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Letter Section

On zero-finding methods of the fourth order

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Abstract

Using the iteration formulas of the third order for solving the single equation $f(z) = 0$ and a procedure for the acceleration of convergence, three new methods of the fourth order are derived. The comparison with other methods is given.

Keywords: Solving equations; Iteration methods; Multiple zeros; Acceleration of convergence

1. Acceleration of iteration methods

Let us consider *one-point* iteration process

$$z_{n+1} = h(z_n) \quad (n = 0, 1, \ldots), \tag{1}$$

which converges to the zero r of the single equation $f(z) = 0$ with the order of convergence k, $k \geqslant 2$. Then, as it is known from the theory of iteration processes, the modified iteration method

$$z_{n+1} = h(z_n) + \frac{1}{k} h'(z_n)[h(z_n) - z_n] \quad (n = 0, 1, \ldots) \tag{2}$$

has the order of convergence $k + 1$ (see, e.g., [3]). This means that the method (2) accelerates the convergence of the basic method (1) from k to $k + 1$ ($k \geqslant 2$). Most frequently, the iteration method (1) has the form $z_{n+1} = h(z_n) := z_n - \delta(z_n)$, where $z \mapsto \delta(z)$ is a *correction function*. Then the iteration method (2) can be written in the following equivalent form:

$$z_{n+1} = z_n - \delta(z_n)\left[1 + \frac{1}{k} h'(z_n)\right] \quad (n = 0, 1, \ldots). \tag{3}$$

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2. New methods of the fourth order

First we give a list of the known iteration third-order methods ($k = 3$) of the form (1) for finding a zero of the single equation $f(z) = 0$ in the case with requiring the knowledge of multiplicity m:

$$z_{n+1} = z_n - \frac{1}{2} m(m+1) \frac{f(z_n)}{f'(z_n)} + \frac{1}{2}(m-1)^2 \frac{f'(z_n)}{f''(z_n)} \quad [4], \tag{4}$$

$$z_{n+1} = z_n - \frac{1}{\dfrac{m+1}{2m} \dfrac{f'(z_n)}{f(z_n)} - \dfrac{f''(z_n)}{2f'(z_n)}} \quad [2], \tag{5}$$

$$z_{n+1} = z_n - m \frac{f(z_n)}{f'(z_n)} \left[\frac{1}{2}(3-m) + m \frac{f''(z_n) f(z_n)}{2f'(z_n)^2} \right] \quad [5, \text{p. } 139]. \tag{6}$$

The application of the accelerating procedure (2) (or (3)) to the cubically convergent methods (4), (5) and (6) produces the three new methods of the fourth order.

2.1. *First method based on Osada's formula*

First, we consider the iteration method (4). Following the above notations, we have

$$\delta_1(z) = \frac{1}{2} m(m+1) \frac{f(z)}{f'(z)} - \frac{1}{2}(m-1)^2 \frac{f'(z)}{f''(z)} \tag{7}$$

and (starting from $h_1(z) = z - \delta_1(z)$)

$$h_1'(z) = \frac{3(1-m)}{2} + \frac{1}{2} m(m+1) \frac{f(z) f''(z)}{f'(z)^2} - \frac{1}{2}(m-1)^2 \frac{f'(z) f'''(z)}{f''(z)^2}. \tag{8}$$

Then, in view of (3), the new iteration method of the fourth order reads

$$z_{n+1} = z_n - \delta_1(z_n) \left[1 + \frac{1}{3} h_1'(z_n) \right] \quad (n = 0, 1, \ldots), \tag{9}$$

where the functions δ_1 and h_1' are given by (7) and (8), respectively.

2.2. *Second method based on Farmer–Loizou's formula*

From the iteration formula (4) we find

$$\delta_2(z) = \frac{2}{\dfrac{m+1}{m} \dfrac{f'(z)}{f(z)} - \dfrac{f''(z)}{f'(z)}} = \frac{2}{v(z)} \quad \text{with} \quad v(z) = \frac{m+1}{m} \frac{f'(z)}{f(z)} - \frac{f''(z)}{f'(z)}.$$

Now we find

$$h_2'(z) = \frac{\mathrm{d}}{\mathrm{d}z}(z - \delta_2(z)) = 1 + \frac{2v'(z)}{v(z)^2},$$

where

$$v'(z) = \frac{m+1}{m}\left[\frac{f''(z)}{f(z)} - \left(\frac{f'(z)}{f(z)}\right)^2\right] - \frac{f'''(z)}{f'(z)} + \left(\frac{f''(z)}{f'(z)}\right)^2.$$

Using the expressions for the functions $v(z)$ and $v'(z)$, according to (3) we obtain the new iteration method of the fourth order in the form

$$z_{n+1} = z_n - \frac{8}{3v(z_n)} - \frac{4v'(z_n)}{3v(z_n)^3} \quad (n = 0, 1, \ldots). \tag{10}$$

2.3. *Third method based on Traub's formula*

Let $u(z) = f(z)/f'(z)$ and $A_k(z) = f^{(k)}(z)/(k!f'(z))$. From (6) we have

$$\delta_3(z) = \frac{m(3-m)u(z)}{2} + m^2 A_2(z)u(z)^2 \tag{11}$$

and

$$h_3'(z) = \frac{\mathrm{d}}{\mathrm{d}z}(z - \delta_3(z)) = \frac{(m-1)(m-2)}{2} - 3m(m-1)u(z)A_2(z) + 3m^2u(z)^2(2A_2(z)^2 - A_3(z)). \tag{12}$$

Then, by (3), the new iteration method of the fourth order has the form

$$z_{n+1} = z_n - \delta_3(z_n)\left[1 + \frac{1}{3}h_3'(z_n)\right] \quad (n = 0, 1, \ldots), \tag{13}$$

where δ_3 and h_3' are given by (11) and (12).

3. Numerical results

The newly established methods (9), (10) and (13) of the fourth order were tested in the examples of algebraic and transcendental equations having a multiple zero of the known multiplicity. All three methods have shown very fast convergence even in the case of relatively rough starting approximations. Beside the new methods we implemented Farmer–Loizou's method [1]

$$z_{n+1} = z_n - \frac{mu(z_n)[(m+1)/2 - mA_2(z_n)u(z_n)]}{(m+1)(2m+1)/6 - m(m+1)A_2(z_n)u(z_n) + m^2A_3(z_n)u(z_n)^2} \tag{14}$$

and Traub's method [5]

$$z_{n+1} = z_n - mu(z_n)\left[\frac{m^2 - 6m + 11}{6} + m(2-m)A_2(z_n)u(z_n) + m^2(2A_2(z_n)^2 - A_3(z_n))u(z_n)^2\right]. \tag{15}$$

Both of these methods are of the fourth order of convergence.

Table 1

Method	3	4	5	6	7	8	9	10	11–15	16–20	>20	a.n.i.	CPU time (μs)
(9)	15	35	10	3	3	4	5	4	13	7	1	7.10	218
(10)	12	53	7	6	3	2	6	1	8	2	0	5.52	235
(13)	28	33	3	5	3	3	2	2	8	5	8	7.81	220
(14)	7	13	23	27	4	3	2	1	8	11	1	7.45	217
(15)	24	38	4	3	3	4	2	3	8	5	6	7.32	215

We have tested 20 algebraic polynomials with the degree N in the range [6, 20] and 5 various rather crude initial values. The stopping criterion has been given by $|P_N(z_n)| \leqslant 10^{-15}$. Numerical computations have been performed using MS-FORTRAN Version 5.1 in double precision.

The number $C([n_1, n_2], (\mathrm{M}))$ $(n_1 \leqslant n_2)$ which denotes that the method (M) has satisfied this criterion by $n \in [n_1, n_2]$ iteration steps is given in Table 1 for all tested methods and 100 experiments. The number (or range) of iterations is displayed in the first horizontal line, while the average number of iterations (a.n.i.) considering all 100 experiments and the total number of iterations is given in the column a.n.i. Although this feature is not a genuine measure of convergence, it offers a close look at the efficiency of methods belonging to the same class. From this column we see that the new method (10) is the fastest, while the convergence rate of the remaining four methods (9), (13), (14) and (15) is almost the same. From Table 1 we also can see that all methods converges very fast, mainly in less than 6 steps. For few number of particular examples only the new method (13) and Traub's method (15) required a great number of iterations. The number of numerical operations is almost the same for all five methods which has been confirmed by measuring the CPU time on PC 486DX2/66 (the last column, the average times are displayed).

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