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Asymptotic expansions for multivariate polynomial approximation

Guido Walz *

Department of Mathematics and Computer Science, University of Mannheim, D-68131 Mannheim, Germany

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Abstract

In this paper the approximation of multivariate functions by (multivariate) Bernstein polynomials is considered. Building on recent work of Lai (J. Approx. Theory 70 (1992) 229–242), we can prove that the sequence of these Bernstein polynomials possesses an asymptotic expansion with respect to the index n. This generalizes a corresponding result due to Costabile et al. (BIT 36 (1996) 676–687) on univariate Bernstein polynomials, providing at the same time a new proof for it. After having shown the existence of an asymptotic expansion we can apply an extrapolation algorithm which accelerates the convergence of the Bernstein polynomials considerably; this leads to a new and very efficient method for polynomial approximation of multivariate functions. Numerical examples illustrate our approach.

Keywords: Asymptotic expansion; Bernstein operator; Convergence acceleration; Extrapolation; Multivariate polynomial approximation

1. Introduction and preliminaries

One of the fundamental questions in extrapolation theory is the following one: Can the convergence of a given sequence be accelerated by a suitable extrapolation algorithm or not? The oldest and up the present day most widespread criterion for a positive answer to this question is the existence of an asymptotic expansion for the sequence to be accelerated (see the next section for exact definitions).

This is the reason why the terms asymptotic expansion and extrapolation are so deeply connected. Now, the next question is: Where in Applied Analysis do exist sequences with this property? It is the main reason of this paper, which is both a survey and a research paper, to convince the reader that this is the case also in a field where this was not so well-known until now: Approximation of multivariate functions by polynomials.

* Tel.: +49-621-292-5341; fax: +49-621-292-1064.

E-mail address: walz@math.uni-mannheim.de (G. Walz)

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We first shortly review what kind of asymptotic expansion we are looking at, and what the corresponding extrapolation process looks like. For more details on these topics, see [7].

Definition. Let there be given a sequence of real or complex numbers $\{\sigma_n\}$ and natural numbers m and N. The sequence $\{\sigma_n\}$ is said to possess an asymptotic expansion of order m, if each σ_n for $n > N$ can be written in the form

$$\sigma_n = \sum_{\mu=0}^{m} \frac{c_\mu}{n^{\rho_\mu}} + \mathrm{o}(n^{-\mathrm{Re}\,\rho_m}) = c_0 + \sum_{\mu=1}^{m} \frac{c_\mu}{n^{\rho_\mu}} + \mathrm{o}(n^{-\mathrm{Re}\,\rho_m}) \quad \text{for } n \to \infty. \tag{1}$$

Here, the exponents $\{\rho_\mu\}$ are real or complex numbers with the property

$$\rho_0 = 0 \quad \text{and} \quad \mathrm{Re}\,\rho_\mu < \mathrm{Re}\,\rho_{\mu+1} \quad \text{for all } \mu \in \mathbb{N}_0.$$

Moreover, if a sequence $\{\sigma_n\}$ possesses an expansion of type (1) for all $m \in \mathbb{N}$, then we say that the expansion is of arbitrary order, and write

$$\sigma_n = c_0 + \sum_{\mu=1} \frac{c_\mu}{n^{\rho_\mu}} \tag{2}$$

for short.

Asymptotic expansions if the type (1) are sometimes also denoted in more detail as logarithmic asymptotic expansions (see [5] or [7]). In this paper, we will use the abbrevatied notation asymptotic expansion only.

It is well known (cf., e.g., [1,7]) that the basic idea of extrapolation applied to such sequences is to compute the values of σ_n for several choices of n, say $n=n_0 < n_1 < n_2 < \cdots$, and to combine them in order to obtain new sequences, which converge faster than the original ones. For many applications it is convenient to choose the sequence $\{n_i\}$ not just anyhow, but as a geometric progression: With natural numbers n_0 and b, $b \geqslant 2$, we put

$$n_i := n_0 b^i, \qquad i = 0, 1, 2, \ldots\ . \tag{3}$$

Then the extrapolation process reads as follows (cf. (4)):

Lemma 1. *Let there be given a sequence $\{\sigma_n\}$, which possesses an asymptotic expansion of the form* (1), *and a sequence of natural numbers $\{n_i\}$, satisfying* (3). *Furthermore, choose some $K \in \mathbb{N}$, $K \leqslant m$ and define for $k = 0, \ldots, K$ new sequences $\{y_i^{(k)}\}_{i\in\mathbb{N}}$ through the process*

$$y_i^{(0)} = \sigma_{n_i}, \quad i = 0, 1, \ldots,$$
$$y_i^{(k)} = \frac{b^{\rho_k} \cdot y_{i+1}^{(k-1)} - y_i^{(k-1)}}{b^{\rho_k} - 1} \qquad \begin{cases} k = 1, 2, \ldots, K, \\ i = 0, 1, \ldots\ . \end{cases} \tag{4}$$

Then each of the sequences $\{y_i^{(k)}\}_{i\in\mathbb{N}}$ possesses an asymptotic expansion of the form

$$y_i^{(k)} = c_0 + \sum_{\mu=k+1}^{m} \frac{c_\mu^{(k)}}{n_i^{\rho_\mu}} + \mathrm{o}(n_i^{-\mathrm{Re}\,\rho_m}) \quad \textit{for } n_i \to \infty \tag{5}$$

with coefficients $c_\mu^{(k)}$ independent of n_i. In particular, each of the sequences $\{y_i^{(k)}\}$ converges faster to the limit c_0 than its precedessor.

So, the message is: If one has a convergent numerical process of whatever kind, say, a discretized differential equation or a quadrature formula, one should always check whether the output of this process has a asymptotic expansion. Experience says that this is indeed the case in much more situations than commonly expected or known.

To illustrate and to support this remark, we consider in this paper the Bernstein polynomial operators (or Bernstein polynomials), which are in Approximation Theory well known as a tool for polynomial approximation of functions. It will be shown that the sequence of these operators also possesses an asymptotic expansion, and thus that their order of convergence can be improved considerably using extrapolation. In the univariate case, this result was proved quite recently in [2] (see Theorem 2). As the main new contribution, we develop an anloguous result for the multivariate case. Since the proof in [2] cannot be adopted for the multivariate case, we had to develop a new approach, building on results published in [3]. This provides at the same time a new proof also for the univariate case.

2. Asymptotic expansion for the Bernstein operator

We first briefly review some results on the univariate case and then prove our main result (Theorem 5) on the multivariate one.

The sequence of Bernstein operators

$$B_n(f;x) := \sum_{v=0}^{n} f\left(\frac{v}{n}\right) B_{n,v}(x), \tag{6}$$

defined for any $f \in C[0,1]$, converges uniformly to f on $[0,1]$. Here, $B_{n,v}(x)$ denotes the (univariate) Bernstein polynomial

$$B_{n,v}(x) := \binom{n}{v} x^v (1-x)^{n-v}.$$

However, as shown by Voronowskaja [6], we have

$$\lim_{n\to\infty} n\,(B_n(f;x) - f(x)) = \frac{x(1-x)}{2} f^{(2)}(x) \tag{7}$$

in each point $x \in [0,1]$ where $f^{(2)}(x)$ exists.

This means that already quadratic polynomials are not reproduced by $B_n(f;\cdot)$, and that the order of convergence is not better than $\mathrm{O}(1/n)$. Therefore, several attempts have been made to improve this order of convergence, see [2] for an overview and some references.

In view of asymptotic expansion and extrapolation theory, a big step was done recently in [2], who established the asymptotic expansion for the Bernstein operator B_n. Their main result can be stated as follows:

Theorem 2. *Let $f \in C^{2k}[0,1]$ with some $k \in \mathbb{N}$. Then the sequence $\{B_n(f;x)\}$, defined in* (6), *possesses an asymptotic expansion of the form*

$$B_n(f;x) = f(x) + \sum_{v=1}^{k} \frac{c_v(x)}{n^v} + \mathrm{o}(n^{-k}) \quad \textit{for } n \to \infty.$$

It is our goal to develop an analoguous result for the multivariate case.

However, we do *not* generalize the proof given in [2], which could by the way be shortened considerably by using the asymptotic results already to be found in [4]. Instead, we will make use of some asymptotic relations for multivariate Bernstein polynomials, established quite recently in [3].

Let $v^0,\dots,v^s$ be $(s+1)$ distinct points in $\mathbb{R}^s$, such that the volume of the s-simplex $T:=\langle v^0,\dots,v^s\rangle$ is positive. For each point $\mathbf{x}\in T$, we denote by $(\lambda_0,\dots,\lambda_s)$ the barycentric coordinates of $\mathbf{x}$ w.r.t. T.

It is well known that any polynomial $p_n(\mathbf{x})$ of total degree n can be expressed by using the basic functions

$$B_\alpha(\lambda):=\frac{|\alpha|!}{\alpha!}\lambda^\alpha,\quad \alpha\in\mathbb{N}_0^{s+1}\quad\text{with } |\alpha|=n$$

in the form

$$p_n(\mathbf{x})=\sum_{\substack{|\alpha|=n\\ \alpha\in\mathbb{N}_0^{s+1}}} c_\alpha B_\alpha(\lambda),\quad \mathbf{x}\in T.$$

Here, as usual, for any $\alpha=(\alpha_0,\dots,\alpha_s)\in\mathbb{N}_0^{s+1}$, we set $|\alpha|=\alpha_0+\cdots+\alpha_s$ and $\alpha!=\alpha_0!\cdots\alpha_s!$. Also, it is $\lambda^\alpha=\lambda_0^{\alpha_0}\cdots\lambda_s^{\alpha_s}$.

For each $\alpha\in\mathbb{N}_0^{s+1}$, denote by $\mathbf{x}_\alpha$ the point

$$\mathbf{x}_\alpha:=\frac{1}{|\alpha|}\sum_{i=0}^{s}\alpha_i v^i.$$

We consider the approximation of a given function $f\in C(T)$ by the multivariate Bernstein polynomial

$$B_n(f;\mathbf{x}):=\sum_{\substack{|\alpha|=n\\ \alpha\in\mathbb{N}_0^{s+1}}} f(\mathbf{x}_\alpha)B_\alpha(\lambda). \tag{8}$$

As in [3], we introduce the auxiliary polynomials

$$S_\gamma^n(\mathbf{x})=n^{|\gamma|}\sum_{\substack{|\alpha|=n\\ \alpha\in\mathbb{N}_0^{s+1}}}(\mathbf{x}_\alpha-\mathbf{x})^\gamma B_\alpha(\lambda)$$

for $\gamma\in\mathbb{N}_0^s$. The following results, which we will make use of below, were proved in [3].

Theorem 3. *The polynomials S_γ^n possess the explicit representations*

$$S_\gamma^n\equiv 0\quad \textit{for } |\gamma|\leqslant 1,$$

$$S_\gamma^n(\mathbf{x})=n\sum_{j=0}^{s}\lambda_j(v^j-\mathbf{x})^\gamma\quad \textit{for } |\gamma|=2,$$

and

$$S_\gamma^n(\mathbf{x})=\sum_{\mu=1}^{|\gamma|-2}\sum_{\substack{\beta^1,\dots,\beta^\mu\in\mathbb{N}_0^s\\ \beta^1+\cdots+\beta^\mu=\gamma\\ |\beta^i|\geqslant 2,\, i=1,\dots,\mu}} n(n-1)\cdots(n-\mu+1)\prod_{i=1}^{\mu}\left(\sum_{j=0}^{s}\lambda_j(v^j-\mathbf{x})^{\beta^i}\right)\quad \textit{for } |\gamma|\geqslant 3.$$

Theorem 4. *For $k \in \mathbb{N}$ and $f \in C^{2k}(T)$, we have*

$$\lim_{n\to\infty} n^k \left[B_n(f;\mathbf{x}) - f(\mathbf{x}) - \sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma|\leqslant 2k-1}} \frac{1}{\gamma!}\frac{S_\gamma^n(\mathbf{x})}{n^{|\gamma|}} D^\gamma f(\mathbf{x}) \right]$$

$$= \sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma|=2k}} \frac{1}{\gamma!} \sum_{\substack{\beta^1,\ldots,\beta^k\in\mathbb{N}_0^s \\ \beta^1+\cdots+\beta^k=\gamma \\ |\beta^i|=2,\, i=1,\ldots,k}} \prod_{i=1}^{k} \left(\sum_{j=0}^{s} \lambda_j (v^j - \mathbf{x})^{\beta^i} \right) D^\gamma f(\mathbf{x}). \tag{9}$$

Building on these auxiliary results, we can now state and prove our main theorem. Note that although Theorem 4 is a deep and nice result on the asymptotic behavior of the multivariate Bernstein approximants, it does still not yet prove the asymptotic expansion. To do this, a careful analysis of the coefficient functions in (9) is necessary.

Theorem 5. *Let $f \in C^{2k}(T)$ with some $k \in \mathbb{N}$. Then the sequence of Bernstein approximants $\{B_n(f;\mathbf{x})\}$, defined in* (8), *possesses an asymptotic expansion of the form*

$$B_n(f;\mathbf{x}) = f(\mathbf{x}) + \sum_{v=1}^{k} \frac{c_v(\mathbf{x})}{n^v} + \mathrm{o}(n^{-k}) \quad \textit{for } n\to\infty. \tag{10}$$

The coefficient functions $c_v(\mathbf{x})$ can be given explicitly; *we have*

$$c_v(\mathbf{x}) = \sum_{\substack{\gamma\in\mathbb{N}_0^s \\ v+1\leqslant|\gamma|\leqslant 2v}} \frac{1}{\gamma!} \sum_{\mu=|\gamma|-v}^{\lfloor|\gamma|/2\rfloor} \alpha_{|\gamma|-v,\mu} \sum_{\substack{\beta^1,\ldots,\beta^\mu\in\mathbb{N}_0^s \\ \beta^1+\cdots+\beta^\mu=\gamma \\ |\beta^i|\geqslant 2,\, i=1,\ldots,\mu}} \prod_{i=1}^{\mu} \left(\sum_{j=0}^{s} \lambda_j (v^j - \mathbf{x})^{\beta^i} \right) \tag{11}$$

with recursively computable numbers $\alpha_{i,\mu}$, see (14) *below.*

Proof. We use the following result, to be found for example in [7].

A sequence $\{B_n\}$ possesses an asymptotic expansion of the desired form, if and only if for $m = 1,\ldots,k$,

$$\lim_{n\to\infty} n^m \left\{ B_n - f - \sum_{v=1}^{m-1} \frac{c_v}{n^v} \right\} =: c_m \tag{12}$$

exists and is different from zero. (Here and below, we set empty sums equal to zero.)

From (12), it is clear that the results due to Lai, as quoted above, are a big step towards the proof of our Theorem, but as be seen below, there is still something to do.

We first have to make a further analysis of the functions S_γ^n. It is clear that if we have points $\beta^1,\ldots,\beta^\mu \in \mathbb{N}_0^s$ with $|\beta^i|\geqslant 2$, $i=1,\ldots,\mu$, and if $\mu > |\gamma|/2$, then

$$|\beta^1 + \cdots + \beta^\mu| > |\gamma|.$$

This means that

$$\frac{S_\gamma^n(\mathbf{x})}{n^{|\gamma|}} = \sum_{\mu=1}^{|\gamma|-2} \frac{n(n-1)\cdots(n-\mu+1)}{n^{|\gamma|}} \sum_{\substack{\beta^1,\ldots,\beta^\mu\in\mathbb{N}_0^s \\ \beta^1+\cdots+\beta^\mu=\gamma \\ |\beta^i|\geqslant 2,\, i=1,\ldots,\mu}} \prod_{i=1}^{\mu}\left(\sum_{j=0}^{s}\lambda_j(v^j-\mathbf{x})^{\beta^i}\right)$$

$$= \sum_{\mu=1}^{\lfloor|\gamma|/2\rfloor} \frac{n(n-1)\cdots(n-\mu+1)}{n^{|\gamma|}} \sum_{\substack{\beta^1,\ldots,\beta^\mu\in\mathbb{N}_0^s \\ \beta^1+\cdots+\beta^\mu=\gamma \\ |\beta^i|\geqslant 2,\, i=1,\ldots,\mu}} \prod_{i=1}^{\mu}\left(\sum_{j=0}^{s}\lambda_j(v^j-\mathbf{x})^{\beta^i}\right) \tag{13}$$

with

$$\left\lfloor \frac{|\gamma|}{2} \right\rfloor = \begin{cases} \frac{|\gamma|}{2}, & |\gamma| \text{ even}, \\ \frac{|\gamma|-1}{2}, & |\gamma| \text{ odd}. \end{cases}$$

Next, we observe that the expression

$$n(n-1)\cdots(n-\mu+1)$$

is a polynomial of exact degree μ in n, say

$$n(n-1)\cdots(n-\mu+1) = \sum_{i=1}^{\mu} \alpha_{i,\mu} n^i,$$

with coefficients $\alpha_{i,\mu}$, which can be computed by the recursion

$$\alpha_{1,1} = 1, \quad \alpha_{i,1} = 0, \;\; i \neq 1,$$

and

$$\alpha_{i,\mu+1} := \alpha_{i-1,\mu} - \mu\alpha_{i,\mu}, \quad \mu \geqslant 1, \quad 1 \leqslant i \leqslant \mu+1. \tag{14}$$

In particular,

$$\alpha_{\mu,\mu} = 1 \quad \text{and} \quad \alpha_{1,\mu} = (-1)^{\mu-1}(\mu-1)! \tag{15}$$

for all μ.

Together with (13), it follows that

$$\frac{S_\gamma^n(\mathbf{x})}{n^{|\gamma|}} = \sum_{\mu=1}^{\lfloor|\gamma|/2\rfloor} \sum_{i=1}^{\mu} \frac{\alpha_{i,\mu}}{n^{|\gamma|-i}} \sum_{\substack{\beta^1,\ldots,\beta^\mu\in\mathbb{N}_0^s \\ \beta^1+\cdots+\beta^\mu=\gamma \\ |\beta^i|\geqslant 2,\, i=1,\ldots,\mu}} \prod_{i=1}^{\mu}\left(\sum_{j=0}^{s}\lambda_j(v^j-\mathbf{x})^{\beta^i}\right). \tag{16}$$

Rearranging this according to powers of n, we obtain

$$\frac{S_\gamma^n(\mathbf{x})}{n^{|\gamma|}} = \sum_{l=\lfloor|\gamma|+1/2\rfloor}^{|\gamma|-1} \frac{\delta_{l,\gamma}(\mathbf{x})}{n^l} \tag{17}$$

with coefficient functions $\delta_{l,\gamma}$, which do not depend on n.

For later use, we note that for $|\gamma|$ even, say $|\gamma| = 2v$, the coefficient of the lowest power of n, $\delta_{v,\gamma}$, can be given explicitly: From (16), we deduce that

$$\delta_{v,\gamma}(\mathbf{x}) = \alpha_{v,v} \sum_{\substack{\beta^1,\dots,\beta^\mu \in \mathbb{N}_0^s \\ \beta^1+\cdots+\beta^\mu=\gamma \\ |\beta^i| \geqslant 2,\, i=1,\dots,\mu}} \prod_{i=1}^{\mu} \left(\sum_{j=0}^{s} \lambda_j (v^j - \mathbf{x})^{\beta^i} \right)$$

$$= \sum_{\substack{\beta^1,\dots,\beta^v \in \mathbb{N}_0^s \\ \beta^1+\cdots+\beta^v=2v \\ |\beta^i|=2,\, i=1,\dots,v}} \prod_{i=1}^{v} \left(\sum_{j=0}^{s} \lambda_j (v^j - \mathbf{x})^{\beta^i} \right). \tag{18}$$

From (17), it follows that the sum over all these expressions itself is of the form

$$\sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma| \leqslant 2k-1}} \frac{1}{\gamma!} \frac{S_\gamma^n(\mathbf{x})}{n^{|\gamma|}} D^\gamma f(\mathbf{x}) = \frac{d_{1,k}(\mathbf{x})}{n} + \frac{d_{2,k}(\mathbf{x})}{n^2} + \cdots + \frac{d_{k,k}(\mathbf{x})}{n^k} + \mathrm{O}(n^{-(k+1)}). \tag{19}$$

We now make the

Claim. *For all* $v \leqslant k$, *the coefficient functions* $d_{j,v}$ *in* (19) *satisfy*

$$d_{j,v}(\mathbf{x}) = d_{j,v-1}(\mathbf{x}) \quad j = 1,\dots,v-2$$

and

$$d_{v-1,v}(\mathbf{x}) = d_{v-1,v-1}(\mathbf{x}) + \sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma|=2v-2}} \delta_{v-1,\gamma}(\mathbf{x}) \tag{20}$$

with $\delta_{v-1,2v-1}$ *from* (17).

Proof of Claim. The proof is by induction. For $k = 1$, there is nothing to show, while for $k = 2$, the only relation to prove is

$$d_{1,2}(\mathbf{x}) = d_{1,1}(\mathbf{x}) + \sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma|=2}} \delta_{1,\gamma}(\mathbf{x}).$$

But this is true, since $d_{1,1} = 0$.

Now we assume that the claim is true for v, and prove it for $v + 1$.

From (17) and (19) and the induction hypothesis,

$$\sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma| \leqslant 2v+1}} \frac{1}{\gamma!} \frac{S_\gamma^n}{n^{|\gamma|}} D^\gamma f(\mathbf{x})$$

$$= \frac{d_{1,v}(\mathbf{x})}{n} + \cdots + \frac{d_{v,v}(\mathbf{x})}{n^v} + \mathrm{O}(n^{-(v+1)}) + \sum_{l=v}^{2v-1} \sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma|=2v}} \frac{\delta_{l,\gamma}(\mathbf{x})}{n^l} + \sum_{l=v+1}^{2v} \sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma|=2v+1}} \frac{\delta_{l,\gamma}(\mathbf{x})}{n^l}$$

$$= \frac{d_{1,v+1}(\mathbf{x})}{n} + \cdots + \frac{d_{v,v+1}(\mathbf{x})}{n^v} + \frac{d_{v+1,v+1}(x)}{n^{v+1}} + \mathrm{O}(n^{-(v+2)})$$

and comparing coefficients on both sides of this equation proves the claim.

We now define, for $v = 1,\ldots,k$, coefficient functions $\tilde{c}_v(\mathbf{x})$ by

$$\tilde{c}_v(\mathbf{x}) := d_{v,v}(\mathbf{x}) + \sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma|=2v}} \frac{1}{\gamma!}\delta_{v,\gamma}(\mathbf{x}) D^\gamma f(\mathbf{x}). \tag{21}$$

We now claim: For $m = 1,\ldots,k$, it is

$$\lim_{n\to\infty} n^m \left\{ B_n(f;\mathbf{x}) - f(\mathbf{x}) - \sum_{v=1}^{m-1} \frac{\tilde{c}_v(\mathbf{x})}{n^v} \right\} = \tilde{c}_m(\mathbf{x}). \tag{22}$$

For $m = 1$, this was established in [3] as a corollary to Theorem 3.

Now let $2 \leqslant m \leqslant k$. From (19) in connection with Theorem 4, we get

$$\lim_{n\to\infty} n^m \left(B_n(f;\mathbf{x}) - f(\mathbf{x}) - \left(\frac{d_{1,m}(\mathbf{x})}{n} + \cdots + \frac{d_{m,m}(\mathbf{x})}{n^m} \right) \right)$$

$$= \sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma|=2m}} \frac{1}{\gamma!} \sum_{\substack{\beta^1,\ldots,\beta^m\in\mathbb{N}_0^s \\ \beta^1+\cdots+\beta^m=\gamma \\ |\beta^i|\geqslant 2,\, i=1,\ldots,m}} \prod_{i=1}^m \left(\sum_{j=0}^s \lambda_j (v^j - \mathbf{x})^{\beta^i} \right) D^\gamma f(\mathbf{x}).$$

Together with (21), (20) and (18), this gives

$$\lim_{n\to\infty} n^m \left(B_n(f;\mathbf{x}) - f(\mathbf{x}) - \sum_{v=1}^{m-1} \frac{\tilde{c}_v(\mathbf{x})}{n^v} - \frac{d_{m,m}(\mathbf{x})}{n^m} \right) = \sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma|=2m}} \frac{1}{\gamma!}\delta_{v,\gamma}(\mathbf{x}) D^\gamma f(\mathbf{x}),$$

and so, using (21) once more, (22) is proved.

This also completes the proof of the existence of the asymptotic expansion, as stated in (10).

To verify (11) (i.e., to prove that $c_v = \tilde{c}_v$), we once again analyse the sum in (19). Using (16) gives

$$\sum_{\substack{\gamma\in\mathbb{N}_0^s \\ |\gamma|\leqslant 2v-1}} \frac{1}{\gamma!} \sum_{\mu=1}^{\lfloor \frac{|\gamma|}{2} \rfloor} \sum_{i=1}^{\mu} \frac{\alpha_{i,\mu}}{n^{|\gamma|-i}} \sum_{\substack{\beta^1,\ldots,\beta^\mu\in\mathbb{N}_0^s \\ \beta^1+\cdots+\beta^\mu=\gamma \\ |\beta^i|\geqslant 2,\, i=1,\ldots,\mu}} \prod_{i=1}^{\mu} \left(\sum_{j=0}^s \lambda_j (v^j - \mathbf{x})^{\beta^i} \right) D^\gamma f(\mathbf{x}).$$

Table 1a
Errors in approximating f_1

$0.2083e(00)$			
	$0.1042e(-1)$		
$0.9896e(-1)$		$0.5208e(-2)$	
	$0.6510e(-2)$		$0.0000e(1)$
$0.4622e(-1)$		$0.6510e(-3)$	
	$0.2116e(-2)$		$0.0000e(1)$
$0.2205e(-1)$		$0.8138e(-4)$	
	$0.5900e(-3)$		$0.0000e(1)$
$0.1073e(-1)$		$0.1017e(-4)$	
	$0.1551e(-3)$		$0.0000e(1)$
$0.5288e(-2)$		$0.1272e(-5)$	
	$0.3974e(-4)$		
$0.2624e(-2)$			

Table 1b
Quotients of the entries of Table 1a

2.105		
	1.600	
2.141		8.000
	3.077	
2.096		8.000
	3.586	
2.055		8.000
	3.803	
2.029		8.000
	3.904	
2.015		8.000
	3.953	
2.008		

Collecting in this expression all terms containing $1/n^v$ shows that the coefficient of this power of n is

$$\sum_{\substack{\gamma\in\mathbb{N}_0^s\\ |\gamma|\leqslant 2v-1}} \frac{1}{\gamma!} \sum_{\mu=1}^{\lfloor\frac{|\gamma|}{2}\rfloor} \alpha_{|\gamma|-v,\mu} \sum_{\substack{\beta^1,\dots,\beta^\mu\in\mathbb{N}_0^s\\ \beta^1+\cdots+\beta^\mu=\gamma\\ |\beta^i|\geqslant 2,\, i=1,\dots,\mu}} \prod_{i=1}^{\mu} \left(\sum_{j=0}^{s} \lambda_j (v^j-\mathbf{x})^{\beta^i}\right) D^\gamma f(\mathbf{x}).$$

Since $\alpha_{i,\mu}=0$ for $i\leqslant 0$ and $i>\mu$, this is equal to

$$\sum_{\substack{\gamma\in\mathbb{N}_0^s\\ v+1\leqslant|\gamma|\leqslant 2v-1}} \frac{1}{\gamma!} \sum_{\mu=|\gamma|-v}^{\lfloor\frac{|\gamma|}{2}\rfloor} \alpha_{|\gamma|-v,\mu} \sum_{\substack{\beta^1,\dots,\beta^\mu\in\mathbb{N}_0^s\\ \beta^1+\cdots+\beta^\mu=\gamma\\ |\beta^i|\geqslant 2,\, i=1,\dots,\mu}} \prod_{i=1}^{\mu} \left(\sum_{j=0}^{s} \lambda_j (v^j-\mathbf{x})^{\beta^i}\right) D^\gamma f(\mathbf{x}).$$

Now using once more relation (18) completes the proof of Theorem 5. □

Table 2a
Errors in approximating f_2

0.1097e(00)				
	0.2113e(−2)			
0.5378e(−1)		0.1500e(−4)		
	0.5396e(−3)		0.2869e(−6)	
0.2662e(−1)		0.1624e(−5)		0.1575e(−9)
	0.1361e(−3)		0.1779e(−7)	
0.1324e(−1)		0.1875e(−6)		0.7047e(−11)
	0.3417e(−4)		0.1105e(−8)	
0.6603e(−2)		0.2247e(−7)		0.2520e(−12)
	0.8559e(−5)		0.6883e(−10)	
0.3297e(−2)		0.2748e(−8)		0.8359e(−14)
	0.2142e(−5)		0.4294e(−11)	
0.1648e(−2)		0.3398e(−9)		0.2687e(−15)
	0.5357e(−6)		0.2681e(−12)	
0.8235e(−3)		0.4224e(−10)		
	0.1340e(−6)			
0.4117e(−3)				

Table 2b
Quotients of the entries of Table 2a

2.039				
	3.917			
2.020		9.237		
	3.964		16.133	
2.010		8.664		22.345
	3.984		16.096	
2.005		8.344		27.966
	3.992		16.055	
2.003		8.175		30.146
	3.996		16.029	
2.001		8.088		31.111
	3.998		16.015	
2.001		8.044		
	3.999			
2.000				

3. Numerical results

Having proved the existence of the asymptotic expansion (10), we can now apply the extrapolation process (4) to the sequence of Bernstein approximants. It follows from (10) that $\rho_k = k$ for all k. In order to illustrate the numerical effect of extrapolation, we show in this section a small selection of a number of numerical tests that have been examined, and all of which showed the asymptotic behaviour that was predicted.

The results shown below were obtained for $s=2$ on the triangle T with vertices

$$v^0 = \begin{pmatrix}1\\0\end{pmatrix}, \quad v^1 = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{and} \quad v^2 = \begin{pmatrix}0\\2\end{pmatrix}$$

in euclidean coordinates. We computed the absolute values of the error functions in the barycenter of T, the point

$$\beta = \tfrac{1}{3}(v^0 + v^1 + v^2).$$

As a first test, we applied the method to the bivariate polynomial

$$f_1(x,y) := xy^3.$$

The *errors* of the approximations $y_i^{(k)}$ of the true value $f_1(\beta) = \frac{1}{3}$, computed by extrapolation with $K=3$, $n_0=2$, and $i=0,\ldots,6$, are shown in Table 1a. As expected, the entries of the third column are identically zero, since f_1 is a polynomial of total degree 4, and therefore the third extrapolation step already gives the exact result. Note in this connection that the Bernstein approximants themselves do *not* reproduce the polynomial f_1 exactly, however high their degree might be.

As a second example, we consider approximation of the function

$$f_2(x,y) := \exp(x+y)$$

and again compare our numerical approximations with the true value of f_2 in β, which is $\exp(\frac{4}{3})$. This time, the errors (in absolute value) of the approximations computed by our method with $K=4$, $n_0=4$, and $i=0,\ldots,8$ are shown, see Table 2a.

In Tables 1b and 2b, finally, we have the *quotients* of two subsequent values in the columns of Table 1a (resp. Table 2a). As predicted, the entries of the kth column (starting to count with $k=0$) converge to 2^{k+1}.

4. Conclusion

In contrast to the univariate case, the approximation of multivariate functions by polynomials is still a very difficult task, and many problems are open. Up to now, there exist very few numerical methods for the computation of good polynomial approximations. Therefore, we are convinced that the approach developed in this paper provides a very efficient new method for polynomial approximation of multivariate functions.

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