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Multilevel methods for the *h*-, *p*-, and *hp*-versions of the boundary element method ☆

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Abstract

In this paper we give an overview on the definition of finite element spaces for the *h*-, *p*-, and *hp*-version of the BEM along with preconditioners of additive Schwarz type. We consider screen problems (with a hypersingular or a weakly singular integral equation of first kind on an open surface Γ) as model problems. For the hypersingular integral equation and the *h*-version with piecewise bilinear functions on a coarse and a fine grid we analyze a preconditioner by iterative substructuring based on a non-overlapping decomposition of Γ. We prove that the condition number of the preconditioned linear system behaves polylogarithmically in H/h. Here H is the size of the subdomains and h is the size of the elements. For the *hp*-version and the hypersingular integral equation we comment in detail on an additive Schwarz preconditioner which uses piecewise polynomials of high degree on the fine grid and yields also a polylogarithmically growing condition number. For the weakly singular integral equation, where no continuity of test and trial functions across the element boundaries has to been enforced, the method works for nonuniform degree distributions as well. Numerical results supporting our theory are reported.

MSC: 65N55; 65N38

Keywords: *h-p* version of the boundary element method; Schwarz methods; Preconditioning

1. Introduction

The use of piecewise polynomials of high degree guarantees high accuracy of Galerkin solutions for elliptic boundary value problems even with singularities [8]. This holds both for the finite element method (FEM) [9] as well as for the boundary element method (BEM) [41], i.e., for Galerkin schemes to solve corresponding integral equations. The convergence analysis of the

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hp-version of the BEM for integral equations on polygons is analyzed in [1,11,20,21,23] within the framework of Mellin convolution operators. For three-dimensional problems, i.e., integral equations on polyhedral/open surfaces see [19,26,27,29]. The solution of the weakly singular integral equation of the first kind with the single-layer potential belongs to the countably normed spaces $B^1_\beta(\Gamma)$ when Γ is a polyhedron. The solution of the hypersingular integral equation with the operator of the normal derivative of the double-layer potential belongs to $B^2_\beta(\Gamma)$ when the given data are piecewise analytic on Γ. The Dirichlet problem for the Laplace operator in a polyhedral domain, which is converted into the above weakly singular integral equation, is analyzed in [29], whereas the corresponding Neumann problem leads to the hypersingular integral equation considered in [30]. We show in both cases that the solution of the boundary integral equation can be approximated exponentially fast by appropriately chosen piecewise polynomials on a geometric mesh which is refined towards the edges and corners of the polyhedral surface. The trial functions can be chosen as tensor products of Legendre polynomials and their antiderivatives, respectively. For further reference compare the survey article [41].

The development of efficient adaptive refinement strategies for BEM to solve 3D problems is of high practical importance. Residual error estimators for the *h*-version have been studied in [5,6] extending to BEM the Eriksson/Johnson approach for FEM. Another strategy to define error indicators uses hierarchical multilevel decompositions of the trial spaces (for curves see [4,46], and for weakly singular integral equations on surfaces see [32]). The framework of adaptive multilevel decompositions seems also to be suitable for the construction of *p*- and *hp*-adaptive methods (for numerical experiments of the BEM see [28]). For a complete theoretical study the corresponding multilevel decompositions need to be analyzed which do not only localize the subspaces containing the trial functions with high degrees but these subspaces must be further decomposed. To the authors' knowledge this is still an open problem. Nevertheless, a sequence of preliminary work has been developed recently, examining the preconditioners for domain decomposition techniques belonging to *h*-, *p*-, and the *hp*-versions BEM. A lot of work has been done for preconditioning techniques for the pure *h*-version [22,42,47]; for the *p*-version see [13,14,16,48] and for the *hp*-version see [24,25]. Here in Section 2 we will report on [24]. So far there seems to be no theoretical results available for domain decomposition methods for the *hp*-version of the BEM for 3D problems on nonuniform meshes with anisotropic elements. First results for two-level decompositions with respect to the polynomial degree are in [28] which can be used for adaptive refinements. The above-mentioned references deal with symmetric positive definite problems. Domain decompositions for nonsymmetric or indefinite systems for the BEM on curves (*h*- and *p*-version) are investigated in [44,45]. These techniques can also be applied to 3D problems [15] and be used for adaptive steering of indefinite boundary element problems [28]. There is a rapidly growing literature on the above topic for the FEM. For brevity the given references address only the BEM (only some papers for the FEM are cited). For the *h*-version BEM there are further preconditioning techniques which however do not use subspace decompositions of the boundary element space (see [39] and the references therein).

The paper is organized as follows. In Section 2 we present the additive Schwarz method for the *hp*-version of the Galerkin boundary element method applied to first kind integral equations on surfaces. In Section 3 for the *h*-version we prove the polylogarithmic growth of the condition number for the preconditioned system of the hypersingular integral equation. In Section 4 we give some numerical experiments showing the influence of preconditioning for various *p*-versions.

2. Schwarz methods for boundary integral equations of first kind

Additive Schwarz methods for the h- and p-versions of the BEM applied to weakly singular and hypersingular integral equations of first kind in $\mathbb{R}^2$ are studied in [14,25,47,48]. For the p-version [14,48] it could be shown that the condition number of the additive Schwarz operator grows at most like $O(\log^2 p)$ where p denotes the polynomial degree (for the hp-version see below this section). For the h-version we could show in [47] that the condition number of the additive Schwarz operator grows at most like $h^{-\varepsilon}$ for $\varepsilon > 0$ arbitrarily small where h denotes the mesh size. The corresponding result for methods with hierarchical basis functions in $\mathbb{R}^2$ is derived in [46] with growth $O(|\log h|)$. For a summary of the results compare the survey article [42]. The multilevel method from [47] could be generalized to hypersingular integral operators on surfaces in [15]. The results show bounded condition number of the preconditioned system for closed surfaces and an upper bound $O(|\log^{1/2} h|)$ for the condition number in case of open surfaces. Additive Schwarz decompositions with two levels and independent coarse grid and fine grids were analyzed in [22] for hypersingular integral operators on surfaces. As described below the corresponding condition numbers of additive Schwarz operators grow at most like $O(|\log^2 H/h|)$ where H denotes the size of the subdomains.

Domain decompositions and additive Schwarz methods for the p-version of the BEM in $\mathbb{R}^3$ are discussed in [13,16–18]. For nonoverlapping decompositions and weakly singular integral operators it could be shown in [18] that the condition number of the corresponding additive Schwarz operator grows at most like $O(\log^2 Hp/h)$. In [16] special, nonhierarchical basis functions have been used to define decompositions for hypersingular operators where the condition number of the additive Schwarz operator grows at most polylogarithmically in p. Overlapping decompositions are analyzed for 2D problems in [49] and for 3D in [17].

As a model problem we consider the weak form of the hypersingular integral equation

$$\langle Du, v\rangle_{L^2(\Gamma)} = \langle f, v\rangle_{L^2(\Gamma)} \quad \text{for all } v \in \tilde{H}^{1/2}(\Gamma) \tag{1}$$

on a plane rectangular surface piece $\Gamma \subset \mathbb{R}^3$ where $f \in H^{-1/2}(\Gamma)$ is a given function. Here D is the hypersingular integral operator

$$Du(x) = \frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,\mathrm{d}S_y, \quad x \in \Gamma$$

which is a continuous and positive-definite mapping from $\tilde{H}^{1/2}(\Gamma)$ onto $H^{-1/2}(\Gamma)$, cf. [40]. Hence, there holds the equivalence of norms

$$\langle Dv, v\rangle_{L^2(\Gamma)} \simeq ||v||^2_{\tilde{H}^{1/2}(\Gamma)} \quad \text{for all } v \in \tilde{H}^{1/2}(\Gamma).$$

The Sobolev spaces $\tilde{H}^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ are defined in the next section. The solution u of (1) is the jump across Γ of the solution of a Neumann problem for the Laplacian in $\mathbb{R}^3\backslash\bar{\Gamma}$, cf. [40]. The extension of our results to hypersingular integral equations on closed, polyhedral surfaces [37] and to more practical problems like exterior traction problems in linear elasticity is essentially straightforward and for ease of presentation we concentrate on the generic model problem for the Laplacian.

The Galerkin scheme for (1) reads as follows. Given a finite-dimensional subspace $\Psi \subset \tilde{H}^{1/2}(\Gamma)$ with $\dim \Psi = N$ find $u_N \in \Psi$ such that

$$\langle Du_N, v\rangle_{L^2(\Gamma)} = \langle f, v\rangle_{L^2(\Gamma)} \quad \text{for all } v \in \Psi. \tag{2}$$

The solution u of (1) behaves singularly at the edges and corners of Γ, cf. [37,43]. Due to these singularities the standard h- and p-versions of the Galerkin method converge at a rather low rate. On the other hand, when appropriately combining mesh refinements and polynomial-degree distributions in a nonuniform fashion, even an exponential rate of convergence is achievable, cf. [19,29].

The approach from [24] described here is a first step towards preconditioning methods for the general hp-version of the boundary element method in three-dimensions. We consider nonuniform meshes as well as nonuniform degree distributions. However, we require that the elements are shape regular, i.e., they are not too distorted, and locally quasi-uniform. Moreover, we assume that the polynomial degrees vary not too much within elements, i.e., the ratio of maximum and minimum polynomial degrees is bounded on individual elements. Since the polynomial degrees on neighboring elements are coupled by the continuity of the basis functions this boundedness of the ratio then holds also on patches of adjacent elements. Therefore, we call this nonuniform p version locally uniform.

In any case, the stiffness matrices for the hp-version in (2) are ill-conditioned and a preconditioner is necessary for an efficient solution. The method of choice for solving positive definite linear systems is the conjugate gradient method. Let A denote the stiffness matrix of the linear system with spectral condition number κ. Then a bound on the decrease of the energy norm of the error, after k steps, is given by

$$2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^k \quad \text{where } \kappa = \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}.$$

The goal, now, is to investigate a preconditioner for A which yields good bounds for κ. Provided the stiffness matrix is given in a suitable basis with appropriate numbering of basis functions, we present a preconditioner which amounts to a block-Jacobi step where some of the blocks may overlap with others. Each block of this method corresponds to a discretization of the integral operator for a given subspace of the full approximation space Ψ. Therefore, the preconditioner is related to a decomposition of Ψ into subspaces. The main structure of this decomposition is given by a three-level method. There are two levels for the piecewise polynomials of lowest degree corresponding to a coarse and a fine mesh. The fine level subspace then is further decomposed into a couple of subspaces associated with the wire basket and individual elements of the coarse mesh. Finally, the third level is given by the full space of piecewise polynomials of high degrees and is also further decomposed. The subspaces of the latter decomposition belong to the wire basket and individual elements of the fine mesh.

The three-level decomposition of our method is analogous to that proposed by Guo and Cao [10] for the finite element method in three-dimensions. However, we have to deal with a hypersingular integral operator on surfaces that means we have to consider trace spaces of H^1. We take polynomials with minimal L^2-norm as given in [36] and use the L^2-bilinear form on the wire basket.

Let us note that [24] extends the results in [22] where the pure h version of the boundary element method on quasi-uniform meshes is considered. Indeed, the first two levels of our preconditioner are identical with the subspace decomposition of the h version in [22] where, however, in each case

the original bilinear form is used as preconditioner. In our previous paper [25] we also deal with preconditioners for the *hp*-version of the boundary element method where, in particular, geometrically graded meshes and nonuniform polynomial degrees are considered. However, that paper only deals with two-dimensinal problems on polygonal domains, i.e., with integral operators on curves, and the subspace decompositions and technical tools are not as sophisticated as for three-dimensional problems.

The ansatz space Ψ consists of piecewise polynomials of varying degree on a locally quasi-uniform mesh. This mesh is defined by two levels.

One level is the *coarse mesh* which is given by a regular family of triangles or quadrilaterals, $\bar{\Gamma} = \bigcup_{j=1}^{J} \bar{\Gamma}_j$. The elements Γ_j of this level are called *subdomains* and its nodal points are referred to as *vertices* whereas the lines between the vertices are the *edges*. The subdomains must be shape regular, i.e., they are not too distorted. The coarse mesh can be nonuniform and the diameter of Γ_j is denoted by H_j.

The second level of the mesh is the *fine mesh*. It is given by partitioning each subdomain into a number of quasi uniform quadrilaterals Γ_{ji} (the *elements*) which are shape regular and of diameter h_j on Γ_j. The nodal points of the fine mesh are called *nodes* and the lines between the nodes are the *sides*.

Having defined the mesh on Γ the h–p approximation space is completely determined by defining basis functions locally on the reference element $\Gamma_{\mathrm{ref}} := (-1,1)^2$ and by specifying polynomial degrees on the elements of the mesh.

On the reference element we use the vector of polynomial degrees

$$\mathscr{P} = (p_{\gamma_1}, \ldots, p_{\gamma_4}, p_{I_1}, p_{I_2}),$$

where p_{γ_j} and p_{I_j} are the degrees associated with the sides and the interior of Γ_{ref} (x_1- and x_2-direction), respectively, in a certain order. The maximum polynomial degree on Γ_{ref} is denoted by $p_{\max}$. The elements Γ_{ji} are associated with degree vectors $\mathscr{P}_{ji}$, and we assume that the ratio of the maximum and minimum polynomial degrees on individual elements is bounded uniformly on the fine mesh.

Now let us define the basis functions. As in the standard p version we make a distinction between nodal, side, and internal shape functions. Let φ_0^p denote the pth degree polynomial on $I := (-1,1)$ with $\varphi_0^p(-1) = 0$ and $\varphi_0^p(1) = 1$ which minimizes the $L^2(I)$-norm over the space of all pth degree polynomials (subject to the same boundary conditions).

- One of the four nodal shape functions, the one for node $V_1 = (-1,-1)$, is given by

$$\varphi_0^{p_{\gamma_1}}(-x_1)\varphi_0^{p_{\gamma_4}}(-x_2).$$

- One of the sets of side shape functions, for the edge $E_1 = \{(x_1, x_2);\ x_2 = -1\}$, is given by

$$\psi^{p_{\gamma_1}}(x_1)\varphi_0^{p_{I_2}}(-x_2)$$

  where $\psi^{p_{\gamma_1}}$ is a polynomial of degree p_{γ_1} such that $\psi^{p_{\gamma_1}}(-1) = \psi^{p_{\gamma_1}}(1) = 0$.
- The interior shape functions are polynomials of degree p_{I_1} in x_1 and of degree p_{I_2} in x_2 which vanish at the sides of Γ_{ref}.

Using the shape functions defined above we introduce a polynomial space on Γ_{ref} by

$$\Psi_{\mathscr{P}}(\Gamma_{\mathrm{ref}}) = \Psi_{\mathscr{P}}^{[N]}(\Gamma_{\mathrm{ref}}) + \bigcup_{l=1}^{4} \Psi_{\mathscr{P}}^{[\gamma_l]}(\Gamma_{\mathrm{ref}}) + \Psi_{\mathscr{P}}^{[I]}(\Gamma_{\mathrm{ref}}).$$

Here, $\Psi_{\mathscr{P}}^{[N]}(\Gamma_{\mathrm{ref}})$ is the space of nodal shape functions on the reference element Γ_{ref}, $\Psi_{\mathscr{P}}^{[\gamma_l]}(\Gamma_{\mathrm{ref}})$ is the space of side shape functions on the side γ_l of Γ_{ref}, and $\Psi_{\mathscr{P}}^{[I]}(\Gamma_{\mathrm{ref}})$ denotes the space of interior shape functions on Γ_{ref}.

The full h–p approximation space on Γ is now defined by taking affine transformations onto the elements Γ_{ji} of the polynomial space $\Psi_{\mathscr{P}}(\Gamma_{\mathrm{ref}})$. Using the notation previously introduced we have on each element the representation

$$\Psi_{\mathscr{P}_{ji}}(\Gamma_{ji}) = \Psi_{\mathscr{P}_{ji}}^{[N]}(\Gamma_{ji}) + \bigcup_{l=1}^{4} \Psi_{\mathscr{P}_{ji}}^{[\gamma_l]}(\Gamma_{ji}) + \Psi_{\mathscr{P}_{ji}}^{[I]}(\Gamma_{ji})$$

and the full space

$$\Psi(\Gamma) = \{\psi;\ \psi|_{\Gamma_{ji}} \in \Psi_{\mathscr{P}_{ji}}(\Gamma_{ji})\} \cap \tilde{H}^{1/2}(\Gamma).$$

We decompose the approximation space by

$$\Psi(\Gamma) = \Psi_H(\Gamma) + \Psi_{\mathscr{W}}(\Gamma) + \bigcup_j \Psi_{\Gamma_j} + \Psi_W(\Gamma) + \bigcup_{j,i} \Psi_{\mathscr{P}_{ji}}^{[I]}(\Gamma_{ji}). \tag{3}$$

Here, $\Psi_H(\Gamma) = \Pi_h \Psi_H^*(\Gamma)$ where $\Psi_H^*(\Gamma)$ is the space of piecewise linear/bilinear functions on the coarse mesh and

$$\Pi_h : \Psi(\Gamma) \to \Psi_h(\Gamma)$$

is the interpolation operator onto the space of piecewise bilinear functions $\Psi_h(\Gamma)$ on the fine mesh. Further, $\Psi_{\mathscr{W}}(\Gamma)$ is the space of piecewise bilinear functions on the fine mesh which are zero at the nodes which are not on the wire basket $\mathscr{W}$. The piecewise bilinear functions on the fine mesh which are nonzero only on the subdomain Γ_j span the space Ψ_{Γ_j}. The remaining spaces, $\Psi_W(\Gamma)$ and $\Psi_{\mathscr{P}_{ji}}^{[I]}(\Gamma_{ji})$, represent a decomposition of the space of high-degree polynomials. $\Psi_W(\Gamma)$ is spanned by all the side and nodal functions and $\Psi_{\mathscr{P}_{ji}}^{[I]}(\Gamma_{ji})$ comprises all polynomials of the specified degrees on Γ_{ji} that vanish on $\Gamma \backslash \Gamma_{ji}$.

Note that the above decomposition amounts to a three-level method. The first two levels, $\Psi_H(\Gamma)$ and $\Psi_{\mathscr{W}}(\Gamma) + \bigcup_j \Psi_{\Gamma_j}$, represent a two-level decomposition of the piecewise polynomials of lowest degree whereas the third level, $\Psi_W(\Gamma) + \bigcup_{j,i} \Psi_{\mathscr{P}_{ji}}^{[I]}(\Gamma_{ji})$, contains all piecewise polynomials of higher degrees.

For ease of presentation we use, instead of (3), also the notation

$$\Psi(\Gamma) = H_1 + \cdots + H_k,$$

for the three-level decomposition of $\Psi(\Gamma)$ where the number of subspaces k equals to three plus the number of subdomains Γ_j plus the number of elements Γ_{ji} (if the polynomial degrees are large enough such that all the subspaces are nonempty).

The additive Schwarz method consists in solving, by an iterative method, the equation

$$Pu_N := (P_1 + P_2 + \cdots + P_k)u_N = f_N, \tag{4}$$

where the projections $P_j : \Psi(\Gamma) \to H_j,\ j = 1,\dots,k$, are defined for any $v \in \Psi(\Gamma)$ by

$$a_j(P_j v, \varphi) = \langle Dv, \varphi\rangle_{L^2(\Gamma)} \quad \text{for any } \varphi \in H_j.$$

Here, $a_j,\ j = 1,\dots,k$, are given bilinear forms. On all but the wire basket spaces $\Psi_{\mathcal{W}}(\Gamma)$ and $\Psi_W(\Gamma)$ we use the original bilinear form given by the integral operator, i.e.,

$$a_j(v,w) := \langle Dv, w\rangle_{L^2(\Gamma)} \quad \text{for } v,w \text{ both in } \Psi_H(\Gamma), \Psi_{\Gamma_j} \text{ or } \Psi^{[I]}_{\mathcal{P}_{ji}}(\Gamma_{ji}).$$

On the wire basket spaces $\Psi_{\mathcal{W}}(\Gamma)$ and $\Psi_W(\Gamma)$ we use the L^2-bilinear form over the wire baskets, i.e.,

$$a_j(v,w) := \langle v, w\rangle_{L^2(\mathcal{W})} \quad \text{for } v,w \in \Psi_{\mathcal{W}}(\Gamma) \tag{5}$$

and

$$a_j(v,w) := \langle v, w\rangle_{L^2(W)} \quad \text{for } v,w \in \Psi_W(\Gamma). \tag{6}$$

The right-hand side of (4), $f_N = \sum_{j=1}^k P_j u_N$, can be computed without knowing the solution u_N of (2) by

$$a_j(P_j u_N, \varphi) = \langle f, \varphi\rangle_{L^2(\Gamma)} \quad \text{for any } \varphi \in H_j,\ j = 1,\dots,k.$$

Eq. (4) is the preconditioned linear system and an estimate of its condition number is given by the next theorem.

Theorem 1. *There exist positive constants c_1, c_2 which are independent of H_j, h_j, and p_j such that for all $v \in \Psi$ there holds*

$$c_1 \min_j \left(1 + \log\frac{H_j}{h_j} p_j\right)^{-2} \langle Dv, v\rangle_{L^2(\Gamma)} \leqslant \langle DPv, v\rangle_{L^2(\Gamma)} \leqslant c_2 \langle Dv, v\rangle_{L^2(\Gamma)}.$$

P is the additive Schwarz operator defined by the decomposition of the ansatz space Ψ and by the given bilinear forms.

The proof of Theorem 1 needs a far amount of technical details (see [24]). For simplicity we present in the next section the proof for the pure h-version, i.e., $p_j = 1$, from [22]. Nevertheless, this simpler case should still suffice to highlighten the various building blocks of our analysis.

A prototype of a weakly singular integral equation is

$$\langle Vu, v\rangle = \langle f, v\rangle \quad \forall v \in \tilde{H}^{-1/2}(\Gamma), \tag{7}$$

with the single-layer potential operator

$$Vu(x) := \frac{1}{4\pi} \int_\Gamma \frac{u(y)}{|x-y|}\, \mathrm{d}s_y.$$

This pseudodifferential operator has order -1, and the corresponding energy space is the dual space $\tilde{H}^{-1/2}(\Gamma)$ of $H^{1/2}(\Gamma)$ where the latter is an interpolation space between $L^2(\Gamma)$ $H^1(\Gamma)$. Hence to obtain bounds for the condition number of the additive Schwarz operator for weakly singular operators inequalities of the form

$$c_1 \sum_{i=1}^N ||v_i||^2_{\tilde{H}^{-1/2}(\Gamma)} \leqslant ||v||^2_{\tilde{H}^{-1/2}(\Gamma)} \leqslant c_2 \sum_{i=1}^N ||v_i||^2_{\tilde{H}^{-1/2}(\Gamma)}$$

are of central importance. The function v belongs to the BEM trialspace which is now a subspace of $\tilde{H}^{-1/2}(\Gamma)$ and the representation $v=\sum_{i=1}^{N} v_i$ belongs to an appropriate subspace splitting. In contrary to the hypersingular operator the conformity in case of the weakly singular operator requires no continuity of the piecewise polynomial trial functions and the components of v can, e.g., be defined by restrictions.

Let us consider the Galerkin scheme for (7) and concentrate on the p-version of BEM for which we introduce a nonoverlapping method which is almost optimal. To define the additive Schwarz preconditioners for our model problem let $\bar{\Gamma}_h=\bigcup_{j=1}^{J}\bar{\Gamma}_j$ be a given mesh of J rectangles which define implicitly the subspace $\Psi\subset\tilde{H}^{-1/2}(\Gamma)$ of piecewise polynomials on Γ_h by specifying the polynomial degrees. For the decomposition of Ψ we choose a coarse mesh $\bar{\Gamma}_H=\bigcup_{j=1}^{n}\bar{G}_j$ of size $H\geqslant h$, assuming that Γ_H is compatible with the boundary element mesh Γ_h. We decompose

$$\Psi = H_0 \oplus H_1 \oplus \cdots \oplus H_n, \tag{8}$$

where H_0 is the space of piecewise constant functions on the coarse mesh Γ_H and

$$H_j:=\{v_{|G_j}\colon v\in\Psi\backslash H_0, \langle v,1\rangle_{L^2(G_j)}=0\},\quad j=1,\ldots,n.$$

Theorem 2 (Heuer [15,18]). *There exists a constant $c>0$ independent of h, H, and p such that for the condition number of the additive Schwarz operator P implicitly defined by the decomposition* (8) *there holds*

$$\kappa(P)\leqslant c\left(1+\log\left(\frac{H}{h}(p+1)\right)\right)^2.$$

Remark 1. Above the same degree p is used everywhere, for simplicity, but here the method works for nonuniform degree distributions as well. The above Theorem can be directly applied to the h-version (using piecewise constant trial functions), cf. [32]. Since the boundary element functions for the weakly singular integral equations need not to be continuous across the inner element boundaries the proof of Theorem 2 only consists of a detailed analysis of the Sobolev norms involved. No special care has to be taken of the basis functions. In contrast, Theorem 1 covers the hypersingular integral operator and there the trial space Ψ is a subspace of $\tilde{H}^{1/2}(\Gamma)$ and therefore continuity of the boundary element functions across the element boundaries is required.

In the two-dimensional situation, when dealing with integral equations on curves, additive Schwarz methods for weakly singular operators directly correspond to additive Schwarz methods for hypersingular operators and viceversa. This is due to the existence of simple isomorphisms between $\tilde{H}^{1/2}(\Gamma)$ and

$$\tilde{H}_0^{-1/2}(\Gamma):=\left\{\psi\in\tilde{H}^{-1/2}(\Gamma)\colon \int_\Gamma \psi\,\mathrm{d}s=0\right\},$$

which are the energy spaces of operators of orders 1 and -1, respectively.

The extensions of standard differentiation and integration, which preserve polynomials, onto $\tilde{H}^{1/2}(\Gamma)$ and $\tilde{H}_0^{-1/2}(\Gamma)$, respectively can be taken. By these mappings, any subspace decomposition of an ansatz space for hypersingular operators gives a related subspace decomposition of the ansatz space

of differentiated functions for weakly singular operators, and vice versa. Both decompositions then provide the same spectral properties of the corresponding additive Schwarz methods.

Such an easy isomorphism which preserves polynomials on surfaces in $\mathbb{R}^3$ is not known. For example $(-\triangle)^{1/2}$ and its inverse would be candidates but they are only pseudo-differential operators which in general do not map polynomials onto polynomials. Therefore, on surfaces we use different tools to analyze Schwarz preconditioners for operators of order one and of order -1.

Let us mention some other approaches for preconditioning linear systems arising from the h-version of BEM. Im [34,35] norm equivalences are proved for finite element multilevel splittings both in $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$ which yield estimates for multilevel additive Schwarz preconditioners applied to BEM [32].

Further, we mention the method by Steinbach [39] who uses operators of opposite orders to construct preconditioners. This method is especially worth being considered when one deals with systems where all the needed operators occur. Then there is no extra work to construct the needed stiffness matrices. In the framework of domain decomposition this approach has also been proposed by Xu and Zhang, see [51]. Here, the explicit representation of the inverse of the Steklov–Poincaré operator by a weakly singular operator, which is well-known in the boundary element literature, see, e.g., [38], is used to precondition the Steklov–Poincaré operator which is hypersingular. Finally, we mention that multiplicative Schwarz methods for the BEM are studied in [12,31].

3. Proof of Theorem 1 for the *h*-version

We return to the Galerkin scheme (2) and analyze for piecewise linear elements ($p=1$) its additive Schwarz preconditioner belonging to (3). But now we use on all subspaces the energy bilinear form $a_j(\cdot,\cdot)=\langle D\cdot,\cdot\rangle$. For simplicity we restrict our considerations to uniform rectangular meshes Γ_h. The decomposition of Γ is given by a uniform rectangular mesh Γ_H which is assumed to be compatible with Γ_h, i.e., the nodes of Γ_H are also nodes of Γ_h. Then the decomposition (3) becomes

$$S_h^1(\Gamma)=S_H^1(\Gamma)\cup S_{h,H}^1(\Gamma)\cup\bigcup_{j=1}^{J}S_h^1(\Gamma_j). \tag{9}$$

The spaces $S_H^1(\Gamma)$ and $S_h^1(\Gamma)$ consist of the usual continuous piecewise bilinear functions on the meshes Γ_H and Γ_h of size H and h on Γ. $S_{h,H}^1(\Gamma)$ is the so-called wire basket space which is spanned by the piecewise bilinear hat functions of $S_h^1(\Gamma)$ which are concentrated at the nodes lying on the element boundaries of the mesh of size H. The spaces $S_h^1(\Gamma_j)$ are spanned by the piecewise bilinear hat functions concentrated at the nodes interior to the restricted meshes $\Gamma_h|_{\Gamma_j}=\Gamma_{j,h}$, $j=1,\ldots,J$.

We are now in the position to state and prove Theorem 1 which says that the preconditioner implicitly defined by the decomposition (9) is almost optimal for general mesh sizes h and H and that it is optimal if one fixes the ratio H/h and the polynomial degree.

The following two lemmas present abstract bounds for the minimum and maximum eigenvalues of the additive Schwarz operator corresponding to the h-version, i.e., we choose

$$\Psi(\Gamma)=S_h^1(\Gamma)=H_1+\cdots+H_k$$

according to (9) and we take the energy bilinear form $a(u,v)=\langle Du,v\rangle$. The proofs can be found, e.g., in [33,50].

Lemma 1. *If there exists a constant C_1 such that for any $\varphi \in S_h^1(\Gamma)$ there exist $\varphi_j \in H_j$, $j=1,\dots,k$, satisfying $\varphi = \sum_{j=1}^k \varphi_j$ and*

$$\sum_{j=1}^{k} a(\varphi_j, \varphi_j) \leqslant C_1^{-1} a(\varphi, \varphi),$$

then

$$\lambda_{\min}(P) \geqslant C_1.$$

Lemma 2. *If there exists a constant C_2 such that for any $\varphi \in S_h^1(\Gamma)$ and $\varphi_j \in H_j$, $j = 1,\dots,k$, satisfying $\varphi = \sum_{j=1}^k \varphi_j$ and*

$$a(\varphi, \varphi) \leqslant C_2 \sum_{j=1}^{k} a(\varphi_j, \varphi_j)$$

then

$$\lambda_{\max}(P) \leqslant C_2.$$

Let us define the Sobolev spaces that are in use. The space $H^1(\Gamma)$ is endowed with the usual norm

$$||\cdot||^2_{H^1(\Gamma)} = c||\cdot||^2_{L^2(\Gamma)} + ||\partial_{x_1}\cdot||^2_{L^2(\Gamma)} + ||\partial_{x_2}\cdot||^2_{L^2(\Gamma)}$$

where ∂_{x_1} and ∂_{x_2} denote the partial derivatives with respect to the Cartesian coordinates x_1 and x_2 on Γ. For fixed domains Γ or Ω the constant $c = 1$ is taken. However, if we consider subdomains of diameter H the constant $c = 1/H^2$ is taken. This is to ensure appropriate scaling properties of the norms. The space $H_0^1(\Gamma)$ is the completion of $C_0^\infty(\Gamma)$ with respect to the norm $||\cdot||_{H^1(\Gamma)}$. For nonintegral s we use the K-method of the interpolation theory as described in [2]. For two normed spaces A_0 and A_1 the interpolation space $A_s = [A_0, A_1]_s$ $(0 < s < 1)$ is equipped with the norm

$$||a||_{[A_0,A_1]_s} := \left(\int_0^\infty \left(t^{-s} \inf_{a=a_0+a_1} (||a_0||_{A_0} + t||a_1||_{A_1}) \right)^2 \frac{\mathrm{d}t}{t} \right)^{1/2}.$$

For $0 < s < 1$ we define

$$H^s(\Gamma) = [L^2(\Gamma), H^1(\Gamma)]_s, \qquad \tilde{H}^s(\Gamma) = [L^2(\Gamma), H_0^1(\Gamma)]_s.$$

The spaces $H^{-s}(\Gamma)$ and $\tilde{H}^{-s}(\Gamma)$ are the dual spaces with respect to the L^2-inner product

$$H^{-s}(\Gamma) = (\tilde{H}^s(\Gamma))', \qquad \tilde{H}^{-s}(\Gamma) = (H^s(\Gamma))'.$$

The spaces $H^s(\Gamma)$ and $\tilde{H}^s(\Gamma)$ for $|s| > 1$ can be defined analogously by interpolating between $H^{m-1}(\Gamma)$ and $H^m(\Gamma)$ or $H_0^m(\Gamma)$ for the smallest integer $m > |s|$. The following lemma is used for estimating the largest eigenvalue of the additive Schwarz operator.

Lemma 3 (Heuer [13, Lemma 4]). *Let $\{\Gamma_j;\ j = 1,\dots,J\}$ be a finite covering of Γ by subdomains Γ_j with Lipschitz boundary,*

$$\bar{\Gamma} = \bigcup_{j=1}^{J} \bar{\Gamma}_j,$$

with a covering constant J_c, i.e., we can color $\{\Gamma_j;\ j=1,\dots,J\}$ using at most J_c colors in such a way that subdomains of the same color are disjoint. Let $\varphi=\sum_{j=1}^{J}\varphi_j\in\tilde{H}^s(\Gamma)$ for real s with $\varphi_j\in\tilde{H}^s(\Gamma_j)$, $j=1,\dots,J$. Then there holds

$$||\varphi||^2_{\tilde{H}^s(\Gamma)}\leqslant J_c\sum_{j=1}^{J}||\varphi_j||^2_{\tilde{H}^s(\Gamma_j)}.$$

It is crucial for substructuring techniques to split global norms into norms over subdomains. This is straightforward for Sobolev norms of integral order. For norms of nonintegral order which typically appear in the boundary element method for first kind integral equations this is not trivial. The following lemma is used in the proof of the main theorem.

Lemma 4 (Heuer [16, Lemma 3.3]). *Let $s>0$ and $\varphi\in\tilde{H}^s(\Gamma)$ with $\varphi_j:=\varphi|_{\Gamma_j}\in\tilde{H}^s(\Gamma_j)$, $j=1,\dots,J$. There exist constants $C_1,C_2>0$ which are independent of φ and J such that*

$$C_1\sum_{j=1}^{J}||\varphi_j||^2_{H^s(\Gamma_j)}\leqslant||\varphi||^2_{\tilde{H}^s(\Gamma)}\leqslant C_2\sum_{j=1}^{J}||\varphi_j||^2_{\tilde{H}^s(\Gamma_j)}.$$

To bound the maximum eigenvalue of P we take for a given $\phi\in S^1_h(\Gamma)$ an arbitrary representation

$$\phi=\phi_H+\phi_{h,H}+\sum_{j=1}^{J}\phi_{j,h}$$

according to the decomposition (9) By the triangle inequality and by applying a colouring argument (Lemma 3) to the third component $\sum_{j=1}^{J}\phi_{j,h}$ we obtain

$$||\phi||^2_{\tilde{H}^{1/2}(\Gamma)}\leqslant C\left(||\phi_H||^2_{\tilde{H}^{1/2}(\Gamma)}+||\phi_{h,H}||^2_{\tilde{H}^{1/2}(\Gamma)}+\sum_{j=1}^{J}||\phi_{j,h}||^2_{\tilde{H}^{1/2}(\Gamma_j)}\right).$$

Thus, due to Lemma 3, we proved the boundedness of the maximum eigenvalue,

$$\lambda_{\max}(P)\leqslant C.$$

In order to derive a lower bound for the minimum eigenvalue of P we apply Lemma 1 to a specific representation for an arbitrary function $\phi\in S^1_h(\Gamma)$. We choose

$$\phi_H:=(Q_H\mathscr{E}\phi)|_\Gamma\in S^1_H(\Gamma),$$

where Q_H is the L^2-projector onto $S^1_H(\Omega)$. Here, $\mathscr{E}$ is the discretely harmonic extension operator from Γ onto $\Omega:=(-H,H)\times\Gamma$ (cf. (11)). By the trace theorem, the stability of Q_H in $H^1(\Omega)$, and the extension theorem for discretely harmonic functions we obtain

$$||\phi_H||^2_{\tilde{H}^{1/2}(\Gamma)}\leqslant C|Q_H\mathscr{E}\phi|^2_{H^1(\Omega)}\leqslant C|\mathscr{E}\phi|^2_{H^1(\Omega)}\leqslant C||\phi||^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{10}$$

Here, we made use of the fact that piecewise trilinear functions in Ω with respect to the mesh size H are discretely harmonic.

Let us use the notation

$$w_h:=\phi-\phi_H.$$

To define the component $\phi_{h,H}$ of ϕ which belongs to $S^1_{h,H}(\Gamma)$ we need some more notations.

By W we denote the wire basket of the mesh Γ_H of size H on Γ, i.e. the union of the edges of the elements of Γ_H. We neglect the edges which are on the boundary of Γ since we need zero boundary conditions for subspaces of $\tilde{H}^{1/2}(\Gamma)$. The nodes of the mesh Γ_h of size h which belong to the wire basket W are denoted by W_{nodes}. The nodes of Γ_h which do not belong to the boundary of Γ are denoted by Γ_{nodes}. Of course, these sets depend on the mesh sizes H and h. Now we define the component $\phi_{h,H}$ by the following relations:

$$\phi_{h,H} \in S_h^1(\Gamma), \qquad \phi_{h,H}(x) := \begin{cases} w_h(x) & \text{for all } x \in W_{\text{nodes}}, \\ 0 & \text{for all } x \in \Gamma_{\text{nodes}} \setminus W_{\text{nodes}}. \end{cases}$$

Obviously, the function $\phi_{h,H}$ belongs to $S_{h,H}^1(\Gamma)$. Let $\mathscr{E}\phi_{h,H}$ denote the discretely harmonic extension of $\phi_{h,H}$ onto $\Omega := (-H,H)\times\Gamma$. More precisely we embed the mesh Γ_h in the three-dimensional mesh Ω_h of cubes of size h which is defined on $\Omega = (-H,H)\times\Gamma$. We identify $\Gamma = \Omega|_{z=0}$ and use the notations $\Omega_1 := \Omega|_{z<0}$ and $\Omega_2 := \Omega|_{z>0}$. Then we define

$$\mathscr{E}\phi_{h,H} \in S_h^1(\Omega), \qquad \mathscr{E}\phi_{h,H}|_\Gamma = \phi_{h,H},$$

$$\int_\Omega \nabla \mathscr{E}\phi_{h,H} \nabla\varphi \, \mathrm{d}(x,y,z) = 0 \quad \text{for all } \varphi \in S_h^1(\Omega_i), \ \ i = 1,2.$$

Using the trace theorem and [7, Lemma 4.7] we deduce

$$||\phi_{h,H}||^2_{\tilde{H}^{1/2}(\Gamma)} \leqslant C|\mathscr{E}\phi_{h,H}|^2_{H^1(\Omega)} \leqslant C||\phi_{h,H}||^2_{L^2(W)}. \tag{11}$$

Since $\phi_{h,H} = w_h$ on the wire basket W we obtain by again using the discretely harmonic extension operator $\mathscr{E}$ and by [7, Lemma 4.3]

$$||\phi_{h,H}||^2_{L^2(W_j)} \leqslant C\left(1 + \log\frac{H}{h}\right)||\mathscr{E}w_h||^2_{H^1(\Omega_{i,j})}, \quad i = 1,2, \ \ j = 1,\ldots,J. \tag{12}$$

Here $\bar{\Omega}_i = \bigcup_{j=1}^J \bar{\Omega}_{i,j}$ is a covering of Ω_i, $i = 1,2$, which is compatible with the decomposition of Γ into subdomains Γ_j, $j = 1,\ldots,J$, and $W_j := \partial\Gamma_j$. By the approximation property of the projection operator Q_H, and by using the identity $\mathscr{E}((Q_H\mathscr{E}\phi)|_\Gamma) = Q_H\mathscr{E}\phi$, there holds

$$||\mathscr{E}w_h||^2_{L^2(\Omega_{i,j})} = ||\mathscr{E}\phi - Q_H\mathscr{E}\phi||^2_{L^2(\Omega_{i,j})} \leqslant CH^2|\mathscr{E}\phi|^2_{H^1(\Omega_{i,j})},$$

and by [3, (3.11)] we obtain for discretely harmonic functions φ

$$|\varphi|^2_{H^1(\Omega_i)} \leqslant C||\varphi||^2_{\tilde{H}^{1/2}(\partial\Omega_i)}.$$

Therefore, together with (10), we obtain

$$\begin{aligned} \sum_{j=1}^J ||\mathscr{E}w_h||^2_{H^1(\Omega_{i,j})} &= \sum_{j=1}^J (H^{-2}||\mathscr{E}w_h||^2_{L^2(\Omega_{i,j})} + |\mathscr{E}w_h|^2_{H^1(\Omega_{i,j})}) \\ &\leqslant C(|\mathscr{E}\phi|^2_{H^1(\Omega_i)} + |\mathscr{E}w_h|^2_{H^1(\Omega_i)}) \\ &\leqslant C(||w_h||^2_{\tilde{H}^{1/2}(\Gamma)} + ||\phi||^2_{\tilde{H}^{1/2}(\Gamma)}) \leqslant C||\phi||^2_{\tilde{H}^{1/2}(\Gamma)}. \end{aligned} \tag{13}$$

Combining (11)–(13) we obtain

$$||\phi_{h,H}||^2_{\tilde{H}^{1/2}(\Gamma)} \leqslant C\left(1 + \log\frac{H}{h}\right)||\phi||^2_{\tilde{H}^{1/2}(\Gamma)}. \tag{14}$$

In the last step we define the components of ϕ belonging to the spaces $S_h^1(\Gamma_j)$, $j=1,\ldots,J$, by

$$\phi_{j,h}:=\begin{cases} w_h-\phi_{h,H} & \text{on } \Gamma_j,\\ 0 & \text{elsewhere.}\end{cases}$$

Since $w_h=\phi_{h,H}$ on the boundaries of the subdomains the functions $\phi_{j,h}$ are continuous on Γ and therefore belong to $S_h^1(\Gamma)$ and the corresponding subspace $S_h^1(\Gamma_j)$ as well. Thus we have

$$\phi_h:=\sum_{j=1}^{J}\phi_{j,h}\in S_h^1(\Gamma)\subset\tilde{H}^{1/2}(\Gamma)$$

and

$$\phi_h|_{\Gamma_j}=\phi_{j,h}\in S_h^1(\Gamma_j)\subset\tilde{H}^{1/2}(\Gamma_j),\quad j=1,\ldots,J.$$

By Lemma 4 in [24] there holds

$$||\phi_{j,h}||_{\tilde{H}^{1/2}(\Gamma_j)}\leqslant C\left(1+\log\frac{H}{h}\right)||w_h||_{\tilde{H}^{1/2}(\Gamma_j)}.$$

Therefore, we obtain by Lemma 4 and (10)

$$\begin{aligned}\sum_{j=1}^{J}||\phi_{j,h}||^2_{\tilde{H}^{1/2}(\Gamma_j)}&\leqslant C\left(1+\log\frac{H}{h}\right)^2||w_h||^2_{\tilde{H}^{1/2}(\Gamma)}\\&\leqslant C\left(1+\log\frac{H}{h}\right)^2||\phi||^2_{\tilde{H}^{1/2}(\Gamma)}.\end{aligned}\tag{15}$$

Since

$$\begin{aligned}\phi_H+\phi_{h,H}+\sum_{j=1}^{J}\phi_{j,h}&=\phi_H+\phi_{h,H}+w_h-\phi_{h,H}\\&=\phi_H+\phi-\phi_H=\phi\end{aligned}$$

we defined a representation of ϕ and using (10), (14) and (15) we proved that

$$\begin{aligned}&||\phi_H||^2_{\tilde{H}^{1/2}(\Gamma)}+||\phi_{h,H}||^2_{\tilde{H}^{1/2}(\Gamma)}+\sum_{j=1}^{J}||\phi_{j,h}||^2_{\tilde{H}^{1/2}(\Gamma)}\\&\leqslant C||\phi||^2_{\tilde{H}^{1/2}(\Gamma)}+C\left(1+\log\frac{H}{h}\right)^2||\phi||^2_{\tilde{H}^{1/2}(\Gamma)}\leqslant C\left(1+\log\frac{H}{h}\right)^2||\phi||^2_{\tilde{H}^{1/2}(\Gamma)}.\end{aligned}$$

Therefore, due to Lemma 1,

$$\lambda_{\min}(P)\geqslant C\left(1+\log\frac{H}{h}\right)^{-2}$$

and

$$\kappa(P)=\lambda_{\max}(P)/\lambda_{\min}(P)\leqslant C\left(1+\log\frac{H}{h}\right)^2.$$

4. Numerical results

To demonstrate the efficiency of our preconditioning method and to underline the theoretical estimates we present some experimental results for the extremum eigenvalues and the condition numbers of the preconditioned systems belonging to the Galerkin p-version for the hypersingular integral equation (1).

We emphasize that the boundary element method produces stiffness matrices which are in general fully occupied which means that even functions with disjoint supports are coupled via the integral operator. Therefore, when performing a domain decomposition to create a preconditioner, one not only decouples adjacent subdomains but also neglects the coupling of functions in subdomains which are not adjacent. The latter coupling is not present in the finite element method. Therefore, in the boundary element method, the theoretical bounds for the extremum eigenvalues are most often just asymptotically obtained and are not as obvious as in the finite element method from the experimental results.

Due to Theorem 1 we expect for the uniform and locally uniform methods bounded maximum eigenvalues and minimum eigenvalues which behave like $(1+\log p_{\max}H/h)^{-2}$.

For our model problem we choose the domain $\Gamma=(-1/2,1/2)^2\times\{0\}$ and take a uniform mesh of squares with length h. For the concrete choice of the trial spaces see [24]. The polylogarithmic behavior in p of the condition number is checked with Fig. 1. Here we consider the mesh $h=1/3$ and $H/h=1$. Both cases, uniform and locally uniform p-version, as well as the results for the nonuniform p-version are shown. In the uniform case $p=7$ corresponds to $N=400$ number of unknowns. In the locally uniform case we have only $N=202$ for $p=7$ and in the nonuniform example $p=7$ means $N=82$. However, we observe that all curves are quite close which means that the condition number essentially depends on the maximum polynomial degree. The efficiency of the preconditioner seems

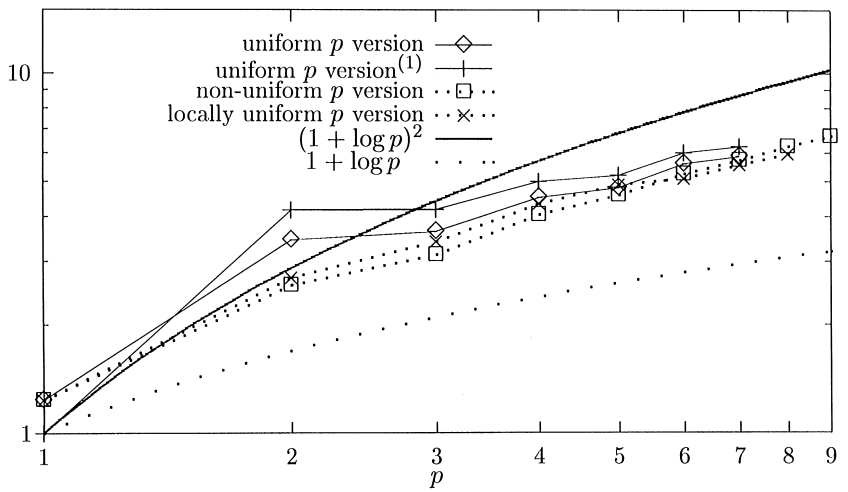

Fig. 1. Condition numbers for the preconditioned uniform and nonuniform/locally uniform p-version ($1/h=3$, $H/h=1$). The results marked by (1) (only the uniform p-version) are obtained by using the original bilinear form instead of the L^2-bilinear form in the definition of the preconditioner.

to be independent of the actual distribution of the polynomial degrees, which is restricted in our theory. Moreover, in all cases the theoretical bound $(1+\log p)^2$ is numerically fulfilled. Further, let us note that sometimes it is natural, e.g., when the full stiffness matrix is available, to use the original bilinear form instead of the L^2-bilinear form on the wire baskets, cf. (5) and (6). This replacement yields a different preconditioner whose implementation does not require additional inner products. Although this method is not covered by our theory the results in Fig. 1 for this preconditioner (indicated by (1)) show the same asymptotic behavior as the theoretically justified method.

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