented in Figures 2-3 with $h = 1/20$ and $\Delta t = 10^{-6}$. We can see that the proposed WG method gives quite accurate results for the pressure approximation, and captures the Mandel-Creyer effect around $x = a$ ($a = 1$). This agrees with the results in [1, 25], and indicates that the proposed WG method is capable of overcoming the numerical oscillations of pressure.

References

- [1] Y. Abousleiman, A. D. Cheng, L. Cui, E. Detournay, J. Roegiers, Mandel's problem revisited. *Géotechnique*. 46 (1996) 187-195
- [2] G. Aguilar, F. Gaspar, F. Lisbona and C. Rodrigo, Numerical stabilization of Biot's consolidation model by a perturbation on the flow equation. *Int. J. Numer. Meth. Engng.* 75 (2008) 1282-1300
- [3] D. N. Arnold, An interior penalty finite element method with discontinuous elements. *SIAM J. Numer. Anal.* 19 (1982) 742-760
- [4] C. Bi, V. Ginting, A posteriori error estimates of discontinuous Galerkin method for nonmonotone quasi-linear elliptic problems. *J. Sci. Comput.* 55 (2013) 659-687
- [5] M. Biot, General theory of three dimensional consolidation. *J. Appl. Phys.* 12(2) (1941) 155-164
- [6] J. R. Booker, J. C. Small, An investigation of the stability of numerical solutions of Biot's equations of consolidation. *Int. J. Solids Struct.* 11 (1975) 907-917
- [7] G. Chen, M. Feng, and X. Xie, Robust globally divergence-free weak galerkin methods for Stokes equations. *J. Comput. Math.*, 34 (2016) 549-572

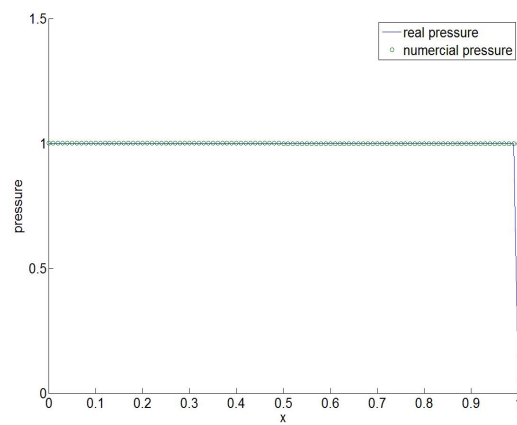


Figure 2: pressure at $t = 10^{-6}$, when $l = 1$

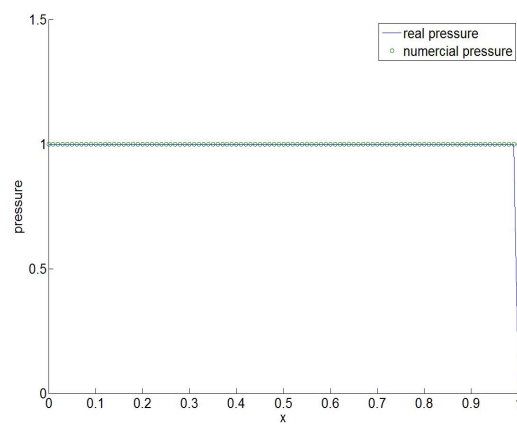


Figure 3: pressure at $t = 10^{-6}$, when $l = 2$

- [8] G. Chen, M. Feng, and X. Xie, A class of robust WG/HDG finite element method for convection-diffusion-reaction equations. *J. Comput. Appl. Math.*, 315 (2017) 107-125
- [9] G. Chen, X. Xie, A robust weak Galerkin finite element method for linear elasticity with strong symmetric stresses. *Comput. Methods Appl. Math.* 16 (2016) 389-408
- [10] L. Chen, J. Wang, Y. Wang, X. Ye, An auxiliary space multigrid preconditioner for the weak Galerkin method. *Comput. Math. Appl.* 70 (2015) 330-334
- [11] Y. Chen, Y. Luo, M. Feng, Analysis of a discontinuous Galerkin method for the Biot's consolidation problem. *Appl. Math. Comput.* 219 (2013) 9043-9056
- [12] X. Hu, C. Rodrigo, F. J. Gaspar, L. T. Zikatanov, A nonconforming finite element method for the Biot's Consolidation model in poroelasticity. *J. Comput. Appl. Math.* 310 (2016) 143-154
- [13] J. Korsawe, G. Starke, A least-squares mixed finite element method for Biot's consolidation problem in porous media. *SIAM J. Numer. Anal.* 43(1) (2005) 318-339
- [14] J. J. Lee, Robust error analysis of coupled mixed methods for Biot's consolidation model. *J. Sci. Comput.* 69 (2015) 1-23
- [15] R. W. Lewis, B. A. Schrefler, The finite element method in the static and dynamic deformation and consolidation of porous media. Wiley, Chichester, UK, 1998.
- [16] B. Li, X. Xie, A two-level algorithm for the weak Galerkin discretization of diffusion problems. *J. Comput. Appl. Math.* 287 (2015) 179-195
- [17] Q. H. Li, J. Wang, Weak Galerkin finite element methods for parabolic equations. *Numer. Meth. Part. Diff. Equ.* 29 (2013) 2004-2024
- [18] J. Mandel, Consolidation des sols (étude mathématique). *Géotechnique* 30 (1953) 287-299
- [19] L. Mu, J. Wang, X. Ye, Weak Galerkin finite element methods for the biharmonic equation on polytopal meshes. *Numer. Meth. Part. Diff. Equ.* 30 (2014) 1003-1029
- [20] L. Mu, J. Wang, X. Ye, S. Zhang, A weak Galerkin finite element method for the Maxwell equations. *J. Sci. Comput.* 65(1) (2015) 363-386
- [21] M. A. Murad, A. F. D. Loula, Improved accuracy in finite element analysis of Biot's consolidation problem. *Comput. Methods Appl. Mech. Eng.* 95 (1992) 359-382
- [22] M. A. Murad, A. F. D. Loula, On stability and convergence of finite element approximations of Biot's consolidation problem. *Internat. J. Numer. methods Engrg.* 37 (1994) 645-667.

- [23] M. A. Murad, V. Thomée, A. F. D. Loula, Asymptotic behavior of semi discrete finite-element approximations of Biot's consolidation problem. *SIAM J. Numer. Anal.* 33 (1996) 1065-1083
- [24] J. Nečas, Equations aux Dérivées Partielles. Presses de l'Université de Montréal, Montréal, Canada, 1966.
- [25] P. J. Phillips, M. F. Wheeler, A coupling of mixed and continuous Galerkin finite element methods for poroelasticity I: the continuous in time case. *Comput. Geosci.* 11(2) (2007) 131-144
- [26] P. J. Phillips, M. F. Wheeler, A coupling of mixed and discontinuous Galerkin finite-element methods for poroelasticity. *Comput. Geosci.* 12(4) (2008) 417-435
- [27] P. J. Phillips, M. F. Wheeler, Overcoming the problem of locking in linear elasticity and poroelasticity: an heuristic approach. *Comput. Geosci.* 13(1) (2009) 5-12
- [28] C. Rodrigo, F. J. Gaspar, X. Hu, L. T. Zikatanov, Stability and monotonicity for some discretizations of the Biot's consolidation model. *Comput. Methods Appl. Mech. Engrg.* 298 (2016) 183-204
- [29] V. Thomée, Galerkin finite element methods for parabolic problems, 2nd Edition, Vol. 25 of Springer Series in Computational Mathematics. Springer-Verlag, Berlin, 2006.
- [30] C. Wang, J. Wang, An efficient numerical scheme for the biharmonic equation by weak Galerkin finite element methods on polygonal or polyhedral meshes. *Adv. Comput. Part. Diff. Equ.* 68 (2014) 2314-2330
- [31] C. Wang, J. Wang, R. Wang, R. Zhang, A locking-free weak Galerkin finite element method for elasticity problems in the primal formulation. *J. Comput. Appl. Mathe.* 307 (2016) 346-366
- [32] J. Wang, X. Ye, A weak Galerkin finite element method for second-order elliptic problems. *J. Comput. Appl. Math.* 241 (2013) 103-115
- [33] J. Wang, X. Ye, A weak Galerkin finite element method for the Stokes equations. *Adv. Comput. Math.* 275(1) (2016) 1-20
- [34] Y. Yokoo, K. Yamagata, H. Nagaoka, Variational principles for consolidation. *Soils Found.* 11 (1971) 25-36
- [35] Y. Yokoo, K. Yamagata, H. Nagaoka, Finite element methods applied to Biot's consolidation theory. *Soils Found.* 11 (1971) 29-46
- [36] S.-Y. Yi, A coupling of nonconforming and mixed finite element methods for Biot's consolidation model. *Numer. Meth. Part. Diff. Equ.* 29(5) (2013) 1749-1777
- [37] S.-Y. Yi, Convergence analysis of a new mixed finite element method for Biot's consolidation model. *Numer. Meth. Part. Diff. Equ.* 30(4) (2014) 1189-1210