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An inverse problem for a fractional diffusion equation

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ABSTRACT

In this paper, we consider an inverse problem for a fractional diffusion equation which is highly ill-posed. Such a problem is obtained from the classical diffusion equation by replacing the first-order time derivative by the Caputo fractional derivative of order α $(0 < \alpha < 1)$. We show that the problem is severely ill-posed and further apply an optimal regularization method to solve it based on the solution in the frequency domain. We can prove the optimal convergence estimate, which shows that the regularized solution depends continuously on the data and is a good approximation to the exact solution. Numerical examples show that the proposed method works well.

1. Introduction

In recent years, fractional calculus and derivatives have encountered much success in many fields of science, for instance, dissipation [1], heat conduction [2–4]. Fractional derivatives also have been found to be quite flexible in describing viscoelastic behaviors. Hence, in the last decades, the study of fractional diffusion equations has attracted intense attention.

If the initial concentration distribution and boundary conditions are given, a complete recovery of the unknown solution is attainable from solving a well-posed forward problem. However, in some practical problems, the boundary data can only be measured on a portion of the boundary or some points in the solution domain. This leads to an ill-posed problem of the fractional heat diffusion equation, which means the solution does not depend continuously on the given known conditions, see [5]. In this paper, we investigate an inverse heat conduction problem (IHCP) for the fractional diffusion equation. This kind of ill-posed problem is important in many branches of engineering sciences [6–8]. This is usually referred to as the ill-posed backward determination problem, which is in nature unstable because the unknown solution and its derivatives have to be determined from indirect observable data which contain measurement errors. The major difficulty in establishing any numerical algorithm for approximating the solution is due to the severe ill-posedness of the problem.

Due to difficulty of the fractional derivative and the ill-posedness, to the authors' knowledge, the results on inverse problems for fractional diffusion equations are very few. The uniqueness of an inverse problem for a one-dimensional fractional diffusion equation was given in [9]. Zheng and Wei [10] gave a regularization method for a Cauchy problem of the time fractional advection–dispersion equation in a space-unbounded domain. Numerical results by using difference methods were given in [11,12]. Recently, Tuan [13,14] investigated the inverse spectral problems for the fractional diffusion equation.

In this paper, we focus on a fractional inverse heat conduction problem (FIHCP). This is a semi-unbounded problem, in which the Fourier transform is a powerful tool to analyze. Many of the current researches on the ill-posed problem are based on solving the corresponding direct problem iteratively. The iterations and the initial guess value are important and sensitive in these iterative computational methods. By using the direct optimal regularization method, in this paper, a one-stage direct computational method for this ill-posed problem is obtained.

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In the present paper, motivated by [15,16], we construct a stable approximate solution for the problem and present convergence results under suitable choices of the regularization parameter.

Our paper is divided into three sections: in Section 2, we analyze the ill-posedness of the problem and propose an optimal filtering regularization method. In Section 3, optimal error estimate is given based on an a-priori assumption for the exact solution.

2. Ill-posedness of the problem and regularization

2.1. Formulation of the problem

In several engineering contexts, it is sometimes necessary to estimate the surface temperature or heat flux in a body from a measured temperature history at a fixed location inside a body. This is the so-called "inverse heat conduction problem". For some numerical methods, we can refer to [17–19] and the references therein.

Let us consider the following IHCP for the fractional heat equation when anomalous diffusion occurs:

$$\begin{aligned} &\frac{\partial^\beta u}{\partial t^\beta} - u_{xx} = 0, \quad x > 0,\ t > 0, \\ &u(x,0) = 0, \quad x > 0, \\ &u(1,t) = g(t), \quad t > 0, \\ &u(x,t)|_{x\to\infty} \text{ bounded} \end{aligned} \tag{2.1}$$

where the time fractional derivative $\frac{\partial^\beta u}{\partial t^\beta}$ is the Caputo fractional derivative of order β $(0 < \beta \le 1)$ defined by (see [20])

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u(x,s)}{\partial s} \frac{ds}{(t-s)^\beta}, \quad 0 < \beta < 1, \tag{2.2}$$

$$\frac{\partial^\beta u}{\partial t^\beta} = \frac{\partial u(x,t)}{\partial t}, \quad \beta = 1. \tag{2.3}$$

Here, we wish to determine the temperature $u(x,t)$ for $0 \le x < 1$ from temperature measurements $g_\delta(t)$.

The above FIHCP is an inverse problem and is severely ill-posed. That means the solution does not depend continuously on the given data and any small perturbation in the given data may cause large a change to the solution. In this study, we use an optimal filtering method to solve the FIHCP.

2.2. Ill-posedness

In order to use the Fourier transform, we extend the functions $u(x,\cdot)$, $g(\cdot)$, and $g_\delta(\cdot)$ to the whole line $-\infty < t < \infty$ by defining them to be zero for $t < 0$. Here and in the following sections, $\|\cdot\|$ denotes the L_2 norm, i.e.,

$$\|f(\cdot)\| = \left(\int_{\mathbb{R}} |f(t)|^2 dt \right)^{\frac{1}{2}}.$$

Since the measurements usually contain an error, we now could assume that the measured data function $g_\delta(t)$ satisfies

$$\|g(\cdot) - g_\delta(\cdot)\| \le \delta, \tag{2.4}$$

where the constant $\delta > 0$ represents a bound on the measurement error.

If we take the Laplace transform of both sides of (2.1) with respect to t, according to the properties of Laplace transform of the Caputo derivative (see [20, p. 106]), we get

$$s^\beta U(x,s) - s^{\beta-1} u(x,0) - U_{xx}(x,s) = 0, \tag{2.5}$$

where s is the variable of Laplace transform on t. Applying the homogenous initial condition, we have

$$U_{xx}(x,s) = s^\beta U(x,s), \tag{2.6}$$

which is a second-order ordinary differential equation. Now using the boundary conditions, we can get

$$U(x,s) = \exp(\sqrt{s^\beta}(1-x))G(s), \tag{2.7}$$

where $\sqrt{s^\beta}$ denotes the principal square root of s^β.

Throughout this paper, we extend all the functions to the whole line $-\infty < t < \infty$. Let

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt$$

be the Fourier transform of the function $f(t) \in L^2(\mathbb{R})$.

For functions $f(t)$ which vanish on the negative t axis, the Fourier and Laplace transforms are related via

$$F(i\xi) = \sqrt{2\pi}\hat{f}(\xi).$$

Therefore, from (2.6), setting $s = i\xi$, the solution of the problem can be formulated in the frequency domain:

$$\hat{u}(x, \xi) = \exp(x\eta)\hat{g}(\xi), \tag{2.8}$$

where

$$\eta := (i\xi)^{\frac{\beta}{2}} = |\xi|^{\frac{\beta}{2}} \left(\cos\left(\frac{\beta\pi}{4}\right) + i\,\mathrm{sign}(\xi)\sin\left(\frac{\beta\pi}{4}\right)\right). \tag{2.9}$$

Since $|e^{(1-x)(i\xi)^{\frac{\beta}{2}}}|$ is unbounded with respect to variable ξ for fixed $0 < x < 1$, and the solution $\hat{u}(x, \xi)$ with respect to ξ is assumed to be in $L^2(\mathbb{R})$, for $0 \le x < 1$, the exact data function, $\hat{g}(\xi)$, must decay rapidly as $|\xi| \to \infty$. However, the data $g(t)$ in problem (2.1) are generally based on observations, and we only have the noisy data $g_\delta(t) \in L^2(\mathbb{R})$ with $\|g(t) - g_\delta(t)\|_{L^2(\mathbb{R})} \le \delta$. Since we cannot expect the measurement data $\hat{g_\delta}(\xi)$ to have the same decay in the frequency domain as the exact data $\hat{g}(\xi)$, the solution $\hat{u}^\delta(x, \xi) = e^{(1-x)(i\xi)^{\frac{\beta}{2}}}\hat{g_\delta}(\xi)$ will not, in general, be in $L^2(\mathbb{R})$ for fixed $0 \le x < 1$. Thus, if we try to solve problem (2.1) numerically, high frequency components in the error δ are magnified and can destroy the solution.

2.3. Regularization

From the analysis of Section 2.2, we know that the real cause of ill-posedness is that the noise of data in the high frequency components blows up the solution. Also from (2.8), we have two ways to stably solve the inverse problem. One way is to eliminate the noise in the high frequency components through mollifying the noisy data. The other way is to eliminate the high frequency effect through modifying the "kernel" $e^{(1-x)(i\xi)^{\frac{\beta}{2}}}$. In the present paper, we are interested in the optimal regularization method and the optimal convergence estimate. Motivated by [18] where the authors cut off the high frequency components directly, here we discuss a regularization method which preserves the information of high frequency components partially according to

$$\hat{u}_\alpha^\delta(x, \xi) = k_\alpha(x, \xi)\hat{g_\delta}(\xi), \tag{2.10}$$

where

$$k_\alpha(x, \xi) = \begin{cases} e^{(1-x)(i\xi)^{\frac{\beta}{2}}}, & |e^{(1-x)(i\xi)^{\frac{\beta}{2}}}| \le \alpha(x), \\ \alpha(x)e^{i(1-x)\mathrm{sign}(\xi)|\xi|^{\frac{\beta}{2}}\sin\frac{\beta\pi}{4}}, & |e^{(1-x)(i\xi)^{\frac{\beta}{2}}}| > \alpha(x), \end{cases} \tag{2.11}$$

where $\alpha(x)$ will be determined in Section 3, and can be considered as a regularization parameter.

To obtain the convergent rate between the regularized solution (2.10) and the exact one (2.8), a-priori knowledge about the true solution is an essential element in the successful computation of ill-posed inverse problems [21,22]:

$$\|u(0, \cdot)\|_p \le E, \quad p \ge 0, \tag{2.12}$$

where $E > 0$ is a constant and $\|\cdot\|_p$ denotes the norm in Sobolev space $H^p(\mathbb{R})$ defined by

$$\|u(0, \cdot)\|_p := \left(\int_{\mathbb{R}} (1 + \xi^2)^p |\hat{u}(0, \xi)|^2 d\xi\right)^{\frac{1}{2}}.$$

When $p = 0$, $H^p(\mathbb{R}) = H^0(\mathbb{R}) = L^2(\mathbb{R})$, and formula (2.11) is bounded in the $L^2(\mathbb{R})$-norm. Obviously, the larger the p, the more restrictive the assumption (2.12).

3. Optimal estimate

In the following, we study the properties of (2.10) considered as a regularized solution of problem (2.1). We can give the optimal convergence estimate which shows that formula (2.10) is really an effective approximation.

By previous analysis, the cause of ill-posedness of the problem lies in the amplified factor $|e^{(1-x)(i\xi)^{\frac{\beta}{2}}}|$ of data. To stabilize the problem, a natural idea is to construct a function that approaches the amplified factor. The constructed function must satisfy the following properties:

(I) The modulus of constructed function should be less than or equal to that of the amplified factor.

(II) For the reason of stability, we need to preserve the information of low-frequency components and eliminate partially or completely the information of high-frequency components as well.

Based on the above idea, we have (2.10).

We have the following main conclusion.

Theorem 3.1. *Let $u(x,t)$ be the solution of problem (2.1) with the exact data $g(t)$, which can be expressed as formula (2.8) in the frequency domain. Let $u_\alpha^\delta(x,t)$ be the regularized solution with the measured data $g_\delta(x,t)$, which can be expressed as formula (2.10) in the frequency domain. Assume that the measured data at $x=1, g_\delta(t)$ satisfies $\|g(t)-g_\delta(t)\|\le\delta$, and the a-priori bound (2.12) holds. Then, if choosing*

$$\alpha(x)=x\left(\frac{E}{\delta}\right)^{1-x}, \tag{3.1}$$

we have the optimal error estimate for $p=0$ and the fixed $0<x<1$:

$$\|u(x,\cdot)-u_\alpha^\delta(x,\cdot)\|\le E^{1-x}\delta^x. \tag{3.2}$$

Proof. We can rewrite solution (2.8) as

$$\hat{u}(x,\xi)=e^{(1-x)(a+bi)}\hat{g}(\xi), \tag{3.3}$$

where

$$a:=|\xi|^{\frac{\beta}{2}}\cos\frac{\beta\pi}{4},\qquad b:=|\xi|^{\frac{\beta}{2}}\operatorname{sign}(\xi)\sin\frac{\beta\pi}{4},$$

and rewrite the regularized solution (2.10) as

$$\hat{u}_\alpha^\delta(x,\xi)=k_\alpha(x,\xi)\hat{g_\delta}(\xi), \tag{3.4}$$

where

$$k_\alpha(x,a,b)=\begin{cases}e^{(1-x)(a+bi)}, & e^{(1-x)a}\le\alpha(x)\\ \alpha(x)e^{(1-x)bi}, & e^{(1-x)a}>\alpha(x).\end{cases} \tag{3.5}$$

From (3.3) we have

$$\hat{g}(\xi)=e^{-(a+bi)}\hat{u}(0,\xi). \tag{3.6}$$

Now using the Parseval equality, (3.3) and (3.4), we have

$$\|u(x,\cdot)-u_\alpha^\delta(x,\cdot)\|=\|\hat{u}(x,\cdot)-\hat{u}_\alpha^\delta(x,\cdot)\|=\|e^{(1-x)(a+bi)}\hat{g}-k_\alpha(x,a,b)\hat{g_\delta}\|. \tag{3.7}$$

Adding and subtracting $k_\alpha(x,a,b)\hat{g}$, and using the triangle inequality, we get

$$\|u(x,\cdot)-u_\alpha^\delta(x,\cdot)\|\le\|(e^{(1-x)(a+bi)}-k_\alpha(x,a,b))\hat{g}\|+\|k_\alpha(x,a,b)(\hat{g}-\hat{g_\delta})\|. \tag{3.8}$$

The second term on the right-hand side of (3.8) is easy, i.e.,

$$\|k_\alpha(x,a,b)(\hat{g}-\hat{g_\delta})\|\le\delta\sup_{\xi\in\mathbb{R}}|k_\alpha(x,a,b)|\le\delta\alpha(x), \tag{3.9}$$

where we have also used the error bound $\|g-g_\delta\|\le\delta$.

We now estimate the first term on the right hand side of (3.8), note that (3.5) and (3.6),

$$\begin{aligned}(e^{(1-x)(a+bi)}-k_\alpha(x,a,b))\hat{g} &= \frac{e^{(1-x)(a+bi)}-k_\alpha(x,a,b)}{e^{a+bi}}\hat{u}(0,\xi)\\ &= \frac{e^{(1-x)(a+bi)}-\min\{e^{(1-x)a},\alpha(x)\}e^{(1-x)bi}}{e^{a+bi}}\hat{u}(0,\xi)\\ &= \frac{e^{(1-x)a}-\min\{e^{(1-x)a},\alpha(x)\}e^{(1-x)bi}}{e^{a+bi}}\hat{u}(0,\xi).\end{aligned}$$

Consequently, we have

$$\|(e^{(1-x)(a+bi)} - k_\alpha(x,a,b))\hat{g}\| \le \sup_{\xi\in\mathbb{R}} \left| \frac{(e^{(1-x)a} - \min\{e^{(1-x)a}, \alpha(x)\})e^{(1-x)bi}}{e^{a+bi}} \right| E$$
$$= \sup_{\xi\in\mathbb{R}, e^{(1-x)a}>\alpha(x)} \frac{e^{(1-x)a} - \alpha(x)}{e^a} E,$$

where we have used the a-priori bound (2.12) for $p = 0$.

Let $h(a) = (e^{(1-x)a} - \alpha(x))/e^a$. Differentiating $h(a)$ and setting the derivative equal to zero, we find

$$e^{(1-x)a} = \frac{1}{x}\alpha(x). \tag{3.10}$$

When formula (3.10) holds, the function $h(a)$ arrives at the maximum value. Therefore, we have

$$\|(e^{(1-x)(a+bi)} - k_\alpha(x,a,b))\hat{f}\| \le \frac{\left(\frac{1}{x}-1\right)\alpha(x)}{\left(\frac{1}{x}\alpha(x)\right)^{\frac{1}{1-x}}} E = \frac{1-x}{x}\left(\frac{1}{x}\right)^{\frac{-1}{1-x}} \alpha(x)^{\frac{-x}{1-x}} E. \tag{3.11}$$

Combining (3.8), (3.9) and (3.11) with the choice of $\alpha(x)$ in (3.1), we arrive at optimal estimate (3.2), i.e.,

$$\begin{aligned}
\|u(x,\cdot) - u_\alpha^\delta(x,\cdot)\| &\le \frac{1-x}{x}\left(\frac{1}{x}\right)^{\frac{-1}{1-x}} \alpha(x)^{\frac{-x}{1-x}} E + \delta\alpha(x) \\
&= \left(\frac{1-x}{x}\right)\left(\frac{1}{x}\right)^{\frac{-1}{1-x}} \left(x\left(\frac{E}{\delta}\right)^{1-x}\right)^{\frac{-x}{1-x}} E + \delta x\left(\frac{E}{\delta}\right)^{1-x} \\
&= \left(\frac{1-x}{x}\right)(x)^{\frac{1}{1-x}}(x)^{\frac{-x}{1-x}}\left(\frac{E}{\delta}\right)^{(-x)} E + \delta x\left(\frac{E}{\delta}\right)^{1-x} \\
&= (1-x)\left(\frac{E}{\delta}\right)^{(-x)} E + \delta x\left(\frac{E}{\delta}\right)^{1-x} \\
&= \left(\frac{E}{\delta}\right)^{(-x)}\left((1-x)E + \delta x\frac{E}{\delta}\right) \\
&= \left(\frac{E}{\delta}\right)^{(-x)} E = E^{1-x}\delta^x. \quad \square
\end{aligned}$$

Remark 3.2. In our application $\|u(x,\cdot)\|_p$ is usually not known, therefore we have no exact a-priori bound E in (2.12) and cannot choose the parameter $\alpha(x)$ according to (3.1). However, if selecting

$$\alpha^*(x) = x\left(\frac{1}{\delta}\right)^{1-x}, \tag{3.12}$$

we can also obtain the convergent rate

$$\|u(x,\cdot) - u_{\alpha^*}^\delta(x,\cdot)\| \le c\delta^x,$$

where the constant $c = 1/((1-x)\|u(0,\cdot)\|_p + x)$. This choice is helpful in our realistic computation.

Although the above argument provides an approximation only for $0 < x < 1$, we may use this construction to obtain an estimate for $f(t) := u(0,t)$. Although we will not have such a nice estimate as for $0 < x < 1$, we do obtain the convergence as $\delta \to 0$. In the following discussion, not only do we obtain the convergence at $x = 0$, but we also obtain the explicit convergent rate.

In fact, since we are now especially concerned with endpoint $x = 0$, we rewrite the regularization formula instead of (3.4):

$$\hat{f}_\alpha^\delta(\xi) = k_\alpha(a,b)\hat{f}_\delta(\xi), \tag{3.13}$$

where

$$k_\alpha(a,b) = \begin{cases} e^{a+bi}, & e^a \le \alpha \\ \alpha\, e^{bi}, & e^a > \alpha, \end{cases}$$

and α which will be given in the following theorem, can be considered as a regularization parameter.

Theorem 3.3. *Let $u(0, t)$ be the solution of problem (2.1) with the exact data $g(t)$, which can be expressed as formula (3.3) in the frequency domain. Let $f_\alpha^\delta(t)$ be the regularized solution with the measure data $g_\delta(t)$, which can be expressed as formula (3.13) in the frequency domain. Assume that the measured data at $x = 1$, g_δ, satisfy $\|g - g_\delta\| \le \delta$, and the a-priori bound (2.12) holds with $p > 0$. Then, if choosing*

$$\alpha = \frac{1}{c(a_0)}\delta^{-r}, \tag{3.14}$$

where the constant $c(a_0) > 1$, $0 < r < 1$, a_0 is a constant, there holds the convergence estimate:

$$\|f(\cdot) - f_\alpha^\delta(\cdot)\| \le c\left(\ln\frac{1}{\delta}\right)^{\frac{-2p}{\beta}}, \quad p > 0. \tag{3.15}$$

Proof. Following the process of proof of Theorem 3.1, we have

$$\begin{aligned}
\|f - f_\alpha^\delta\| &= \|\hat{f} - \hat{f}_\alpha^\delta\| = \|e^{(a+bi)}\hat{g} - k_\alpha\hat{g_\delta}\| \le \|e^{(a+bi)}\hat{g} - k_\alpha\hat{g}\| + \|k_\alpha\hat{g} - k_\alpha\hat{g_\delta}\| \\
&= \left\|\frac{e^{a+bi} - k_\alpha}{(1+\xi^2)^{\frac{p}{2}}e^{a+bi}}(1+\xi^2)^{\frac{p}{2}}e^{a+bi}\hat{g}\right\| + \|k_\alpha(\hat{g} - \hat{g_\delta})\| \\
&= \left\|\frac{e^{a+bi} - k_\alpha}{(1+\xi^2)^{\frac{p}{2}}e^{a+bi}}(1+\xi^2)^{\frac{p}{2}}\hat{u}(0,\cdot)\right\| + \|k_\alpha(\hat{g} - \hat{g_\delta})\| \\
&\le \sup_{\xi\in\mathbb{R}}\left|\frac{e^{a+bi} - k_\alpha}{(1+\xi^2)^{\frac{p}{2}}e^{a+bi}}\right|E + \sup_{\xi\in\mathbb{R}}|k_\alpha|\delta = \sup_{\xi\in\mathbb{R}}\left|\frac{(e^a - \min\{e^a, \alpha\})e^{bi}}{(1+\xi^2)^{\frac{p}{2}}e^{a+bi}}\right|E + \sup_{\xi\in\mathbb{R}}|k_\alpha|\delta \\
&\le \sup_{\xi\in\mathbb{R}, e^a>\alpha}\frac{e^a - \alpha}{(1+\xi^2)^{\frac{p}{2}}e^a}E + \delta\alpha \le \sup_{\xi\in\mathbb{R}, e^a>\alpha}\frac{e^a - \alpha}{|\xi|^p e^a}E + \delta\alpha \le \sup_{\xi\in\mathbb{R}, e^a>\alpha}\frac{e^a - \alpha}{a^{\frac{2p}{\beta}}e^a}E + \delta\alpha,
\end{aligned}$$

where we have used the following fact

$$a = |\xi|^{\frac{\beta}{2}}\cos\frac{\beta\pi}{4} \le |\xi|^{\frac{\beta}{2}},$$

therefore,

$$|\xi|^p \ge a^{\frac{2p}{\beta}}.$$

Let $h(a) = (e^a - \alpha)/(a^{\frac{2p}{\beta}}e^a)$. Differentiating $h(a)$ and setting the derivative equals to zero, we find

$$\frac{2p}{\beta}\alpha + a_0\alpha - e^{a_0}\frac{2p}{\beta} = 0,$$

i.e.,

$$e^{a_0} = c(a_0)\alpha, \qquad c(a_0) = \left(1 + \frac{\beta a_0}{2p}\right) > 1.$$

Therefore,

$$\begin{aligned}
\|f - f_\alpha^\delta\| &\le \frac{c(a_0) - 1}{c(a_0)}(\ln(c(a_0)\alpha))^{\frac{-2p}{\beta}}E + \delta\alpha \\
&= \frac{c(a_0) - 1}{c(a_0)}\left(r\ln\frac{1}{\delta}\right)^{\frac{-2p}{\beta}}E + \frac{1}{c(a_0)}\delta^{1-r} \\
&\approx c\left(\ln\frac{1}{\delta}\right)^{\frac{-2p}{\beta}}, \quad \delta \to 0,
\end{aligned}$$

where we have also used formula (3.14). □

Our Theorem 3.1 shows that we not only obtain the convergence but also give the optimal convergent rate for $0 < x < 1$. Theorem 3.3 shows that we really get the convergence for the endpoint $x = 0$. However, the convergence rate is very slow.

Table 1
The error behaviors for different noisy levels where $\alpha = x(E/\delta)^{1-x}$.

δ	1%	3%	5%
α	2.45	1.97	1.80
RMSE$(u - u_\alpha^\delta)$	0.010	0.015	0.020

Table 2
The error behaviors for different noisy levels where $\alpha = \delta^{-1/3}$.

δ	1%	3%	5%
α	2.86	2.32	2.13
RMSE$(u - u_\alpha^\delta)$	0.040	0.060	0.080

4. Numerical examples

To demonstrate the effectiveness and stability of the proposed numerical methods, two examples, smooth and stepwise, are presented in this section. To test the example, firstly we consider the following direct problem for the given data $f(t)$:

$$\begin{aligned} &\frac{\partial^\beta u}{\partial t^\beta} - u_{xx} = 0, \quad x > 0, t > 0, \\ &u(0, t) = f(t), \quad t > 0, \\ &u(x, 0) = 0, \quad x > 0. \end{aligned} \tag{4.1}$$

This problem is a well-posed problem and its solution at $x = 1$ is given by

$$g(t) := u(1, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\xi t} e^{-(i\xi)^{\frac{\beta}{2}}} \hat{f}(\xi) d\xi. \tag{4.2}$$

In numerical implementation, we give the data $f(t)$ and sample at an equidistant grid, then carry out discrete Fourier transformation. We obtain the data $g(t)$ via inverse discrete Fourier transformation according to (4.2), then generate the noisy data g_δ:

$$g_\delta = g + g_{\max} * \delta \text{ rand } (\text{size}(g)), \tag{4.3}$$

δ indicates that the error level of g, $g_{\max}$ is the maximum of sampled data g, RMS denotes the root mean square for a sampled function W, which is defined by

$$\text{RMS}(W) = \sqrt{\frac{1}{s} \sum_{j=1}^{s} (W(t_j))^2}, \tag{4.4}$$

s is the total number of test points. Similarly, we can define the root mean square error (RMSE) for the computed data and exact data. The symbol rand(size$(\cdot)$) is a random number between $[-1, 1]$.

Numerical implementation is completed by Matlab in IEEE double precision with unit round-off $1.1 \cdot 10^{-16}$. The regularized solutions were computed by the discrete Fast Fourier Transform (FFT) and inverse discrete Fast Fourier Transform technique according to formula (2.10) in Section 2. In the numerical experiment, we fix the total number of test points $s = 101$.

Firstly we consider a smooth function.

Example 1. Let $u(0, t) := f(t) = \sin(\pi t) e^{-t^2}$.

Fig. 1 shows the input exact data $g(t)$.

Fig. 2 shows the results with three noise levels $\delta = 1\%, 3\%, 5\%$, $x = 0.8$ and $\beta = 0.1$. From Fig. 2, we can see that the regularized solutions approach the exact solution well. The error behaviors are summarized in Table 1 where the regularization parameters are computed by the formula $\alpha = x\left(\frac{E}{\delta}\right)^{1-x}$.

In general, $\|u(0, \cdot)\|_p$ is very difficult to obtain. From Theorem 3.3, we get the choice rule of regularization parameter for $x = 0$, i.e., $\alpha = O(\delta^{-r})$, with $0 < r < 1$. For this example, we take $r = 1/3$. Fig. 3 shows the results with three noise levels $\delta = 1\%, 3\%, 5\%$, $x = 0$ and $\beta = 0.1$. From Fig. 3, we can see that the regularized solutions approach the exact solution well. The error behaviors are summarized in Table 2 where the regularization parameters are computed by the formula $\alpha = \delta^{-1/3}$.

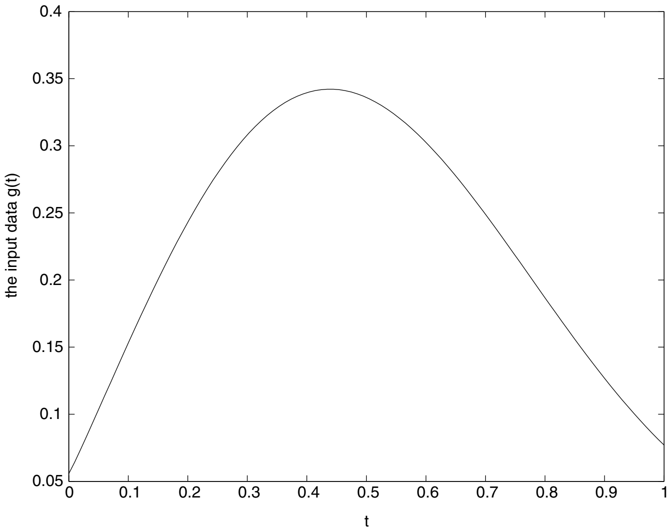

Fig. 1. The exact input data $g(t)$.

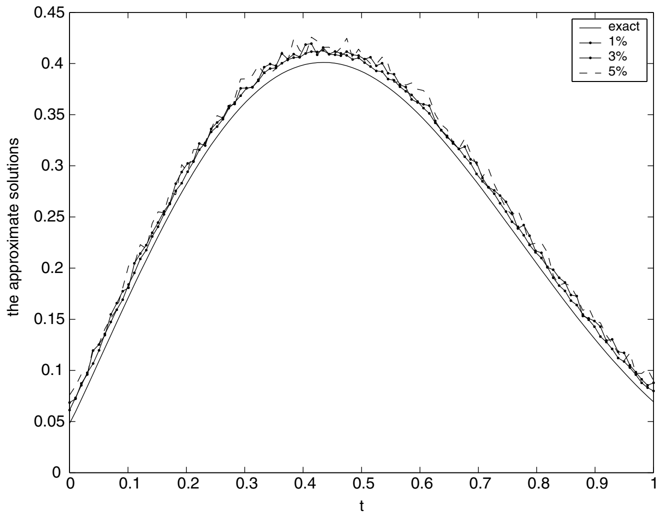

Fig. 2. The reconstruction with $x = 0.8$, $\beta = 0.1$ with different noise levels.

Table 3
The error behaviors for different noisy levels where $\alpha = x(E/\delta)^{1-x}$.

δ	1%	3%	5%
α	2.47	2.00	1.80
RMSE$(u - u_\alpha^\delta)$	0.015	0.020	0.025

Example 2. Consider the following direct problem

$$\begin{aligned} &\frac{\partial^\beta u}{\partial t^\beta} - u_{xx} = 0, \quad x > 0,\ t > 0, \\ &u(x, 0) = 0, \quad t > 0, \\ &u(0, t) = H(t - 0.2) - H(t - 0.6), \quad t > 0, \end{aligned}$$

where $H(t)$ denotes the Heaviside function.

Fig. 4 shows the input data $g(t)$.

Fig. 5 shows the results with three noise levels $\delta = 1\%, 3\%, 5\%$, $x = 0.8$ and $\beta = 0.5$. The error behaviors are summarized in Table 3.

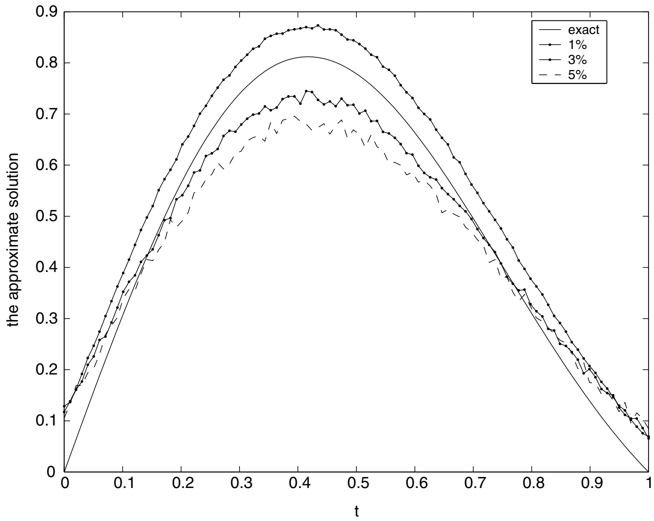

Fig. 3. The reconstruction with $x = 0$, $\beta = 0.1$ with different noise levels.

Table 4
The error behaviors for different noisy levels where $\alpha = \delta^{-2/3}$.

δ	1%	3%	5%
α	7.60	5.34	4.46
$\mathrm{RMSE}(u - u_\alpha^\delta)$	0.12	0.14	0.16

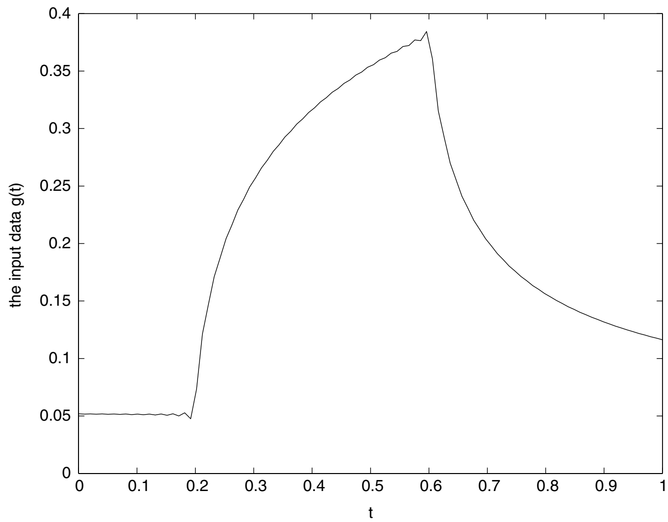

Fig. 4. The exact input data $g(t)$.

We take $\alpha = O(\delta^{-r})$, with $0 < r < 1$ for the reconstruction at $x = 0$ with $r = 2/3$ in this numerical example. Fig. 6 shows the results with three noise levels $\delta = 1\%, 3\%, 5\%$, $x = 0$ and $\beta = 0.5$. From Fig. 6, we can see that the regularized solutions approach the exact solution well. The error behaviors are summarized in Table 4 where the regularization parameters are computed by the formula $\alpha = \delta^{-2/3}$.

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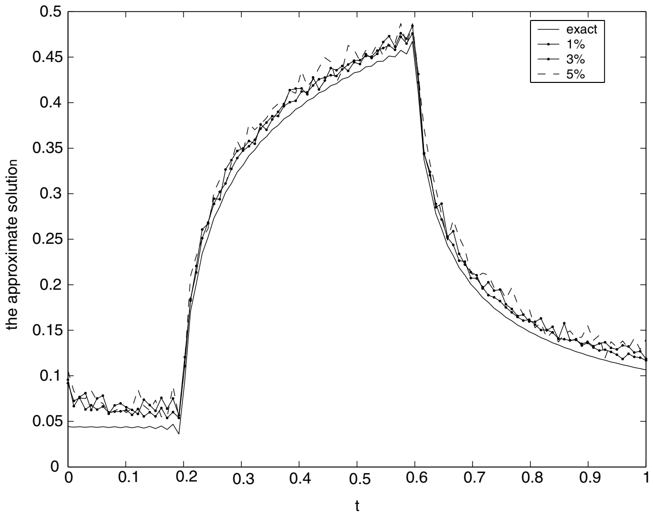

Fig. 5. The reconstruction with $x = 0.8$, $\beta = 0.5$ with different noise levels.

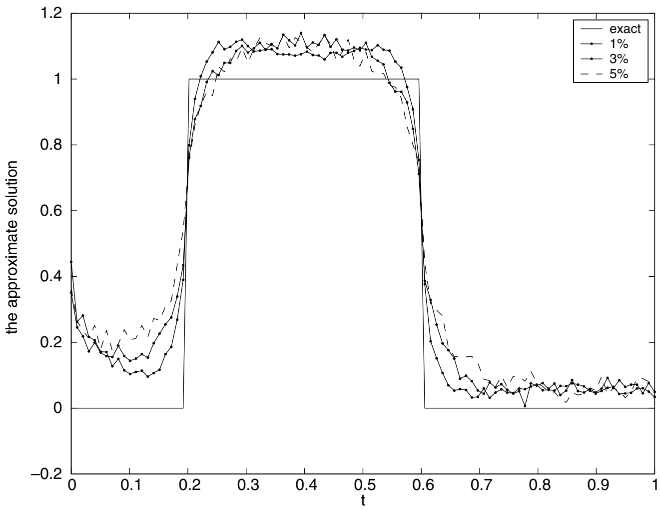

Fig. 6. The reconstruction with $x = 0$, $\beta = 0.5$ with different noise levels.

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