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PII: S0377-0427(17)30103-6
DOI: http://dx.doi.org/10.1016/j.cam.2017.02.038
Reference: CAM 11040

To appear in: *Journal of Computational and Applied Mathematics*

Received date: 19 December 2016

A convergence analysis of a fourth-order method for computing all zeros of a polynomial simultaneously

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Abstract

In 2011, Petković, Rančić and Milošević [6] introduced and studied a new fourth-order iterative method for finding all zeros of a polynomial simultaneously. They obtained a semilocal convergence theorem for their method with computationally verifiable initial conditions, which is of practical importance. In this paper, we provide new local as well as semilocal convergence results for this method over an algebraically closed normed field. Our semilocal results improve and complement the result of Petković, Rančić and Milošević in several directions. The main advantage of the new semilocal results are: weaker sufficient convergence conditions, computationally verifiable a posteriori error estimates, and computationally verifiable sufficient conditions for all zeros of a polynomial to be simple. Furthermore, several numerical examples are provided to show some practical applications of our semilocal results.

Keywords: Simultaneous methods, Polynomial zeros, Local convergence, Semilocal convergence, Error estimates, Cone metric space
2000 MSC: 65H04, 12Y05

1. Introduction

Throughout this paper $(\mathbb{K}, |\cdot|)$ denotes an algebraically closed normed field and $\mathbb{K}[z]$ denotes the ring of polynomials over $\mathbb{K}$. Let the vector space $\mathbb{K}^n$ be endowed with the p-norm $\|\cdot\|_p \colon \mathbb{K}^n \to \mathbb{R}$ defined by $\|x\|_p = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ for some $1 \le p \le \infty$.

The function $d \colon \mathbb{K}^n \to \mathbb{R}^n$ is defined by $d(x) = (d_1(x), \dots, d_n(x))$, where

$$d_i(x) = \min_{j \neq i} |x_i - x_j| \qquad (i = 1, \dots, n),$$

and the function $\delta \colon \mathbb{K}^n \to \mathbb{R}_+$ is defined by

$$\delta(x) = \min_{i \neq j} |x_i - x_j|.$$

In the sequel, for two vectors $x \in \mathbb{K}^n$ and $y \in \mathbb{R}^n$ we denote by $\dfrac{x}{y}$ the vector in $\mathbb{R}^n$ defined by

$$\frac{x}{y} = \left(\frac{|x_1|}{y_1}, \cdots, \frac{|x_n|}{y_n}\right)$$

provided that y has only nonzero components. Throughout the paper, $\mathcal{D}$ denotes the set of all vectors in $\mathbb{K}^n$ with pairwise distinct components, i.e.

$$\mathcal{D} = \{x \in \mathbb{K}^n \colon \delta(x) > 0\}.$$

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \ge 2$. A vector $\xi \in \mathbb{K}^n$ is called a *root vector* of polynomial f if $f(z) = a_0 \prod_{i=1}^n (z - \xi_i)$ for all $z \in \mathbb{K}$, where $a_0 \in \mathbb{K}$. The first iterative method for simultaneous computation of all

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zeros of f was presented by Weierstrass [22] in 1891. The *Weierstrass method* has second order of convergence and is defined in $\mathbb{K}^n$ by the following iteration:

$$x^{k+1} = x^k - W_f(x^k), \qquad k = 0, 1, 2, \dots, \tag{1.1}$$

where the correction $W_f \colon \mathcal{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ is given by

$$W_f(x) = (W_1(x), \dots, W_n(x)) \quad \text{with} \quad W_i(x) = \frac{f(x_i)}{a_0 \prod_{j \neq i} (x_i - x_j)}, \tag{1.2}$$

where a_0 is the leading coefficient of f.

In 2011, Petković, Rančić and Milošević [6] introduced and studied a new fourth-order simultaneous method defined as follows:

$$x^{k+1} = T x^k, \qquad k = 0, 1, 2, \dots, \tag{1.3}$$

where the iteration function $T \colon \mathscr{D} \subset \mathbb{K}^n \to \mathbb{K}^n$ is defined by $T x = (T_1(x), \dots, T_n(x))$ with

$$T_i(x) = x_i - u_i - \frac{u_i^2 \left(\dfrac{f''(x_i)}{f'(x_i)} - u_i (S_i^2 - G_i) \right)}{2(1 - u_i S_i)^2} \qquad (i = 1, \dots, n), \tag{1.4}$$

where

$$u_i = \frac{f(x_i)}{f'(x_i)}, \quad S_i = \sum_{j \neq i} \frac{1}{x_i - x_j} \quad \text{and} \quad G_i = \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}.$$

Obviously, the domain $\mathscr{D}$ of the iteration function T is the set

$$\mathscr{D} = \{x \in \mathcal{D} \colon\ f'(x_i) \neq 0,\ 1 - u_i S_i \neq 0 \ \text{ for all } i \in I_n\}. \tag{1.5}$$

Here and throughout the paper I_n denotes the set of indices $1, \dots, n$, that is $I_n = \{1, \dots, n\}$.

In 2016, Proinov [12] obtained relationships between different types of initial conditions that guarantee the convergence of iterative methods for simultaneously finding all zeros of a polynomial. In the next definition, we give his classification of the initial conditions of convergence theorems for simultaneous methods.

Definition 1.1 ([12]). Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$, $\xi \in \mathbb{K}^n$ be a root-vector of f, $x^0 \in \mathbb{K}^n$ be an initial guess of an iterative method for simultaneously finding all zeros of f, $1 \leq p \leq \infty$, and $R = R(n, p)$ be a positive number which depends only on n and p. An initial condition is said to be:

(a) *of the first type* if it can be represented in the form

$$\left\| \frac{x^0 - \xi}{d(\xi)} \right\|_p \leq R \quad \text{or} \quad \frac{\|x^0 - \xi\|_p}{\delta(\xi)} \leq R; \tag{1.6}$$

(b) *of the second type* if it can be represented in the form

$$\left\| \frac{x^0 - \xi}{d(x^0)} \right\|_p \leq R \quad \text{or} \quad \frac{\|x^0 - \xi\|_p}{\delta(x^0)} \leq R; \tag{1.7}$$

(c) *of the third type* if it can be represented in the form

$$\left\| \frac{W_f(x^0)}{d(x^0)} \right\|_p \leq R \quad \text{or} \quad \frac{\|W_f(x^0)\|_p}{\delta(x^0)} \leq R. \tag{1.8}$$

Petković, Rančić and Milošević [6, Theorem 1] proved that if a polynomial $f \in \mathbb{C}[z]$ of degree $n \geq 3$ has only simple zeros and the initial guess $x^0 \in \mathbb{C}^n$ is sufficiently close to a root vector $\xi \in \mathbb{C}^n$, then the iterative method (1.3) converges to ξ with the order of convergence four. This asymptotic result was improved by Cholakov and Petkova [3, Theorem 4.1]. They provided a local convergence theorem of the first type (with a priori and a posteriori error estimates) for the method (1.3). In particular, they obtained a lower estimate for the radius of the convergence ball of this method (see Corollary 4.2 of [3]).

The main result of Petković, Rančić and Milošević [6, Theorem 5] is the following convergence theorem of the third type:

Theorem 1.2 (Petković, Rančić and Milošević [6]). *Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 3$ which has only simple zeros. If an initial guess $x^0 \in \mathbb{C}^n$ with distinct components satisfies the initial condition*

$$\|W_f(x^0)\|_\infty < \frac{\delta(x^0)}{3n+1}, \tag{1.9}$$

then the iteration (1.3) *converges to a root vector of f with the order of convergence four.*

In this paper, we provide new local as well as semilocal convergence results for the iterative method (1.3). Our semilocal results improve and complement Theorem 1.2 in several directions. The main advantage of the new semilocal results are weaker sufficient convergence conditions as well as computationally verifiable a posteriori error estimates. Another important aspect of our semilocal results is that the initial conditions give computationally verifiable sufficient conditions for all zeros of a polynomial to be simple.

The paper is structured as follows: In Section 2, we present some auxiliary results. In Section 3, we obtain a second type local convergence result (Theorem 3.3) for the method (1.3). In Section 4, we provide new semilocal convergence results for the method (1.3) (Theorem 4.1 and Theorem 4.2). In Section 5, we provide several numerical examples to show the applicability of our semilocal convergence results.

2. Preliminaries

Recently, Proinov [9–13] has developed a general convergence theory for iterative methods of the type

$$x_{k+1} = T x_k, \qquad k = 0, 1, 2, \ldots, \tag{2.1}$$

where $T\colon D \subset X \to X$ is an iteration function in a metric space X, cone metric space X or n-dimensional vector space over $\mathbb{K}$ ($X = \mathbb{K}^n$). A central role in this theory is played by the concept *a function of initial conditions* of T. In this theory, the convergence of an iterative process of the type (2.1) is always studied with respect to a function of initial conditions.

Recall that a function $E\colon D \to \mathbb{R}_+$ is called a function of initial conditions of an iteration function $T\colon D \subset X \to X$ if there exists an interval $J \subset \mathbb{R}_+$ containing 0 and a gauge function $\varphi\colon J \to J$ such that $E(Tx) \leq \varphi(E(x))$ for all $x \in D$ with $Tx \in D$ and $E(x) \in J$. Some examples of functions of initial conditions can be found in [2, 3, 7–20].

Throughout this paper we follow the terminology from Proinov [13]. In particular, we refer to this paper for the definitions of the following notions: quasi-homogeneous function of degree $r \geq 0$, gauge function of order $r \geq 1$; initial point of a map; solid vector space; cone normed space.

Let $(\mathbb{R}^n, \|\cdot\|_p)$ be equipped with coordinate-wise ordering $\preceq$ defined by

$$x \preceq y \quad \text{if and only if} \quad x_i \leq y_i \;\text{ for each }\; i \in I_n$$

for $x, y \in \mathbb{R}^n$. Then $(\mathbb{R}^n, \preceq)$ is a solid vector space. Furthermore, we define in $\mathbb{K}^n$ a cone norm $\|\cdot\|\colon \mathbb{K}^n \to \mathbb{R}^n$ by

$$\|x\| = (|x_1|, \ldots, |x_n|).$$

Then $(\mathbb{K}^n, \|\cdot\|)$ becomes a cone norm space over $\mathbb{R}^n$. Given p with $1 \leq p \leq \infty$, we always denote by q the conjugate exponent of p, i.e. q is defined by means of

$$1 \leq q \leq \infty \quad \text{and} \quad 1/p + 1/q = 1.$$

For the sake of brevity, for a given n and p, we use the following notation

$$a = (n-1)^{1/q}, \quad b = 2^{1/q}. \tag{2.2}$$

First, we need the following three lemmas.

Lemma 2.1 ([14]). *Let $u, v \in \mathbb{K}^n$ and $1 \le p \le \infty$. If v is a vector with distinct components then for all $i, j \in I_n$,*

$$|u_i - v_j| \ge \left(1 - \left\|\frac{u-v}{d(v)}\right\|_p\right)|v_i - v_j|, \ |u_i - u_j| \ge \left(1 - b\left\|\frac{u-v}{d(v)}\right\|_p\right)|v_i - v_j|,$$

where b is defined by (2.2).

Lemma 2.2 ([20]). *Let $u, v, \xi \in \mathbb{K}^n$, $\alpha \ge 0$ and $1 \le p \le \infty$. If v is a vector with distinct components such that*

$$\|u - \xi\| \le \alpha \|v - \xi\|, \tag{2.3}$$

then for all $i, j \in I_n$,

$$|u_i - u_j| \ge \left(1 - b\,(1+\alpha)\left\|\frac{v-\xi}{d(v)}\right\|_p\right)|v_i - v_j|, \tag{2.4}$$

where b is defined by (2.2).

Lemma 2.3 ([16]). *Let $\mathbb{K}$ be an arbitrary field, and let $f \in \mathbb{K}[z]$ be polynomial of degree $n \ge 1$ which splits over $\mathbb{K}$. Suppose that $\xi_1, \xi_2, \ldots, \xi_n$ are all the zeros of f counting with their multiplicities.*

(i) *If $z \in \mathbb{K}$ is not a zero of f, then*

$$\frac{f'(z)}{f(z)} = \sum_{j=1}^{n} \frac{1}{z - \xi_j}\,.$$

(ii) *If $z \in \mathbb{K}$ is not a zero of both f and f', then*

$$\frac{f''(z)}{f'(z)} = \frac{f'(z)}{f(z)} - \frac{f(z)}{f'(z)}\left(\frac{1}{(z-\xi_i)^2} + \sum_{j \ne i} \frac{1}{(z-\xi_j)^2}\right). \tag{2.5}$$

We need the following three theorems of Proinov [12, 13], which play a crucial role in the proofs of the main results of the present work.

Theorem 2.4 ([13, Theorem 3.3]). *Let $(X, \|\cdot\|)$ is a cone normed space over a solid vector space $(Y, \preceq)$. Let $T\colon D \subset X \to X$ be an operator on X and $E\colon D \to \mathbb{R}_+$ be a function of initial conditions of T with a gauge function φ of order $r \ge 1$ on an interval J. Suppose $\xi \in D$ is such that $E(\xi) \in J$ and*

$$\|Tx - \xi\| \preceq \beta(E(x))\,\|x - \xi\| \quad \textit{for all } \ x \in D \ \textit{ with } \ E(x) \in J, \tag{2.6}$$

where $\beta\colon J \to \mathbb{R}_+$ is a nondecreasing function such that

$$t\beta(t) \ \text{ is a strict gauge function of order } \ r \text{ on } J \tag{2.7}$$

and

$$\text{for } \ t \in J : \phi(t) = 0 \ \text{ implies } \ \beta(t) = 0, \tag{2.8}$$

where $\phi\colon J \to \mathbb{R}_+$ is a nondecreasing function such that $\varphi(t) = t\phi(t)$ for all $t \in J$. Then ξ is a unique fixed point of T in $U = \{x \in D : E(x) \in J\}$, and for each initial point x_0 of T the following statements hold:

(i) *The iteration* (2.1) *remains in the set U and converges to ξ.*

(ii) *For all $k \geq 0$ we have the error estimates*

$$\|x_{k+1} - \xi\| \leq \theta \lambda^{r^k} \|x_k - \xi\| \quad \textit{and} \quad \|x_k - \xi\| \leq \theta^k \lambda^{S_k(r)} \|x_0 - \xi\|, \tag{2.9}$$

where $S_k(r) = 1 + r + r^2 + \cdots + r^{k-1}$, $\lambda = \phi(E(x_0))$, $\theta = \psi(E(x_0))$, and $\psi \colon J \to \mathbb{R}_+$ is such that $\beta(t) = \phi(t)\psi(t)$ for all $t \in J$.

Recently Proinov [12] has shown that there is a deep relationship between local and semilocal theorems for simultaneous root-finding methods. It turns out that from any local convergence theorem for a simultaneous method one can obtain as a consequence a semilocal theorem for the same method. In particular, Proinov [12] has presented two theorems for converting any local convergence theorem of the second type into a theorem with computationally verifiable initial conditions.

Theorem 2.5 ([12, Theorem 5.1]). *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. Suppose there exists a vector $x \in \mathbb{K}^n$ with distinct components such that*

$$E_f(x) = \left\| \frac{W_f(x)}{d(x)} \right\|_p \leq \tau = \frac{1}{(1 + \sqrt{a})^2} \tag{2.10}$$

for some $1 \leq p \leq \infty$, where the operator W_f is defined by (1.2) *and a with* (2.2). *In the case $n = 2$ and $p = \infty$ we assume that the inequality* (2.10) *is strict. Then f has only simple zeros and there exists a root-vector $\xi \in \mathbb{K}^n$ of f such that*

$$\|x - \xi\| \leq \alpha(E_f(x)) \|W_f(x)\| \quad \textit{and} \quad \left\| \frac{x - \xi}{d(x)} \right\|_p \leq h(E_f(x)), \tag{2.11}$$

where the functions $\alpha, h \colon [0, \tau] \to \mathbb{R}_+$ are defined by

$$\alpha(t) = \frac{2}{1 - (a-1)t + \sqrt{(1 - (a-1)t)^2 - 4t}} \quad \textit{and} \quad h(t) = t\alpha(t). \tag{2.12}$$

Moreover, if the inequality (2.10) *is strict, then the second inequality in* (2.11) *is strict too.*

Theorem 2.6 ([12, Theorem 5.2]). *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. Suppose there exists a vector $x \in \mathbb{K}^n$ with distinct component such that*

$$\left\| \frac{W_f(x)}{d(x)} \right\|_p < \frac{R(1-R)}{1 + (a-1)R} \tag{2.13}$$

for some $1 \leq p \leq \infty$ and $0 \leq R \leq 1/(1 + \sqrt{a})$, where W_f is defined by (1.2) *and a with* (2.2). *Then polynomial f has only simple zeros in $\mathbb{K}$ and there exists a root-vector $\xi \in \mathbb{K}^n$ of f such that*

$$\|x - \xi\| \leq \alpha(E_f(x)) \|W_f(x)\| \quad \textit{and} \quad \left\| \frac{x - \xi}{d(x)} \right\|_p < R, \tag{2.14}$$

where the function α is defined by (2.12).

3. Local convergence analysis

The main purpose of this section is to prove a local convergence theorem of the second type with error estimates.

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has only simple zeros in $\mathbb{K}$ and let $\xi \in \mathbb{K}^n$ be a root vector of the polynomial f. We study the convergence of the iterative method (1.3) with respect to the function of initial conditions $E \colon \mathcal{D} \to \mathbb{R}_+$ defined by

$$E(x) = \left\| \frac{x - \xi}{d(x)} \right\|_p \qquad (1 \leq p \leq \infty). \tag{3.1}$$

Define the real functions $\beta, \psi, \Psi \colon [0, 1/n) \to \mathbb{R}$ as follows

$$\beta(t) = \frac{(2an - 3a - 1)t^2 - (2n - 1 - a)t + 2n}{2(1 - nt)(1 - t - at^2)^2} \, a\, t^3, \tag{3.2}$$

$$\psi(t) = 1 - bt(1 + \beta(t)), \tag{3.3}$$

$$\Psi(t) = \psi(t) - \beta(t) = 1 - bt - \beta(t)(1 + bt), \tag{3.4}$$

where a, b are defined by (2.2). It is easy to show that β is an increasing function and ψ and Ψ are decreasing functions. Besides, β is a quasi-homogeneous function of the third degree on $[0, 1/n)$. Let R be the unique solution of the equation $\Psi(t) = 0$ in $(0, 1/n)$. Then we can define the functions $\phi, \varphi \colon [0, R] \to \mathbb{R}_+$ by

$$\phi(t) = \frac{\beta(t)}{\psi(t)} \quad \text{and} \quad \varphi(t) = t\,\phi(t). \tag{3.5}$$

Note that ϕ is a quasi-homogeneous function of the third degree on $[0, R]$ and φ is a control function of the fourth order on $[0, R]$.

Lemma 3.1. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has only simple zeros in $\mathbb{K}$ and let $\xi \in \mathbb{K}^n$ be a root vector of f. Suppose $x \in \mathscr{D}$ is a vector such that $f(x_i) \neq 0$ for some $1 \leq i \leq n$. Then*

$$T_i(x) - \xi_i = \frac{A_i^2 + 2\,\sigma_i\, A_i^2 - 2\,\sigma_i\, A_i - B_i}{2(1 + \sigma_i)(1 - A_i)^2}\,(x_i - \xi_i), \tag{3.6}$$

where σ_i, A_i and B_i are defined by

$$\sigma_i = (x_i - \xi_i) \sum_{j \neq i} \frac{1}{x_i - \xi_j}\,, \quad A_i = (x_i - \xi_i) \sum_{j \neq i} \frac{x_j - \xi_j}{(x_i - \xi_j)(x_i - x_j)} \tag{3.7}$$

and

$$B_i = (x_i - \xi_i)^2 \sum_{j \neq i} \frac{(x_j - \xi_j)(2x_i - x_j - \xi_j)}{(x_i - \xi_j)^2 (x_i - x_j)^2}\,. \tag{3.8}$$

Proof. From Lemma 2.3(i), taking into account that ξ is a root-vector of f and the fact that $f(x_i) \neq 0$ for some $1 \leq i \leq n$, we get

$$\frac{1}{u_i} = \frac{f'(x_i)}{f(x_i)} = \sum_{j=1}^{n} \frac{1}{x_i - \xi_j} = \frac{1}{x_i - \xi_i} + \sum_{j \neq i} \frac{1}{x_i - \xi_j} = \frac{1 + \sigma_i}{x_i - \xi_i}\,. \tag{3.9}$$

Again from Lemma 2.3(ii) for $z = x_i$ and $f'(x_i) \neq 0$, we get

$$\begin{aligned}\frac{f''(x_i)}{f'(x_i)} &= \frac{f'(x_i)}{f(x_i)} - \frac{f(x_i)}{f'(x_i)} \left(\frac{1}{(x_i - \xi_i)^2} + \sum_{j \neq i} \frac{1}{(x_i - \xi_j)^2} \right) \\ &= \frac{1 + \sigma_i}{x_i - \xi_i} - \frac{1 + \tau_i}{(1 + \sigma_i)(x_i - \xi_i)} = \frac{\sigma_i^2 + 2\sigma_i - \tau_i}{(1 + \sigma_i)(x_i - \xi_i)}\,,\end{aligned} \tag{3.10}$$

where

$$\tau_i = (x_i - \xi_i)^2 \sum_{j \neq i} \frac{1}{(x_i - \xi_j)^2}\,.$$

From (1.4), (3.9) and (3.10), we obtain

$$\begin{aligned}T_i(x) - \xi_i &= x_i - \xi_i - \frac{x_i - \xi_i}{1 + \sigma_i} - \frac{(x_i - \xi_i)(\sigma_i^2 + 2\sigma_i - \tau_i - \mu_i^2 + \nu_i)}{2(1 + \sigma_i)(1 + \sigma_i - \mu_i)^2} \\ &= \left(1 - \frac{2(1 + \sigma_i - \mu_i)^2 + \sigma_i^2 + 2\sigma_i - \tau_i - \mu_i^2 + \nu_i}{2(1 + \sigma_i)(1 + \sigma_i - \mu_i)^2} \right) (x_i - \xi_i) \\ &= \left(1 - \frac{1}{1 + \sigma_i - \mu_i} \left(1 + \frac{(1 + \sigma_i - \mu_i)^2 - 1 + \nu_i - \tau_i}{2(1 + \sigma_i)(1 + \sigma_i - \mu_i)} \right) \right) (x_i - \xi_i),\end{aligned}$$

where

$$\mu_i = (x_i - \xi_i) \sum_{j \neq i} \frac{1}{x_i - x_j} \quad \text{and} \quad \nu_i = (x_i - \xi_i)^2 \sum_{j \neq i} \frac{1}{(x_i - x_j)^2}\,.$$

Now taking into account that $\sigma_i - \mu_i = -A_i$ and $\nu_i - \tau_i = B_i$, we get (3.6). □

Lemma 3.2. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has only simple zeros in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ be a root vector of f and $1 \leq p \leq \infty$. Suppose a vector $x \in \mathbb{K}^n$ with distinct components satisfies*

$$E(x) < 1/n \qquad \text{and} \qquad \Psi(E(x)) \geq 0, \tag{3.11}$$

where the functions E and Ψ are defined by (3.1) *and* (3.4) *respectively. Then the following statements hold true:*

(i) *$x \in \mathscr{D}$, where $\mathscr{D}$ is defined by* (1.5)*;*
(ii) *$\|Tx - \xi\| \leq \beta(E(x))\,\|x - \xi\|$, where β is defined by* (3.2)*;*
(iii) *$E(Tx) \leq \varphi(E(x))$, where φ is defined by* (3.5).

Proof. (i) First we have to prove that $f'(x_i) \neq 0$ for every $i \in I_n$. Let $i \in I_n$ be fixed. If $x_i = \xi_i$, then $f(x_i) = 0$ and $f'(x_i) \neq 0$, since f has only simple zeros. Let $x_i \neq \xi_i$. From Lemma 2.1 with $u = \xi$ and $v = x$, and (3.11), we get

$$|x_i - \xi_j| \geq \left(1 - \left\|\frac{x - \xi}{d(x)}\right\|_p\right)|x_i - x_j| \geq (1 - E(x))\,d_i(x) > 0, \tag{3.12}$$

for every $i \neq j$. Hence x_i is not a zero of f and (3.9) is well defined. We define σ_i as in (3.7). It follows from (3.9) that $f'(x_i) \neq 0$ is equivalent to $\sigma_i \neq -1$. From (3.7), the triangle inequality and (3.12), we obtain

$$|\sigma_i| \leq |x_i - \xi_i| \sum_{j \neq i} \frac{1}{|x_i - \xi_j|} \leq \frac{|x_i - \xi_i|}{(1 - E(x))d_i(x)} \sum_{j \neq i} 1 \leq \frac{(n-1)E(x)}{1 - E(x)} < 1\,, \tag{3.13}$$

which yields $\sigma_i \neq -1$. In view of the definition of $\mathscr{D}$ (see (1.5)), it remains to prove that

$$1 - \frac{f(x_i)}{f'(x_i)} \sum_{j \neq i} \frac{1}{x_i - x_j} \neq 0. \tag{3.14}$$

From (3.9) and (3.7) it follows that

$$1 - \frac{f(x_i)}{f'(x_i)} \sum_{j \neq i} \frac{1}{x_i - x_j} = \frac{1 - A_i}{1 + \sigma_i}, \tag{3.15}$$

where A_i is defined by (3.7). This means that (3.14) holds true if and only if $A_i \neq 1$. By (3.7), the triangle inequality, (3.12), Hölder's inequality, and the first part of (3.11), we obtain

$$\begin{aligned} |A_i| &\leq |x_i - \xi_i| \sum_{j \neq i} \frac{|x_j - \xi_j|}{|x_i - \xi_j||x_i - x_j|} \\ &\leq \frac{|x_i - \xi_i|}{(1 - E(x))d_i(x)} \sum_{j \neq i} \frac{|x_j - \xi_j|}{d_j(x)} \leq \frac{aE(x)^2}{1 - E(x)} < \frac{1}{n} \end{aligned} \tag{3.16}$$

which proves that $A_i \neq 1$.

(ii) We have to prove that

$$|T_i(x) - \xi_i| \leq \beta(E(x))\,|x_i - \xi_i| \quad \text{for all } i \in I_n. \tag{3.17}$$

Let $i \in I_n$ be fixed. If $x_i = \xi_i$, then $T_i(x) = \xi_i$ and so (3.17) becomes an equality. Suppose $x_i \neq \xi_i$. From Lemma 3.1 and the triangle inequality, we get

$$|T_i(x) - \xi_i| \leq \frac{|A_i|^2 + 2\,|\sigma_i|\,|A_i|^2 + 2\,|\sigma_i|\,|A_i| + |B_i|}{2(1 - |\sigma_i|)(1 - |A_i|)^2}\,|x_i - \xi_i|. \tag{3.18}$$

From (3.8), the triangle inequality and (3.12) it follows that

$$\begin{aligned} |B_i| &\leq |x_i - \xi_i|^2 \sum_{j \neq i} \frac{|x_j - \xi_j|\,|2x_i - x_j - \xi_j|}{|x_i - \xi_j|^2\,|x_i - x_j|^2} \\ &\leq \frac{|x_i - \xi_i|^2}{(1 - E(x))^2 d_i(x)^2} \sum_{j \neq i} \frac{|x_j - \xi_j|\,(|x_i - x_j| + |x_i - \xi_j|)}{|x_i - x_j|^2}\,. \end{aligned} \tag{3.19}$$

Using the triangle inequality, we obtain

$$|x_i - \xi_j| \le |x_i - x_j| + |x_j - \xi_j| \le \left(1 + \frac{|x_j - \xi_j|}{d_j(x)}\right) |x_i - x_j| \le (1 + E(x))\, |x_i - x_j|. \tag{3.20}$$

From (3.19), (3.20) and Hölder's inequality, we obtain the following estimate

$$|B_i| \le \frac{a(2 + E(x))\, E(x)^3}{(1 - E(x))^2}\,. \tag{3.21}$$

Combining (3.18), (3.16), (3.13) and (3.21), we get (3.17).

(iii) Inequality in (ii) allow us to apply Lemma 2.2 with $u = Tx$, $v = x$ and $\alpha = \beta(E(x))$. By Lemma 2.2 and (3.3), we deduce

$$|T_i(x) - T_j(x)| \ge (1 - bE(x)(1 + \beta(E(x)))\, |x_i - x_j| = \psi(E(x))\, |x_i - x_j|.$$

By taking the minimum over all $j \in I_n$ such that $j \neq i$, we obtain

$$d_i(Tx) \ge \psi(E(x)) d_i(x) > 0 \tag{3.22}$$

which implies that Tx has distinct components. It follows from (3.17), (3.22) and (3.5) that

$$\frac{|T_i(x) - \xi_i|}{d_i(Tx)} \le \frac{\beta(E(x))}{\psi(E(x))} \frac{|x_i - \xi_i|}{d_i(x)} = \phi(E(x)) \frac{|x_i - \xi_i|}{d_i(x)}\,.$$

By taking the p-norm and taking into account (3.5), we obtain

$$E(Tx) \le \phi(E(x))\, E(x) = \varphi(E(x))$$

which completes the proof. □

Now, we are ready to state and prove the main result in this section.

Theorem 3.3. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \ge 2$ which has only simple zeros in $\mathbb{K}$, $\xi \in \mathbb{K}^n$ be a root vector of f and $1 \le p \le \infty$. Suppose $x^0 \in \mathbb{K}^n$ is an initial guess with distinct components which satisfies the following conditions*

$$E(x^0) < 1/n \qquad \text{and} \qquad \Psi(E(x^0)) \ge 0, \tag{3.23}$$

where the functions E and Ψ are defined by (3.1) *and* (3.4) *respectively. Then the iteration* (1.3) *is well defined and converges to ξ with error estimates*

$$\|x^{k+1} - \xi\| \le \theta \lambda^{4^k} \|x^k - \xi\| \quad \text{and} \quad \|x^k - \xi\| \le \theta^k \lambda^{(4^k - 1)/3} \|x^0 - \xi\| \quad \text{for all } k \ge 0, \tag{3.24}$$

where $\lambda = \phi(E(x^0))$, $\theta = \psi(E(x^0))$ and ϕ and ψ are defined by (3.5) *and* (3.3)*, respectively. Moreover, if the second inequality in* (3.23) *is strict, then the rate of convergence is of order four.*

Proof. We shall apply Theorem 2.4 to the iteration function $T\colon D \subset \mathbb{K}^n \to \mathbb{K}^n$ defined by (1.4).

Consider the interval $J = [0, R]$, where R is the unique solution of the equation $\Psi(t) = 0$ in $(0, 1/n)$, and define the function $\varphi\colon J \to \mathbb{R}_+$ by (3.5). As we have noticed above, φ is a gauge function of order $r = 4$ on J. On the other hand, condition (3.23) is equivalent to $E(x^0) \in J$. Now from Lemma 3.2, we conclude that $E\colon D \to \mathbb{R}_+$ is a function of initial condition of T with gauge function φ of order $r = 4$ on J. Furthermore, Lemma 3.2 implies that T satisfies the contractive condition (2.6) with $\beta\colon J \to \mathbb{R}_+$ defined by (3.2).

Now we shall show that x^0 is an initial point of T. According to the assumptions of the theorem, we have $x^0 \in \mathcal{D}$ and $E(x^0) \in J$. Hence, by Lemma 3.2 we conclude that $x^0 \in \mathscr{D}$. According to Proposition 4.1 of [10] we have to prove that $x \in D$ with $E(x) \in J$ implies $Tx \in D$. Since $x \in D$, then $Tx \in \mathbb{K}^n$. From (3.22), we get $Tx \in \mathcal{D}$. From Lemma 3.2 and condition (3.23), we get $E(Tx) \le E(x) < 1/n$. This yields $\Psi(E(Tx)) \ge \Psi(E(x)) \ge 0$ since Ψ is decreasing on $[0, 1/n)$. Thus, we have $Tx \in \mathcal{D}$, $E(Tx) < 1/n$ and $\Psi(E(Tx)) \ge 0$. Applying Lemma 3.2 to the vector Tx, we conclude that $Tx \in D$.

Hence, all the assumptions of Theorem 2.4 are satisfied. Applying Theorem 2.4 to the operator $T\colon D \subset \mathbb{K}^n \to \mathbb{K}^n$ and the function of initial conditions $E\colon D \to \mathbb{R}_+$, we conclude that the iteration (1.3) is well defined and converges to ξ with error estimates (3.24). This completes the proof. □

Setting $p = \infty$ in Theorem 3.3, we get the following result.

Theorem 3.4. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has only simple zeros in $\mathbb{K}$ and let $\xi \in \mathbb{K}^n$ be a root vector of f. Suppose $x^0 \in \mathbb{K}^n$ is an initial guess with distinct components which satisfies the following conditions*

$$E(x^0) = \left\| \frac{x^0 - \xi}{d(x^0)} \right\|_\infty < \frac{1}{n} \qquad \text{and} \qquad \Psi(E(x^0)) \geq 0, \tag{3.25}$$

where the function Ψ is defined by

$$\Psi(t) = 1 - 2t - \beta(t)(1 + 2t), \quad \text{where} \quad \beta(t) = \frac{(2n-1)(n-2)t^2 - nt + 2n}{2(1-nt)(1-t-(n-1)t^2)^2}\,(n-1)t^3. \tag{3.26}$$

Then the iteration (1.3) *is well defined and converges to ξ with error estimates*

$$\|x^{k+1} - \xi\| \leq \theta \lambda^{4^k} \|x^k - \xi\| \quad \text{and} \quad \|x^k - \xi\| \leq \theta^k \lambda^{(4^k-1)/3} \|x^0 - \xi\| \quad \text{for all } k \geq 0, \tag{3.27}$$

where $\theta = \psi(E(x^0))$, $\lambda = \phi(E(x^0))$, the functions ψ and ϕ are defined by $\psi(t) = \Psi(t) + \beta(t)$ and $\phi(t) = \beta(t)/\psi(t)$. Moreover, if the second inequality in (3.25) *is strict, then the rate of convergence is of order four.*

Corollary 3.5. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ which has only simple zeros in $\mathbb{K}$ and let $\xi \in \mathbb{K}^n$ be a root vector of f. Suppose $x^0 \in \mathbb{K}^n$ is an initial guess with distinct components which satisfies the following conditions*

$$E(x^0) = \left\| \frac{x^0 - \xi}{d(x^0)} \right\|_\infty \leq \frac{4}{7n}. \tag{3.28}$$

Then the iteration (1.3) *is well defined and converges to ξ with order of convergence four and with error estimates* (3.27).

Proof. In view of Theorem 3.4 we have to prove that x^0 satisfies the initial conditions (3.25). The first part of (3.25) is obvious. To prove the second part of (3.25) it is sufficient to show that $\Psi(4/(7n)) > 0$. This inequality is equivalent to

$$\beta\left(\frac{4}{7n}\right) < \frac{7n-8}{7n+8}. \tag{3.29}$$

Note that the function $\beta(4/7n)$ is decreasing and $(7n-8)/(7n+8)$ is increasing for $n \geq 2$. Then for every $n \geq 2$,

$$\beta\left(\frac{4}{7n}\right) = \frac{64(n-1)(49n^3 + 2n^2 - 40n + 16)}{3n(49n^2 - 44n + 16)^2} \leq \frac{224}{961} < \frac{3}{11} \leq \frac{7n-8}{7n+8}.$$

This completes the proof. □

4. Semilocal convergence analysis

In this section, we establish semilocal convergence theorems for the iterative method (1.3). The results improve Theorem 1.2 in several directions.

Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$. We study the convergence of the method (1.3) with respect to the function of initial conditions $E_f \colon \mathcal{D} \to \mathbb{R}_+$ defined by

$$E_f(x) = \left\| \frac{W_f(x)}{d(x)} \right\|_p \qquad (1 \leq p \leq \infty). \tag{4.1}$$

For given n and p define the number μ by

$$\mu = \frac{n-1}{n(a+n-1)}, \tag{4.2}$$

where a is defined by (2.2). Consider the function $\Omega\colon [0,\mu) \to \mathbb{R}_+$ defined by

$$\Omega(t) = \Psi(h(t)), \tag{4.3}$$

where the functions Ψ and h are defined by (3.4) and (2.12), respectively. Note that the definition of Ω is correct since $\mu \in [0,\tau]$, $h(\mu) = 1/n$ and $h([0,\mu)) = [0,1/n)$, where τ is defined in (2.10).

Now, we can state and prove the main result of this paper.

Theorem 4.1. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ and $1 \leq p \leq \infty$. Suppose $x^0 \in \mathbb{K}^n$ is an initial guess with distinct components satisfying*

$$E_f(x^0) < \mu \quad \textit{and} \quad \Omega(E_f(x^0)) \geq 0, \tag{4.4}$$

where μ is defined by (4.2) *and the functions E_f and Ω are defined by* (4.1) *and* (4.3)*, respectively. Then f has only simple zeros in $\mathbb{K}$ and the iteration* (1.3) *is well-defined and converges to a root-vector ξ of f with order of convergence four and with error estimate*

$$\|x^k - \xi\| \leq \alpha(E_f(x^k))\|W_f(x^k)\|, \tag{4.5}$$

for all $k \geq 0$ such that $E_f(x^k) < \mu$ and $\Omega(E_f(x^k)) \geq 0$, where the function α is defined by (2.12)*.*

Proof. Let $x^0 \in \mathbb{K}^n$ satisfy (4.4). From the first inequality of (4.4) and Theorem 2.5, we conclude that f has only simple zeros and there exists a root-vector $\xi \in \mathbb{K}^n$ of f such that

$$\left\| \frac{x^0 - \xi}{d(x^0)} \right\|_p < h(E_f(x^0)) < \frac{1}{n}. \tag{4.6}$$

The function Ψ is decreasing on $[0, 1/n)$. Hence, from (4.6), (4.3) and the second inequality in (4.4), we obtain

$$\Psi\left(\left\| \frac{x^0 - \xi}{d(x^0)} \right\|_p \right) > \Psi(h(E_f(x^0))) = \Omega(E_f(x^0)) \geq 0.$$

It follows from Theorem 3.3 that the iteration (1.3) is well-defined and converges to ξ with fourth-order of convergence. It remains to prove the estimate (4.5). Suppose that for some $k \geq 0$,

$$E_f(x^k) < \mu \quad \text{and} \quad \Omega(E_f(x^k)) \geq 0. \tag{4.7}$$

Then it follows from the first inequality in (4.7) and Theorem 2.5 that there exists a root-vector $\eta \in \mathbb{K}^n$ of f such that

$$\|x^k - \eta\| \leq \alpha(E_f(x^k))\, \|W_f(x^k)\| \quad \text{and} \quad \left\| \frac{x^k - \eta}{d(x^k)} \right\|_p < h(E_f(x^k)) < \frac{1}{n}. \tag{4.8}$$

From the second inequality in (4.8) and the second inequality in (4.7), we get

$$\Psi\left(\left\| \frac{x^k - \eta}{d(x^k)} \right\|_p \right) > \Psi(h(E_f(x^k))) = \Omega(E_f(x^k)) \geq 0.$$

By Theorem 3.3, we conclude that the iteration (1.3) converges to η. By the uniqueness of the limit, $\eta = \xi$. Hence, the error estimate (4.5) follows from the first inequality in (4.8). □

Using Corollary 3.5 and Theorem 2.6, we obtain the next semilocal result.

Theorem 4.2. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ and $x^0 \in \mathbb{K}^n$ be an initial guess with distinct components satisfying*

$$E_f(x^0) = \left\| \frac{W_f(x^0)}{d(x^0)} \right\|_\infty \leq R'_n = \frac{4(7n-4)}{7n(11n-8)}. \tag{4.9}$$

Then f has only simple zeros in $\mathbb{K}$ and the iteration (1.3) *is well-defined and converges to a root-vector ξ of f with order of convergence four and with error estimate*

$$\|x^k - \xi\| \leq \alpha(E_f(x^k))\|W_f(x^k)\|, \tag{4.10}$$

for all $k \geq 0$ such that $E_f(x^k) \leq R'_n$, where the function α is defined in (2.12) *with $p = \infty$.*

Proof. Let $R = 4/(7n)$ then (4.9) can be written in the form

$$\left\| \frac{W_f(x^0)}{d(x^0)} \right\|_\infty < \frac{R(1-R)}{1+(n-2)R}\,.$$

Then it follows from Theorem 2.6 that f has only simple zeros in $\mathbb{K}$ and there exists a root-vector $\xi \in \mathbb{K}^n$ of f such that

$$\left\| \frac{x^0 - \xi}{d(x^0)} \right\|_\infty < R.$$

Now Corollary 3.5 implies that the iteration (1.3) converges to ξ with order of convergence four. The proof of the error estimate (4.10) is similar as in the the proof of Theorem 4.1. This ends the proof. □

We end this section with a corollary of Theorem 4.2 which improves the result of Petković, Rančić and Milošević [6] (see Theorem 1.2 above) in several directions.

Corollary 4.3. *Let $f \in \mathbb{K}[z]$ be a polynomial of degree $n \geq 2$ and $x^0 \in \mathbb{K}^n$ is an initial guess with distinct components satisfying*

$$E_f(x^0) = \left\| \frac{W_f(x^0)}{d(x^0)} \right\|_\infty \leq R_n'' = \frac{1}{An - B}\,, \tag{4.11}$$

where $A = 11/4$ and $B = 3/7$. Then f has only simple zeros in $\mathbb{K}$ and the iteration (1.3) *is well-defined and converges to a root-vector ξ of f with order of convergence four and with error estimate* (4.10)*.*

Proof. Define the function $E_f \colon \mathcal{D} \to \mathbb{R}_+$ by (4.1) with $p = \infty$. If x^0 satisfy the initial condition (4.11), then

$$E_f(x^0) = \left\| \frac{W_f(x^0)}{d(x^0)} \right\|_\infty \leq \frac{1}{An - B} < \frac{4(7n-4)}{7n(11n-8)}\,.$$

Hence, x^0 satisfies the initial condition (4.9) of Theorem 4.2 and the statement of the corollary follows. □

Remark 4.4. We shall compare the convergence domains of Theorem 4.1 ($p = \infty$), Theorem 4.2, Corollary 4.3 and Theorem 1.2. Denote these domains by Δ, Δ', Δ'' and $\tilde{\Delta}$, respectively. It is easy to prove that

$$\Delta \supset \Delta' \supset \Delta'' \supset \tilde{\Delta}. \tag{4.12}$$

Indeed, let R_n be the unique solution of the equation $\Omega(t) = 0$ in the interval $(0, \mu)$, where Ω and μ are defined by (4.3) and (4.2), respectively. Then the initial conditions (4.4) of Theorem 4.1 can be written in the form $E_f(x^0) \leq R_n$. Hence, the convergence domain Δ of Theorem 4.1 is the set

$$\Delta = \left\{ x \in \mathcal{D} : E_f(x) \leq R_n \right\}.$$

The convergence domains Δ' and Δ'' can be written in the same form replacing R_n by R_n' and R_n'', respectively. The convergence domains $\tilde{\Delta}$ of Theorem 1.2 is the set

$$\tilde{\Delta} = \left\{ x \in \mathcal{D} : \frac{\|W_f(x)\|}{\delta(x)} < \tilde{R}_n \right\},$$

where $\tilde{R}_n = 1/(3n+1)$. Now the inclusions (4.12) follow from the inequalities

$$R_n \geq R_n' > R_n'' > \tilde{R}_n\,. \tag{4.13}$$

In Figure 1, one can see the comparison of R_n, R_n', R_n'' and $\tilde{R}_n$ for $n = 2, \ldots, 11$.

Remark 4.5. Theorem 4.1 improves Theorem 1.2 in several directions. The main of them are the following:

- larger convergence domain (see Remark 4.4);
- computationally verifiable a posteriori error estimates right from the first iteration;
- computationally verifiable sufficient condition for all zeros of a polynomial to be simple.

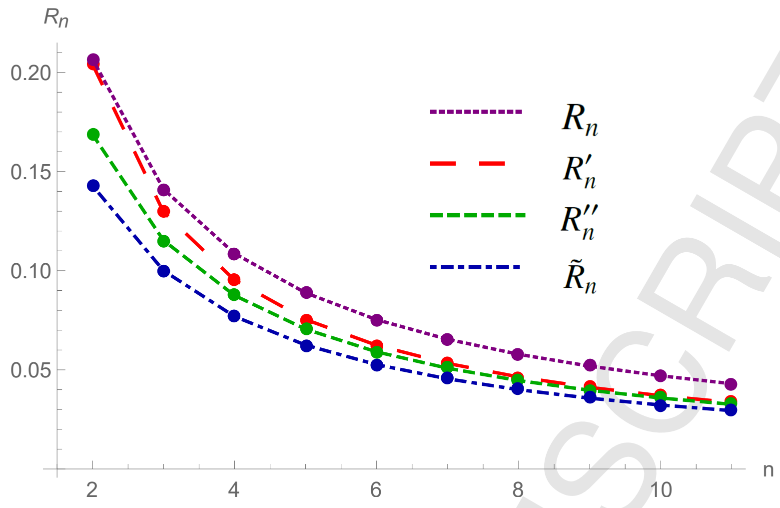

Figure 1: Values of R_n

5. Numerical examples

In this section, we provide several numerical examples to show the applicability of Theorem 4.1 in the case $p = \infty$. Let $f \in \mathbb{C}[z]$ be a polynomial of degree $n \geq 2$ and let $x^{(0)} \in \mathbb{C}^n$ be an initial guess. We consider the function of initial conditions $E_f : \mathcal{D} \to \mathbb{R}_+$ defined by

$$E_f(x) = \left\| \frac{W_f(x)}{d(x)} \right\|_\infty . \tag{5.1}$$

Furthermore, we define the function $\Omega : [0, \mu) \to \mathbb{R}$ by $\Omega(t) = \Psi(h(t))$, where $\mu = 1/(2n)$ and the functions Ψ and h are defined by (3.26) and (2.12), respectively.

It follows from Theorem 4.1 that if there exists an integer $m \geq 0$ such that

$$E_f(x^m) < \mu \quad \text{and} \quad \Omega(E_f(x^m)) \geq 0, \tag{5.2}$$

then f has only simple zeros and the iteration (1.3) starting from x^0 is well-defined and converges to a root-vector ξ of f with order of convergence four. Besides, the following a posteriori error estimate holds:

$$\|x^k - \xi\|_\infty \leq \varepsilon_k, \quad \text{where} \quad \varepsilon_k = \alpha(E_f(x^k))\, \|W_f(x^k)\|_\infty \tag{5.3}$$

for all $k \geq m$ such that

$$E_f(x^k) \leq \mu \quad \text{and} \quad \Omega(E_f(x^k)) \geq 0. \tag{5.4}$$

In the examples below, we apply the iterative method method (1.3) using the stopping criterion

$$\varepsilon_k < 10^{-15} \qquad (k \geq m) \tag{5.5}$$

together with (5.4). For each example we calculate the smallest $m \geq 0$ which satisfies the convergence condition (5.2), the smallest $k \geq m$ for which the stopping criterion (5.5) is satisfied, as well as the value of ε_k for the last k. From these data it follows that:

- f has only simple zeros;
- the iteration (1.3) starting from $x^{(0)}$ is well-defined and converges to a root-vector of f with order of convergence four;
- at kth iteration the zeros of f are calculated with an accuracy at least ε_k.

In Table 2 the values of iterations are given to 15 decimal places. The values of other quantities (μ, $E_f(x^m)$, etc.) are given to 6 decimal places.

Example 5.1. We consider the polynomial

$$f(z) = z^5 - 15z^4 + 22z^3 + 438z^2 - 1175z - 1575$$

with zeros ± 5, -1, 7, 9 and initial guess

$$x^0 = (-5.7,\ -1.8,\ 4.1,\ 6.2,\ 9.8),$$

which are taken form Nedzhibov and Petkov [4]. For this polynomial $\mu = 0.1$. It can be seen from Table 1 that the convergence condition (5.2) is satisfied for $m = 3$. This guarantees that f has only simple zeros and the iteration (1.3) (starting from x^0) is well-defined and converges to a root-vector ξ of f with order of convergence four. Also, it shows that the stopping criterion (5.5) is satisfied for $k = 4$. Moreover, we can see from this table that at the fifth iteration we have calculated the zeros of f with accuracy less than 10^{-155}. In Table 2, we present numerical results for Example 5.1 giving the first four iterations.

Table 1: Values of m, $E_f(x^m)$, $\Omega(E_f(x^m))$, k, ε_k for Example 5.1

m	$E_f(x^m)$	$\Omega(E_f(x^m))$	ε_m	k	ε_k	ε_{k+1}
3	0.000000	0.999999	1.678603×10^{-7}	4	8.650052×10^{-36}	$5.288490 \times 10^{-156}$

Table 2: Numerical results for Example 5.1.

s	$x_1^{(s)}$	$x_2^{(s)}$	$x_3^{(s)}$
0	−5.7	−1.8	4.1
1	−4.990616790202758	−1.006790776418849	5.048737791535741
2	−5.000000003395358	−0.999999998746670	4.999944962410054
3	−5.000000000000000	−0.999999999999999	4.999999999999989
4	−5.000000000000000	−0.999999999999999	4.999999999999999

s	$x_4^{(s)}$	$x_5^{(s)}$
0	6.2	9.8
1	6.062075553270243	9.036744753761113
2	7.290004092874400	9.000010142803904
3	7.000000167860284	8.999999999999999
4	7.000000000000000	9.000000000000000

Example 5.2. Let us consider the polynomial

$$f(z) = z^9 + 3z^8 - 3z^7 - 9z^6 + 3z^5 + 9z^4 + 99z^3 + 297z^2 - 100z - 300 \tag{5.6}$$

with the zeros -3, ± 1, $\pm 2i$, $\pm 2 \pm i$. For this polynomial $\mu = 0.055555$. Petković, Ilić and Tričković [5] considered this polynomial with Aberth's initial approximation $x^0 \in \mathbb{C}^n$ (see [1]) given by

$$x_\nu^0 = -\frac{a_1}{n} + r_0 \exp(i\theta_\nu), \quad \theta_\nu = \frac{\pi}{n}\left(2\nu - \frac{3}{2}\right), \quad \nu = 1, \dots, n, \tag{5.7}$$

where $n = 9$ and $r_0 = 100$.

One can see from Table 3 that at the eighteen iteration ($m = 18$) we prove the convergence of the method, and at the nineteen iteration ($k = 19$) we obtain the zeros of f with necessary accuracy. Moreover, at the next iteration we get the zeros of f with accuracy less than 10^{-104}.

Table 3: Values of m, $E_f(x^m)$, $\Omega(E_f(x^m))$, k, ε_k for Example 5.2.(a)

m	$E_f(x^m)$	$\Omega(E_f(x^m))$	ε_m	k	ε_k	ε_{k+1}
18	0.000007	0.999984	1.114636×10^{-5}	19	1.230716×10^{-24}	$1.739414 \times 10^{-105}$

Now, we shall apply the method (1.3) again to the polynomial (5.6) with the initial guess (5.7) but with $r_0 = 10$. We can see from Table 4 that at the tenth iteration ($m = 10$) we prove the convergence of the method, and at the twelve iteration ($k = 12$) we obtain the zeros of f with accuracy less than 10^{-42}. In Figure 2, we present the trajectories of approximations generated after 12 iterations.

Table 4: Values of m, $E_f(x^m)$, $\Omega(E_f(x^m))$, k, ε_k for Example 5.2.(b)

m	$E_f(x^m)$	$\Omega(E_f(x^m))$	k	ε_{k-1}	ε_k	ε_{k+1}
10	0.017657	0.957888	12	9.952478×10^{-10}	6.711502×10^{-43}	$1.133974 \times 10^{-182}$

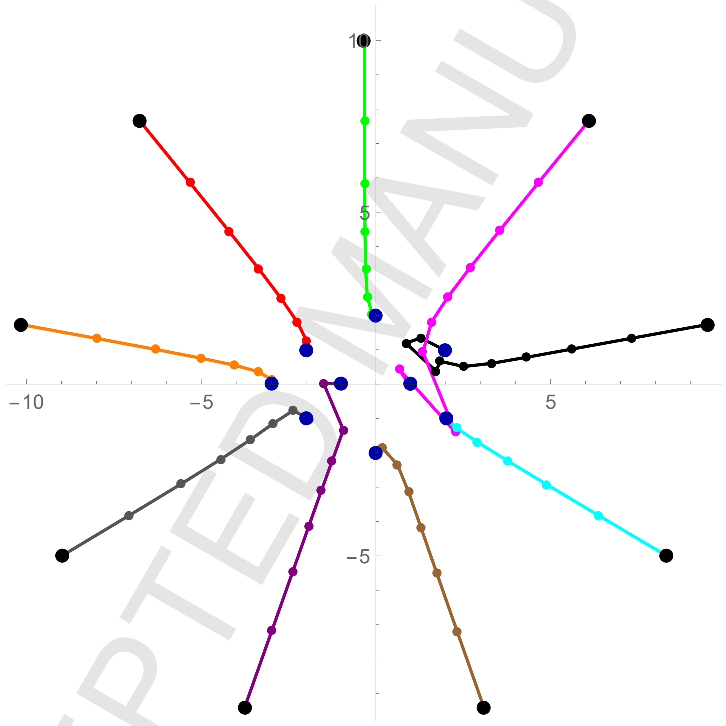

Figure 2: Trajectories of approximations

Example 5.3. Consider the polynomial

$$\begin{aligned} f(z) &= z^{12} - (2+5i)z^{11} - (1-10i)z^{10} + (12-25i)z^9 - 30z^8 \\ &\quad -z^4 + (2+5i)z^3 + (1-10i)z^2 - (12-25i)z + 30 \end{aligned}$$

with zeros ± 1, $\pm i$, $\sqrt{2}/2 \pm i\sqrt{2}/2$, $-\sqrt{2}/2 \pm i\sqrt{2}/2$, $2i$, $3i$, $1 \pm 2i$ and the initial guess

$$\begin{aligned} x^0 &= (1.3+0.2i,\ -1.3+0.2i,\ -0.3-1.2i,\ -0.3+1.2i,\ 0.5+0.5i,\ 0.5-0.5i, \\ &\quad -0.5+0.5i,\ -0.5-0.5i,\ 0.2+2.2i,\ 0.2+3.2i,\ 1.3+2.2i,\ 1.3-2.2i) \end{aligned}$$

which are taken from Sakurai and Petković [21]. In this case $\mu = 0.041666$. For this example, we get that at the third iteration ($m = 3$) we prove the convergence of the method and that at the forth iteration ($k = 4$) we obtain the zeros of f with necessary accuracy (see Table 5).

Table 5: Values of m, $E_f(x^m)$, $\Omega(E_f(x^m))$, k, ε_k for Example 5.3

m	$E_f(x^m)$	$\Omega(E_f(x^m))$	ε_m	k	ε_k	ε_{k+1}
3	0.000001	0.999996	1.157111×10^{-6}	4	4.232015×10^{-25}	$4.465326 \times 10^{-100}$

Acknowledgements

The authors wish to express their gratitude to Professor Petko D. Proinov for his valuable comments and suggestions on this work. This research is supported by the project NI15-FMI-004 of Plovdiv University.

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