Available at
www.ElsevierMathematics.com
POWERED BY SCIENCE DIRECT®

Journal of Computational and Applied Mathematics 166 (2004) 453–463

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

www.elsevier.com/locate/cam

Periodic solution of a delayed predator–prey system with Michaelis–Menten type functional response

Hai-Feng Huo[a,*], Wan-Tong Li[b]

[a]Department of Applied Mathematics, Gansu University of Technology, Lanzhou, Gansu 730050, PR China
[b]Department of Mathematic, Lanzhou University, Lanzhou, Gansu 730000, PR China

Received 12 September 2002; received in revised form 4 August 2003

Abstract

By using the continuation theorem base on Gaines and Mawhin's coincidence degree, sufficient and realistic conditions are obtained for the global existence of positive periodic solution for a delayed predator–prey system with Michaelis–Menten type functional response. Indeed, our results are applicable to state-dependent delays.

MSC: 34K15; 92D25; 34C25

Keywords: Predator–prey system; Positive periodic solution; Coincidence degree

1. Introduction

Recently, Xu and Chaplain [11] considered the following delayed predator–prey model with Michaelis–Menton type functional response:

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1(t)\left(a_1 - a_{11}x_1(t-\tau_{11}) - \frac{a_{12}x_2(t)}{m_1+x_1(t)}\right),\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_2(t)\left(-a_2 + \frac{a_{21}x_1(t-\tau_{21})}{m_1+x_1(t-\tau_{21})} - a_{22}x_2(t-\tau_{22}) - \frac{a_{23}x_3(t)}{m_2+x_2(t)}\right),\\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= x_3(t)\left(-a_3 + \frac{a_{32}x_2(t-\tau_{32})}{m_2+x_2(t-\tau_{32})} - a_{33}x_3(t-\tau_{33})\right),
\end{aligned} \tag{1}$$

where a_i and a_{ij} $(i,j=1,2,3)$ are positive constants, and τ_{ij} $(i,j=1,2,3)$ are nonnegative constants.

* Corresponding author.
E-mail addresses: hfhuo@lut.cn (H.-F. Huo), wtli@lzu.edu.cn (W.-T. Li).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2003.08.042

In [11], the authors proved that system (1) is uniformly persistent under appropriate conditions and obtained sufficient conditions for global asymptotic stability of the positive equilibrium of system (1).

Since the variation of the environment plays an important role in many biological and ecological systems. In particular, the effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption of periodicity of the parameters in the way (in a way) incorporates the periodicity of the environment (e.g., seasonal affects of weather, food supplies, mating habits, etc.). In fact, it has been suggested in [8] that any periodic change of climate tends to impose its period upon oscillations of internal origin or to cause such oscillations to have a harmonic relation to periodic climatic changes. In view of this it is realistic to assume that the parameters in the models are periodic functions of period ω. Thus, the modification of (1) according to the environmental variation is the nonautonomous delay differential system

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1(t)\left(a_1(t) - a_{11}(t)x_1(t-\tau_{11}) - \frac{a_{12}(t)x_2(t)}{m_1 + x_1(t)}\right),\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_2(t)\left(-a_2(t) + \frac{a_{21}(t)x_1(t-\tau_{21})}{m_1 + x_1(t-\tau_{21})} - a_{22}(t)x_2(t-\tau_{22}) - \frac{a_{23}(t)x_3(t)}{m_2 + x_2(t)}\right),\\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= x_3(t)\left(-a_3(t) + \frac{a_{32}(t)x_2(t-\tau_{32})}{m_2 + x_2(t-\tau_{32})} - a_{33}(t)x_3(t-\tau_{33})\right),
\end{aligned} \tag{2}$$

where $a_i(t)$ and $a_{ij}(t)$ $(i,j=1,2,3)$ are continuous, bounded, and strictly positive functions on $[0,\infty)$, and τ_{ij} $(i,j=1,2,3)$ are nonnegative constants.

A very basic and important ecological problem associated with the study of multispecies population interaction in a periodic environment is the global existence of positive periodic solution which plays the role played by the equilibrium of the autonomous models. The main purpose of this paper is to derive easily verifiable sufficient conditions for the global existence of a positive periodic solution of systems (2). The method used here will be the coincidence degree theory developed by Gaines and Mawhin [2]. Such approach was adopted in [1,4,6,7].

Finally, we remark that in recent years periodic population dynamics has become a very popular subject. In fact, several different periodic models have been studied in [5,9,10,12].

2. Main results

In order to obtain the existence of a positive periodic solution of system (2), we first make the following preparations.

Let Y and Z be two Banach spaces. Consider an operator equation

$$Ly = \lambda N y, \quad \lambda \in (0,1),$$

where $L : \operatorname{Dom} L \cap Y \to Z$ is a linear operator and λ is a parameter. Let P and Q denote two projectors such that

$$P : Y \cap \operatorname{Dom} L \to \operatorname{Ker} L \quad \text{and } Q : Z \to Z/\operatorname{Im} L.$$

In the sequel, we will use the following result of Mawhin [2, p. 40].

Lemma 2.1. *Let Y and Z be two Banach spaces and L a Fredholm mapping of index zero. Assume that $N:\bar{\Omega}\to Z$ is L-compact on $\bar{\Omega}$ with Ω open bounded in Y. Furthermore, assume*:

(a) *for each* $\lambda\in(0,1)$, $y\in\partial\Omega\cap\operatorname{Dom}L$,

$$Ly\neq\lambda Ny;$$

(b) *for each* $y\in\partial\Omega\cap\operatorname{Ker}L$,

$$QNy\neq 0$$

and

$$\deg\{JQNy,\Omega\cap\operatorname{Ker}L,0\}\neq 0.$$

Then the equation $Ly=Ny$ has at least one solution in $\bar{\Omega}\cap L$.

Recall that a linear mapping $L:\operatorname{Dom}L\cap y\to Z$ with $\operatorname{Ker}L=L^{-1}(0)$ and $\operatorname{Im}L=L(\operatorname{Dom}L)$, will be called a Fredholm mapping if the following two conditions hold:
(i) $\operatorname{Ker}L$ has a finite dimension;
(ii) $\operatorname{Im}L$ is closed and has a finite codimension.

Recall also that the codimension of $\operatorname{Im}L$ is the dimension of $Z/\operatorname{Im}L$, i.e., the dimension of the cokernel coker L of L.

When L is a Fredholm mapping, its index is the integer $\operatorname{Ind}L=\dim\ker L-\operatorname{codim}\operatorname{Im}L$.

We shall say that a mapping N is L-compact on Ω if the mapping $QN:\bar{\Omega}\to Z$ is continuous, $QN(\bar{\Omega})$ is bounded, and $K_p(I-Q)N:\bar{\Omega}\to Y$ is compact, i.e., it is continuous and $K_p(I-Q)N(\bar{\Omega})$ is relatively compact, where $K_p:\operatorname{Im}L\to\operatorname{Dom}L\cap\operatorname{Ker}P$ is a inverse of the restriction L_p of L to $\operatorname{Dom}L\cap\operatorname{Ker}P$, so that $LK_p=I$ and $K_pL=I-P$.

For convenience, we shall introduce the notation:

$$\bar{u}=\frac{1}{\omega}\int_0^{\omega}u(t)\,\mathrm{d}t,$$

where u is a periodic continuous function with period ω.

In system (2), we always assume the following:

(H_1) $a_i(t)$ and $a_{ij}(t)$ $(i,j=1,2,3)$ are continuous, bounded, and strictly positive periodic functions with $\omega>0$.

Now we state our first theorem for the existence of a positive ω-periodic solution of system (2).

Theorem 2.1. *In addition to* (H_1), *assume further that system* (2) *satisfies*

(H_2)

$$\min\left\{\frac{\bar{a}_{32}(\bar{a}_1/\bar{a}_{12})\exp\{2\bar{a}_2\omega\}}{m_2+(\bar{a}_1/\bar{a}_{12})\exp\{2\bar{a}_2\omega\}},\frac{\bar{a}_{32}}{m_2+1}\right\}>\bar{a}_3;$$

(H_3)

$$\bar{a}_{21}>\bar{a}_2;$$

(H_4) *The system of the equations*

$$\bar{a}_1 - \bar{a}_{11}v_1 - \frac{\bar{a}_{12}v_2}{m_1+v_1} = 0,$$

$$-\bar{a}_2 + \frac{\bar{a}_{21}v_1}{m_1+v_1} - \bar{a}_{22}v_2 - \frac{\bar{a}_{23}v_3}{m_2+v_2} = 0,$$

$$-\bar{a}_3 + \frac{\bar{a}_{32}v_2}{m_2+v_2} - \bar{a}_{33}v_3 = 0 \tag{3}$$

has a unique positive solution $(v_1, v_2, v_3) \in R^3$. *Then system* (2) *has at least one positive* ω*-periodic solution.*

Proof. Since

$$x_1(t) = x_1(0)\exp\left\{\int_0^t \left[a_1(s) - a_{11}(s)x_1(s-\tau_{11}) - \frac{a_{12}(s)x_2(s)}{m_1+x_1(s)}\right]\mathrm{d}s\right\},$$

$$x_2(t) = x_2(0)\exp\int_0^t \left(-a_2(s) + \frac{a_{21}(s)x_1(s-\tau_{21})}{m_1+x_1(s-\tau_{21})} - a_{22}(s)x_2(s-\tau_{22}) - \frac{a_{23}(s)x_3(s)}{m_2+x_2(s)}\right)\mathrm{d}s,$$

$$x_3(t) = x_3(0)\exp\left\{\int_0^t \left[-a_3(s) + \frac{a_{32}(s)x_2(s-\tau_{32})}{m_2+x_2(s-\tau_{32})} - a_{33}(s)x_3(s-\tau_{33})\right]\mathrm{d}s\right\}$$

the solution of system (2) remains positive for $t \geqslant 0$, we can let

$$x_1(t) = \exp\{y_1(t)\}, \quad x_2(t) = \exp\{y_2(t)\}, \quad x_3(t) = \exp\{y_3(t)\} \tag{4}$$

and derive that

$$\frac{\mathrm{d}y_1}{\mathrm{d}t} = a_1(t) - a_{11}(t)\exp\{y_1(t-\tau_{11})\} - \frac{a_{12}(t)\exp\{y_2(t)\}}{m_1+\exp\{y_1(t)\}},$$

$$\frac{\mathrm{d}y_2}{\mathrm{d}t} = -a_2(t) + \frac{a_{21}(t)\exp\{y_1(t-\tau_{21})\}}{m_1+\exp\{y_1(t-\tau_{21})\}} - a_{22}(t)\exp\{y_2(t-\tau_{22})\} - \frac{a_{23}(t)\exp\{y_3(t)\}}{m_2+\exp\{y_2(t)\}},$$

$$\frac{\mathrm{d}y_3}{\mathrm{d}t} = -a_3(t) + \frac{a_{32}(t)\exp\{y_2(t-\tau_{32})\}}{m_2+\exp\{y_2(t-\tau_{32})\}} - a_{33}(t)\exp\{y_3(t-\tau_{33})\}. \tag{5}$$

In order to use Lemma 2.1 to system (2), we take

$$Y = Z = \{y(t) = (y_1(t), y_2(t), y_3(t))^{\mathrm{T}} \in C(R, R^3) : y(t+\omega) = y(t)\}$$

and denote

$$\|y\| = \|(y_1(t), y_2(t), y_3(t))^{\mathrm{T}}\| = \max_{t\in[0,\omega]}|y_1(t)| + \max_{t\in[0,\omega]}|y_2(t)| + \max_{t\in[0,\omega]}|y_3(t)|.$$

Then Y and Z are Banach spaces when they are endowed with the norms $\|\cdot\|$.

Set

$$Ny = \begin{bmatrix} a_1(t) - a_{11}(t)\exp\{y_1(t-\tau_{11})\} - \dfrac{a_{12}(t)\exp\{y_2(t)\}}{m_1+\exp\{y_1(t)\}} \\ -a_2(t) + \dfrac{a_{21}(t)\exp\{y_1(t-\tau_{21})\}}{m_1+\exp\{y_1(t-\tau_{21})\}} - a_{22}(t)\exp\{y_2(t-\tau_{22})\} - \dfrac{a_{23}(t)\exp\{y_3(t)\}}{m_2+\exp\{y_2(t)\}} \\ -a_3(t) + \dfrac{a_{32}(t)\exp\{y_2(t-\tau_{32})\}}{m_2+\exp\{y_2(t-\tau_{32})\}} - a_{33}(t)\exp\{y_3(t-\tau_{33})\} \end{bmatrix}$$

and

$$Ly = y', \quad Py = \frac{1}{\omega}\int_0^{\omega} y(t)\,\mathrm{d}t, \;\; y\in Y, \quad Qz = \frac{1}{\omega}\int_0^{\omega} z(t)\,\mathrm{d}t, \;\; z\in Z.$$

Evidently, $\operatorname{Ker} L = \{y \mid y\in Y, y = R^3\}$, $\operatorname{Im} L = \{z \mid z\in Z, \int_0^{\omega} z(t)\,\mathrm{d}t = 0\}$ is closed in Z and $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 3$. Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_p : \operatorname{Im} L \to \operatorname{Ker} P \cap \operatorname{dom} L$ has the form

$$K_p(z) = \int_0^t z(s)\,\mathrm{d}s - \frac{1}{\omega}\int_0^{\omega}\int_0^t z(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Thus,

$$QNy = \begin{bmatrix} \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\left[a_1(t) - a_{11}(t)\exp\{y_1(t-\tau_{11})\} - \dfrac{a_{12}(t)\exp\{y_2(t)\}}{m_1+\exp\{y_1(t)\}}\right]\mathrm{d}t \\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\left[-a_2(t) + \dfrac{a_{21}(t)\exp\{y_1(t-\tau_{21})\}}{m_1+\exp\{y_1(t-\tau_{21})\}} - a_{22}(t)\exp\{y_2(t-\tau_{22})\} \right. \\ \left. -\dfrac{a_{23}(t)\exp\{y_3(t)\}}{m_2+\exp\{y_2(t)\}}\right]\mathrm{d}t \\ \dfrac{1}{\omega}\displaystyle\int_0^{\omega}\left[-a_3(t) + \dfrac{a_{32}(t)\exp\{y_2(t-\tau_{32})\}}{m_2+\exp\{y_2(t-\tau_{32})\}} - a_{33}(t)\exp\{y_3(t-\tau_{33})\}\right]\mathrm{d}t \end{bmatrix}$$

and

$$K_p(I-Q)Ny$$
$$= \begin{bmatrix} \displaystyle\int_0^{t}\left[a_1(s) - a_{11}(s)\exp\{y_1(s-\tau_{11})\} - \dfrac{a_{12}(s)\exp\{y_2(s)\}}{m_1+\exp\{y_1(s)\}}\right]\mathrm{d}s \\ \displaystyle\int_0^{t}\left[-a_2(s) + \dfrac{a_{21}(s)\exp\{y_1(s-\tau_{21})\}}{m_1+\exp\{y_1(s-\tau_{21})\}} - a_{22}(s)\exp\{y_2(s-\tau_{22})\} \right. \\ \left. -\dfrac{a_{23}(s)\exp\{y_3(s)\}}{m_2+\exp\{y_2(s)\}}\right]\mathrm{d}s \\ \displaystyle\int_0^{t}\left[-a_3(s) + \dfrac{a_{32}(s)\exp\{y_2(s-\tau_{32})\}}{m_2+\exp\{y_2(s-\tau_{32})\}} - a_{33}(s)\exp\{y_3(s-\tau_{33})\}\right]\mathrm{d}s \end{bmatrix}$$

$$
-\begin{bmatrix}
\frac{1}{\omega}\int_0^{\omega}\int_0^{t}\left[a_1(s)-a_{11}(s)\exp\{y_1(s-\tau_{11})\}-\frac{a_{12}(s)\exp\{y_2(s)\}}{m_1+\exp\{y_1(s)\}}\right]\mathrm{d}s\,\mathrm{d}t \\
\frac{1}{\omega}\int_0^{\omega}\int_0^{t}\left[-a_2(s)+\frac{a_{21}(s)\exp\{y_1(s-\tau_{21})\}}{m_1+\exp\{y_1(s-\tau_{21})\}}-a_{22}(s)\exp\{y_2(s-\tau_{22})\}\right. \\
\left.\quad -\frac{a_{23}(s)\exp\{y_3(s)\}}{m_2+\exp\{y_2(s)\}}\right]\mathrm{d}s\,\mathrm{d}t \\
\frac{1}{\omega}\int_0^{\omega}\int_0^{t}\left[-a_3(s)+\frac{a_{32}(s)\exp\{y_2(s-\tau_{32})\}}{m_2+\exp\{y_2(s-\tau_{32})\}}-a_{33}(s)\exp\{y_3(s-\tau_{33})\}\right]\mathrm{d}s\,\mathrm{d}t
\end{bmatrix}
-\begin{bmatrix}
\left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}\left[a_1(t)-a_{11}(t)\exp\{y_1(t-\tau_{11})\}-\frac{a_{12}(t)\exp\{y_2(t)\}}{m_1+\exp\{y_1(t)\}}\right]\mathrm{d}t \\
\left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}\left[-a_2(t)+\frac{a_{21}(t)\exp\{y_1(t-\tau_{21})\}}{m_1+\exp\{y_1(t-\tau_{21})\}}-a_{22}(t)\exp\{y_2(t-\tau_{22})\}\right. \\
\left.\quad -\frac{a_{23}(t)\exp\{y_3(t)\}}{m_2+\exp\{y_2(t)\}}\right]\mathrm{d}t \\
\left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}\left[-a_3(t)+\frac{a_{32}(t)\exp\{y_2(t-\tau_{32})\}}{m_2+\exp\{y_2(t-\tau_{32})\}}-a_{33}(t)\exp\{y_3(t-\tau_{33})\}\right]\mathrm{d}t
\end{bmatrix}.
$$

Clearly, QN and $K_p(I-Q)N$ are continuous and, moreover, $QN(\bar{\Omega}), K_p(I-Q)N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega\subset X$. Hence, N is L-compact on $\bar{\Omega}$, here Ω is any open bounded set in X.

Now, we reach the position to search for an appropriate open bounded subset Ω for the application of Lemma 2.1. Corresponding to equation $Ly=\lambda Ny$, $\lambda\in(0,1)$, we have

$$
\begin{aligned}
\frac{\mathrm{d}y_1}{\mathrm{d}t} &= \lambda\left(a_1(t)-a_{11}(t)\exp\{y_1(t-\tau_{11})\}-\frac{a_{12}(t)\exp\{y_2(t)\}}{m_1+\exp\{y_1(t)\}}\right),\\
\frac{\mathrm{d}y_2}{\mathrm{d}t} &= \lambda\left(-a_2(t)+\frac{a_{21}(t)\exp\{y_1(t-\tau_{21})\}}{m_1+\exp\{y_1(t-\tau_{21})\}}-a_{22}(t)\exp\{y_2(t-\tau_{22})\}-\frac{a_{23}(t)\exp\{y_3(t)\}}{m_2+\exp\{y_2(t)\}}\right),\\
\frac{\mathrm{d}y_3}{\mathrm{d}t} &= \lambda\left(-a_3(t)+\frac{a_{32}(t)\exp\{y_2(t-\tau_{32})\}}{m_2+\exp\{y_2(t-\tau_{32})\}}-a_{33}(t)\exp\{y_3(t-\tau_{33})\}\right).
\end{aligned}
\tag{6}
$$

Suppose that $y(t)=(y_1,y_2,y_3)\in Y$ is a solution of system (6) for a certain $\lambda\in(0,1)$. By integrating (6) over the interval $[0,\omega]$, we obtain

$$
\begin{aligned}
&\int_0^{\omega}\left[a_1(t)-a_{11}(t)\exp\{y_1(t-\tau_{11})\}-\frac{a_{12}(t)\exp\{y_2(t)\}}{m_1+\exp\{y_1(t)\}}\right]\mathrm{d}t=0,\\
&\int_0^{\omega}\left[-a_2(t)+\frac{a_{21}(t)\exp\{y_1(t-\tau_{21})\}}{m_1+\exp\{y_1(t-\tau_{21})\}}-a_{22}(t)\exp\{y_2(t-\tau_{22})\}-\frac{a_{23}(t)\exp\{y_3(t)\}}{m_2+\exp\{y_2(t)\}}\right]\mathrm{d}t=0,\\
&\int_0^{\omega}\left[-a_3(t)+\frac{a_{32}(t)\exp\{y_2(t-\tau_{32})\}}{m_2+\exp\{y_2(t-\tau_{32})\}}-a_{33}(t)\exp\{y_3(t-\tau_{33})\}\right]\mathrm{d}t=0.
\end{aligned}
$$

Hence,

$$\int_0^{\omega}\left[a_{11}(t)\exp\{y_1(t-\tau_{11})\}+\frac{a_{12}(t)\exp\{y_2(t)\}}{m_1+\exp\{y_1(t)\}}\right]\mathrm{d}t=\bar{a}_1\omega, \tag{7}$$

$$\int_0^{\omega}\left[\frac{a_{21}(t)\exp\{y_1(t-\tau_{21})\}}{m_1+\exp\{y_1(t-\tau_{21})\}}-a_{22}(t)\exp\{y_2(t-\tau_{22})\}-\frac{a_{23}(t)\exp\{y_3(t)\}}{m_2+\exp\{y_2(t)\}}\right]\mathrm{d}t=\bar{a}_2\omega \tag{8}$$

and

$$\int_0^{\omega}\left[\frac{a_{32}(t)\exp\{y_2(t-\tau_{32})\}}{m_2+\exp\{y_2(t-\tau_{32})\}}-a_{33}(t)\exp\{y_3(t-\tau_{33})\}\right]\mathrm{d}t=\bar{a}_3\omega. \tag{9}$$

From (6)–(9), we obtain

$$\int_0^{\omega}|y_1'(t)|\,\mathrm{d}t<\int_0^{\omega}\left[a_{11}(t)\exp\{y_1(t-\tau_{11})\}+\frac{a_{12}(t)\exp\{y_2(t)\}}{m_1+\exp\{y_1(t)\}}\right]\mathrm{d}t$$

$$+\int_0^{\omega}|a_1(t)|\,\mathrm{d}t=2\bar{a}_1\omega, \tag{10}$$

$$\int_0^{\omega}|y_2'(t)|\,\mathrm{d}t<2\bar{a}_2\omega \tag{11}$$

and

$$\int_0^{\omega}|y_3'(t)|\,\mathrm{d}t<2\bar{a}_3\omega. \tag{12}$$

Note that $(y_1(t),y_2(t),y_3(t))\in Y$, then there exists $\xi_i,\eta_i\in[0,\omega]$, $i=1,2,3$ such that

$$y_i(\xi_i)=\min_{t\in[0,\omega]}y_i(t),\; y_i(\eta_i)=\max_{t\in[0,\omega]}y_i(t),\quad i=1,2,3. \tag{13}$$

By (7) and (13), we have

$$\bar{a}_1\omega\geqslant\bar{a}_{11}\omega\exp\{y_1(\xi_1)\}.$$

That is

$$y_1(\xi_1)\leqslant\ln\left\{\frac{\bar{a}_1}{\bar{a}_{11}}\right\}.$$

Then

$$y_1(t)\leqslant y_1(\xi_1)+\int_0^{\omega}|y_1'(t)|\,\mathrm{d}t<\ln\left\{\frac{\bar{a}_1}{\bar{a}_{11}}\right\}+2\bar{a}_1\omega. \tag{14}$$

By virtue of (7), (13) and (14), we also have

$$\bar{a}_1\omega\geqslant\bar{a}_{12}\omega\frac{\exp\{y_2(\xi_2)\}}{m_1+\exp\{y_1(t)\}}>\bar{a}_{12}\omega\frac{\exp\{y_2(\xi_2)\}}{m_1+(\bar{a}_1/\bar{a}_{11})\exp\{2\bar{a}_1\omega\}}.$$

So

$$y_2(\xi_2)\leqslant\ln\left\{\frac{\bar{a}_1(m_1+(\bar{a}_1/\bar{a}_{11})\exp\{2\bar{a}_1\omega\})}{\bar{a}_{12}}\right\}.$$

Then

$$y_2(t) \leqslant y_2(\xi_2) + \int_0^{\omega} |y_2'(t)|\,\mathrm{d}t < \ln\left\{\frac{\bar{a}_1(m_1 + (\bar{a}_1/\bar{a}_{11})\exp\{2\bar{a}_1\omega\})}{\bar{a}_{12}}\right\} + 2\bar{a}_2\omega. \tag{15}$$

By (8) and (13), we have

$$\frac{\bar{a}_{21}\omega\exp\{y_1(\eta_1)\}}{m_1 + \exp\{y_1(\eta_1)\}} \geqslant \bar{a}_2\omega. \tag{16}$$

So

$$y_1(\eta_1) \geqslant \ln\left\{\frac{\bar{a}_2 m_1}{\bar{a}_{21} - \bar{a}_2}\right\}.$$

Then

$$y_1(t) \geqslant y_1(\eta_1) - \int_0^{\omega} |y_1'(t)|\,\mathrm{d}t \geqslant \ln\left\{\frac{\bar{a}_2 m_1}{\bar{a}_{21} - \bar{a}_2}\right\} - 2\bar{a}_1\omega. \tag{17}$$

It follows from (9) and (13) that

$$\frac{\bar{a}_{32}\omega\exp\{y_2(\eta_2)\}}{m_2 + \exp\{y_2(\eta_2)\}} \geqslant \bar{a}_3\omega. \tag{18}$$

So

$$y_2(\eta_2) \geqslant \ln\left\{\frac{\bar{a}_3 m_2}{\bar{a}_{32} - \bar{a}_3}\right\}.$$

Then

$$y_2(t) \geqslant y_2(\eta_2) - \int_0^{\omega} |y_2'(t)|\,\mathrm{d}t \geqslant \ln\left\{\frac{\bar{a}_3 m_2}{\bar{a}_{32} - \bar{a}_3}\right\} - 2\bar{a}_2\omega. \tag{19}$$

Hence, (14), (15), (17) and (19) imply that

$$\max_{t\in[0,\omega]} |y_1(t)| \leqslant \max\left\{\left|\ln\left\{\frac{\bar{a}_1}{\bar{a}_{11}}\right\} + 2\bar{a}_1\omega\right|, \left|\ln\left\{\frac{\bar{a}_2 m_1}{\bar{a}_{21} - \bar{a}_2}\right\} - 2\bar{a}_1\omega\right|\right\} := B_1 \tag{20}$$

and

$$\begin{aligned}\max_{t\in[0,\omega]} |y_2(t)| \leqslant \max\Bigg\{&\left|\ln\left\{\left(\bar{a}_1\left(m_1 + \frac{\bar{a}_1}{\bar{a}_{11}}\exp\{2\bar{a}_1\omega\}\right)\right)\Big/\bar{a}_{12}\right\} + 2\bar{a}_2\omega\right|,\\ &\left|\ln\left\{\frac{\bar{a}_3 m_2}{\bar{a}_{32} - \bar{a}_3}\right\} - 2\bar{a}_2\omega\right|\Bigg\}\\ :=\ & B_2.\end{aligned} \tag{21}$$

By (9), (13) and (15), we also obtain

$$\frac{\bar{a}_{32}(\bar{a}_1/\bar{a}_{12})\exp\{2\bar{a}_2\omega\}}{m_2 + (\bar{a}_1/\bar{a}_{12})\exp\{2\bar{a}_2\omega\}} - \bar{a}_{33}\exp\{y_3(\xi_3)\} \geqslant \bar{a}_3.$$

Thus

$$y_3(\xi_3) \leqslant \ln\left\{\left(\frac{\bar{a}_{32}(\bar{a}_1/\bar{a}_{12})\exp\{2\bar{a}_2\omega\}}{m_2+(\bar{a}_1/\bar{a}_{12})\exp\{2\bar{a}_2\omega\}} - \bar{a}_3\right)\Big/\bar{a}_{33}\right\}.$$

Then

$$y_3(t) \leqslant y_3(\xi_3) + \int_0^\omega |y_3'(t)|\,\mathrm{d}t \leqslant \ln\left\{\left(\frac{\bar{a}_{32}(\bar{a}_1/\bar{a}_{12})\exp\{2\bar{a}_2\omega\}}{m_2+(\bar{a}_1/\bar{a}_{12})\exp\{2\bar{a}_2\omega\}} - \bar{a}_3\right)\Big/\bar{a}_{33}\right\} + 2\bar{a}_3\omega. \quad (22)$$

Furthermore, in view of (9) and (13), we obtain

$$\frac{\bar{a}_{32}}{m_2+1} - \bar{a}_{33}\exp\{y_3(\eta_3)\} \leqslant \bar{a}_3.$$

That is

$$y_3(\eta_3) \geqslant \ln\left\{\frac{[\bar{a}_{32}/(m_2+1)] - \bar{a}_3}{\bar{a}_{33}}\right\}.$$

Then

$$y_3(t) \geqslant y_3(\eta_3) - \int_0^\omega |y_3'(t)|\,\mathrm{d}t \geqslant \ln\left\{\frac{[\bar{a}_{32}/(m_2+1)] - \bar{a}_3}{\bar{a}_{33}}\right\} - 2\bar{a}_3\omega. \quad (23)$$

(22) and (23) imply that

$$\begin{aligned}\max_{t\in[0,\omega]}|y_3(t)| &\leqslant \max\left\{\left|\ln\left\{\left(\frac{\bar{a}_{32}(\bar{a}_1/\bar{a}_{12})\exp\{2\bar{a}_2\omega\}}{m_2+(\bar{a}_1/\bar{a}_{12})\exp\{2\bar{a}_2\omega\}} - \bar{a}_3\right)\Big/\bar{a}_{33}\right\} + 2\bar{a}_3\omega\right|,\right.\\ &\qquad\left.\left|\ln\left\{\frac{[\bar{a}_{32}/(m_2+1)] - \bar{a}_3}{\bar{a}_{33}}\right\} - 2\bar{a}_3\omega\right|\right\}\\ &:= B_3. \end{aligned} \quad (24)$$

Clearly, B_i, $i=1,2,3$ are independent of λ, Under the assumption in Theorem 2.1, it is easy to show that the system of algebraic equations (3) has a unique solution $(v_1^*, v_2^*, v_3^*)^{\mathrm{T}} \in \operatorname{int} R_+^3$ with $v_i^* > 0, i=1,2,3$. Denote $B = B_1+B_2+B_3+B_4$, where $B_4>0$ is taken sufficiently large such that $\|(\ln\{v_1^*\}, \ln\{v_2^*\}, \ln\{v_3^*\})\| = |\ln\{v_1^*\}| + |\ln\{v_2^*\} + \ln\{v_3^*\}| < B_4$, and define

$$\Omega = \{y(t) \in Y : \|y\| < B\}.$$

It is clear that Ω satisfies condition (a) of the Lemma 2.1. When $y=(y_1,y_2,y_3)^{\mathrm{T}} \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^3$, y is a constant vector in R^3 with $\|y\| = M$. Then

$$QNx = \begin{bmatrix} \bar{r}_1 - \bar{a}_{11}\exp\{y_1\} - \dfrac{\bar{a}_{12}\exp\{y_2\}}{m_1+\exp\{y_1\}} \\ -\bar{r}_2 + \dfrac{\bar{a}_{21}\exp\{y_1\}}{m_1+\exp\{y_1\}} - \bar{a}_{22}\exp\{y_2\} - \dfrac{\bar{a}_{23}\exp\{y_3\}}{m_2+\exp\{y_2\}} \\ -\bar{r}_3 + \dfrac{\bar{a}_{32}\exp\{y_2\}}{m_2+\exp\{y_2\}} - \bar{a}_{33}\exp\{y_3\} \end{bmatrix} \neq 0.$$

Furthermore, in view of assumption in Theorem 2.1, it can easily be seen that

$$\deg\{JQNx, \Omega\cap\operatorname{Ker} L, 0\} \neq 0.$$

By now we know that Ω verifies all the requirements of Lemma 2.1 and then system (5) has at least one ω-periodic solution. By the medium of (4), we derive that (2) has at least one positive ω-periodic solution. The proof is complete. □

Remark 2.1. From [11] we know that, under certain conditions, system (2) is uniformly persistent, then system (2) must have a positive equilibrium [3], which also implies that system (3) has a positive solution.

Next, we consider the following predator–prey systems with state dependent delays:

$$\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}t} &= x_1(t)\left(a_1(t) - a_{11}(t)x_1(t-\tau_{11}(t,x_1(t),x_2(t),x_3(t))), \frac{a_{12}(t)\exp\{y_2(t)\}}{m_1+\exp\{y_1(t)\}}\right),\\
\frac{\mathrm{d}x_2}{\mathrm{d}t} &= x_2(t)\left(-a_2(t) + \frac{a_{21}(t)x_1(t-\tau_{21}(t,x_1(t),x_2(t),x_3(t)))}{m_1+x_1(t-\tau_{21}(t,x_1(t),x_2(t),x_3(t)))}\right.\\
&\quad \left. - a_{22}(t)x_2(t-\tau_{22}(t,x_1(t),x_2(t),x_3(t))) - \frac{a_{23}(t)x_3(t)}{m_2+x_2(t)}\right),\\
\frac{\mathrm{d}x_3}{\mathrm{d}t} &= x_3(t)\left(-a_3(t) + \frac{a_{32}(t)x_2(t-\tau_{32}(t,x_1(t),x_2(t),x_3(t)))}{m_2+x_2(t-\tau_{32}(t,x_1(t),x_2(t),x_3(t)))}\right)\\
&\quad - a_{33}(t)x_3(t-\tau_{33}(t,x_1(t),x_2(t),x_3(t))). \qquad (25)
\end{aligned}$$

Theorem 2.2. *If the assumptions of Theorem* 2.1 *hold, then the system* (25) *has at least one positive ω-periodic solution.*

Proof. The proof is similar to that of Theorem 2.1 and hence is omitted here. □

Acknowledgements

This work is supported by the NNSF of China (10171040), the NSF of Gansu Province of China (ZS011-A25-007-Z), the Foundation for University Key Teacher by the Ministry of Education of China, the Teaching and Research Award Program for Outstanding Young Teachers in Higher Education Institutions of Ministry of Education of China, the Key Research and Development Program for Outstanding Groups of Gansu University of Technology and the Development Program for Outstanding Young Teachers in Gansu University of Technology.

References

[1] M. Fan, K. Wang, Periodicity in a delayed ratio-dependent predator–prey system, J. Math. Anal. Appl. 262 (2001) 179–190.
[2] R.E. Gaines, J.L. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Springer, Berlin, 1977.
[3] V. Hotson, The existence of an equilibrium for permanent systems, Rocky Mountain J. Math. 20 (1990) 1033–1040.
[4] H.F. Huo, W.T. Li, Periodic solution of a periodic two-species competition model with delays, Internat. J. Appl. Math. 12 (2003) 13–21.

[5] H.F. Huo, W.T. Li, S.S. Cheng, Periodic solutions of two-species diffusion models with continuous time delays, Demonstratio Math. 35 (2) (2002) 433–446.
[6] Y.K. Li, Periodic solutions of a periodic delay predator–prey system, Proc. Amer. Math. Soc. 127 (1999) 1331–1335.
[7] Y.K. Li, Y. Kuang, Periodic solutions of periodic delay Lotka–Volterra equations and systems, J. Math. Anal. Appl. 255 (2001) 260–280.
[8] A.J. Nicholson, The balance of animal population, J. Anim. Ecol. 2 (1993) 132–178.
[9] S.H. Saker, S. Agarwal, Oscillation and global attractivity in a nonlinear delay periodic model of resiratory dynamics, Comput. Math. Appl. 44 (2002) 623–632.
[10] S.H. Saker, Sheba Agarwal, Oscillation and global attractivity in a periodic Nichlson's blowflies model, Math. Comput. Modelling 35 (2002) 719–731.
[11] R. Xu, M.A.J. Chaplain, Persistence and global stability in a delayed predator–prey system with Michaelis–Menten type functional response, Appl. Math. Comput. 130 (2002) 441–455.
[12] J. Yan, Q. Feng, Global attractivity and oscillation in a nonlinear delay equation, Nonlinear Anal. 43 (2001) 101–108.