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On developing a stable and quadratic convergent method for solving absolute value equation

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ABSTRACT

We modify the generalized Newton method, proposed by Mangasarian (2007), for solving NP-complete absolute value equation, so that it is numerically stable and has convergence order two. Moreover, the convergence conditions are weaker than already iterative methods, hence this method can be applied to a broad range of problems. Applicability of the proposed method is tested for various examples.

1. Introduction

We consider the following absolute value equation (AVE)

$$G(x) = Ax - |x| - b = 0, \tag{1}$$

where $A \in R^{n\times n}$, $b \in R^n$, and $|.|$ denotes absolute value. Mangasarian has proved that the general linear complementarity problem (LCP) is equivalent to an absolute value equation such as (1) (see Proposition in 1 [1]). To solve (1), Mangasarian applies the generalized Newton method for solving the AVE (1) provided that the singular values of A are not less than one (see Lemma 6 in [2]). Although, the generalized Newton method is a linear convergent method, a quadratically convergent method under the same condition has been developed [3]. When the singular values of A exceed 1, the AVE (1) has a unique solution [4,5]. It is worth pointing out that this condition has some equivalences [6]. Hence it seems that under this limitation, such iterative methods converge globally [7]. On the other hand, Prokopyev proves that checking whether the AVE (1) has a unique or multiple solutions is an NP-complete problem [8]. Therefore, it is not generally possible to construct a polynomial algorithm for solvability of AVE. It is worth noting that to avoid the assumption of having singular values greater than one, some other iterative methods have been developed in which all of them converge linearly [9–12].

We develop an iterative method to overcome the two limitations suggested by Mangasarian [1] of the generalized Newton method for solving the NP-complete AVE (1). First, since we are dealing with an NP-complete problem, we cannot generally assume that the singular values of A exceed one. As a vivid example in R, the generalized Newton–Mangasarian method fails to solve the AVE $x-|x| = 1$, because it has no solution. Consequently, as a limitation, this assumption can undoubtedly reduce the number of the real problems and applications that occur in LCP. Second, we focus on modifying the generalized method in such a way that has convergence order two, and it is a numerically stable method. Here our method converges locally because of the nonlinear and NP-complete nature of the problem. If we want to obtain a global quadratically convergent method, we need to make extra assumptions, or, we should consider very special cases. So, we wish to put it aside as an independent

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research problem. This paper is organized as follows: Section 2 deals with construction of the proposed method. Then the convergence analysis and numerical stability are presented. Section 3 is devoted to numerical test problems. The last section concludes the paper.

2. Main results

In this section, reconsidering the generalized Newton method [1], we modify it in such a way that it has convergence order two with some more general conditions compared with the given conditions by Managasarian in [1]. To this end, let the generalized Jacobian of (1) be given by [1]:

$$J_G(x) = A - T_z(x), \tag{2}$$

where $T_z(x)$ =diag(sign(x)). Let x^0 be a suitable starting vector to the exact solution, say x^*, of (1). Then, we propose the following modified Newton–Mangasarian method

$$(A - T_z(x^k))\Delta x^k = -Ax^k + |x^k| + b, \tag{3}$$

$$x^{k+1} = x^k + \Delta x^k, \quad k = 0, 1, 2, \ldots. \tag{4}$$

It should be noted that we first solve the linear system (3), and then, we update the value x^{k+1} from (4). Therefore, we reduce the numerical solution of solving a nonlinear system of equations to the numerical solution of a linear systems of equations. For more details, one can consult [13–15]. We will prove, under weaker conditions than given already, that this method is numerically stable and of convergence order two.

To prove the quadratic convergence order of the method (3)–(4), we need the following lemma:

Lemma 2.1. *Let D be an open convex set in R^n, and let J_G be Lipschitz continuous at x in the neighborhood D. Then, for any $t \in [0, 1]$ and $x + t\Delta x \in D$,*

$$\|G(x + \Delta x) - G(x) - J_G(x)\Delta x\| \le \frac{L_{J_G}}{2}\|\Delta x\|^2, \tag{5}$$

where L_{J_G} is the Lipschitz constant for J_G at x, in other words,

$$\|J_G(x + t\Delta x) - J_G(x)\| \le L_{J_G}\|t\Delta x\|.$$

Proof. By the use of integral mean value theorem, we have

$$\begin{aligned} G(x + t\Delta x) - G(x) - J_G(x)\Delta x &= \int_0^1 J_G(x + t\,\Delta x)\,\Delta x\,dt - J_G(x)\,\Delta x \\ &= \int_0^1 (J_G(x + t\,\Delta x) - J_G(x))\;\Delta x\,dt. \end{aligned}$$

Taking norm and considering the Lipschitz condition on J_G, we have

$$\begin{aligned} \|G(x + t\Delta x) - G(x) - J_G(x)\Delta x\| &= \|\int_0^1 \big(J_G(x + t\,\Delta x) - J_G(x)\big)\,\Delta x\,dt\| \\ &\le \int_0^1 \|J_G(x + t\,\Delta x) - J_G(x)\|\,\|\Delta x\|\,dt \\ &\le \int_0^1 \|L_{J_G} t\,\Delta x\|\,\|\Delta x\|\,dt = \frac{L_{J_G}}{2}\|\Delta x\|^2. \end{aligned}$$

Now, we can prove the quadratic convergence of the proposed method (3)–(4). Let $N_r(x^*) = \{x \in R^n : \|x - x^*\| < r\}$, and $r_k = \|x^k - x^*\|$.

Theorem 2.2. *Suppose that x^* is a solution of the AVE (1), i.e., $G(x^*) = 0$. In addition, suppose that the assumptions of the Lemma 2.1 hold, G is a continuously differentiable for all $x^k \in N_r(x^*) \subset D$, and $\|J_G(x)^{-1}\| < 1$. Then, the sequence $\{x^k\}$, $k > 0$, generated by (3)–(4) satisfies*

$$\|x^{k+1} - x^*\| \le \frac{L_{J_G}}{2}\|x^k - x^*\|^2. \tag{6}$$

Proof. Considering the iterative algorithm (3)–(4) and $G(x^*) = 0$, we have

$$\begin{aligned} x^{k+1} - x^* &= x^k - J_G(x^k)^{-1}G(x^k) - x^* \\ &= x^k - x^* - J_G(x^k)^{-1}(G(x^k) - G(x^*)) \\ &= J_G(x^k)^{-1}\big(G(x^*) - G(x^k) - J_G(x^k)(x^* - x^k)\big). \end{aligned}$$

Hence

$$\begin{aligned} \|x^{k+1} - x^*\| &= \|J_G(x^k)^{-1}\big(G(x^*) - G(x^k) - J_G(x^k)(x^* - x^k)\big)\| \\ &\le \|J_G(x^k)^{-1}\|\|\big(G(x^*) - G(x^k) - J_G(x^k)(x^* - x^k)\big)\|. \end{aligned}$$

Thus, by Lemma 2.1 the assertion follows.

The next theorem addresses that the sequence $\{x^k\}$ converges to the solution x^*.

Theorem 2.3. *Let the assumptions of Theorem 2.2 hold. If $0 < r_k \le r_0 < 2/L_{J_G}$, and $N_{r_0}(x^*) \subset D$. Then, (3)–(4) generate the sequence $\{x^k\}$ such that $x^k \in N_{r_0}(x^*)$, and $x^k \to x^*$ as $k \to \infty$.*

Proof. Since $0 < r_k \le r_0$, then by assumption we have $\frac{r_0 L_{J_G}}{2} < 1$. Consequently, Theorem 2.2 gives

$$\begin{aligned} \|x^{k+1} - x^*\| &< \frac{L_{J_G}}{2}\|x^k - x^*\|\|x^k - x^*\| \\ &< \frac{r_0 L_{J_G}}{2}\|x^k - x^*\| < \|x^k - x^*\|. \end{aligned}$$

Hence the sequence $\{x^k\}$ converges to x^*.

The uniqueness of the solution is established in the next theorem.

Theorem 2.4. *If the conditions of Theorem 2.3 hold, then the solution x^* is unique in $N_{2/L_{J_G}}(x^*)$.*

Proof. By contradiction suppose that y^* is another solution in $N_{2/L_J}(x^*)$. Therefore, $G(y^*) = 0$, and $\|y^* - x^*\| < 2/L_{J_G}$. In the proof of Theorem 2.2 substitute y^* for x^{k+1}, and use Lemma 2.1, we have

$$\begin{aligned} \|y^* - x^*\| &= \|J_G(x^*)^{-1}G(y^*) - G(x^*) - J_G(x^*)(y^* - x^*)\| \\ &\le \frac{L_{J_G}}{2}\|y^* - x^*\|\|y^* - x^*\| < \|y^* - x^*\|. \end{aligned}$$

This is impossible only if $y^* = x^*$.

2.1. Numerical stability analysis

In practice, we have to use floating point arithmetic with finite digits accuracy. Consequently rounding errors occur, and we compute $\hat{x}^k$ or x^k_ε instead of x^k [13,16]. Actually, the computed $\hat{x}^k$ satisfies

$$(A - T_z(\hat{x}^k) + E_1^k)\Delta\hat{x}^k = -Ax^k + |x^k| + b + E_2^k, \tag{7}$$

$$\hat{x}^{k+1} = \hat{x}^k + \Delta\hat{x}^k + E_3^k, \quad k = 0, 1, 2, \dots. \tag{8}$$

where E_1^k, E_2^k, and E_3^k are the errors that are made in computing $G(\hat{x}^k) = A\hat{x}^k - |\hat{x}^k| - b$, forming $J_G(\hat{x}^k)$ and solving the linear system (3), and updating (4), respectively. For more details one can consult the Chapter one of [16].

Now we concern with studying numerical stability of (2)–(3). For this purposes, similar to Wozniakowski [13,15], we need to consider and study $G(x) = G(x; d) = 0$ which means that G depends on an input data d. The condition number of $G(x; d)$, which plays an important role in our study, is defined by

$$\mathrm{cond}(G; d) = \|G'_x(x^*; d)^{-1}G'_d(x^*; d)\|\frac{\|d\|}{\|x^*\|}, \tag{9}$$

where G'_x and G'_d stand for the Frechet derivatives with respect to x and d. Let $G(x; A) = Ax - |x| - b$, where the data vector is supposed to be the given matrix A. For the sake of simplicity we do not consider the vector b as a part of the data vector. Hence, $G'_x(x; A) = A - T_z(x)$, and $G'_A(x; A) = \mathrm{diag}(x)$ where $\mathrm{diag}(x)$ denotes an $n \times n$ diagonal matrix whose elements are

$d_{i,i} = x_i$ for $i = 1, \ldots, n$. Then,

$$\begin{aligned}
\mathrm{cond}(G; A) &= \|G'_x(x^*; A)^{-1} G'_A(x^*; A)\| \frac{\|A\|}{\|x^*\|} \\
&= \|(A - T_z(x^*))^{-1} \mathrm{diag}(x^*)\| \frac{\|A\|}{\|x^*\|} \\
&= \|(A - T_z(x^*))^{-1}\| \|x^*\| \frac{\|A\|}{\|x^*\|} \\
&= \|(A - T_z(x^*))^{-1}\| \|A\|.
\end{aligned}$$

We have proved

Lemma 2.5. *Let the matrix A be the data vector in (1). Then,*

$$cond(G; A) = \|(A - T_z(x^*))^{-1}\| \|A\|. \tag{10}$$

As can be seen from (10), if $T_z(x^*) = 0$, then $G(x; A) = Ax - b = 0$, and we obtain the classic condition number of the matrix A, say $K(A) = \|A^{-1}\| \|A\|$.

Now, we suppose that x_k generated by (3)–(4) is close enough to x^*. It is crucial that the numerical accuracy of x_{k+1} highly depends on the condition number (10). For a moment we stop and focus on the $G(x^k)$ in (3). Based on our assumption, if k is large, then the value of $G(x^k)$ tends to zero. So, in this case, we have a homogeneous linear system and no matter how ill-conditioned it is. This is the reason why we only pay attention to the condition number $\mathrm{cond}(G; A)$, and not to the condition number $\mathrm{cond}(G'; A)$;

As Wilkinson says [17], we are not generally able to solve the equation $G(x; A) = 0$ exactly because we have to compute in finite digits floating point of arithmetic. Let $\mathrm{eps} = 5 \times 10^{-t}$ and O_ϵ denote the machine precision and round-off error in quantity O, respectively. Let

$$\mathrm{fl}(G(x^k; A)) = (I + \Delta G^k_\epsilon) G(x^k + x^k_\epsilon; A + A_\epsilon) = G(x^k) + \delta G(x^k), \tag{11}$$

where $\|\Delta G^k_\epsilon\| \sim \mathrm{eps}$, $\|x^k_\epsilon\| \sim \mathrm{eps}$, $\|A_\epsilon\| \sim \mathrm{eps}\|A\|$, and

$$\delta G(x^k) = \Delta G^k_\epsilon G(x^k) + (A - T_z(x^k)) x^k_\epsilon + \mathrm{diag}(x^k) \mathrm{D}(A_\epsilon) + O(\mathrm{eps}^2), \tag{12}$$

where D is a vector whose elements are given by $d_i = \max A_\epsilon(i, j),\ \ j = 1, \ldots, n$.

Similarly, let

$$\mathrm{fl}(G'(x^k; A)) = G'(x^k) + \delta G'(x^k), \quad \delta G'(x^k) = O(\mathrm{eps}). \tag{13}$$

Hence, we can assume that the numerical solution of (3) satisfies

$$(G'(x^k) + \delta G'(x^k) + E^k) \Delta \tilde{x}^k = G(x^k) + \delta G(x^k), \tag{14}$$

with $E_k = O(\mathrm{eps})$. For more details on solving (13) consult [13]. Consequently, the next improvement, x^{k+1}, is computed by

$$x^{k+1} = (I + \xi^k)(x^k + \Delta \tilde{x}^k), \tag{15}$$

where ξ^k is a diagonal matrix with $\|\xi^k\| \sim \mathrm{eps}$. Now, we are ready to state the numerical stable features of (14)–(15).

Theorem 2.6. *Let (11)–(13) hold. Then the method (14)–(15) is numerically stable.*

Proof. Under the assumptions (11)–(13), by using the Lemma2, the proof is very similar to the proof of the Theorem 5.1 in [15], and we prefer not to repeat such argument here. Therefore, the result follows.

It is worth mentioning that we can study the numerical stability and basins of attraction by the means introduced in [18].

3. Numerical implementations and comparisons

To show the applicability and check the convergence order of the developed method in this work, we tested it for solving 100 examples. To this end, by the command `randi([−m, m], n, n)` using Matlab, we generated 100 random matrices $A \in R^{n\times n}$ for various n such as $n = 20, 50, 100, 500, 1000$, and $m = 1, 2, \ldots, 20$. Then, we produced 100 sample solutions x^* with the same strategy, but only for $m = 0.5$, and used it to generate the vector b by $b = Ax^* - |x^*|$. For these data, the initial values x^0

Table 1
Numerical results and comparisons for $n = 20$.

Methods	$\|x_1 - x^*\|$	$\|x_2 - x^*\|$	$\|x_3 - x^*\|$	COC	ET
New method (3)–(4)	1.295(−1)	3.7450(−2)	8.0472(−4)	1.99	6
Magasarian's method (8) in [1]	1.5836	2.8509(−1)	8.8639(−2)	0.99	9
Khaksar's method (9) in [10]	7.5082	6.8391 (−1)	8.5375(−2)	1.20	7

Table 2
Numerical results and comparisons for $n = 50$.

Methods	$\|x_1 - x^*\|$	$\|x_2 - x^*\|$	$\|x_3 - x^*\|$	COC	ET
New method (3)–(4)	5.0839(−1)	3.507(−2)	4.4175(−4)	2.00	15
Magasarian's method (8) in [1]	3.7501	1.5837(−1)	2.0751(−2)	1.00	24
Khaksar's method (9) in [10]	2.5846	6.8310(−1)	4.8390(−2)	1.00	16

Table 3
Numerical results and comparisons for $n = 100$.

Methods	$\|x_1 - x^*\|$	$\|x_2 - x^*\|$	$\|x_3 - x^*\|$	COC	ET
New method (3)–(4)	5.3810(−2)	1.9630(−4)	3.6259(−9)	2.00	30
Magasarian's method (8) in [1]	3.9572(−1)	3.6289(−2)	8.3885(−2)	1.01	47
Khaksar's method (9) in [10]	2.9630(−1)	8.3820 (−2)	5.5302(−3)	1.01	35

Table 4
Numerical results and comparisons for $n = 500$.

Methods	$\|x_1 - x^*\|$	$\|x_2 - x^*\|$	$\|x_3 - x^*\|$	COC	ET
New method (3)–(4)	4.4439(−2)	5.0079(−4)	8.4161(−9)	2.100	100
Magasarian's method (8) in [1]	6.8204(−1)	4.3920(2−)	3.9208(−3)	1.10	153
Khaksar's method (9) in [10]	5.6392(−1)	3.8460(−2)	3.5027(−4)	1.21	200

Table 5
Numerical results and comparisons for $n = 1000$.

Methods	$\|x_1 - x^*\|$	$\|x_2 - x^*\|$	$\|x_3 - x^*\|$	COC	ET
New method (3)–(4)	6.3901(−3)	6.4820(−6)	4.9561(−12)	2.04	201
Magasarian's method (8) in [1]	4.8601(−2)	6.8410(−3)	5.9739(−4)	1.13	281
Khaksar's method (9) in [10]	4.0851(−3)	5.9847 (−4)	5.9817(−5)	1.04	307

were chosen so that they satisfied our hypotheses. For comparison purposes, we have used the computational convergence order (COC) defined by [19,20].

$$\mathrm{COC} = \frac{\ln \|x^{k+1} - x^k\|}{\ln \|x^k - x^{k-1}\|}, \quad k = 0, 2, \ldots .$$

In tables, $a(b)$ denotes $a \times 10^b$ and the last columns show elapsed time (ET) in second. Although all of the tested problems by our method were succeeded, however, since all our sample matrices did not satisfy the condition of the methods [1] and [10], and for the save of space, in the following we reported some of the results in which all the mentioned methods accomplished. As can be seen in Tables 1–5, our developed method (3)–(4) supports the given theory and has convergence order two and is competitive rather than the compared iterative methods.

4. Concluding remarks

We have modified the generalized Newton method [1] for solving AVE so that it works under weaker convergence, hence, it can be used for wider range of the problems. In addition, it is numerically stable with local convergence order two while similar iterative methods are linear convergent. Only one linear system of equation needs to be solved for each iteration which shows another effectiveness of the developed method. This method was successfully applied to solve 100 samples. It is a single step method and it seems that this method can be exploited for developing multi step methods for solving AVE. These kinds of solvers are more economic and have been well studied in the literature [14,21,22,23].

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