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Multiresolution analysis over triangles, based on quadratic Hermite interpolation [☆]

M. Dæhlen, T. Lyche, K. Mørken ^{*}, R. Schneider, H-P. Seidel*Department of Informatics, University of Oslo, P.O. Box 1080 Blindern, 0316 Oslo, Norway*

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Abstract

Given a triangulation T of \mathbb{R}^2 , a recipe to build a spline space $\mathbb{S}(T)$ over this triangulation, and a recipe to refine the triangulation T into a triangulation T' , the question arises whether $\mathbb{S}(T) \subset \mathbb{S}(T')$, i.e., whether any spline surface over the original triangulation T can also be represented as a spline surface over the refined triangulation T' . In this paper we will discuss how to construct such a nested sequence of spaces based on Powell–Sabin 6-splits for a regular triangulation. The resulting spline space consists of piecewise C^1 -quadratics, and refinement is obtained by subdividing every triangle into four subtriangles at the edge midpoints. We develop explicit formulas for wavelet transformations based on quadratic Hermite interpolation, and give a stability result with respect to a natural norm. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [2] we developed a multiresolution setup for quadratic splines based on Hermite interpolation. This was based on the hierarchical basis of Faber [3], see also [9]. In this paper we generalize this univariate, quadratic construction to bivariate, piecewise quadratic functions defined on a given regular triangulation. Multiresolution over triangles is well known, see, e.g., [5,6,9], and there are several approaches that extend to spheres [11] or even surfaces of arbitrary topology [8]. Unfortunately, most

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^{*} Corresponding author.

E-mail address: knutm@ifi.uio.no (K. Mørken).

of the occurring bases result from subdivision algorithms and therefore cannot be represented explicitly. The basis constructed in this paper overcomes that drawback and is also suited to be extended to surfaces of arbitrary topology by adapting the rationale described in [8].

The construction of a multiresolution analysis over a triangulation is closely connected with the construction of nested polynomial spline spaces, see [7]. In that paper there is a distinction between two major approaches: either the triangulation or the polynomial degree is kept fixed. Since we want to extend the univariate method, we choose to keep the degree fixed, i.e., we refine the triangulation to get nested spline spaces

$$V_0 \subset V_1 \subset \cdots \subset V_k \subset \cdots.$$

For piecewise quadratic polynomials that are C^1 , the dimension of the spline space is too low to allow a reasonable basis, so to use quadratic splines we will have to introduce macro patches. We will make use of the Powell–Sabin 6-split [10] which combines nicely with Hermite interpolation.

2. Multiresolution analysis

In this section we briefly summarize what is meant by a multiresolution analysis and a weakly stable basis. We need a fairly general definition of multiresolution analysis, see [1], and also [2] for a specific discussion relevant to the setting in this paper.

Definition 1. A multiresolution analysis consists of

1. A Banach space \mathcal{F} of functions defined on a bounded set $X \subset \mathbb{R}^n$, with $n > 0$ and with associated norm $\|\cdot\|$.
2. A nested sequence of subspaces $V_0 \subset V_1 \subset \cdots \subset V_k \subset \cdots$ that are dense in \mathcal{F} ,

$$\overline{\bigcup_{k=1}^{\infty} V_k} = \mathcal{F}.$$

3. A collection of uniformly bounded operators

$$Q_k : \mathcal{F} \rightarrow V_k$$

with the properties

$$Q_k Q_k = Q_k,$$

$$Q_k Q_{k+1} = Q_k,$$

$$Q_k(\mathcal{F}) = V_k$$

for all integers $k \geq 0$.

With the projectors Q_k given, we can define the complement spaces

$$W_k = \{F \in V_{k+1} \mid Q_k F = 0\}$$

and using the fact that $V_{k+1} = V_k + W_k$ and $V_k \cap W_k = \{0\}$, we get a decomposition of \mathcal{F} as the direct sum

$$\mathcal{F} = V_0 + W_0 + W_1 + W_2 + \cdots.$$

This means that every $F \in \mathcal{F}$ can be decomposed as

$$F = F_0 + G_0 + G_1 + G_2 + \cdots$$

with $G_k = F_{k+1} - F_k$ in W_k . For simplicity, we will refer to the complement spaces W_k as wavelet spaces and functions in W_k as wavelets, although it is common to require from wavelets that they have a number of vanishing moments.

Let $\{\phi_{j,k}\}_{j \in \mathcal{J}_k}$ (often referred to as scaling functions) be a basis for V_k indexed by the index set \mathcal{J}_k , and let $\{\psi_{j,k}\}_{j \in \mathcal{J}_k}$ be a basis for W_k , with index set \mathcal{J}_k . Since nearly all information of a function $F \in \mathcal{F}$ is included in $\bigcup_{k=0}^{\infty} W_k$, it is crucial to know the stability properties of the wavelet functions, which relate the size of a function to the size of its wavelet coefficients. Let $F_n = \sum_{k=1}^n \sum_{j \in \mathcal{J}_k} d_{j,k} \psi_{j,k}$ be the representation of a wavelet function, then we will employ an, as yet unspecified, vector norm

$$\delta_n = \|(d_{j,k})_{k=1, j \in \mathcal{J}_k}^n\|_v \quad (1)$$

to measure the size of the wavelet coefficients.

We will be working with uniform norms, so a weaker form of stability than usual is convenient.

Definition 2. Let \mathcal{F} be a space with a multiresolution analysis and corresponding wavelet bases $\{\psi_{j,k}\}$. The wavelets are said to form a weakly stable basis for $\overline{\bigcup_{k=1}^{\infty} W_k}$ if for each $n \geq 1$ there exist constants $K_{1,n}$ and $K_{2,n}$ such that

$$K_{1,n}^{-1} \delta_n \leq \left\| \sum_{k=1}^n \sum_{j \in \mathcal{J}_k} d_{j,k} \psi_{j,k} \right\| \leq K_{2,n} \delta_n,$$

where δ_n is given by (1), and $K_{1,n}$ and $K_{2,n}$ have at most polynomial growth in n . If the two constants $K_{1,n}$ and $K_{2,n}$ are independent of n , the basis is said to be strongly stable.

3. Multiresolution based on quadratic Hermite interpolation

In the following, we will derive a multiresolution analysis built over an equilateral triangle, but this can be generalized to any regular triangulation. Consider a triangulation defined on the regular hexagon with centre point P_0 at the origin and edge vertices P_l with $P_1 = (2, 0)$, with the other vertices chosen counterclockwise around the circle of radius 2,

$$P_l = 2 \begin{pmatrix} \cos(l-1)\pi/3 \\ \sin(l-1)\pi/3 \end{pmatrix} \quad \text{for } l = 1, \dots, 6.$$

These seven points generate a triangulation consisting of six equilateral triangles. On every triangle we perform a Powell–Sabin 6-split by connecting each vertex with the midpoint of its opposite edge, so we get a new triangulation \mathcal{T}_{HEX} . We denote by $\mathbb{S}_2^1(\mathcal{T}_{\text{HEX}})$ all C^1 -functions that reduce to a quadratic polynomial on each triangle in \mathcal{T}_{HEX} . It is well known that each function in $\mathbb{S}_2^1(\mathcal{T}_{\text{HEX}})$

is uniquely determined by its position and first derivatives at the seven points $\{P_l\}_{l=0}^6$, (see [4]). We can therefore introduce three nodal functions C_1 , C_2 and C_3 that are characterized by the conditions

$$\begin{aligned} C_1(P_l) &= \delta_{l,0}, & C_2(P_l) &= 0, & C_3(P_l) &= 0, \\ \frac{\partial}{\partial x} C_1(P_l) &= 0, & \frac{\partial}{\partial x} C_2(P_l) &= \delta_{l,0}, & \frac{\partial}{\partial x} C_3(P_l) &= 0, \\ \frac{\partial}{\partial y} C_1(P_l) &= 0, & \frac{\partial}{\partial y} C_2(P_l) &= 0, & \frac{\partial}{\partial y} C_3(P_l) &= \delta_{l,0} \end{aligned} \quad (2)$$

for $l = 0, 1, \dots, 6$. The Bézier representation of these functions is shown in Fig. 1.

It is now straightforward to build a multiresolution analysis on an equilateral triangle D using dilates and translates of C_1 , C_2 and C_3 . We choose D to be the subtriangle of our hexagon with vertices P_0 , P_1 , P_2 , and introduce the domain points

$$\Delta_k = \left\{ \Delta_{i,j,k} = i \frac{P_1 - P_0}{2^k} + j \frac{P_2 - P_0}{2^k} = \frac{1}{2^k} \begin{pmatrix} 2i + j \\ \sqrt{3}j \end{pmatrix} \middle| i \geq 0, j \geq 0, i + j \leq 2^k \right\}.$$

For each domain point $\Delta_{i,j,k}$ we define the three scaling functions

$$\Phi_{i,j,k}^l(x, y) = \frac{1}{2^k} C_l(2^k x - 2i - j, 2^k y - \sqrt{3}j) \quad \text{for } l = 1, 2, 3,$$

where the factor $1/2^k$ has been introduced to simplify the arithmetic and stability results following later. We get a sequence of nested spline spaces as the span of these functions

$$V_k = \text{span}_{i \geq 0, j \geq 0, i+j \leq 2^k} \{ \Phi_{i,j,k}^1, \Phi_{i,j,k}^2, \Phi_{i,j,k}^3 \}$$

restricted to the triangle D .

We can now construct Hermite interpolation operators. We choose $Q_k : C^1(D) \rightarrow V_k$, so that for every $F \in C^1(D)$ the projection $F_k = Q_k F$ has the properties

$$\begin{aligned} F_k(\Delta_{i,j,k}) &= F(\Delta_{i,j,k}), \\ \frac{\partial}{\partial x} F_k(\Delta_{i,j,k}) &= \frac{\partial}{\partial x} F(\Delta_{i,j,k}), \\ \frac{\partial}{\partial y} F_k(\Delta_{i,j,k}) &= \frac{\partial}{\partial y} F(\Delta_{i,j,k}) \end{aligned}$$

for all points $\Delta_{i,j,k} \in \Delta_k$. From the inclusion $\Delta_k \subset \Delta_{k+1}$ and uniqueness of interpolation, it follows that Q_k satisfies $Q_k Q_k = Q_k$ and $Q_k Q_{k+1} = Q_k$. By definition, the wavelet space W_k consists of those functions in V_{k+1} whose position and first derivatives vanish at the points of Δ_k ,

$$W_k = \left\{ F_{k+1} \in V_{k+1} \mid F_{k+1}(\Delta_{i,j,k}) = 0, \frac{\partial}{\partial x} F_{k+1}(\Delta_{i,j,k}) = 0, \frac{\partial}{\partial y} F_{k+1}(\Delta_{i,j,k}) = 0, \forall \Delta_{i,j,k} \in \Delta_k \right\}.$$

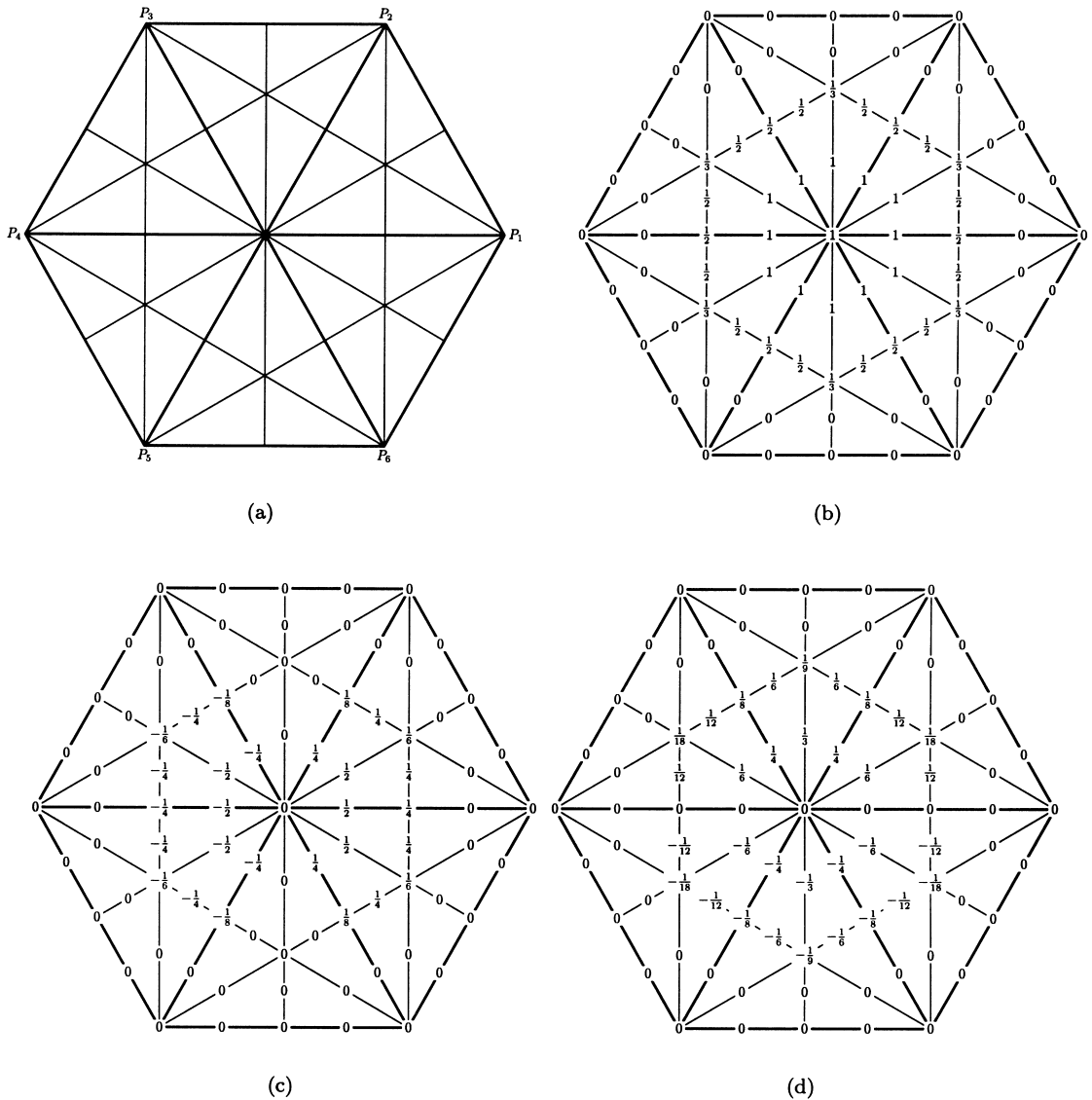


Fig. 1. In (a), the triangulation of the regular hexagon \mathcal{T}_{HEX} is shown, while the Bèzier coefficients of the three piecewise quadratic functions C_1 , C_2 and C_3 are shown in (b), (c), and (d) respectively. Note that the factor $\sqrt{3}$ has been omitted from all the coefficients of C_3 .

This makes it simple to give a basis for W_k . We simply use the scaling functions in V_{k+1} associated with the new vertices in Δ_{k+1} (the vertices that are in Δ_{k+1} but not in Δ_k),

$$W_k = \text{span}_{\Delta_{i,j,k+1} \in \Delta_{k+1} \setminus \Delta_k} \{ \Psi_{i,j,k}^1, \Psi_{i,j,k}^2, \Psi_{i,j,k}^3 \} \quad \text{with } \Psi_{i,j,k}^l = \Phi_{i,j,k+1}^l.$$

It is easy to see that for a basis function $\Psi_{i,j,k}$ in W_k at least one of i and j must be an odd integer.

We now have the basic ingredients for building a multiresolution analysis. As our space \mathcal{F} we take $C^1(D)$, the space of functions defined on the triangle D that are continuous and have continuous first derivatives in D , and the norm we take to be

$$\|F\| = \max \left\{ \|F\|_{L^\infty(D)}, \left\| \frac{\partial F}{\partial x} \right\|_{L^\infty(D)}, \left\| \frac{\partial F}{\partial y} \right\|_{L^\infty(D)} \right\}, \quad (3)$$

where $\|F\|_\infty$ denotes the usual L^∞ -norm for functions defined on D .

Because of (2), it is easy to determine the coefficients of a spline $F_k = \sum_{\Delta_{i,j,k} \in \Delta_k} a_{i,j,k}^l \Phi_{i,j,k}^l$ in V_k ,

$$a_{i,j,k}^1 = 2^k F(\Delta_{i,j,k}),$$

$$a_{i,j,k}^2 = \frac{\partial}{\partial x} F(\Delta_{i,j,k}),$$

$$a_{i,j,k}^3 = \frac{\partial}{\partial y} F(\Delta_{i,j,k}), \quad (4)$$

where $\sum_{\Delta_{i,j,k} \in \Delta_k}$ denotes the sum of all index pairs (i,j) with $\Delta_{i,j,k} \in \Delta_k$. These simple formulas will be useful later.

To verify that we have a multiresolution analysis it remains to check that the operators $\{Q_k\}$ are uniformly bounded and that the spaces $\{V_k\}$ are dense in $C^1(D)$. This is tedious but straightforward, and is postponed until Section 6.

4. Reconstruction and decomposition algorithms

The fundamental algorithms for dealing with wavelets are the wavelet transform and inverse wavelet transform, or the decomposition algorithm and the reconstruction algorithm. The decomposition algorithm starts with a spline F_{k+1} in V_{k+1} and decomposes this into $F_{k+1} = F_k + G_k$ with F_k in V_k and G_k in W_k . This process can be iterated to produce the decomposition $F_{k+1} = F_0 + G_0 + \dots + G_k$, but it suffices to show how the first step is to be performed. The reconstruction algorithm undoes the decomposition and produces F_{k+1} from the two components F_k and G_k . We start by describing the reconstruction algorithm.

4.1. The reconstruction algorithm

Since $V_k \subseteq V_{k+1}$, the scaling functions in V_k can be written as linear combinations of the scaling functions in V_{k+1} . From (4) it follows that the coefficients are given by function values and first derivatives of the scaling functions in V_k at the knots Δ_{k+1} . Since $\Phi_{i,j,k}^l(x, y) = (1/2^k) C_l(2^k x - 2i - j, 2^k y - \sqrt{3}j)$, this means we have to calculate the values of C_l , $(\partial/\partial x)C_l$ and $(\partial/\partial y)C_l$ at the knots $\Delta_{i,j,1}$, and these values can be easily derived from the Bézier representations of $\{C_l\}_{l=1}^3$. The result is some long formulas which we omit here, but with that information we can lift a function F_k in

V_k into V_{k+1} . More specifically, if

$$F_k = \sum_{\substack{(i,j) \in I_k \\ l=1,2,3}} a_{i,j,k}^l \Phi_{i,j,k}^l \Big|_D = \sum_{\substack{(i,j) \in I_{k+1} \\ l=1,2,3}} \tilde{a}_{i,j,k+1}^l \Phi_{i,j,k+1}^l \Big|_D, \quad (5)$$

we obtain formulas for all the refined coefficients ($\tilde{a}_{i,j,k+1}^l$) in terms of the coarser coefficients ($a_{i,j,k}^l$). But this is not the complete reconstruction algorithm. In general, we also have a wavelet component G_k so that $F_{k+1} = F_k + G_k$. But since

$$G_k = \sum_{\substack{\Delta_{i,j,k+1} \in \Delta_{k+1} \setminus \Delta_k \\ l=1,2,3}} b_{i,j,k}^l \Psi_{i,j,k}^l \quad \text{with } \Psi_{i,j,k}^l = \Phi_{i,j,k+1}^l,$$

we obtain the coefficients ($a_{i,j,k+1}^l$) by adding $b_{i,j,k}^l$ to the coefficient $\tilde{a}_{i,j,k+1}^l$ for every $\Delta_{i,j,k+1}$ with at least one odd i or j , i.e.,

$$\begin{aligned} a_{2i,2j,k+1}^1 &= 2a_{i,j,k}^1, \\ a_{2i,2j,k+1}^2 &= a_{i,j,k}^2, \\ a_{2i,2j,k+1}^3 &= a_{i,j,k}^3, \\ a_{2i+1,2j,k+1}^1 &= b_{2i+1,2j,k+1}^1 + (a_{i,j,k}^1 + a_{i+1,j,k}^1) + \frac{1}{2}(a_{i,j,k}^2 - a_{i+1,j,k}^2), \\ a_{2i+1,2j,k+1}^2 &= b_{2i+1,2j,k+1}^2 + (-a_{i,j,k}^1 + a_{i+1,j,k}^1) - \frac{1}{2}(a_{i,j,k}^2 + a_{i+1,j,k}^2), \\ a_{2i+1,2j,k+1}^3 &= b_{2i+1,2j,k+1}^3 + \frac{1}{2}(a_{i,j,k}^3 + a_{i+1,j,k}^3), \\ a_{2i,2j+1,k+1}^1 &= b_{2i,2j+1,k+1}^1 + (a_{i,j,k}^1 + a_{i,j+1,k}^1) + \frac{1}{4}(a_{i,j,k}^2 - a_{i,j+1,k}^2) + \frac{1}{4}\sqrt{3}(a_{i,j,k}^3 - a_{i,j+1,k}^3), \\ a_{2i,2j+1,k+1}^2 &= b_{2i,2j+1,k+1}^2 + \frac{1}{2}(-a_{i,j,k}^1 + a_{i,j+1,k}^1) + \frac{1}{4}(a_{i,j,k}^2 + a_{i,j+1,k}^2) - \frac{1}{4}\sqrt{3}(a_{i,j,k}^3 + a_{i,j+1,k}^3), \\ a_{2i,2j+1,k+1}^3 &= b_{2i,2j+1,k+1}^3 + \frac{1}{2}\sqrt{3}(-a_{i,j,k}^1 + a_{i,j+1,k}^1) - \frac{1}{4}\sqrt{3}(a_{i,j,k}^2 + a_{i,j+1,k}^2) - \frac{1}{4}(a_{i,j,k}^3 + a_{i,j+1,k}^3), \\ a_{2i+1,2j+1,k+1}^1 &= b_{2i+1,2j+1,k+1}^1 + (a_{i,j+1,k}^1 + a_{i+1,j,k}^1) + \frac{1}{4}(a_{i,j+1,k}^2 - a_{i+1,j,k}^2) + \frac{1}{4}\sqrt{3}(-a_{i,j+1,k}^3 + a_{i+1,j,k}^3), \\ a_{2i+1,2j+1,k+1}^2 &= b_{2i+1,2j+1,k+1}^2 + \frac{1}{2}(-a_{i,j+1,k}^1 + a_{i+1,j,k}^1) + \frac{1}{4}(a_{i,j+1,k}^2 + a_{i+1,j,k}^2) + \frac{1}{4}\sqrt{3}(a_{i,j+1,k}^3 + a_{i+1,j,k}^3), \\ a_{2i+1,2j+1,k+1}^3 &= b_{2i+1,2j+1,k+1}^3 + \frac{1}{2}\sqrt{3}(a_{i,j+1,k}^1 - a_{i+1,j,k}^1) + \frac{1}{4}\sqrt{3}(a_{i,j+1,k}^2 + a_{i+1,j,k}^2) - \frac{1}{4}(a_{i,j+1,k}^3 + a_{i+1,j,k}^3). \end{aligned} \quad (6)$$

If we look more carefully at these formulas we note that they can be written in block matrix–vector form as

$$\begin{pmatrix} \mathbf{a}_{k+1}^{\text{even}} \\ \mathbf{a}_{k+1}^{\text{odd}} \end{pmatrix} = \begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{M} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{a}_k \\ \mathbf{b}_k \end{pmatrix}. \quad (7)$$

Here $\mathbf{a}_{k+1}^{\text{even}}$ is the vector of coefficients on level $k+1$ for which both indices are even, while the remaining coefficients on level $k+1$ are grouped together in $\mathbf{a}_{k+1}^{\text{odd}}$. Similarly, the coefficients of F_k are grouped together in \mathbf{a}_k and the coefficients of G_k in \mathbf{b}_k . The matrix \mathbf{D} is a diagonal matrix with 1s and 2s on the diagonal corresponding to the first three equations in (6), while the matrix \mathbf{M} is the matrix that guides how the coefficients in \mathbf{a}_k are combined in the remaining formulas in (6).

4.2. The decomposition algorithm

The decomposition formula is easily obtained by inverting the reconstruction formulas. From (7) we see that \mathbf{a}_k and \mathbf{b}_k can be expressed in terms of the coefficients on level $k+1$ by inverting that relation

$$\begin{pmatrix} \mathbf{a}_k \\ \mathbf{b}_k \end{pmatrix} = \begin{pmatrix} \mathbf{D}^{-1} & \mathbf{0} \\ -\mathbf{MD}^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{a}_{k+1}^{\text{even}} \\ \mathbf{a}_{k+1}^{\text{odd}} \end{pmatrix}.$$

For the coefficients of F_k we then find

$$a_{i,j,k}^1 = \frac{1}{2} a_{2i,2j,k+1}^1,$$

$$a_{i,j,k}^2 = a_{2i,2j,k+1}^2,$$

$$a_{i,j,k}^3 = a_{2i,2j,k+1}^3,$$

while the wavelet coefficients are given by

$$b_{2i+1,2j,k}^1 = a_{2i+1,2j,k+1}^1 - \frac{1}{2}(a_{2i,2j,k+1}^1 + a_{2i+2,2j,k+1}^1) - \frac{1}{2}(a_{2i,2j,k+1}^2 - a_{2i+2,2j,k+1}^2),$$

$$b_{2i+1,2j,k}^2 = a_{2i+1,2j,k+1}^2 - \frac{1}{2}(-a_{2i,2j,k+1}^1 + a_{2i+2,2j,k+1}^1) - \frac{1}{2}(a_{2i,2j,k+1}^2 + a_{2i+2,2j,k+1}^2),$$

$$b_{2i+1,2j,k}^3 = a_{2i+1,2j,k+1}^3 - \frac{1}{2}(a_{2i,2j,k+1}^3 + a_{2i+2,2j,k+1}^3),$$

$$\begin{aligned} b_{2i,2j+1,k}^1 &= a_{2i,2j+1,k+1}^1 - \frac{1}{2}(a_{2i,2j,k+1}^1 + a_{2i,2j+2,k+1}^1) \\ &\quad - \frac{1}{4}(a_{2i,2j,k+1}^2 - a_{2i,2j+2,k+1}^2) - \frac{1}{4}\sqrt{3}(a_{2i,2j,k+1}^3 - a_{2i,2j+2,k+1}^3), \end{aligned}$$

$$\begin{aligned} b_{2i,2j+1,k}^2 &= a_{2i,2j+1,k+1}^2 - \frac{1}{4}(-a_{2i,2j,k+1}^1 + a_{2i,2j+2,k+1}^1) \\ &\quad - \frac{1}{4}(a_{2i,2j,k+1}^2 + a_{2i,2j+2,k+1}^2) + \frac{1}{4}\sqrt{3}(a_{2i,2j,k+1}^3 + a_{2i,2j+2,k+1}^3), \end{aligned}$$

$$\begin{aligned} b_{2i,2j+1,k}^3 &= a_{2i,2j+1,k+1}^3 - \frac{1}{4}\sqrt{3}(-a_{2i,2j,k+1}^1 + a_{2i,2j+2,k+1}^1) \\ &\quad + \frac{1}{4}\sqrt{3}(a_{2i,2j,k+1}^2 + a_{2i,2j+2,k+1}^2) + \frac{1}{4}(a_{2i,2j,k+1}^3 + a_{2i,2j+2,k+1}^3), \end{aligned}$$

$$\begin{aligned} b_{2i+1,2j+1,k}^1 &= a_{2i+1,2j+1,k+1}^1 - \frac{1}{2}(a_{2i,2j+2,k+1}^1 + a_{2i+2,2j,k+1}^1) \\ &\quad - \frac{1}{4}(a_{2i,2j+2,k+1}^2 - a_{2i+2,2j,k+1}^2) - \frac{1}{4}\sqrt{3}(-a_{2i,2j+2,k+1}^3 + a_{2i+2,2j,k+1}^3), \end{aligned}$$

$$\begin{aligned}
b_{2i+1,2j+1,k}^2 &= a_{2i+1,2j+1,k+1}^2 - \frac{1}{4}(-a_{2i,2j+2,k+1}^1 + a_{2i+2,2j,k+1}^1) \\
&\quad - \frac{1}{4}(a_{2i,2j+2,k+1}^2 + a_{2i+2,2j,k+1}^2) - \frac{1}{4}\sqrt{3}(a_{2i,2j+2,k+1}^3 + a_{2i+2,2j,k+1}^3), \\
b_{2i+1,2j+1,k}^3 &= a_{2i+1,2j+1,k+1}^3 - \frac{1}{4}\sqrt{3}(a_{2i,2j+2,k+1}^1 - a_{2i+2,2j,k+1}^1) \\
&\quad - \frac{1}{4}\sqrt{3}(a_{2i,2j+2,k+1}^2 + a_{2i+2,2j,k+1}^2) + \frac{1}{4}(a_{2i,2j+2,k+1}^3 + a_{2i+2,2j,k+1}^3). \tag{8}
\end{aligned}$$

Note that only the coefficients associated with $\Delta_{i,j,k+1}$ and its two neighbors in Δ_k have an influence on $b_{i,j,k}^l$. This means that decomposition and reconstruction along an edge of the triangle D can be done if we know the corresponding values along that edge. This fact guarantees that there is no need for a special treatment at the boundary of the parameter domain.

We also observe that we get the same formulas for $a_{i,j,k}^1$, $a_{i,j,k}^2$, $b_{2i+1,2j,k}^1$ and $b_{2i+1,2j,k}^2$ as for the corresponding coefficients in the univariate case, taking into account that we introduced the factor $1/2^k$ in the definition of our scaling functions (cf. [2]).

5. Stability

In this section we want to prove that the wavelet basis is weakly stable. Most of the work is done in the next lemma.

Lemma 3. *Let G_k be the wavelet component of F in W_k given by*

$$G_k = Q_{k+1}F - Q_kF = \sum_{\substack{\Delta_{i,j,k+1} \in \Delta_{k+1} \setminus \Delta_k \\ l=1,2,3}} b_{i,j,k}^l \Psi_{i,j,k}^l,$$

where F is assumed to lie in $C^1(D)$. Then the wavelet coefficients are bounded by

$$|b_{2i+1,2j,k}^1| \leq 3 \left\| \frac{\partial}{\partial x} F \right\|_{L^\infty(J^1)},$$

$$|b_{2i+1,2j,k}^2| \leq 4 \left\| \frac{\partial}{\partial x} F \right\|_{L^\infty(J^1)},$$

$$|b_{2i+1,2j,k}^3| \leq 2 \left\| \frac{\partial}{\partial y} F \right\|_{L^\infty(J^1)},$$

$$|b_{2i,2j+1,k}^1| \leq \frac{3}{2}(1 + \sqrt{3}) \max \left\{ \left\| \frac{\partial}{\partial x} F \right\|_{L^\infty(J^2)}, \left\| \frac{\partial}{\partial y} F \right\|_{L^\infty(J^2)} \right\},$$

$$|b_{2i,2j+1,k}^2| \leq (2 + \sqrt{3}) \max \left\{ \left\| \frac{\partial}{\partial x} F \right\|_{L^\infty(J^2)}, \left\| \frac{\partial}{\partial y} F \right\|_{L^\infty(J^2)} \right\},$$

$$\begin{aligned}
|b_{2i,2j+1,k}^3| &\leq (3 + \sqrt{3}) \max \left\{ \left\| \frac{\partial}{\partial x} F \right\|_{L^\infty(J^2)}, \left\| \frac{\partial}{\partial y} F \right\|_{L^\infty(J^2)} \right\}, \\
|b_{2i+1,2j+1,k}^1| &\leq \frac{3}{2} (1 + \sqrt{3}) \max \left\{ \left\| \frac{\partial}{\partial x} F \right\|_{L^\infty(J^3)}, \left\| \frac{\partial}{\partial y} F \right\|_{L^\infty(J^3)} \right\}, \\
|b_{2i+1,2j+1,k}^2| &\leq (2 + \sqrt{3}) \max \left\{ \left\| \frac{\partial}{\partial x} F \right\|_{L^\infty(J^3)}, \left\| \frac{\partial}{\partial y} F \right\|_{L^\infty(J^3)} \right\}, \\
|b_{2i+1,2j+1,k}^3| &\leq (3 + \sqrt{3}) \max \left\{ \left\| \frac{\partial}{\partial x} F \right\|_{L^\infty(J^3)}, \left\| \frac{\partial}{\partial y} F \right\|_{L^\infty(J^3)} \right\}.
\end{aligned}$$

In any subtriangle T with vertices $\Delta_{i,j,k}, \Delta_{i+1,j,k}, \Delta_{i,j+1,k}$ or $\Delta_{i,j,k}, \Delta_{i+1,j,k}, \Delta_{i+1,j-1,k}$ the estimates

$$\begin{aligned}
\|G_k\|_{L^\infty(T)} &\leq \frac{1}{2^{k+1}} \left(\frac{3}{2} + \frac{\sqrt{3}}{6} \right) \max_{\substack{(i,j) \in I \\ l=1,2,3}} \{|b_{i,j,k}^l|\}, \\
\left\| \frac{\partial}{\partial x} G_k \right\|_{L^\infty(T)} &\leq \left(3 + \frac{\sqrt{3}}{3} \right) \max_{\substack{(i,j) \in I \\ l=1,2,3}} \{|b_{i,j,k}^l|\}, \\
\left\| \frac{\partial}{\partial y} G_k \right\|_{L^\infty(T)} &\leq \left(1 + \frac{5}{3} \sqrt{3} \right) \max_{\substack{(i,j) \in I \\ l=1,2,3}} \{|b_{i,j,k}^l|\} \tag{9}
\end{aligned}$$

hold. Here J^1 is the line between $\Delta_{2i,2j,k+1}$ and $\Delta_{2i+2,2j,k+1}$, while J^2 is the line between $\Delta_{2i,2j,k+1}$ and $\Delta_{2i,2j+2,k+1}$ and J^3 is the line between $\Delta_{2i,2j+2,k+1}$ and $\Delta_{2i+2,2j,k+1}$. The index set I consists of the midpoints of the edges of T .

Proof. Note that this proof makes use of Lemma 5 and its proof.

The inequalities for the coefficients are immediate consequences of the decomposition relations. Since the proof is nearly the same in all nine cases, we only verify the inequality for $b_{2i,2j+1,k}^1$. From (8) and (4) we have

$$\begin{aligned}
b_{2i,2j+1,k}^1 &= \frac{1}{2} (a_{2i,2j+1,k+1}^1 - a_{2i,2j,k+1}^1) - \frac{1}{2} (a_{2i,2j+2,k+1}^1 - a_{2i,2j+1,k+1}^1) \\
&\quad - \frac{1}{4} (a_{2i,2j,k+1}^2 - a_{2i,2j+2,k+1}^2) - \frac{1}{4} \sqrt{3} (a_{2i,2j,k+1}^3 - a_{2i,2j+2,k+1}^3) \\
&= 2^k (F(\Delta_{2i,2j+1,k+1}) - F(\Delta_{2i,2j,k+1})) - 2^k (F(\Delta_{2i,2j+2,k+1}) - F(\Delta_{2i,2j+1,k+1})) \\
&\quad - \frac{1}{4} \left(\frac{\partial}{\partial x} F(\Delta_{2i,2j,k+1}) - \frac{\partial}{\partial x} F(\Delta_{2i,2j+2,k+1}) \right) \\
&\quad - \frac{1}{4} \sqrt{3} \left(\frac{\partial}{\partial y} F(\Delta_{2i,2j,k+1}) - \frac{\partial}{\partial y} F(\Delta_{2i,2j+2,k+1}) \right). \tag{10}
\end{aligned}$$

Applying the mean-value theorem we find

$$2^k (F(\Delta_{2i,2j+1,k+1}) - F(\Delta_{2i,2j,k+1})) = \langle \nabla F(\Theta^1), \mathbf{v} \rangle = \frac{1}{2} \frac{\partial}{\partial x} F(\Theta^1) + \frac{1}{2} \sqrt{3} \frac{\partial}{\partial y} F(\Theta^1),$$

$$2^k (F(\Delta_{2i,2j+2,k+1}) - F(\Delta_{2i,2j+1,k+1})) = \langle \nabla F(\Theta^2), \mathbf{v} \rangle = \frac{1}{2} \frac{\partial}{\partial x} F(\Theta^2) + \frac{1}{2} \sqrt{3} \frac{\partial}{\partial y} F(\Theta^2),$$

where Θ^1 and Θ^2 lie in J^2 and $\mathbf{v} = (\frac{1}{2}, \frac{1}{2}\sqrt{3})$. Inserting these two expressions in (10) we obtain the required result.

The inequalities (9) for $\|G_k\|_{L^\infty(T)}$, $\|(\partial/\partial x)G_k\|_{L^\infty(T)}$ and $\|(\partial/\partial y)G_k\|_{L^\infty(T)}$ can be obtained from (22), since $G_k|_T$ can be expressed in terms of only nine nonzero basis functions

$$G_k|_T = \sum_{\substack{(i',j') \in I \\ l=1,2,3}} (b_{i',j',k}^l \Psi_{i',j',k}^l|_T).$$

Here I denotes the midpoint index set for the triangle T , i.e., if T has vertices $\Delta_{2i,2j,k+1}$, $\Delta_{2i+2,2j,k+1}$, $\Delta_{2i,2j+2,k+1}$ we have $I = \{(2i+1, 2j), (2i, 2j+1), (2i+1, 2j+1)\}$ whereas if the vertices of T are $\Delta_{2i,2j,k+1}$, $\Delta_{2i+2,2j,k+1}$, $\Delta_{2i+2,2j-2,k+1}$ we have $I = \{(2i+1, 2j), (2i+1, 2j-1), (2i+2, 2j-1)\}$.

The numerical values for the constants in bounds (9) are taken from (21) in Lemma 5. \square

The two types of triangles referred to in Lemma 3 will often be referred to as triangles of the first and second kind later in the paper.

To express the stability estimates more concisely, we define β_k by

$$\beta_k = \max_{\substack{\Delta_{i,j,k+1} \in \Delta_{k+1} \setminus \Delta_k \\ l=1,2,3}} |b_{i,j,k}^l|.$$

From Lemma 3 we then have

$$\beta_k \leq (3 + \sqrt{3}) \|F\|,$$

$$\|G_k\| \leq K_3 \beta_k,$$

where $\|\cdot\|$ is the Banach space norm defined in (3). For a function $Q_p F = Q_0 F + \sum_{k=0}^{p-1} G_k$ this means that

$$\|Q_p F - Q_0 F\| \leq \sum_{k=0}^{p-1} \|G_k\| \leq \sum_{k=0}^{p-1} K_3 \beta_k.$$

We can therefore sum up the stability as follows.

Theorem 4. *Let F be a function in $C^1(D)$, and let the approximation $Q_p F$ be represented in terms of wavelets as $Q_0 F + \sum_{k=0}^{p-1} G_k$, where*

$$G_k = Q_{k+1} F - Q_k F = \sum_{\substack{\Delta_{i,j,k+1} \in \Delta_{k+1} \setminus \Delta_k \\ l=1,2,3}} b_{i,j,k}^l \Psi_{i,j,k}^l.$$

Then

$$\frac{1}{3 + \sqrt{3}} \max_{k \leq p-1} \beta_k \leq \|Q_p F - Q_0 F\| \leq p \left(1 + \frac{5}{3} \sqrt{3}\right) \max_{k \leq p-1} \beta_k.$$

The wavelets $\{\Psi_{i,j,k}^l\}_{A_{i,j,k+1} \in A_{k+1} \setminus A_k, k \geq 0}$ therefore form a weakly stable basis for $\bigcup_{k=0}^{\infty} W_k$.

6. Uniform boundedness and denseness

In Section 3, there were two properties required of a multiresolution analysis that we did not prove, namely that the space $\bigcup_{k=0}^{\infty} V_k$ is dense in $C^1(D)$, and that the projectors Q_k are uniformly bounded. To prove this, we need some simple properties of the scaling functions.

Let T be a triangle of the first kind, with vertices $A_{i,j,k}$, $A_{i+1,j,k}$, $A_{i,j+1,k}$, and let I denote the index set $I = \{(i,j), (i+1,j), (i,j+1)\}$; then, for every $(x,y) \in T$, we have

$$\sum_{(i',j') \in I} \Phi_{i',j',k}^1(x,y) = \frac{1}{2^k}, \quad (11)$$

$$\sum_{(i',j') \in I} \frac{\partial}{\partial x} \Phi_{i',j',k}^1(x,y) = 0, \quad (12)$$

$$2 \frac{\partial}{\partial x} \Phi_{i+1,j,k}^1(x,y) + \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x,y) + \sum_{(i',j') \in I} \frac{\partial}{\partial x} \Phi_{i',j',k}^2(x,y) = 1, \quad (13)$$

$$\sqrt{3} \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x,y) + \sum_{(i',j') \in I} \frac{\partial}{\partial x} \Phi_{i',j',k}^3(x,y) = 0. \quad (14)$$

Eq. (12) is an immediate consequence of (11), while Eqs. (13) and (14) follow from the fact that there exist constants such that

$$2 \Phi_{i+1,j,k}^1(x,y) + \Phi_{i,j+1,k}^1(x,y) + \sum_{(i',j') \in I} \Phi_{i',j',k}^2(x,y) = x + \text{const},$$

$$\sqrt{3} \Phi_{i,j+1,k}^1(x,y) + \sum_{(i',j') \in I} \Phi_{i',j',k}^3(x,y) = y + \text{const}.$$

Relations similar to (11)–(14) hold for triangles of the second kind as well.

We will also need the estimates

$$|\Phi_{i,j,k}^l(x,y)| \leq \frac{1}{2^k}, \quad (15)$$

$$\left| \frac{\partial}{\partial x} \Phi_{i,j,k}^l(x,y) \right| \leq 1, \quad (16)$$

$$\left| \frac{\partial}{\partial y} \Phi_{i,j,k}^l(x,y) \right| \leq \frac{2}{3} \sqrt{3} \quad \text{for } l = 1, 2, 3 \quad (17)$$

which are immediate consequences of the Bézier representation of C_l , $(\partial/\partial x)C_l$ and $(\partial/\partial y)C_l$.

6.1. Uniform boundedness of the interpolation operators

One demand in our definition of a multiresolution analysis was that the operators Q_k should be uniformly bounded. This will be verified in the next lemma. Recall that the norm of F in $C^1(D)$ is given by

$$\|F\| = \max \left\{ \|F\|_{L^\infty(D)}, \left\| \frac{\partial F}{\partial x} \right\|_{L^\infty(D)}, \left\| \frac{\partial F}{\partial y} \right\|_{L^\infty(D)} \right\}.$$

Lemma 5. *For every $F \in C^1(D)$ and every point $(x, y) \in D$ the inequalities*

$$|(Q_k F)(x, y)| \leq K_1 \|F\|, \quad (18)$$

$$\left| \frac{\partial}{\partial x} (Q_k F)(x, y) \right| \leq (1 + \sqrt{3}) K_2 \|F\|, \quad (19)$$

$$\left| \frac{\partial}{\partial y} (Q_k F)(x, y) \right| \leq (1 + \sqrt{3}) K_3 \|F\| \quad (20)$$

hold, where the constants K_1 , K_2 and K_3 are bounded by

$$K_1 \leq \frac{3}{2} + \frac{1}{6}\sqrt{3}, \quad K_2 \leq 3 + \frac{1}{3}\sqrt{3}, \quad K_3 \leq 1 + \frac{5}{3}\sqrt{3}. \quad (21)$$

The operators Q_k are therefore uniformly bounded with $\|Q_k\| \leq 6 + \frac{8}{3}\sqrt{3}$ for all k .

Proof. Define the constants K_1 , K_2 and K_3 by

$$\begin{aligned} K_1 &= \max_{(x,y) \in D} \sum_{l=1,2,3} |\Phi_{0,0,0}^l(x, y)| + |\Phi_{1,0,0}^l(x, y)| + |\Phi_{0,1,0}^l(x, y)|, \\ K_2 &= \max_{(x,y) \in D} \sum_{l=1,2,3} \left| \frac{\partial}{\partial x} \Phi_{0,0,0}^l(x, y) \right| + \left| \frac{\partial}{\partial x} \Phi_{1,0,0}^l(x, y) \right| + \left| \frac{\partial}{\partial x} \Phi_{0,1,0}^l(x, y) \right|, \\ K_3 &= \max_{(x,y) \in D} \sum_{l=1,2,3} \left| \frac{\partial}{\partial x} \Phi_{0,0,0}^l(x, y) \right| + \left| \frac{\partial}{\partial y} \Phi_{1,0,0}^l(x, y) \right| + \left| \frac{\partial}{\partial y} \Phi_{0,1,0}^l(x, y) \right|, \end{aligned} \quad (22)$$

let T be a triangle with vertices $\Delta_{i,j,k}$, $\Delta_{i+1,j,k}$, and $\Delta_{i,j+1,k}$, and let I denote the index set $\{(i, j), (i+1, j), (i, j+1)\}$. Consider the function $Q_k F|_T$, the restriction of $Q_k F$ to the triangle T . We recall that $Q_k F|_T$ can be represented as a linear combination of nine nonzero basis functions,

$$Q_k F|_T = \sum_{\substack{(i', j') \in I \\ l=1,2,3}} a_{i', j', k}^l \Phi_{i', j', k}^l|_T,$$

where

$$a_{i', j', k}^1 = 2^k F(\Delta_{i', j', k}), \quad a_{i', j', k}^2 = \frac{\partial}{\partial x} F(\Delta_{i', j', k}), \quad a_{i', j', k}^3 = \frac{\partial}{\partial y} F(\Delta_{i', j', k}).$$

To prove (18), we let (x, y) be a point in T . Then

$$\begin{aligned}
 |Q_k F(x, y)| &= \left| \sum_{\substack{(i', j') \in I \\ l=1,2,3}} a_{i', j', k}^l \Phi_{i', j', k}^l(x, y) \right| \leq \sum_{\substack{(i', j') \in I \\ l=1,2,3}} |a_{i', j', k}^l| |\Phi_{i', j', k}^l(x, y)| \\
 &\leq \max_{\substack{(i', j') \in I \\ l=1,2,3}} |a_{i', j', k}^l| \sum_{\substack{(i', j') \in I \\ l=1,2,3}} |\Phi_{i', j', k}^l(x, y)| \\
 &\leq \max_{\substack{(i', j') \in I \\ l=1,2,3}} |a_{i', j', k}^l| \frac{1}{2^k} \max_{(x, y) \in D} \sum_{l=1,2,3} |\Phi_{0,0,0}^l(x, y)| + |\Phi_{1,0,0}^l(x, y)| + |\Phi_{0,1,0}^l(x, y)| \\
 &\leq 2^k \|F\| \frac{1}{2^k} K_1 = K_1 \|F\|.
 \end{aligned}$$

The other two inequalities (19) and (20) are similar, so we only give the proof of the first one. As before, we let (x, y) be a point in T . We have

$$\begin{aligned}
 \left| \frac{\partial}{\partial x} Q_k F(x, y) \right| &= \left| \sum_{\substack{(i', j') \in I \\ l=1,2,3}} a_{i', j', k}^l \frac{\partial}{\partial x} \Phi_{i', j', k}^l(x, y) \right| \\
 &\leq \left| \sum_{(i', j') \in I} a_{i', j', k}^1 \frac{\partial}{\partial x} \Phi_{i', j', k}^1(x, y) \right| + \left| \sum_{\substack{(i', j') \in I \\ l=2,3}} a_{i', j', k}^l \frac{\partial}{\partial x} \Phi_{i', j', k}^l(x, y) \right|. \quad (23)
 \end{aligned}$$

Using the mean-value theorem and (12) we get for the first part

$$\begin{aligned}
 \left| \sum_{(i', j') \in I} a_{i', j', k}^1 \frac{\partial}{\partial x} \Phi_{i', j', k}^1(x, y) \right| &= \left| \sum_{(i', j') \in I} 2^k F(\Delta_{i', j', k}) \frac{\partial}{\partial x} \Phi_{i', j', k}^1(x, y) \right| \\
 &= 2^k \left| (F(\Delta_{i+1, j, k}) - F(\Delta_{i, j, k})) \frac{\partial}{\partial x} \Phi_{i+1, j, k}^1(x, y) \right. \\
 &\quad \left. + (F(\Delta_{i, j+1, k}) - F(\Delta_{i, j, k})) \frac{\partial}{\partial x} \Phi_{i, j+1, k}^1(x, y) \right| \\
 &= 2 \left| \langle \nabla F(\Theta^1), \mathbf{v}_1 \rangle \frac{\partial}{\partial x} \Phi_{i+1, j, k}^1(x, y) \right. \\
 &\quad \left. + \langle \nabla F(\Theta^2), \mathbf{v}_2 \rangle \frac{\partial}{\partial x} \Phi_{i, j+1, k}^1(x, y) \right|,
 \end{aligned}$$

where $\mathbf{v}_1' = (1, 0)$ and $\mathbf{v}_2' = (\frac{1}{2}, \frac{1}{2}\sqrt{3})$. The points Θ^1, Θ^2 lie on the lines between $\Delta_{i,j,k}$ and $\Delta_{i+1,j,k}$, respectively, $\Delta_{i,j,k}$ and $\Delta_{i,j+1,k}$. But this leads to

$$2 \left| \frac{\partial}{\partial x} F(\Theta^1) \frac{\partial}{\partial x} \Phi_{i+1,j,k}^1(x, y) + \left(\frac{1}{2} \frac{\partial}{\partial x} F(\Theta^2) + \frac{1}{2} \sqrt{3} \frac{\partial}{\partial y} F(\Theta^2) \right) \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x, y) \right| \\ \leq 2 \|F\| \left| \frac{\partial}{\partial x} \Phi_{i+1,j,k}^1(x, y) \right| + (1 + \sqrt{3}) \|F\| \left| \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x, y) \right|.$$

The second part in (23) can be expressed as

$$\left| \sum_{(i',j') \in I} \frac{\partial}{\partial x} F(\Delta_{i',j',k}) \frac{\partial}{\partial x} \Phi_{i',j',k}^2(x, y) + \sum_{(i',j') \in I} \frac{\partial}{\partial y} F(\Delta_{i',j',k}) \frac{\partial}{\partial x} \Phi_{i',j',k}^3(x, y) \right| \\ \leq \|F\| \sum_{(i',j') \in I} \left| \frac{\partial}{\partial x} \Phi_{i',j',k}^2(x, y) \right| + \|F\| \sum_{(i',j') \in I} \left| \frac{\partial}{\partial x} \Phi_{i',j',k}^3(x, y) \right|$$

and hence we obtain

$$\left| \frac{\partial}{\partial x} Q_k F(x, y) \right| \leq (1 + \sqrt{3}) K_2 \|F\|.$$

If T is a triangle of the second kind, with vertices $\Delta_{i,j,k}, \Delta_{i+1,j,k}, \Delta_{i+1,j-1,k}$, a similar argument can be used.

Bounds (21) follow from the Bézier representation of C_1, C_2 and C_3 , see (Fig. 1). \square

6.2. The spaces V_k are dense in $C^1(D)$

If we can verify that every function in $C^1(D)$ can be approximated by functions from $\bigcup_{k=0}^{\infty} V_k$ with arbitrarily small error, we have succeeded in showing that the spaces $\{V_k\}_{k=0}^{\infty}$ form a multiresolution analysis. The following lemma confirms this.

Lemma 6. *The spaces $\{V_k\}_{k=0}^{\infty}$ are dense in the Banach space $(C^1(D), \|\cdot\|)$.*

Proof. It is sufficient to show that $\lim_{k \rightarrow \infty} \|F - Q_k F\| = 0$ for every function $F \in C^1(D)$. As in the proof of Lemma 5, let T be a triangle with vertices in Δ_k , and let I denote the index set $I = \{(i, j), (i+1, j), (i, j+1)\}$. For any point $(x, y) \in T$ we have from relation (11) that

$$|F(x, y) - Q_k F(x, y)| = \left| F(x, y) 2^k \sum_{(i',j') \in I} \Phi_{i',j',k}^1(x, y) - \sum_{(i',j') \in I} 2^k F(\Delta_{i',j',k}) \Phi_{i',j',k}^1(x, y) \right. \\ \left. - \sum_{(i',j') \in I} \frac{\partial}{\partial x} F(\Delta_{i',j',k}) \Phi_{i',j',k}^2(x, y) - \sum_{(i',j') \in I} \frac{\partial}{\partial y} F(\Delta_{i',j',k}) \Phi_{i',j',k}^3(x, y) \right|$$

$$\leq 2^k \left| \sum_{(i',j') \in I} (F(x, y) - F(\Delta_{i',j',k})) \Phi_{i',j',k}^1(x, y) \right| \\ + \left| \sum_{(i',j') \in I} \frac{\partial}{\partial x} F(\Delta_{i',j',k}) \Phi_{i',j',k}^2(x, y) \right| + \left| \sum_{(i',j') \in I} \frac{\partial}{\partial y} F(\Delta_{i',j',k}) \Phi_{i',j',k}^3(x, y) \right|.$$

An analogous inequality holds for triangles of the other kind, so from (15) and the uniform continuity of f on D we obtain

$$\lim_{k \rightarrow \infty} \|F - Q_k F\|_{L^\infty(D)} = 0.$$

It remains to show that the two derivatives of $Q_k f$ converge to f ; we will only give the proof for the derivative with respect to x . Let $(x, y) \in T$, then

$$e = \left| \frac{\partial}{\partial x} F(x, y) - \frac{\partial}{\partial x} Q_k F(x, y) \right| \\ \stackrel{(13)}{=} \left| \frac{\partial}{\partial x} F(x, y) \left(2 \frac{\partial}{\partial x} \Phi_{i+1,j,k}^1(x, y) + \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x, y) + \sum_{(i',j') \in I} \frac{\partial}{\partial x} \Phi_{i',j',k}^2(x, y) \right) \right. \\ \left. - \sum_{(i',j') \in I} 2^k F(\Delta_{i',j',k}) \frac{\partial}{\partial x} \Phi_{i',j',k}^1(x, y) - \sum_{(i',j') \in I} \frac{\partial}{\partial x} F(\Delta_{i',j',k}) \frac{\partial}{\partial x} \Phi_{i',j',k}^2(x, y) \right. \\ \left. - \sum_{(i',j') \in I} \frac{\partial}{\partial y} F(\Delta_{i',j',k}) \frac{\partial}{\partial x} \Phi_{i',j',k}^3(x, y) \right|.$$

Using (12) and (14) we can replace $(\partial/\partial x) \Phi_{i,j,k}^1$ and $(\partial/\partial x) \Phi_{i,j,k}^3$ in this expression and obtain

$$e = \left| 2 \frac{\partial}{\partial x} F(x, y) \frac{\partial}{\partial x} \Phi_{i+1,j,k}^1(x, y) + \frac{\partial}{\partial x} F(x, y) \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x, y) \right. \\ \left. + \sum_{(i',j') \in I} \left(\frac{\partial}{\partial x} F(x, y) - \frac{\partial}{\partial x} F(\Delta_{i',j',k}) \right) \frac{\partial}{\partial x} \Phi_{i',j',k}^2(x, y) \right. \\ \left. - 2^k (F(\Delta_{i+1,j,k}) - F(\Delta_{i,j,k})) \frac{\partial}{\partial x} \Phi_{i+1,j,k}^1(x, y) - 2^k (F(\Delta_{i,j+1,k}) - F(\Delta_{i,j,k})) \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x, y) \right. \\ \left. + \sqrt{3} \frac{\partial}{\partial y} F(\Delta_{i,j,k}) \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x, y) - \left(\frac{\partial}{\partial y} F(\Delta_{i+1,j,k}) - \frac{\partial}{\partial y} F(\Delta_{i,j,k}) \right) \frac{\partial}{\partial x} \Phi_{i+1,j,k}^3(x, y) \right. \\ \left. - \left(\frac{\partial}{\partial y} F(\Delta_{i,j+1,k}) - \frac{\partial}{\partial y} F(\Delta_{i,j,k}) \right) \frac{\partial}{\partial x} \Phi_{i,j+1,k}^3(x, y) \right|.$$

Applying the mean-value theorem and introducing the notation $\mathbf{v}_1^t = (1, 0)$ and $\mathbf{v}_2^t = (\frac{1}{2}, \frac{1}{2}\sqrt{3})$ we get

$$\begin{aligned} 2^k(F(\Delta_{i+1,j,k}) - F(\Delta_{i,j,k})) \frac{\partial}{\partial x} \Phi_{i+1,j,k}^1(x, y) &= 2\langle \nabla F(\Theta^1), \mathbf{v}_1 \rangle \frac{\partial}{\partial x} \Phi_{i+1,j,k}^1(x, y) \\ &= 2 \frac{\partial}{\partial x} F(\Theta^1) \frac{\partial}{\partial x} \Phi_{i+1,j,k}^1(x, y), \\ 2^k(F(\Delta_{i,j+1,k}) - F(\Delta_{i,j,k})) \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x, y) &= 2\langle \nabla F(\Theta^2), \mathbf{v}_2 \rangle \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x, y) \\ &= \left(\frac{\partial}{\partial x} F(\Theta^2) + \sqrt{3} \frac{\partial}{\partial y} F(\Theta^2) \right) \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x, y), \end{aligned}$$

where the points Θ^1 and Θ^2 again lie on the lines between $\Delta_{i,j,k}$ and $\Delta_{i+1,j,k}$, respectively, $\Delta_{i,j,k}$ and $\Delta_{i,j+1,k}$. Hence $|(\partial/\partial x)F(x, y) - (\partial/\partial x)Q_k F(x, y)|$, $(x, y) \in T$ can be expressed as

$$\begin{aligned} &\left| 2 \left(\frac{\partial}{\partial x} F(x, y) - \frac{\partial}{\partial x} F(\Theta^1) \right) \frac{\partial}{\partial x} \Phi_{i+1,j,k}^1(x, y) + \left(\frac{\partial}{\partial x} F(x, y) - \frac{\partial}{\partial x} F(\Theta^2) \right) \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x, y) \right. \\ &\quad + \sqrt{3} \left(\frac{\partial}{\partial y} F(\Delta_{i,j,k}) - \frac{\partial}{\partial y} F(\Theta^2) \right) \frac{\partial}{\partial x} \Phi_{i,j+1,k}^1(x, y) \\ &\quad + \sum_{(i',j') \in I} \left(\frac{\partial}{\partial x} F(x, y) - \frac{\partial}{\partial x} F(\Delta_{i',j',k}) \right) \frac{\partial}{\partial x} \Phi_{i',j',k}^2(x, y) \\ &\quad - \left(\frac{\partial}{\partial y} F(\Delta_{i+1,j,k}) - \frac{\partial}{\partial y} F(\Delta_{i,j,k}) \right) \frac{\partial}{\partial x} \Phi_{i+1,j,k}^3(x, y) \\ &\quad \left. - \left(\frac{\partial}{\partial y} F(\Delta_{i,j+1,k}) - \frac{\partial}{\partial y} F(\Delta_{i,j,k}) \right) \frac{\partial}{\partial x} \Phi_{i,j+1,k}^3(x, y) \right|. \end{aligned}$$

Making use of (16) and the uniform continuity of the partial derivatives on D and doing the equivalent work for triangles of the second kind, it follows that

$$\lim_{k \rightarrow \infty} \left\| \frac{\partial}{\partial x} F - \frac{\partial}{\partial x} Q_k F \right\|_{L^\infty(D)} = 0.$$

Together with the analogous result for the derivative with respect to y this completes the proof.

References

- [1] J.M. Carnicer, W. Dahmen, J.M. Peña, Local decomposition of refinable spaces, *Appl. Comp. Harm. Anal.* 3 (1996) 127–153.
- [2] M. Dæhlen, T. Lyche, K. Mørken, R. Schneider, H.-P. Seidel, Multiresolution analysis based on quadratic Hermite interpolation — Part 1: piecewise polynomial Curves, in preparation.
- [3] G. Faber, Über stetige Funktionen, *Math. Ann.* 66 (1909) 81–94.
- [4] G. Farin, Triangular Bernstein–Bézier patches, *Comput. Aided Geom. Design* 3 (1986) 83–127.
- [5] M.S. Floater, E.G. Quak, Piecewise linear prewavelets on arbitrary triangulations, *Numer. Math.* 82 (1999) 221–252.
- [6] T.X. He, C^1 -quadratic macroelements and C^1 -orthogonal multiresolution analyses in 2D, manuscript, Department of Mathematics, Illinois Wesleyan University, 1999.
- [7] A. Le Méhauté, Some Families of triangular finite elements which can provide sequences of nested spaces, Preprint.

- [8] M. Lounsbery, T. DeRose, J. Warren, Multiresolution analysis for surfaces of arbitrary topological type, *ACM Trans. Graphics* 16 (1997) 34–73.
- [9] P. Oswald, *Multilevel Finite Element Approximation: Theory and Applications*, Teubner, Stuttgart, 1994.
- [10] M.J.D. Powell, M.A. Sabin, Piecewise quadratic approximations on triangles, *ACM Trans Math. Software* 3 (4) (1977) 316–325.
- [11] P. Schröder, W. Sweldens, Spherical wavelets: efficiently representing functions on the sphere, *Comput. Graphics (Proc. Siggraph '95)*, pp. 161–172, 1995.