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More series related to the Euler series ☆

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Abstract

We present results for infinite series appearing in Feynman diagram calculations, many of which are similar to the Euler series. These include both one- , two- and three-dimensional series. All these series can be expressed in terms of $\zeta(2)$ and $\zeta(3)$.

Keywords: Euler series; Hypergeometric series; Riemann zeta function; Psi function; Polylogarithms

1. Introduction

In the evaluation of Feynman integrals, one often needs integrals and sums related to the dilogarithm and the Riemann zeta function. This is particularly the case when one considers multi-loop amplitudes (see [12]). There is actually an intriguing connection between Feynman diagrams, topology and number theory, which has recently been elucidated by several authors, in particular by Broadhurst [2,3], Kreimer [4,8] and collaborators (see also Groote et al. [6]).

Many results of this kind have been compiled by Devoto and Duke [5] and by Kölbig et al. [7], in addition to those of the standard tables [11]. In an earlier paper [9] (henceforth referred to as Paper I [1]) we presented results for sums required in the evaluation of Feynman integrals, related to the Euler series. Several of these series involve the digamma or psi function. One such example is the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}[\gamma + \psi(n)],$$

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[1] Often we will refer to results and identities from our first article on this subject. Whenever we quote e.g., Eq. (I.13) or (I.B.2) we are referring to Eq. (13) or (B.2) in [9], respectively.

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which equals $\zeta(3)$ when summed. Here, we present further results of this kind, many of which are obtained using known properties of hypergeometric functions. The sums of these new series are of the form

$$R_3\zeta(3) + R_2\zeta(2) + R_0,$$

where R_i are all rational numbers.

The starting point of this article will be the well-known result:

Series 1

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1. \tag{1}$$

This is easily found by recognizing the sum as $\frac{1}{2}\,{}_2F_1(1,1;3;1)$ and then using (A.1).

Using the definition of the Riemann zeta function along with partial fractioning, we find the following results as immediate corollaries of Series 1:

Series 2–6

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)} = \zeta(2) - 1, \tag{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2} = -\zeta(2) + 2, \tag{3}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^3(n+1)} = \zeta(3) - \zeta(2) + 1, \tag{4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)^2} = 2\zeta(2) - 3, \tag{5}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^3} = -\zeta(3) - \zeta(2) + 3. \tag{6}$$

The generalization of this type of series is well-known, and is found in (5.1.24.8) of [10]. These results will be frequently used throughout the proofs.

2. One-dimensional series

We now turn our attention to some one-dimensional series which bear similarity to the Euler series as well as to those studied in Paper I.

Series 7–15

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}[\gamma + \psi(n)] = 1, \tag{7}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}[\gamma + \psi(1+n)] = \zeta(2), \tag{8}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}[\gamma + \psi(2+n)] = 2, \tag{9}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}[\gamma + \psi(n)] = \zeta(3) - 1, \tag{10}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}[\gamma + \psi(1+n)] = 2\zeta(3) - \zeta(2), \tag{11}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}[\gamma + \psi(2+n)] = 2\zeta(3) + \zeta(2) - 3, \tag{12}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}[\gamma + \psi(n)] = -\zeta(3) - \zeta(2) + 3, \tag{13}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}[\gamma + \psi(1+n)] = -\zeta(3) + \zeta(2), \tag{14}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}[\gamma + \psi(2+n)] = -2\zeta(3) + 3. \tag{15}$$

We prove Series 8. The others follow as corollaries of this result by using the recurrence relation (A.3), partial fractioning, Series 2–6, (I.B.1) and (I.B.2).

Proof of Series 8. We start by using the integral representation (A.2) of the psi function before summing over n

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n(n+1)}[\gamma + \psi(1+n)] &= \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \int_0^1 \mathrm{d}t \frac{1-t^n}{1-t} \\
&= \int_0^1 \mathrm{d}t \frac{1}{1-t}\left[1 - \frac{t}{2}\ {}_2F_1(1,1;3;t)\right] \\
&= \int_0^1 \mathrm{d}t \frac{1}{1-t}\left\{1 - \frac{1}{t}\,[t + (1-t)\log(1-t)]\right\} \\
&= -\int_0^1 \mathrm{d}t \frac{\log(1-t)}{t} = \zeta(2).
\end{aligned}$$

We have used (7.3.2.150) of [11] to rewrite ${}_2F_1$. In the last step we used (3.6.1) of [5]. □

Similar relations can also be found involving the trigamma function (see Appendix A.2):

Series 16–21

$$\sum_{n=1}^{\infty}\frac{1}{n}\psi'(n)=2\zeta(3), \tag{16}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\psi'(1+n)=\zeta(3), \tag{17}$$

$$\sum_{n=1}^{\infty}\frac{1}{n}\psi'(2+n)=\zeta(3)+\zeta(2)-2, \tag{18}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\psi'(n)=1, \tag{19}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\psi'(1+n)=-\zeta(3)+\zeta(2), \tag{20}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\psi'(2+n)=2\zeta(2)-3. \tag{21}$$

We prove Series 17. The others follow by using the recurrence relation (A.6) and partial fractioning, together with Series 1–6.

Proof of Series 17. We start by using the integral representation (A.4) of the trigamma function:

$$\sum_{n=1}^{\infty}\frac{1}{n}\psi'(1+n)=-\sum_{n=1}^{\infty}\frac{1}{n}\int_0^1 \mathrm{d}t\frac{t^n}{1-t}\log t=\int_0^1\frac{\mathrm{d}t}{1-t}\log t\log(1-t)$$
$$=\int_0^1\frac{\mathrm{d}t}{t}\log t\log(1-t)=\zeta(3).$$

In the last step we have used (3.6.21) of [5]. □

Next, we consider series which are quadratic or bilinear in psi functions.

Series 22–27

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}[\gamma+\psi(n)]^2=\zeta(2)+1, \tag{22}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}[\gamma+\psi(n)][\gamma+\psi(1+n)]=\zeta(3)+\zeta(2), \tag{23}$$

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}[\gamma+\psi(n)][\gamma+\psi(2+n)]=3, \tag{24}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}[\gamma + \psi(1+n)]^2 = 3\zeta(3), \tag{25}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}[\gamma + \psi(1+n)][\gamma + \psi(2+n)] = 2\zeta(3) + \zeta(2), \tag{26}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}[\gamma + \psi(2+n)]^2 = \zeta(2) + 3. \tag{27}$$

We prove Series 25. The others are immediate corollaries that follow from using (A.3) along with some of the results derived earlier in this section.

Proof of Series 25. We start by using the integral representation (A.2) of the psi function.

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{1}{n(n+1)}[\gamma + \psi(1+n)]^2 \\
&\quad = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \int_0^1 \mathrm{d}t \frac{1-t^n}{1-t} \int_0^1 \mathrm{d}s \frac{1-s^n}{1-s} \\
&\quad = \int_0^1 \frac{\mathrm{d}t}{1-t} \int_0^1 \frac{\mathrm{d}s}{1-s} \left[1 - \frac{t}{2}\ {}_2F_1(1,1;3;t) - \frac{s}{2}\ {}_2F_1(1,1;3;s) + \frac{st}{2}\ {}_2F_1(1,1;3;st)\right] \\
&\quad = \int_0^1 \frac{\mathrm{d}t}{1-t} \int_0^1 \frac{\mathrm{d}s}{1-s} \left\{1 - \frac{1}{t}[t + (1-t)\log(1-t)] - \frac{1}{s}[s + (1-s)\log(1-s)] \right. \\
&\qquad\qquad \left. + \frac{1}{st}[st + (1-st)\log(1-st)]\right\}.
\end{aligned}$$

Here, we have used (7.3.2.150) of [11]. We continue to simplify this expression:

$$\begin{aligned}
&\int_0^1 \frac{\mathrm{d}t}{1-t} \int_0^1 \frac{\mathrm{d}s}{1-s} \left[(1-st)\frac{\log(1-st)}{st} - (1-t)\frac{\log(1-t)}{t} - (1-s)\frac{\log(1-s)}{s}\right] \\
&\quad = \int_0^1 \frac{\mathrm{d}t}{1-t} \left\{ \int_0^1 \frac{\mathrm{d}s}{1-s} \left[\frac{\log(1-st)}{st} - \frac{\log(1-t)}{t}\right] \right. \\
&\qquad \left. + \int_0^1 \frac{\mathrm{d}s}{1-s}[\log(1-t) - \log(1-st)] - \int_0^1 \mathrm{d}s \frac{\log(1-s)}{s} \right\} \\
&\quad = \int_0^1 \frac{\mathrm{d}t}{1-t} \left\{ \frac{1}{t} \int_0^1 \frac{\mathrm{d}s}{1-s}[\log(1-st) - \log(1-t)] + \frac{1}{t} \int_0^1 \frac{\mathrm{d}s}{s} \log(1-st) \right. \\
&\qquad \left. - \int_0^1 \frac{\mathrm{d}s}{1-s}[\log(1-st) - \log(1-t)] + \zeta(2) \right\}.
\end{aligned}$$

We combine the first and the third of the integrals inside the curly brackets, whereas the second one is evaluated to give

$$\int_0^1 \frac{\mathrm{d}t}{1-t}\left\{\frac{1-t}{t}\int_0^1 \frac{\mathrm{d}s}{1-s}[\log(1-st)-\log(1-t)]-\frac{1}{t}\mathrm{Li}_2(t)+\zeta(2)\right\}.$$

Next, a change of variables yields

$$\int_0^1 \frac{\mathrm{d}t}{1-t}\left\{\frac{1-t}{t}\int_0^1 \frac{\mathrm{d}s}{s}[\log(1-t+st)-\log(1-t)]-\frac{1}{t}\mathrm{Li}_2(t)+\zeta(2)\right\}.$$

We use (3.14.1) and thereafter (2.2.5) of [5]:

$$\int_0^1 \frac{\mathrm{d}t}{1-t}\left[-\frac{1-t}{t}\mathrm{Li}_2\left(\frac{-t}{1-t}\right)-\frac{1}{t}\mathrm{Li}_2(t)+\zeta(2)\right]$$

$$=\int_0^1 \frac{\mathrm{d}t}{1-t}\left\{\frac{1-t}{t}\left[\mathrm{Li}_2(t)+\frac{1}{2}\log^2(1-t)\right]-\frac{1}{t}\mathrm{Li}_2(t)+\zeta(2)\right\}$$

$$=\int_0^1 \frac{\mathrm{d}t}{1-t}[\zeta(2)-\mathrm{Li}_2(t)]+\frac{1}{2}\int_0^1 \frac{\mathrm{d}t}{t}\log^2(1-t)=2\zeta(3)+\zeta(3)=3\zeta(3).$$

In the last step we have used (3.8.9) and (3.6.9) of [5]. □

3. Two-dimensional series

Next, we present results for the sums of several two-dimensional series. Many of these are proved by using results from the one-dimensional series of Section 2 and from Paper I.

Series 28

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{nk(n+k)}=2\zeta(3). \tag{28}$$

Proof.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{nk(n+k)}=\sum_{k=1}^{\infty}\frac{1}{k^2}[\gamma+\psi(1+k)]=2\zeta(3).$$

Here, we have used (I.10) and thereafter (I.B.2). □

Series 29–37

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}=\zeta(2), \tag{29}$$

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)^2(1+n+k)}=2\zeta(3)-\zeta(2), \tag{30}$$

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(k)]=\zeta(3), \tag{31}$$

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(1+k)]=2\zeta(3), \tag{32}$$

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(1+n)]=2\zeta(3), \tag{33}$$

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(n+k)]=\zeta(3)+\zeta(2), \tag{34}$$

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(1+n+k)]=3\zeta(3), \tag{35}$$

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(2+n+k)]=2\zeta(3)+\zeta(2), \tag{36}$$

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(1+n+2k)]=\tfrac{7}{2}\zeta(3). \tag{37}$$

We need to prove most of these results in different ways. Series 34 follows as an immediate corollary of Series 30 and 35 after using (A.3).

Proof of Series 29, 31 and 32.

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}f(k)=\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}\,{}_2F_1(1,k;2+k;1)f(k)=\sum_{k=1}^{\infty}\frac{1}{k^2}f(k).$$

Here, we have used (A.1) to rewrite ${}_2F_1$. By replacing $f(k)$ with the appropriate expression and using the definition of $\zeta(2)$, (I.B.1) or (I.B.2) the proof is complete. □

Proof of Series 30.

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)^2(1+n+k)}=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)^2}-\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k}\psi'(k)-\zeta(2)=2\zeta(3)-\zeta(2).$$

In the last steps we used (A.7) and Series 16. □

Proof of Series 33.

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(1+n)]$$

$$=\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}\ {}_3F_2(1,1,1+n;2,3+n;1)[\gamma+\psi(1+n)]$$

$$=\sum_{n=1}^{\infty}\frac{1}{(n+1)(n+2)}\ {}_3F_2(1,1,1+n;2,3+n;1)[\gamma+\psi(1+n)].$$

Here, we have used the fact that the summand vanishes for $n=0$. Next, we use (7.4.4.40) of [11]. Thereafter we use (6.3.2) of [1] and the recurrence relation for the psi function:

$$\sum_{n=1}^{\infty}\frac{1}{n(n+1)}[\psi(2+n)-\psi(2)][\gamma+\psi(1+n)]$$

$$=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\left[\gamma+\psi(1+n)+\frac{1}{1+n}-1\right][\gamma+\psi(1+n)]$$

$$=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}[\gamma+\psi(1+n)]^2+\sum_{n=1}^{\infty}\frac{1}{n(n+1)^2}[\gamma+\psi(1+n)]$$

$$-\sum_{n=1}^{\infty}\frac{1}{n(n+1)}[\gamma+\psi(1+n)]=2\zeta(3).$$

In the last step we use the results of Series 8, 13 and 25. □

Proof of Series 35. We start by using the recurrence relation for the psi function:

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(1+n+k)]$$

$$=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}\left[\gamma+\psi(2+n+k)-\frac{1}{1+n+k}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(2+n+k)]-\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)^2}$$

$$=2\zeta(3)+\zeta(2)-\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}+\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(1+n+k)^2}$$

$$=2\zeta(3)+\zeta(2)-\zeta(2)+\sum_{k=1}^{\infty}\frac{1}{k}\psi'(1+k)=2\zeta(3)+\zeta(3)=3\zeta(3).$$

Here, we have used (A.7), Series 17 and 29. □

Proof of Series 36.

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(2+n+k)]\\
&\quad=\gamma\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{\Gamma(n+k)}{k\Gamma(2+n+k-x)}\\
&\quad=\gamma\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}\ {}_2F_1(1,k;2+k;1)+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{k=1}^{\infty}\frac{\Gamma(k)}{k\Gamma(2+k-x)}\ {}_2F_1(1,k;2+k-x;1)\\
&\quad=\gamma\sum_{k=1}^{\infty}\frac{1}{k^2}+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{k=1}^{\infty}\frac{\Gamma(k)}{k(1-x)\Gamma(1+k-x)}\\
&\quad=\gamma\sum_{k=1}^{\infty}\frac{1}{k^2}+\sum_{k=1}^{\infty}\frac{1}{k^2}[1+\psi(1+k)]\\
&\quad=\sum_{k=1}^{\infty}\frac{1}{k^2}[\gamma+\psi(1+k)]+\sum_{k=1}^{\infty}\frac{1}{k^2}=2\zeta(3)+\zeta(2).
\end{aligned}
$$

Here, we have used (A.1) to rewrite ${}_2F_1$. In the last step we have also used (I.B.2). □

Proof of Series 37.

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(1+n+2k)]\\
&\quad=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sum_{j=1}^{n+2k}\frac{1}{jk(n+k)(1+n+k)}\\
&\quad=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sum_{j=1}^{2k}\frac{1}{jk(n+k)(1+n+k)}+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=1+2k}^{n+2k}\frac{1}{jk(n+k)(1+n+k)}\\
&\quad=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(1+2k)]\\
&\qquad+\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{j=1}^{n}\frac{1}{k(n+k)(1+n+k)(j+2k)}\\
&\quad=\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}\ {}_2F_1(1,k;2+k;1)[\gamma+\psi(1+2k)]\\
&\qquad+\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{k(n+j+k)(1+n+j+k)(j+2k)}
\end{aligned}
$$

$$=\sum_{k=1}^{\infty}\frac{1}{k^2}[\gamma+\psi(1+2k)]+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{k(j+k)(1+j+k)(j+2k)}\,{}_2F_1(1,j+k;2+j+k;1)$$

$$=\tfrac{11}{4}\zeta(3)+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{k(j+k)(j+2k)}=\tfrac{7}{2}\zeta(3).$$

Here, we have used (A.1) to rewrite ${}_2F_1$. We also used (I.13) and (I.B.4). □

Series (38)–(47)

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}=\zeta(2), \tag{38}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)^2}=3\zeta(3)-2\zeta(2), \tag{39}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(k)]=\zeta(3)+\zeta(2), \tag{40}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(1+k)]=3\zeta(3), \tag{41}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(2+k)]=2\zeta(3)+\zeta(2), \tag{42}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(n)]=2\zeta(3), \tag{43}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(1+n)]=2\zeta(2), \tag{44}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(2+n)]=2\zeta(3)+\tfrac{1}{2}, \tag{45}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(n+k)]=\zeta(3)+2\zeta(2), \tag{46}$$

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(1+n+k)]=4\zeta(3). \tag{47}$$

We use different proofs for most of these series. Series 46 follows as an immediate corollary of Series 39 and 47 after using (A.3).

Proof of Series 38, 40–42.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}f(k)=\sum_{k=1}^{\infty}\frac{1}{k(k+1)}[\gamma+\psi(1+k)]f(k).$$

Here, we have used (I.10). By replacing $f(k)$ by the appropriate expression and using Series 8, 23, 25 or 26 in connection with (A.3), the proof is complete. □

Proof of Series 39.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)^2}=\sum_{k=1}^{\infty}\frac{1}{(k+1)^3}\,{}_4F_3(1,1,1+k,1+k;2,2+k,2+k;1).$$

We rewrite this using (7.5.3.4) of [11]:

$$\sum_{k=1}^{\infty}\frac{1}{(k+1)^3}\left\{\frac{(1+k)^2}{k^2}[\gamma+\psi(1+k)]-\frac{(1+k)^2}{k}\psi'(1+k)\right\}$$
$$=\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}[\gamma+\psi(1+k)]-\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\psi'(1+k)=3\zeta(3)-2\zeta(2).$$

Here, we have made use of Series 11 and 20. □

Proof of Series 43, 44 and 45.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}f(n)$$
$$=\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}f(1)+\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}f(n)$$
$$=[\zeta(2)-1]f(1)+\sum_{n=2}^{\infty}\frac{1}{n(n-1)}[\psi(1+n)-\psi(2)]f(n)$$
$$=[\zeta(2)-1]f(1)+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}[\psi(2+n)-\psi(2)]f(n+1)$$
$$=[\zeta(2)-1]f(1)+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}[\gamma+\psi(2+n)]f(n+1)-\sum_{n=1}^{\infty}\frac{1}{n(n+1)}f(n+1).$$

Here, we first used (I.10). Thereafter we used (6.3.2) of [1]. By replacing $f(n)$ with the appropriate expression and using results derived earlier, the proof is complete. □

Proof of Series 47.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(1+n+k)]$$

$$=\gamma\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n(k+1)(n+k)}+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{\Gamma(n+k)}{n(k+1)\Gamma(1+n+k-x)}$$

$$=\gamma\sum_{k=1}^{\infty}\frac{1}{k(k+1)}[\gamma+\psi(1+k)]$$

$$+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{k=1}^{\infty}\frac{\Gamma(1+k)}{(k+1)\Gamma(2+k-x)}\ {}_3F_2(1,1,1+k;2,2+k-x;1).$$

We have made use of (I.10). Next, we make use of (7.4.4.40) of [11], and obtain

$$\gamma\sum_{k=1}^{\infty}\frac{1}{k(k+1)}[\gamma+\psi(1+k)]+\left.\frac{\mathrm{d}}{\mathrm{d}x}\right|_{x=0}\sum_{k=1}^{\infty}\frac{\Gamma(k)}{(k+1)\Gamma(1+k-x)}[\psi(1+k-x)-\psi(1-x)]$$

$$=\gamma\sum_{k=1}^{\infty}\frac{1}{k(k+1)}[\gamma+\psi(1+k)]$$

$$+\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\{\psi(1+k)[\gamma+\psi(1+k)]+\zeta(2)-\psi'(1+k)\}$$

$$=\sum_{k=1}^{\infty}\frac{1}{k(k+1)}[\gamma+\psi(1+k)]^2+\zeta(2)\sum_{k=1}^{\infty}\frac{1}{k(k+1)}-\sum_{k=1}^{\infty}\frac{1}{k(k+1)}\psi'(1+k)=4\zeta(3).$$

In the final steps we made use of Series 1, 20 and 25. □

Series 48–52

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n(k+1)(n+k)}=2\zeta(2), \tag{48}$$

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(1+k)]=3\zeta(3), \tag{49}$$

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(n)]=3\zeta(3), \tag{50}$$

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(1+n+k)]=6\zeta(3), \tag{51}$$

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n(k+1)(n+k)}[\gamma+\psi(n+k)]=2\zeta(3)+2\zeta(2). \tag{52}$$

These results all follow immediately from Series 38, 41, 43, 46, 47, (I.B.1) and (I.B.2).

Series 53

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n(n+1)(1+n+k)^2}=-\zeta(3)+\zeta(2). \tag{53}$$

Proof.

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\frac{1}{n(n+1)(1+n+k)^2}=\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\psi'(1+n)=-\zeta(3)+\zeta(2).$$

We have used (A.7) and Series 20 in the last steps. □

Series 54

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\frac{\Gamma(n)\Gamma(k)}{\Gamma(1+n+k)}=\zeta(3). \tag{54}$$

Proof.

$$\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k}\frac{\Gamma(n)\Gamma(k)}{\Gamma(1+n+k)}=\sum_{k=1}^{\infty}\frac{1}{k^2(k+1)}\,{}_2F_1(1,1;2+k;1)=\sum_{k=1}^{\infty}\frac{1}{k^3}=\zeta(3).$$

Here, we have used (A.1) to rewrite ${}_2F_1$. □

4. A three-dimensional series

Series 55

$$\sum_{l=0}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{nk(1+l+k)(l+n+k)}=3\zeta(3). \tag{55}$$

Proof.

$$\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{lk(1+n+k)(n+k+l)}=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\frac{1}{k(n+k)(1+n+k)}[\gamma+\psi(1+n+k)]=3\zeta(3).$$

Here, we have used (I.10) and Series 35. □

Appendix A. Useful properties of some special functions

A.1. Three often used identities

During the proofs we will often use the fact that

$${}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{A.1}$$

the integral representation for the psi function

$$\gamma+\psi(z)=\int_0^1 \mathrm{d}t\,\frac{1-t^{z-1}}{1-t}, \tag{A.2}$$

and the recurrence relation

$$\psi(1+z)=\psi(z)+\frac{1}{z}. \tag{A.3}$$

These three results can be found in (7.3.5.2) of [11], (6.3.22) and (6.3.5) of [1], respectively.

A.2. The trigamma function

The trigamma function is the derivative of the psi function (cf. [1, Eq. (6.4.1)]). By taking the derivative of (A.2) with respect to z we find the following integral representation for the trigamma function,

$$\psi'(z)=-\int_0^1 \mathrm{d}t\frac{t^{z-1}}{1-t}\log t. \tag{A.4}$$

In the case of unit argument we get (cf. [1], (6.4.2))

$$\psi'(1)=\frac{\pi^2}{6}. \tag{A.5}$$

This function also satisfies the recurrence relation (cf. [1], (6.4.6))

$$\psi'(1+z)=\psi'(z)-\frac{1}{z^2}. \tag{A.6}$$

One way the trigamma function enters into our series is from the following type of series:

$$\sum_{n=0}^{\infty}\frac{1}{(n+a)^2}=\frac{1}{a^2}\ {}_3F_2(1,a,a;1+a,1+a;1)=\psi'(a), \tag{A.7}$$

where (7.4.4.34) of [11] is used in the last step.

Appendix B. Generalizations

As mentioned in the introduction, all the series presented in this article are of the form

$$R_3\zeta(3)+R_2\zeta(2)+R_0, \tag{B.1}$$

where R_i are rational numbers. We may thus define a class of series which when summed will be of the form (B.1). All the series presented in this article belong to this particular class of series. In Paper I we also presented the results

$$\sum_{n=1}^{\infty}\frac{1}{n^2}[\gamma+\psi(2n)]=\tfrac{9}{4}\zeta(3), \tag{B.2}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}[\gamma+\psi(1+2n)]=\tfrac{11}{4}\zeta(3), \tag{B.3}$$

in (I.B.3) and (I.B.4). These could be obtained from (I.B.1) and (I.B.2) by replacing $\psi(n)$ and $\psi(1+n)$ with $\psi(2n)$ and $\psi(1+2n)$, respectively.

If we make the same replacement in Series 7, 8, 10, 11, 13 and 14, we may sum the series using methods similar to those used in Section 2. We present the results here without proof:

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}[\gamma + \psi(2n)] = 2\log 2 + \tfrac{1}{2}, \tag{B.4}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}[\gamma + \psi(1+2n)] = \tfrac{1}{2}\zeta(2) + 2\log 2, \tag{B.5}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}[\gamma + \psi(2n)] = \tfrac{9}{4}\zeta(3) - 2\log 2 - \tfrac{1}{2}, \tag{B.6}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2(n+1)}[\gamma + \psi(1+2n)] = \tfrac{11}{4}\zeta(3) - \tfrac{1}{2}\zeta(2) - 2\log 2, \tag{B.7}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}[\gamma + \psi(2n)] = -\tfrac{9}{4}\zeta(3) - \tfrac{3}{2}\zeta(2) + 6\log 2 + \tfrac{3}{2}, \tag{B.8}$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)^2}[\gamma + \psi(1+2n)] = -\tfrac{9}{4}\zeta(3) - \tfrac{1}{2}\zeta(2) + 6\log 2. \tag{B.9}$$

These series contain terms with $\log 2$, and are of the form

$$R_3\zeta(3) + R_2\zeta(2) + R_1\log 2 + R_0, \tag{B.10}$$

where R_i are rational numbers. These series do not belong to the class (B.1) mentioned above, but can be considered members of an extension of this class.

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