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IPCDGM AND MULTISCALE IPDPGM FOR THE HELMHOLTZ PROBLEM WITH LARGE WAVE NUMBER

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Abstract. In this paper, based on the two nested spaces of piecewise linear polynomials (on fine and coarse meshes with sizes h and H respectively), which are continuous in macro elements but discontinuous across interior edges/faces of them, an interior penalty continuous-discontinuous Galerkin method (IPCDGM) and a multiscale interior penalty discontinuous Petrov-Galerkin method (MsIPDPGM) are proposed to solve the Helmholtz problem with large wave number k and homogeneous Robin boundary condition. This paper devotes to analyzing the preasymptotic error of the two methods separately. In order to reduce the pollution errors efficiently, the two methods not only include the penalty terms on jumps of function on coarse mesh edges/faces but also add the penalty terms on jumps of first order normal derivatives on fine mesh edges/faces. The error of the IPCDG solution in the broken H^1-norm is bounded by $O(kh+k^3h^2)$. By splitting into coarse and fine scales and basing on the IPCDG variational formulation, the MsIPDPGM's trial and test spaces are constructed with the macro corrected bases. Due to the exponential decay of the correctors, the corrected bases are obtained by solving the local problems on localized subdomains of size LH (L being the oversampling parameter). The preasymptotic error analysis of MsIPDPGM shows that, if kH and $\log k/L$ fall below some constants and if the fine mesh size h is sufficient small, the errors of numerical solutions in the broken H^1-norm can be dominated by $O(H^3)$ without pollution effect. Numerical tests are provided to verify the theoretical findings and advantages of the two methods.

Keywords: Helmholtz Problem, large wave number, interior penalty continuous-discontinuous Galerkin method, Multiscale method, pollution effect, preasymptotic error estimate

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1 Introduction

Let $\Omega \subset R^d (d=2,3)$ be a convex polygonal domain with boundary $\Gamma := \partial\Omega$. The model Helmholtz equation reads as

$$\begin{cases} -\triangle u - k^2 u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \mathrm{i} k u = 0 & \text{on } \Gamma, \end{cases} \tag{1.1}$$

where $i = \sqrt{-1}$ denotes the imaginary unit and n denotes the unit outward normal to Γ. The problem (1.1) is an approximation of the acoustic scattering problem (with time dependence e^{iwt}) . The large positive number k is called the wave number. The Robin boundary condition is known as the first order approximation of the radiation condition.

Due to the high indefiniteness, standard finite element methods for the problem (1.1) at high frequencies ($k \gg 1$) are well known to exhibit the so-called pollution effect [2]. There have been various attempts to reduce or avoid the pollution effect, e.g, high order FEM [24, 25], continuous interior penalty FEM [29, 12, 30], discontinuous Galerkin methods [10, 14, 15, 17, 23] and multiscale FEM [27, 16, 20] among many others.

Discontinuous Galerkin methods are widely used to solve Helmholtz problem owing to their several advantages over other types of numerical methods [14, 15, 17]. The work of [14, 15] shows that the IPDG for the Helmholtz problem can reduce the pollution effect by adjust the penalized parameters, where the authors add the jumps of the normal and tangential derivatives.

On the other hand, the Helmholtz problem of large wave number has the common feature of multiscale problems which are very computationally challenging and often impossible to solve within an acceptable tolerance using standard mesh methods. To overcome the lack of performance, many different so called multiscale methods have been proposed and there are some works about applying the multiscale method to deal with the Helmholtz problem [27, 16]. In the paper [27, 16], by adopting the multiscale method of [20, 21], the authors present and analyze a pollution-free multiscale finite element method for Helmholtz problem with large wave number, where the discrete trial or test spaces are generated from standard mesh-based finite elements by local subscale corrections.

In this paper, based on the space of piecewise linear polynomials on a fine mesh, which are continuous in elements of a coarse mesh, but discontinuous across interior edges/faces of them, an interior penalty continuous-discontinuous Galerkin method (IPCDGM) is firstly proposed to solve the Helmholtz problem with large wave number k and the preasymptotic error analysis is given out.

Then, by adopting the multiscale method of [20, 21], we construct the MsIPDPGM on the macro IPCDGM solver where the basis are obtained by solving the local subscale corrections in spirit of numerical homogenization. The two schemes not only includes the penalty terms on jumps of function on coarse mesh edges/faces but also adds the penalty terms on jumps of first order normal derivatives on fine mesh edges/faces. This is due to that the MsIPDPGM is constructed in the following two discontinuous discrete spaces:

$$V_H = \{v_H|_T \in P_1, \forall T \in \mathcal{T}_H\},$$
$$V_h = \{v_h|_t \in P_1 \text{ and } v_h|_T \in C^0, \forall T \in \mathcal{T}_H \text{ and } t \in \mathcal{T}_h\},$$

where $\mathcal{T}_H$ and $\mathcal{T}_h$ denote two triangulations of Ω with mesh sizes H and h respectively, which are regular and quasi-uniform. P_1 is the set of polynomials of degree ≤ 1. The mesh $\mathcal{T}_h$ is obtained by refining $\mathcal{T}_H$. Obviously $V_H \subset V_h$.

One purpose of this paper is to prove the preasymptotic error bound for the IPCDGM in fine scale mesh $\mathcal{T}_h$. It is shown that if k^3h^2 is sufficient small, the H^1 error of the numerical solution can be bounded by $O(kh + k^3h^2)$. Another purpose is to propose the MsIPDPGM based on our IPCDGM and prove its preasymptotic error bound. It is shown that the error of the MsIPDPG solution is pollution free if $kH \eqsim 1$ is sufficiently small and the solutions of local problems are sufficiently accurate in local patches with sizes $O(\log k)$. In MsIPDPGM, without fully resolving the fine scale problem on the whole domain, we can get the macro-scale numerical solutions due to the correctors' exponential decay. In practice, we can reduce the pollution errors efficiently by adjusting the penalty parameters in the two methods.

Comparing with [27, 16], we use the IPDG as macro solver to construct the MsIPDPGM. By tuning the penalty parameters, the pollution error can be reduced efficiently, which doesn't need h is sufficient small. On the other hand, when k^3h^2 is sufficient small, we show that the solution to the MsIPDPGM is superclose to the IPCDG solution.

The remainder of this paper is organized as follows. The IPCDGM and MsIPDPGM are introduced in Section 2. Section 3 is devoted to the preasymptotic error analysis of IPCDGM and Section 4 is devoted to MsIPDPGM. In Section 5, we simulate a model problem in two dimensions by IPCDGM and MsIPDPGM to verify the theoretical findings.

Throughout the paper, C is used to denote a generic positive constant which is independent of k, h, H, and f. The shorthand notations $A \lesssim B$ and $B \gtrsim A$ mean $A \le CB$ and $B \ge CA$. $A \eqsim B$ is a short notation for the statement $A \lesssim B$ and $B \gtrsim A$. Moreover, the constants $C_j, j = 0,1,2,3$ that appear later are all independent of k, h, H, and f.

2 Formulations of IPCDGM and Multiscale IPDPGM

In this section, we will introduce the IPCDGM and the MsIPDPGM for problem (1.1). We first introduce some notation. We shall use the standard Sobolev space $H^s(\Omega)$, its norm and inner product, and refer to [5, 8] for their definitions. But $(\cdot,\cdot)$ and $\langle\cdot,\cdot\rangle$ are used for the L^2-inner product on the complex-valued spaces $L^2(\Omega)$ and $L^2(\Gamma)$, respectively. For any open set G, we write $\|\cdot\|_{s,G}$ and $|\cdot|_{s,G}$ the norm and semi-norm of $H^s(G)$. We will write briefly that $\|\cdot\|_s = \|\cdot\|_{s,\Omega}$ and $|\cdot|_s = |\cdot|_{s,\Omega}$.

Let $\{\mathcal{T}_H\}$ and $\{\mathcal{T}_h\}$ be two quasi-uniform families of triangulations of Ω such that $\mathcal{T}_h$ is a refinement of $\mathcal{T}_H$. For any element $T \in \mathcal{T}_H$ ($t \in \mathcal{T}_h$), we define $H_T = \mathrm{diam}(T)$ ($h_t = \mathrm{diam}(t)$) and $H = \max_{T\in\mathcal{T}_H} H_T$ ($h = \max_{t\in\mathcal{T}_h} h_t$). Let $\mathcal{E}_H$ and $\mathcal{E}_h$ denote the interior edges/faces of elements in $\mathcal{T}_H$ and $\mathcal{T}_h$, respectively. Denote by $H^s(\mathcal{T}_h)$ and $H^s(\mathcal{T}_H)$ the spaces of piecewise H^s functions on T_h

and T_H, respectively. Let ∇_h be the piecewise gradient on $H^1(\mathcal{T}_h)$, i.e., $(\nabla_h v)|_t = \nabla(v|_t), \forall t \in \mathcal{T}_h$. ∇_H is similarly defined on $H^1(\mathcal{T}_H)$. Clearly, $\nabla_h v = \nabla_H v$ for any $v \in H^1(\mathcal{T}_H)$ and $\nabla_h v = \nabla_H v = \nabla v$ for any $v \in H^1(\Omega)$.

For any interior edge/face $E \in \mathcal{E}_H$ (or $e \in \mathcal{E}_h$) there are two adjacent elements T^- and T^+ (or t^- and t^+) with $T^- \cap T^+ = E$ ($t^- \cap t^+ = e$). We define n to be the unit outward normal vector to edge/face of the element T^+ (or t^+). Define the jump and average of a function v at E (or e) as

$$[v] = v|_{T^+} - v|_{T^-}, \quad \{v\} = \frac{v|_{T^-} + v|_{T^+}}{2} \quad \left(\text{or } [v] = v|_{t^+} - v|_{t^-}, \quad \{v\} = \frac{v|_{t^-} + v|_{t^+}}{2}\right).$$

In addition, denote by $H_E = \operatorname{diam}(E)$, $h_e = \operatorname{diam}(e)$, and by $\langle \cdot, \cdot \rangle_E$ and $\langle \cdot, \cdot \rangle_e$ the inner product on $L^2(E)$ and $L^2(e)$, respectively.

2.1 IPCDGM

We first introduce the approximation space of piecewise linear functions on $\mathcal{T}_h$ which are continuous on each macro-element $T \in \mathcal{T}_H$ but allowed to be discontinuous at $E \in \mathcal{E}_H$:

$$V_h = \left\{ v_h : v_h|_t \in P_1 \;\; \forall t \in \mathcal{T}_h \;\text{ and }\; v_h|_T \in C^0 \;\; \forall T \in \mathcal{T}_H \right\}, \tag{2.2}$$

where P_1 denotes the set of all linear polynomials. Let V be the energy space defined as:

$$V = \left\{ v : v|_t \in H^2(t) \;\; \forall t \in \mathcal{T}_h \;\text{ and }\; v|_T \in H^1(T) \;\; \forall T \in \mathcal{T}_H \right\}.$$

Clearly, $V_h \subset V$. By testing (1.1) by any $v \in V$ and using the magic formula $[vw] = \{v\}[w] + [v]\{w\}$ we conclude that the exact solution u satisfies

$$(\nabla u, \nabla_H v) - \sum_{E \in \mathcal{E}_H} \left\langle \left\{ \frac{\partial u}{\partial n} \right\}, [v] \right\rangle_E - k^2 (u, v) + \mathrm{i} k \langle u, v \rangle = (f, v).$$

Similar to the classical IPDG formulation, we introduce the following sesquilinear form:

$$a_h(u, v) = b_h(u, v) - k^2 (u, v) + i k \langle u, v \rangle \tag{2.3}$$

where

$$b_h(u, v) = (\nabla_h u, \nabla_h v) - \sum_{E \in \mathcal{E}_H} \left(\left\langle \left\{ \frac{\partial u}{\partial n} \right\}, [v] \right\rangle_E + \left\langle [u], \left\{ \frac{\partial v}{\partial n} \right\} \right\rangle_E \right) + J(u, v), \tag{2.4}$$

$$J(u, v) = \sum_{e \subset E \in \mathcal{E}_H} \frac{\gamma_{0,e}}{h_e} \langle [u], [v] \rangle_E + \sum_{e \in \mathcal{E}_h} \gamma_{1,e} h_e \left\langle \left[\frac{\partial u}{\partial n} \right], \left[\frac{\partial v}{\partial n} \right] \right\rangle_e. \tag{2.5}$$

$\gamma_{0,e}$ and $\gamma_{1,e}$ are penalty parameters to be specified later. Therefore, if $u \in H^2(\Omega)$ is the solution of (1.1), then

$$a_h(u, v) = (f, v) \quad \forall v \in V. \tag{2.6}$$

Inspired by the above formulation, we propose the following IPCDGM for problem (1.1): find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h. \tag{2.7}$$

Remark 2.1. (i) *Penalizing the jumps of normal derivatives was used early by Douglas and Dupont [11] for second order PDEs in the context of continuous interior penalty finite element methods (CIPFEM) and by Babuška and Zlámal [3] for fourth order PDEs in the context of C^0 finite element methods, by Baker [4] for fourth order PDEs and by Arnold [1] for second order parabolic PDEs in the context of IPDG methods.*

(ii) *In [14, 15, 12, 30, 6, 29], the IPDGM and the CIPFEM are applied to the Helmholtz equation with high wave number. It turned out that the penalty parameters in IPDG methods and the CIPFEM may be tuned to eliminate the pollution effect in one dimension and greatly reduce the pollution effect in higher dimensions. It will be shown in Section 5 by numerical tests in 2D that the penalty parameters in our IPCDGM can also be tuned to greatly reduce the pollution error.*

(iii) *Note that the approximate functions used in the IPDGM are allowed to be discontinuous at any interface of elements while those used in the CIPFEM are continuous on the whole computational domain just like the FEM. Our IPCDGM is a method between the IPDGM and CIPFEM since the approximate functions in V_h are continuous in each macro-element in $\mathscr{T}_H$ but may be discontinuous at their edges/faces. The IPCDGM use less number of degrees of freedom than the IPDGM while more than the CIPFEM on the same mesh.*

(iv) *The methods and theoretical results can be easily extended to Cartesian meshes. While the extension to higher order methods requires more technical skills (cf. [12]) and will be reported in a future work.*

We introduce the following semi-norms and norms on $H^s(\mathscr{T}_H)$ or $H^s(\mathscr{T}_h)$ for further analysis.

$$\|v\|_H = \left(\|\nabla_H v\|_0^2 + \sum_{E\in\mathscr{E}_H} \frac{1}{H} \|[v]\|_{0,E}^2 + \sum_{E\in\mathscr{E}_H} H \left\| \left[\frac{\partial v}{\partial n} \right] \right\|_{0,E}^2 \right)^{\frac{1}{2}},$$

$$\|v\|_h = \left(\|\nabla_h v\|_0^2 + \sum_{E\in\mathscr{E}_H} \frac{1}{h} \|[v]\|_{0,E}^2 + \sum_{e\in\mathscr{E}_h} h \left\| \left[\frac{\partial v}{\partial n} \right] \right\|_{0,e}^2 \right)^{\frac{1}{2}},$$

$$|||v|||_H = \left(\|v\|_H^2 + k^2 \|v\|_0^2 \right)^{\frac{1}{2}}, \quad |||v|||_h = \left(\|v\|_h^2 + k^2 \|v\|_0^2 \right)^{\frac{1}{2}}.$$

2.2 Multiscale IPDPGM

In this subsection we introduce our multiscale method based on the IPCDGM. The MsIPDPGM uses the same variational formulation as IPCDGM, while the approximation space is defined on the coarse mesh $\mathscr{T}_H$, whose basis functions are obtained by solving some local problems on the fine scale mesh.

We define our multiscale approximation spaces by following the standard procedure in [20, 21, 13, 27, 16, etc.]. First, we introduce the space of piecewise linear functions on $\mathscr{T}_H$:

$$V_H = \{v_H : \ v_H|_T \in P_1, \ \forall T \in \mathscr{T}_H\}, \tag{2.8}$$

and define the local L^2-projector $Q_H : L^2(\Omega) \to V_H$: for any $v \in L^2(\Omega)$ and $T \in \mathscr{T}_H$,

$$(Q_H v - v, v_H)_T = 0 \quad \forall v_H \in V_H. \tag{2.9}$$

Let L be a positive integer. For any macro-element $T \in \mathcal{T}_H$, define the L-layer region surrounding T as

$$\omega_{T,0} = T. \quad \omega_{T,l} = \{T' \in \mathcal{T}_H : T' \cap \omega_{T,l-1} \neq \emptyset\}, \quad l = 1, \cdots, L. \tag{2.10}$$

Introduce the global and local kernel spaces of Q_H as

$$V^f = \{v_h \in V_h : Q_H v_h = 0\}, \quad V^f_{T,L} = \{v_h \in V^f : v_h = 0 \text{ in } \Omega \setminus \omega_{T,L}\}. \tag{2.11}$$

For any $T \in \mathcal{T}_H$, denote by $\lambda_T^j \in V_H (j = 1 \cdots r$ with $r = d+1)$ the nodal basis functions with respect to T. Define the global and localized correctors $(E_\infty \lambda_T^j, E_{T,L}\lambda_T^j)$ and the corresponding adjoint correctors $(E^*_\infty \lambda_T^j, E^*_{T,L}\lambda_T^j)$ of λ_T^j as

$$E_\infty \lambda_T^j \in V^f: \quad a_h(E_\infty \lambda_T^j, w) = a_h(\lambda_T^j, w) \quad \forall w \in V^f, \tag{2.12}$$

$$E^*_\infty \lambda_T^j \in V^f: \quad a_h(w, E^*_\infty \lambda_T^j) = a_h(w, \lambda_T^j) \quad \forall w \in V^f, \tag{2.13}$$

$$E_{T,L} \lambda_T^j \in V^f_{T,L}: \quad a_h(E_{T,L} \lambda_T^j, w_L) = a_h(\lambda_T^j, w_L), \quad \forall w_L \in V^f_{T,L}, \tag{2.14}$$

$$E^*_{T,L} \lambda_T^j \in V^f_{T,L}: \quad a_h(w_L, E^*_{T,L} \lambda_T^j) = a_h(w_L, \lambda_T^j), \quad \forall w_L \in V^f_{T,L}. \tag{2.15}$$

Let x_T^j be the nodal points corresponding to λ_T^j, that is, $\lambda_T^j(x_T^m) = \delta_{jm}, j, m = 1, \cdots, r$. For any $v_H = \sum_{\substack{T\in\mathcal{T}_H \\ j=1\cdots r}} v_H(x_T^j)\lambda_T^j \in V_H$, define

$$E_\infty v_H = \sum_{\substack{T\in\mathcal{T}_H \\ j=1\cdots r}} v_H(x_T^j)\big(E_\infty \lambda_T^j\big), \quad E^*_\infty v_H = \sum_{\substack{T\in\mathcal{T}_H \\ j=1\cdots r}} v_H(x_T^j)\big(E^*_\infty \lambda_T^j\big), \tag{2.16}$$

$$E_L v_H = \sum_{\substack{T\in\mathcal{T}_H \\ j=1\cdots r}} v_H(x_T^j)\big(E_{T,L} \lambda_T^j\big), \quad E^*_L v_H = \sum_{\substack{T\in\mathcal{T}_H \\ j=1\cdots r}} v_H(x_T^j)\big(E^*_{T,L} \lambda_T^j\big). \tag{2.17}$$

Clearly, $E_\infty v_H, E^*_\infty v_H, E_L v_H, E^*_L v_H \in V^f$ for any $v_H \in V_H$ and

$$a_h(E_\infty v_H, w) = a_h(v_H, w), \quad a_h(w, E^*_\infty v_H) = a_h(w, v_H), \quad \forall v_H \in V_H, w \in V^f. \tag{2.18}$$

It is easy to verify that

$$E^*_\infty v_H = \overline{E_\infty \overline{v_H}} \quad \text{and} \quad E^*_L v_H = \overline{E_L \overline{v_H}}, \quad \forall v_H \in V_H. \tag{2.19}$$

Then we introduce the localized MsIPDPG trial and test spaces by correcting these nodal basis functions.

$$V^{ms}_{H,L} = \operatorname{span}\{\lambda_T^j - E_{T,L}\lambda_T^j, T \in \mathcal{T}_H, j = 1\cdots r\},$$

$$V^{ms*}_{H,L} = \operatorname{span}\{\lambda_T^j - E^*_{T,L}\lambda_T^j, T \in \mathcal{T}_H, j = 1\cdots r\}.$$

From (2.19), it is clear that the test space $V^{ms*}_{H,L}$ consists of complex conjugates of functions from the trial space $V^{ms}_{H,L}$.

Now, we are ready to define the MsIPDPGM for problem (1.1) : Find $u^{ms}_{H,L} \in V^{ms}_{H,L}$ such that

$$a_h\big(u^{ms}_{H,L}, v^{ms*}_{H,L}\big) = \big(f, v^{ms*}_{H,L}\big), \quad \forall v^{ms*}_{H,L} \in V^{ms*}_{H,L}. \tag{2.20}$$

Remark 2.2. (i) *The well-posedness of problems (2.12)–(2.15), and (2.20) will be verified in next sections (see Corollary 3.7, Remark 4.7).*

(ii) *The problems* (2.12)–(2.13) *are introduce only for theoretical purpose. In practice, we solve the local problems* (2.14)–(2.15) *and* (2.20) *instead, which are good approximations to* (2.12)–(2.13) *, respectively. As a matter of fact, it can be shown that the global correctors $E_\infty\lambda_{T,j}$ decay exponentially (see Section 4). The local problem (2.14) should be solved on the fine mesh scale, while the problem (2.20) on the macro mesh $\mathcal{T}_H$. On the other hand, the problem (2.14) of each macro basis function can be solved by parallel computing.*

(iii) *We will show in Theorem 4.12 that the solution to the MsIPDPGM is superclose to the IPCDG solution under proper conditions.*

3 Preasymptotic error analysis of IPCDGM

The task of this section is to derive the preasymptotic error estimates between the solution to the IPCDGM (2.7) and the exact solution to (1.1). As by-products, we also give out the stability estimates and the inf-sup condition for the scheme (2.7).

3.1 Preliminary lemmas

In this subsection, we will recall some stability estimates of the continuous problem and some approximation properties of the discrete spaces.

The following stability estimates for the problem (1.1) were proved in [22, 9, 18].

Lemma 3.1. *Suppose $f \in L^2(\Omega)$, for the solution u to the problem* (1.1)*, there holds*

$$k^{-1}\|u\|_2 + \|u\|_1 + k\|u\|_0 \le C\|f\|_0.$$

The following local trace inequality will be used frequently in our analysis, which can be prove by using the standard trace inequality and the scaling argument [5, 7].

$$\|v\|_{0,\partial t} \lesssim h_t^{-\frac{1}{2}}\|v\|_{0,t} + \|v\|_{0,t}^{\frac{1}{2}}\|\nabla v\|_{0,t}^{\frac{1}{2}}, \quad \forall v \in H^1(t), t \in \mathcal{T}_h. \tag{3.21}$$

Let I_h be the Lagrange interpolation operator onto V_h. Denote by $|v|_{2,\mathcal{T}_h} = \left(\sum_{t\in\mathcal{T}_h}|v|_{2,t}^2\right)^{1/2}$ the discrete semi-H^2-norm. We have the following interpolation error estimates.

Lemma 3.2. *Suppose $v \in H^2(\mathcal{T}_h)$. Then*

$$\|v - I_h v\|_0 \lesssim h^2|v|_{2,\mathcal{T}_h}, \quad \|v - I_h v\|_h \lesssim h|v|_{2,\mathcal{T}_h}. \tag{3.22}$$

Moreover, for the solution u to the problem (1.1)*, there hold*

$$\|u - I_h u\|_0 \lesssim kh^2\|f\|_0 \quad \textit{and} \quad \|u - I_h u\|_h \lesssim kh(1+kh)\|f\|_0. \tag{3.23}$$

Proof. The standard interpolation theory [5, 8] says that

$$\|v - I_h v\|_{0,t} + h\|v - I_h v\|_{1,t} \lesssim h^2 |v|_{2,t} \quad \forall t \in \mathscr{T}_h.$$

Clearly, it suffices to prove the second estimate in (3.22). Denote by $w = v - I_h v$. It follows from the local trace inequality that

$$\begin{aligned}
\|v - I_h v\|_h^2 &= \sum_{t\in\mathscr{T}_h} \|\nabla w\|_{0,t}^2 + \sum_{E\in\mathscr{E}_H} \frac{1}{h}\|[w]\|_{0,E}^2 + \sum_{e\in\mathscr{E}_h} h \left\|\left[\frac{\partial w}{\partial n}\right]\right\|_{0,e}^2 \\
&\lesssim \sum_{t\in\mathscr{T}_h} \left(h^{-2}\|w\|_{0,t}^2 + |w|_{1,t}^2 + h^2|w|_{2,t}^2\right) \\
&\lesssim \sum_{t\in\mathscr{T}_h} \left(h^2|v|_{2,t}^2\right) \lesssim h^2|v|_{2,\mathscr{T}_h}^2.
\end{aligned}$$

Then (3.23) is a consequence of (3.22) and Lemma 3.1. This completes the proof of the lemma. □

3.2 Preasymptotic error estimates

First we consider the continuity and coercivity of the sesquilinear form $b_h(\cdot,\cdot)$ defined in (2.4).

Lemma 3.3. *Suppose $|\gamma_{j,e}| \lesssim 1\,(j = 0,1), \forall e \in \mathscr{E}_h$. There hold*

$$|b_h(u,v)| \lesssim \left(\|u\|_h + h|u|_{2,\mathscr{T}_h}\right)\left(\|v\|_h + h|v|_{2,\mathscr{T}_h}\right) \quad \forall u, v \in V, \tag{3.24}$$

$$|b_h(u_h,v_h)| \lesssim \|u_h\|_h \|v_h\|_h \quad \forall u_h, v_h \in V_h. \tag{3.25}$$

Moreover, there exist constants $\alpha_0, \alpha_1 > 0$, such that if $\alpha_0 \le \gamma_{0,e} \lesssim 1$ and $-\alpha_1 \le \gamma_{1,e} \lesssim 1$, then

$$b_h(v_h, v_h) \gtrsim \|v_h\|_h^2 \quad \forall v_h \in V_h. \tag{3.26}$$

Proof. From (2.4)–(2.5) and the local trace inequality (3.21) we obtain

$$\begin{aligned}
|b_h(u,v)| &\lesssim \|\nabla_h u\|_0 \|\nabla_h v\|_0 + \sum_{E\in\mathscr{E}_H} \left(\left\|\left\{\frac{\partial u}{\partial n}\right\}\right\|_{0,E} \|[v]\|_{0,E} + \|[u]\|_{0,E} \left\|\left\{\frac{\partial v}{\partial n}\right\}\right\|_{0,E} \right) \\
&\quad + \sum_{e\subset E\in\mathscr{E}_H} h^{-1}\|[u]\|_{0,E}\|[v]\|_{0,E} + \sum_{e\in\mathscr{E}_h} h \left\|\left[\frac{\partial u}{\partial n_e}\right]\right\|_{0,e} \left\|\left[\frac{\partial v}{\partial n_e}\right]\right\|_{0,e} \\
&\lesssim \left(\|u\|_h^2 + h \sum_{e\subset E\in\mathscr{E}_H} \left\|\left\{\frac{\partial u}{\partial n}\right\}\right\|_{0,E}^2\right)^{\frac12} \left(\|v\|_h^2 + h \sum_{e\subset E\in\mathscr{E}_H} \left\|\left\{\frac{\partial v}{\partial n}\right\}\right\|_{0,E}^2\right)^{\frac12} \\
&\lesssim \left(\|u\|_h^2 + h \sum_{t\in\mathscr{T}_h} \left(h^{-1}\|\nabla u\|_{0,t}^2 + h|\nabla u|_{1,t}^2\right)\right)^{\frac12} \\
&\quad \times \left(\|v\|_h^2 + h \sum_{t\in\mathscr{T}_h} \left(h^{-1}\|\nabla v\|_{0,t}^2 + h|\nabla v|_{1,t}^2\right)\right)^{\frac12}
\end{aligned}$$

which implies that (3.24) holds. (3.25) is a consequence of (3.24) since $|v_h|_{2,\mathscr{T}_h} = 0, \forall v_h \in V_h$.

It remains to prove (3.26). Suppose $\gamma_{0,e} \geq \alpha_0 > 0$ and $\gamma_{1,e} \geq -\alpha_1$ for some constants $\alpha_0, \alpha_1 > 0$. For any $v_h \in V_h$, we have

$$
\begin{aligned}
b_h(v_h, v_h) &= \|\nabla_h v_h\|_0^2 - 2\mathrm{Re}\sum_{E\in\mathscr{E}_H}\left\langle\left\{\frac{\partial v_h}{\partial n}\right\}, [v_h]\right\rangle_E \\
&\quad + \sum_{e\subset E\in\mathscr{E}_H}\frac{\gamma_{0,e}}{h_e}\|[v_h]\|_{0,E}^2 + \sum_{e\in\mathscr{E}_h}\gamma_{1,e}h_e\left\|\left[\frac{\partial v_h}{\partial n}\right]\right\|_{0,e}^2 \\
&\geq \|\nabla_h v_h\|_0^2 + \frac{\alpha_0}{2}\sum_{e\subset E\in\mathscr{E}_H}\frac{1}{h_e}\|[v_h]\|_{0,E}^2 \\
&\quad - \sum_{e\in\mathscr{E}_h}\left(\alpha_1 h_e\left\|\left[\frac{\partial v_h}{\partial n}\right]\right\|_{0,e}^2 + \frac{2}{\alpha_0}h_e\left\|\left\{\frac{\partial v_h}{\partial n}\right\}\right\|_{0,e}^2\right)
\end{aligned}
$$

Noting from the local trace inequality that

$$
\sum_{e\in\mathscr{E}_h}\left(h_e\left\|\left[\frac{\partial v_h}{\partial n}\right]\right\|_{0,e}^2 + h_e\left\|\left\{\frac{\partial v_h}{\partial n}\right\}\right\|_{0,e}^2\right) \lesssim \|\nabla_h v_h\|_0^2,
$$

there exists a constant $\tilde{C}$ such that

$$
b_h(v_h, v_h) \geq \|\nabla_h v_h\|_0^2 + \frac{\alpha_0}{2}\sum_{e\subset E\in\mathscr{E}_H}\frac{1}{h_e}\|[v_h]\|_{0,E}^2 - \tilde{C}\max\left(\alpha_1, \frac{2}{\alpha_0}\right)\sum_{t\in\mathscr{T}_h}\|\nabla v_h\|_{0,t}^2.
$$

Thus (3.26) holds if $\alpha_1 \leq 1/(2\tilde{C})$ and $\alpha_0 \geq 4\tilde{C}$. This completes the proof of the lemma. □

Remark 3.4. *Denote by $\tilde{V}_h$ the space of piecewise linear functions on $\mathscr{T}_h$. It is clear that the continuity (3.25) and the coercivity (3.26) hold also on $\tilde{V}_h$.*

Following the analyses in [12, 30, 29], we introduce the following elliptic projection $P_h : V \mapsto V_h$ that will be used in the error estimates.

$$
b_h(P_h\varphi, v_h) + (P_h\varphi, v_h) = b_h(\varphi, v_h) + (\varphi, v_h) \quad \forall v_h \in V_h. \tag{3.27}
$$

The elliptic projection satisfies the following estimates [12].

Lemma 3.5. *Under the conditions of Lemma 3.3, there holds for any $\varphi \in H^2(\Omega)$ that*

$$
\|\varphi - P_h\varphi\|_0 + h\left\|\varphi - P_h\varphi\right\|_h \lesssim h^2|\varphi|_2.
$$

Next we present the main theorem of this section, which gives the preasymptotic error estimates of the IPCDGM.

Theorem 3.6. *Let u and u_h be the solutions to problems (1.1) and (2.7), respectively. Suppose that $f \in L^2(\Omega)$ and the conditions of Lemma 3.3 hold. There exists a constant $C_0 > 0$ such that, if $k^3h^2 \leq C_0$, then*

$$
\|u - u_h\|_h \lesssim (kh + k^3h^2)\|f\|_0 \quad \text{and} \quad \|u - u_h\|_0 \lesssim (kh^2 + k^2h^2)\|f\|_0.
$$

Proof. Denote $e_h = u - u_h = (u - P_h u) + (P_h u - u_h) := \rho + \theta_h$. From (2.6)–(2.7), we have the Galerkin orthogonality $a_h(e_h, v_h) = 0$. Use the definition of P_h (3.27) to get

$$b_h(\theta_h, v_h) - k^2(\theta_h, v_h) + \mathrm{i}k\langle\theta_h, v_h\rangle = (k^2+1)(\rho, v_h) - \mathrm{i}k\langle\rho, v_h\rangle. \tag{3.28}$$

Step 1: Bounding $\|\theta_h\|_{0,\Gamma}$ by $\|\theta_h\|_0$. By letting $v_h = \theta_h$ in (3.28) and taking the imaginary part we conclude that

$$k\|\theta_h\|_{0,\Gamma}^2 \lesssim k^2\|\rho\|_0\|\theta_h\|_0 + k\|\rho\|_{0,\Gamma}\|\theta_h\|_{0,\Gamma}.$$

The trace inequality, Lemma 3.5, and Lemma 3.1 imply that

$$\|\rho\|_0 + h\|\rho\|_h \lesssim kh^2\left\|f\right\|_0 \quad \text{and} \quad \|\rho\|_{0,\Gamma} \lesssim H^{-\frac{1}{2}}\|\rho\|_0 + \|\rho\|_0^{\frac{1}{2}}\|\nabla_H\rho\|_0^{\frac{1}{2}} \lesssim kh^{\frac{3}{2}}\left\|f\right\|_0. \tag{3.29}$$

The Young's inequality yields

$$\begin{aligned}\|\theta_h\|_{0,\Gamma}^2 &\lesssim k\|\rho\|_0\|\theta_h\|_0 + \|\rho\|_{0,\Gamma}^2 \lesssim k^2h^2\left\|f\right\|_0\|\theta_h\|_0 + k^2h^3\left\|f\right\|_0^2\\ &\lesssim k^2h\|\theta_h\|_0^2 + k^2h^3\left\|f\right\|_0^2.\end{aligned} \tag{3.30}$$

Step 2: Bounding $|||\theta_h|||_h$ by $\|\theta_h\|_0$. By letting $v_h = \theta_h$ in (3.28), taking the real part, and using (3.29)–(3.30), we conclude that

$$\begin{aligned}|||\theta_h|||_h^2 &\lesssim b_h(\theta_h,\theta_h) + k^2\|\theta_h\|_0^2 = 2k^2\|\theta_h\|_0^2 + \mathrm{Re}\big((k^2+1)(\rho,\theta_h) - \mathrm{i}k\langle\rho,\theta_h\rangle\big)\\ &\lesssim k^2\|\theta_h\|_0^2 + k^2\|\rho\|_0^2 + k\|\rho\|_{0,\Gamma}\|\theta_h\|_{0,\Gamma}\\ &\lesssim k^2\|\theta_h\|_0^2 + k^4h^4\left\|f\right\|_0^2 + k^2h^{\frac{3}{2}}\left\|f\right\|_0\big(kh^{\frac{1}{2}}\|\theta_h\|_0 + kh^{\frac{3}{2}}\left\|f\right\|_0\big).\end{aligned}$$

which implies by the Young's inequality that

$$|||\theta_h|||_h \lesssim k\|\theta_h\|_0 + (kh)^{\frac{3}{2}}\left\|f\right\|_0. \tag{3.31}$$

Step 3: Estimating $\|\theta_h\|_0$. We will use the modified duality argument [30, 12]. Consider the following adjoint problem:

$$\begin{aligned}-\triangle w - k^2 w &= \theta_h \ \text{ in } \ \Omega\\ \frac{\partial w}{\partial n} - \mathrm{i}kw &= 0 \ \text{ on } \ \partial\Omega\end{aligned}$$

Similar to Lemma 3.1, we have $\|w\|_2 \lesssim k\|\theta_h\|_0$. Testing the above problem by $e_h = \rho + \theta_h$ and using the Galerkin orthogonality, we get

$$\begin{aligned}&\|\theta_h\|_0^2 + (\rho,\theta_h) = a_h(e_h, w) = a_h(e_h, w - P_h w)\\ =&b_h(e_h, w - P_h w) + (e_h, w - P_h w) - (k^2+1)(e_h, w - P_h w) + \mathrm{i}k\,\langle e_h, w - P_h w\rangle\\ =&b_h(\rho, w - P_h w) + (\rho, w - P_h w) - (k^2+1)(\rho + \theta_h, w - P_h w) + \mathrm{i}k\big\langle\rho + \theta_h, w - P_h w\big\rangle.\end{aligned}$$

Noting that $|v_h|_{2,\mathscr{T}_h} = 0$ for any $v_h \in V_h$, from Lemma 3.3, (3.29)–(3.31) we conclude that

$$\begin{aligned}\|\theta_h\|_0^2 \lesssim & (\|\rho\|_h + h|u|_2)(\|w - P_h w\|_h + h|w|_2) + k^2(\|\rho\|_0 + \|\theta_h\|_0)\|w - P_h w\|_0 \\ & + k(\|\rho\|_{0,\Gamma} + \|\theta_h\|_{0,\Gamma})\|w - P_h w\|_{0,\Gamma} \\ \lesssim & kh\|f\|_0 kh\|\theta_h\|_0 + k^2(kh^2\|f\|_0 + \|\theta_h\|_0)kh^2\|\theta\|_0 \\ & + k(kh^{\frac{3}{2}}\|f\|_0 + kh^{\frac{1}{2}}\|\theta_h\|_0 + kh^{\frac{3}{2}}\|f\|_0)kh^{\frac{3}{2}}\|\theta\|_0,\end{aligned}$$

which implies that

$$\|\theta_h\|_0 \lesssim k^2h^2\|f\|_0 + k^3h^2\|\theta_h\|_0.$$

and hence the following estimate holds if k^3h^2 is small enough.

$$\|\theta_h\|_0 \lesssim k^2h^2\|f\|_0. \tag{3.32}$$

The proof of the theorem follows by combining (3.29), (3.31), (3.32), and the triangle inequality. □

Corollary 3.7. *Under the conditions of Theorem 3.6, there holds the following estimate*

$$\|\!|u_h|\!\|_h \lesssim \|f\|_0,$$

and hence the IPCDGM is well-posed.

Proof. The proof is a direct consequence of Theorem 3.6 and Lemma 3.1. □

Remark 3.8. (i) *The stability estimates of the IPCDG solution in L^2- and H^1-norms are the same as those of the exact solution in Lemma 3.1.*

(ii) *It can be shown that the IPCDGM is absolute stable if the imaginary part of the penalty parameters are positive (cf. [14, 15, 30, 29]). While for simplicity we only consider the real penalty parameters in this paper.*

3.3 Discrete inf-sup condition

The following inf-sup condition of $a_h(\cdot,\cdot)$ is useful in the analysis of the multiscale IPCDGM.

Lemma 3.9. *Under the conditions of Theorem 3.6, there holds*

$$\inf_{u_h \in V_h \setminus \{0\}} \sup_{v_h \in V_h \setminus \{0\}} \frac{\mathrm{Re}\, a_h(v_h, u_h)}{\|\!|v_h|\!\|_h \|\!|u_h|\!\|_h} \gtrsim \frac{1}{k}. \tag{3.33}$$

Proof. For any given $u_h \in V_h$, we define the following problem: find $w_h \in V_h$

$$a_h(w_h, v_h) = 2k^2(u_h, v_h), \quad \forall v_h \in V_h.$$

Corollary 3.7 implies that $\|\!|w_h|\!\|_h \lesssim k^2\|u_h\|_0 \lesssim k\|\!|u_h|\!\|_h$. Set $v_h = u_h + w_h$ to get

$$\begin{aligned}\mathrm{Re}\, a_h(v_h, u_h) &= \mathrm{Re}\, a_h(u_h, u_h) + \mathrm{Re}\, a_h(w_h, u_h) \\ &= \mathrm{Re}\, a_h(u_h, u_h) + 2k^2(u_h, u_h) \gtrsim \|\!|u_h|\!\|_h^2\end{aligned}$$

where we have used (3.26) to derive the last inequality. On the other hand $\|\!|v_h|\!\|_h \lesssim \|\!|u_h|\!\|_h + \|\!|w_h|\!\|_h \lesssim k\|\!|u_h|\!\|_h$, which together with the above estimate implies that (3.33) holds. This completes the proof of the lemma. □

4 Preasymptotic error analysis of MsIPDPGM

In this section, we first list some useful lemmas about L^2-projector and cut-off functions. Then we prove the exponential decay of the global corrector. In the last subsection, we present the preasymptotic error analysis of the MsIPDPGM.

4.1 Preliminary lemmas

Recall that $\tilde{V}_h$ is the space of piecewise linear functions on $\mathcal{T}_h$ (see Remark 3.4). The following lemma lists some useful estimates on two interpolation operators: the Oswald interpolation operator [26, 19] (denoted by I_h^c) and the Scott-Zhang interpolation operator [28, 5] (denoted by Π_h).

Lemma 4.1. *There exist two interpolation operators $I_h^c : \tilde{V}_h \to \tilde{V}_h \cap H^1(\Omega)$ and $\Pi_H : H^1(\Omega) \to V_H \cap H^1(\Omega)$ satisfying the following estimates:*

$$h^{-2}\left\|v_h - I_h^c v_h\right\|_{0,T}^2 + \left\|\nabla(v_h - I_h^c v_h)\right\|_{0,T}^2 \lesssim \sum_{e\in\mathcal{E}_h, e\cap T\neq\emptyset} h^{-1}\|[v_h]\|_{0,e}^2, \quad \forall v_h \in \tilde{V}_h, T\in\mathcal{T}_H, \tag{4.34}$$

$$\|v - \Pi_H v\|_0 \lesssim H\|\nabla v\|_0, \quad \forall v \in H^1(\Omega). \tag{4.35}$$

The following lemma gives some estimates of the L^2-projector Q_H.

Lemma 4.2. *The following estimates hold for any $v \in H^2(\Omega)$, $v_h \in V_h$ and $T \in \mathcal{T}_H$:*

$$\|v - Q_H v\|_0 \lesssim H^2|v|_2, \quad \|v_h - Q_H v_h\|_{0,T} \lesssim H\|\nabla v_h\|_{0,T}, \quad \textit{and} \quad \|Q_H v_h\|_H \lesssim \|v_h\|_h.$$

Proof. The first two estimates are well-known (see e.g. [5]).

Denote by $v_h^c = I_h^c v_h$. It follows from (4.34)–(4.35), the local trace inequality, the inverse inequality, and the stability estimates $|Q_H v_h|_{j,T} \lesssim |v_h|_{j,T}\,(j = 0,1)$ that

$$\begin{aligned}
\|Q_H v_h\|_H^2 &= \|\nabla_H Q_H v_h\|_0^2 + \sum_{E\in\mathcal{E}_H} \frac{1}{H}\|[Q_H v_h]\|_{0,E}^2 + \sum_{E\in\mathcal{E}_H} H\left\|\left[\frac{\partial Q_H v_h}{\partial n}\right]\right\|_{0,E}^2 \\
&\lesssim \|\nabla_H Q_H v_h\|_0^2 + \sum_{E\in\mathcal{E}_H} \frac{1}{H}\left\|[Q_H v_h - \Pi_H v_h^c]\right\|_{0,E}^2 \\
&\lesssim \|\nabla_H Q_H v_h\|_0^2 + \sum_{T\in\mathcal{T}_H} \frac{1}{H^2}\left\|Q_H v_h - \Pi_H v_h^c\right\|_{0,T}^2 \\
&\lesssim \|\nabla_H Q_H v_h\|_0^2 + \frac{1}{H^2}\left(\left\|Q_H(v_h - v_h^c)\right\|_0^2 + \left\|Q_H v_h^c - v_h^c\right\|_0^2 + \left\|v_h^c - \Pi_H v_h^c\right\|_0^2\right) \\
&\lesssim \|\nabla_H v_h\|_0^2 + \frac{1}{H^2}\left\|v_h - v_h^c\right\|_0^2 + \|\nabla_H v_h^c\|_0^2 \\
&\lesssim \|\nabla_H v_h\|_0^2 + \sum_{E\in\mathcal{E}_H} \frac{1}{h}\|[v_h]\|_{0,E}^2 \lesssim \|v_h\|_h^2 .
\end{aligned}$$

This completes the proof of the lemma. $\square$

From [13, Lemma 6] and the fact that $\sum_{e\in\mathcal{E}_h} h\big\|[\frac{\partial v_h}{\partial n}]\big\|_{0,e} \lesssim \sum_{t\in\mathcal{T}_h}\|\nabla v_h\|_{0,t}^2$, we have the following lemma which shows that given some $v_H \in V_H$ there exists a H^1-conforming preimage $b \in Q_H^{-1} v_H \subset V_h$ with comparable support. The proof is omitted.

Lemma 4.3. *For each $v_H \in V_H$, there exists $b \in V_h \cap H^1(\Omega)$, such that $Q_H b = v_H$, $\mathrm{supp}(b) \subseteq \mathrm{supp}(I_H^c v_H)$ and*

$$\|b\|_h \lesssim \|v_H\|_H. \tag{4.36}$$

Remark 4.4. *Note that I_H^c may be appropriately defined such that $\mathrm{supp}(b) \subseteq \omega_{T,L}$, $\overline{\Omega\setminus\omega_{T,L}}$, or $\overline{\omega_{T,L}\setminus\omega_{T,L-1}}$, if so is $\mathrm{supp}\, v_H$.*

For any $T \in \mathcal{T}_H$, let η_T be a cut-off function of piecewise constant on $\mathcal{T}_h$ with values in the interval $[0,1]$, which satisfies

$$\eta_T = \begin{cases} 1 & \text{in } \omega_{T,L-1}, \\ 0 & \text{in } \Omega\setminus\omega_{T,L}, \end{cases} \quad \text{and} \quad \big\|[\eta_T]\big\|_{L^\infty(e)} \begin{cases} \lesssim \frac{h}{H}, & \text{if } e\in\mathcal{E}_h, e\subset\big(\omega_{T,L}\setminus\omega_{T,L-1}\big)^\circ, \\ = 0, & \text{otherwise.} \end{cases} \tag{4.37}$$

Here $^\circ$ denotes the interior of a point set. Note that for any $v_h \in V^f$, $\eta_T v_h$ may not be in V_h since it may be discontinuous in the macro-elements contained in $\omega_{T,L}\setminus\omega_{T,L-1}$. The following lemma says that there exists a proper approximation of $\eta_T v_h$ in V_h.

Lemma 4.5. *For any $v_h \in V^f$, there exists $I_{h,T}^c(\eta_T v_h) \in V_h$ satisfying the following estimates.*

$$h^{-1}\big\|\eta_T v_h - I_{h,T}^c(\eta_T v_h)\big\|_0 + \big\|\eta_T v_h - I_{h,T}^c(\eta_T v_h)\big\|_h \lesssim \|v_h\|_{h,\,\omega_{T,L}\setminus\omega_{T,L-1}}, \tag{4.38}$$

$$H^{-1}\big\|v_h - I_{h,T}^c(\eta_T v_h)\big\|_0 + \big\|v_h - I_{h,T}^c(\eta_T v_h)\big\|_h \lesssim \|v_h\|_{h,\,\Omega\setminus\omega_{T,L-1}}. \tag{4.39}$$

Here $\|v\|_{h,G} = \Big(\sum_{t\in\mathcal{T}_h, t\subset\bar G}\|\nabla v\|_{0,t}^2 + \sum_{E\in\mathcal{E}_H, E\subset G}\frac{1}{h}\|[v]\|_{0,E}^2 + \sum_{e\in\mathcal{E}_h, e\subset G} h\big\|[\frac{\partial v}{\partial n}]\big\|_{0,e}^2\Big)^{\frac12}$ for $G\subset\Omega$ and $\bar G$ is the closure of G.

Proof. Define

$$I_{h,T}^c(\eta_T v_h) = \begin{cases} I_h^c(\eta_T v_h), & \text{in } \omega_{T,L}\setminus\omega_{T,L-1}, \\ \eta_T v_h, & \text{otherwise,} \end{cases} \tag{4.40}$$

where I_h^c is the Oswald interpolation operator satisfying (4.34). Clearly $I_{h,T}^c(\eta_T v_h) \in V_h$ and there holds

$$\big\|\eta_T v_h - I_{h,T}^c(\eta_T v_h)\big\|_0^2 \lesssim \sum_{\substack{e\in\mathcal{E}_h \\ e\subset(\omega_{T,L}\setminus\omega_{T,L-1})^\circ}} h\big\|[\eta_T v_h]\big\|_{0,e}^2. \tag{4.41}$$

Since $v_h = v_h - Q_H v_h$, from Lemma 4.2 we have

$$\|v_h\|_{0,T} \lesssim H\|\nabla v_h\|_{0,T}. \tag{4.42}$$

Therefore, from the local trace and inverse inequalities, (4.41), the magic formula, (4.37), and (4.42), we conclude that

$$
\begin{aligned}
\left\|\eta_T v_h - I^c_{h,T}(\eta_T v_h)\right\|_h^2 &\lesssim h^{-2}\left\|\eta_T v_h - I^c_{h,T}(\eta_T v_h)\right\|_0^2 \\
&\lesssim \sum_{\substack{e\in\mathcal{E}_h \\ e\subset(\omega_{T,L}\backslash\omega_{T,L-1})^\circ}} h^{-1}\left\|\{\eta_T\}[v_h]+[\eta_T]\{v_h\}\right\|_{0,e}^2 \\
&\lesssim \sum_{\substack{e\in\mathcal{E}_h \\ e\subset(\omega_{T,L}\backslash\omega_{T,L-1})^\circ}} h^{-1}\left\|[v_h]\right\|_{0,e}^2 + \sum_{\substack{t\in\mathcal{T}_h \\ t\subset\overline{\omega_{T,L}\backslash\omega_{T,L-1}}}} h^{-1}\frac{h^2}{H^2}h^{-1}\left\|v_h\right\|_{0,t}^2 \\
&\lesssim \sum_{\substack{e\in\mathcal{E}_h \\ e\subset(\omega_{T,L}\backslash\omega_{T,L-1})^\circ}} h^{-1}\left\|[v_h]\right\|_{0,e}^2 + \sum_{\substack{T\in\mathcal{T}_H \\ T\subset\overline{\omega_{T,L}\backslash\omega_{T,L-1}}}} \left\|\nabla v_h\right\|_{0,T}^2
\end{aligned}
$$

which together with (4.41) implies that (4.38) holds.

It remains to prove (4.39). Denote by $w_h = v_h - \eta_T v_h$. Noting that $\eta_T = 1$ and $[\eta_T] = 0$ in $\omega_{T,L-1}$, similarly as above, we have

$$
\begin{aligned}
\|w_h\|_h^2 =&\|\nabla_h w_h\|_{0,\Omega\backslash\omega_{T,L-1}}^2 + \sum_{\substack{E\in\mathcal{E}_H \\ E\subset\Omega\backslash\omega_{T,L-1}}} \frac{1}{h}\|[w_h]\|_{0,E}^2 + \sum_{\substack{e\in\mathcal{E}_h \\ e\subset\Omega\backslash\omega_{T,L-1}}} h\left\|\left[\frac{\partial w_h}{\partial n}\right]\right\|_{0,e}^2 \\
\lesssim&\|\nabla_h w_h\|_{0,\Omega\backslash\omega_{T,L-1}}^2 + \sum_{\substack{E\in\mathcal{E}_H \\ E\subset\Omega\backslash\omega_{T,L-1}}} \frac{1}{h}\|[(1-\eta_T)v_h]\|_{0,E}^2 \\
\lesssim&\|(1-\eta_T)\nabla_h v_h\|_{0,\Omega\backslash\omega_{T,L-1}}^2 + \sum_{\substack{e\in\mathcal{E}_h \\ e\subset\Omega\backslash\omega_{T,L-1}}} h^{-1}\left\|[v_h]\right\|_{0,e}^2 + \left\|\nabla_h v_h\right\|_{0,\Omega\backslash\omega_{T,L-1}}^2 \\
\lesssim&\|v_h\|_{h,\Omega\backslash\omega_{T,L-1}}.
\end{aligned}
$$

On the other hand, from Lemma 4.2, we have

$$
\|w_h\|_0 \lesssim \|v_h\|_{0,\Omega\backslash\omega_{T,L-1}} \lesssim H\|\nabla_h v_h\|_{0,\Omega\backslash\omega_{T,L-1}} \lesssim H\|v_h\|_{h,\Omega\backslash\omega_{T,L-1}}
$$

Then (4.39) follows by combining the above two estimates and (4.38) and using the triangle inequality. This completes the proof of the lemma. □

4.2 Exponential closeness between the localized and global correctors

In this subsection, the main task is to prove that the localized corrector $E_{T,L}\lambda_T^j$ is exponentially close to the global corrector $E_\infty\lambda_T^j$.

Lemma 4.6.

(i) *The norms* $\|\cdot\|_h$ *and* $|||\cdot|||_h$ *are equivalent on the space* V^f *if* $kH \lesssim 1$.

(ii) *Under the conditions of Lemma 3.3, there exist a positive constant* C_1*, such that, if* $kH \le C_1$ *then* $\mathrm{Re}\, a_h(v_h, v_h) \gtrsim \|v_h\|_h^2$, $\forall v_h \in V^f$.

(iii) *Suppose* $kh \lesssim 1$. *Then* $|a_h(u_h, v_h)| \lesssim |||u_h|||_h\, |||v_h|||_h$, $\forall u_h, v_h \in V_h$.

Proof. Obviously $\|v_h\|_h \lesssim |||v_h|||_h$. Using Lemma 4.2 and $Q_H v_h = 0$,we have

$$\begin{aligned}|||v_h|||_h^2 = \|v_h\|_h^2 + k^2\|v_h\|_0^2 = \|v_h\|_h^2 + k^2\|v_h - Q_H v_h\|_0^2\\ \lesssim \|v_h\|_h^2 + (Hk)^2\|\nabla_h v_h\|_0^2 \lesssim \|v_h\|_h^2\end{aligned}$$

which implies that (i) holds.

From (2.3), (3.26), and Lemma 4.2, we have

$$\mathrm{Re} a_h(v_h, v_h) = b_h(v_h, v_h) - k^2\|v_h\|_0^2 \gtrsim \|v_h\|_h^2 - C(kH)^2\|\nabla_h v_h\|_0^2$$

which implies that (ii) holds.

For $v_h \in V_h$, let $v_h^c = I_h^c v_h \in V_h \cap H^1(\Omega)$. It follows from the trace inequality, the local trace inequality, the inverse inequality, and (4.34) that

$$\begin{aligned}k\|v_h\|_{0,\Gamma}^2 &\lesssim k\left\|v_h^c\right\|_{0,\Gamma}^2 + k\left\|v_h - v_h^c\right\|_{0,\Gamma}^2\\ &\lesssim k\left\|v_h^c\right\|_0\left\|v_h^c\right\|_1 + kh^{-1}\left\|v_h - v_h^c\right\|_0^2\\ &\lesssim \left\|\nabla v_h^c\right\|_0^2 + k^2\left\|v_h^c\right\|_0^2 + kh^{-1}\left\|v_h - v_h^c\right\|_0^2\\ &\lesssim \|\nabla_h v_h\|_0^2 + k^2\|v_h\|_0^2 + \left\|\nabla_h(v_h^c - v_h)\right\|_0^2 + \left(k^2 + kh^{-1}\right)\left\|v_h - v_h^c\right\|_0^2\\ &\lesssim \|\nabla_h v_h\|_0^2 + k^2\|v_h\|_0^2 + \left(1 + k^2h^2 + kh\right)\sum_{e\in\mathcal{E}_h} h^{-1}\|[v_h]\|_{0,e}^2 \lesssim |||v_h|||_h,\end{aligned}$$

which implies that $k\|u_h\|_{0,\Gamma}\|v_h\|_{0,\Gamma} \lesssim |||u_h|||_h\, |||v_h|||_h$ and hence (iii) follows from (2.3) and (3.25). This completes the proof of the lemma. □

Remark 4.7. *From the Lax-Milgram lemma, the corrector problems (2.12)–(2.15) are well-posed under the conditions of Lemma 4.6.*

Similar to [13, Lemma 8] for coercive elliptic problems, we have the following stability result for the corrected basis functions for the highly indefinite Helmholtz problem.

Lemma 4.8. *Under the conditions of Lemma 4.6, there hold*

$$\left|\left|\left|v_H - E_\infty v_H\right|\right|\right|_h \lesssim \left|\left|\left|v_H\right|\right|\right|_H \quad \forall v_H \in V_H \quad \textit{and} \quad \left|\left|\left|\lambda_T^j - E_{T,L}\lambda_T^j\right|\right|\right|_h \lesssim \left|\left|\left|\lambda_T^j\right|\right|\right|_H. \tag{4.43}$$

Proof. We first prove the first inequality. From Lemma 4.3, for any $v_H \in V_H$, there exists $b \in V_h \cap H^1(\Omega)$ such that $Q_H b = v_H$ and $\|b\|_h \lesssim \left\|v_H\right\|_H$. Noting that $v_H - E_\infty v_H - b \in V^f$, from Lemma 4.6 and (2.18), we have

$$\begin{aligned}\left|\left|\left|v_H - E_\infty v_H - b\right|\right|\right|_h^2 &\lesssim a_h\left(v_H - E_\infty v_H - b, v_H - E_\infty v_H - b\right) = a_h\left(-b, v_H - E_\infty v_H - b\right)\\ &\lesssim |||b|||_h\left|\left|\left|v_H - E_\infty v_H - b\right|\right|\right|_h.\end{aligned}$$

Thus $\left|\left|\left|v_H - E_\infty v_H - b\right|\right|\right|_h \lesssim |||b|||_h$ and hence

$$\left|\left|\left|v_H - E_\infty v_H\right|\right|\right|_h \lesssim |||b|||_h.$$

On the other hand, from Lemma 4.2,

$$\begin{aligned}\|\!|b|\!\|_h^2 &= \|b\|_h^2 + k^2\|b\|_0^2 \lesssim \|v_H\|_H^2 + k^2\|v_H\|_0^2 + k^2\|b-v_H\|_0^2\\ &\lesssim \|v_H\|_H^2 + k^2\|v_H\|_0^2 + k^2H^2\|\nabla b\|_0^2\\ &\lesssim \|\!|v_H|\!\|_H^2.\end{aligned}$$

The proof of the first inequality in (4.43) follows by combining the above two estimates.

Note that $\operatorname{supp}(b) \subseteq \operatorname{supp}\big(I_H^c\lambda_T^j\big) = \omega_{T,1}$ and hence $\lambda_T^j - E_\infty\lambda_T^j - b \in V_{T,L}^f$. The proof of the second inequality in (4.43) can be proved by following the lines for the first one. We omit the details. □

The following theorem says that the localized corrector $E_{T,L}\lambda_T^j$ is exponentially close to the global corrector $E_\infty\lambda_T^j$, which is the main result of this subsection.

Theorem 4.9. *Under the conditions of Lemma 4.6, there exists a positive constant C_2 such that if $kH < C_2$ then*

$$\big\|E_\infty\lambda_T^j - E_{T,L}\lambda_T^j\big\|_h \lesssim \beta^L \|\!|\lambda_T^j|\!\|_H, \quad \forall T \in \mathcal{T}_H, j = 1\cdots r, \tag{4.44}$$

where $\beta := \Big(\frac{(kH)^2+1}{C_2^2+1}\Big)^{\frac14} < 1$ and E_∞ and $E_{T,L}$ are defined in (2.12) and (2.14), respectively.

Proof. First, from Lemma 4.8 we have

$$\|\!|\, E_\infty\lambda_T^j - E_{T,L}\lambda_T^j|\!\|_h \lesssim \|\!|\lambda_T^j|\!\|_H. \tag{4.45}$$

Therefore, it suffices to prove (4.44) for sufficiently large L. Without lose of generality, we suppose $L > 2$ in the rest of the proof which is divided into two steps.

Step 1: Denote by $\phi_\infty = E_\infty\lambda_T^j$, $\phi_L = E_{T,L}\lambda_T^j$, $\tau = \phi_\infty - \phi_L \in V^f$. From (2.12), (2.14), and Lemma 4.6, we have

$$\|\tau\|_h^2 \lesssim \operatorname{Re} a_h(\tau,\tau) = \operatorname{Re} a_h(\tau, \phi_\infty - \psi_L) \lesssim \|\tau\|_h\|\phi_\infty - \psi_L\|_h, \quad \forall \psi_L \in V_{T,L}^f.$$

From Lemma 4.3, for $Q_H\big(I_{h,T}^c(\eta_T\phi_\infty)\big)$ with η_T in (4.37) and $I_{h,T}^c$ in (4.40), there exists a $b \in V_h \cap H^1(\Omega)$ which satisfies

$$\|b\|_h \lesssim \big\|Q_H\big(I_{h,T}^c(\eta_T\phi_\infty)\big)\big\|_H \quad\text{and}\quad Q_H\big(b - I_{h,T}^c(\eta_T\phi_\infty)\big) = 0.$$

Note that $\operatorname{supp}(\eta_T\phi_\infty) \subseteq \omega_{T,L}$, so is $\operatorname{supp}\big(I_{h,T}^c(\eta_T\phi_\infty)\big)$ and $\operatorname{supp}(b)$. Setting $\psi_L := I_{h,T}^c(\eta_T\phi_\infty) - b \in V_{T,L}^f$, by using Lemmas 4.2 and 4.5, we obtain

$$\begin{aligned}\|\tau\|_h &\lesssim \big\|\phi_\infty - I_{h,T}^c(\eta_T\phi_\infty)\big\|_h + \|b\|_h\\ &\lesssim \big\|\phi_\infty - I_{h,T}^c(\eta_T\phi_\infty)\big\|_h + \big\|Q_H\big(I_{h,T}^c(\eta_T\phi_\infty)\big)\big\|_H\\ &= \big\|\phi_\infty - I_{h,T}^c(\eta_T\phi_\infty)\big\|_h + \big\|Q_H\big(I_{h,T}^c(\eta_T\phi_\infty) - \phi_\infty\big)\big\|_H\\ &\lesssim \big\|\phi_\infty - I_{h,T}^c(\eta_T\phi_\infty)\big\|_h\\ &\lesssim \|\phi_\infty\|_{h,\Omega\setminus\omega_{T,L-1}}.\end{aligned} \tag{4.46}$$

Step 2: We introduce the following piecewise constant cut-off function on $\mathcal{T}_h$:

$$\xi_T = \begin{cases} 0 & \text{in } \omega_{T,L-2}, \\ 1 & \text{in } \Omega\setminus\omega_{T,L-1}, \end{cases} \quad \text{and} \quad \|[\xi_T]\|_{L^\infty(e)} \begin{cases} \lesssim \frac{h}{H}, & \text{if} e\in\mathcal{E}_h, e\subset\left(\omega_{T,L-1}\setminus\omega_{T,L-2}\right)^\circ, \\ =0, & \text{otherwise.} \end{cases} \tag{4.47}$$

Therefore, from Remark 3.4, the identity $\{vw\}=\{v\}\{w\}+\frac{1}{4}[v][w]$, and the magic formula, we have

$$\begin{aligned}
\|\phi_\infty\|^2_{h,\Omega\setminus\omega_{T,L-1}} \le \|\xi_T\phi_\infty\|^2_h \lesssim \mathrm{Re}a_h(\xi_T\phi_\infty,\xi_T\phi_\infty)+k^2\left(\xi_T\phi_\infty,\xi_T\phi_\infty\right)& \\
=\Big(\mathrm{Re}a_h(\phi_\infty,\xi_T^2\phi_\infty)+k^2\|\xi_T\phi_\infty\|_0^2\Big)& \\
+\frac{1}{2}\mathrm{Re}\sum_{E\in\mathcal{E}_H}\left(\left(\left\{\frac{\partial\phi_\infty}{\partial n}\right\},[\xi_T]^2[\phi_\infty]\right)_E-\left(\left[\frac{\partial\phi_\infty}{\partial n}\right],[\xi_T]^2\{\phi_\infty\}\right)_E\right)& \\
+\sum_{E\in\mathcal{E}_H}\frac{\gamma_0}{h}\left(\|[\xi_T]\{\phi_\infty\}\|^2_{0,E}-\frac{1}{4}\|[\xi_T][\phi_\infty]\|^2_{0,E}\right)& \\
+\sum_{e\in\mathcal{E}_h}\gamma_1 h\left(\left\|[\xi_T]\left\{\frac{\partial\phi_\infty}{\partial n}\right\}\right\|^2_{0,e}-\frac{1}{4}\left\|[\xi_T]\left[\frac{\partial\phi_\infty}{\partial n}\right]\right\|^2_{0,e}\right)& \\
:=I+II+III+IV.&
\end{aligned} \tag{4.48}$$

Next we estimate the each term in (4.48). First we estimate I. Similar to Lemma 4.5, there exists $I^c_{h,T}(\xi_T^2\phi_\infty)\in V_h$ defined similarly as in (4.40) but with L replaced by $L-1$ and $\eta_T v_h$ replaced by $\xi_T^2\phi_\infty$, which satisfies the following estimates:

$$h^{-1}\left\|\xi_T^2\phi_\infty-I^c_{h,T}(\xi_T^2\phi_\infty)\right\|_0+\left\|\xi_T^2\phi_\infty-I^c_{h,T}(\xi_T^2\phi_\infty)\right\|_h\lesssim\|\phi_\infty\|_{h,\,\omega_{T,L-1}\setminus\omega_{T,L-2}}, \tag{4.49}$$

$$H^{-1}\left\|\phi_\infty-I^c_{h,T}(\xi_T^2\phi_\infty)\right\|_{0,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}+\left\|\phi_\infty-I^c_{h,T}(\xi_T^2\phi_\infty)\right\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}\lesssim\|\phi_\infty\|_{h,\,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}. \tag{4.50}$$

Moreover, from Lemma 4.3, there exists a $b\in V_h\cap H^1(\Omega)$ satisfying

$$Q_H b=Q_H\big(I^c_{h,T}(\xi_T^2\phi_\infty)\big),\quad\text{and}\quad\|b\|_h\lesssim\left\|Q_H\big(I^c_{h,T}(\xi_T^2\phi_\infty)\big)\right\|_H. \tag{4.51}$$

We write

$$\begin{aligned}
I=&\mathrm{Re}a_h\big(\phi_\infty,\xi_T^2\phi_\infty-I^c_{h,T}(\xi_T^2\phi_\infty)\big)+\mathrm{Re}a_h\big(\phi_\infty,I^c_{h,T}(\xi_T^2\phi_\infty)-b\big)\\
&+\mathrm{Re}a_h\big(\phi_\infty,b\big)+k^2\|\xi_T\phi_\infty\|_0^2\\
:=&I_1+I_2+I_3+I_4,
\end{aligned}$$

Noting that $\xi_T^2\phi_\infty-I^c_{h,T}(\xi_T^2\phi_\infty)$ is supported in $\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}$, from (2.3), Remark 3.4, we have

$$I_1\lesssim\|\!|\phi_\infty|\!\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}\|\!|\xi_T^2\phi_\infty-I^c_{h,T}(\xi_T^2\phi_\infty)|\!\|_h+k\|\phi_\infty\|_{0,\Gamma\cap\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}\|\xi_T^2\phi_\infty-I^c_{h,T}(\xi_T^2\phi_\infty)\|_{0,\Gamma}.$$

In case of $\Gamma\cap\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}\neq\emptyset$, it follows from the local trace inequality and $\phi_\infty\in V^f$ that

$$\begin{aligned}
\|\phi_\infty\|_{0,\Gamma\cap\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}&\lesssim H^{-\frac{1}{2}}\|\phi_\infty\|_{0,\omega_{T,L-1}\setminus\omega_{T,L-2}}+H^{\frac{1}{2}}\|\nabla_h\phi_\infty\|_{0,\omega_{T,L-1}\setminus\omega_{T,L-2}}\\
&\lesssim H^{\frac{1}{2}}\|\nabla_h\phi_\infty\|_{0,\omega_{T,L-1}\setminus\omega_{T,L-2}}.
\end{aligned}$$

From (4.49) we conclude that

$$
\begin{aligned}
I_1 \lesssim & \|\phi_\infty\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}} \|\xi_T^2\phi_\infty - I_{h,T}^c(\xi_T^2\phi_\infty)\|_h \\
& + kH^{\frac{1}{2}}h^{-\frac{1}{2}}\|\nabla_h\phi_\infty\|_{0,\omega_{T,L-1}\setminus\omega_{T,L-2}}\|\xi_T^2\phi_\infty - I_{h,T}^c(\xi_T^2\phi_\infty)\|_0 \\
\lesssim & \|\phi_\infty\|^2_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}} + k(Hh)^{\frac{1}{2}}\|\phi_\infty\|^2_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}} \\
\lesssim & \|\phi_\infty\|^2_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}.
\end{aligned}
$$

Since $Q_H\big(I_{h,T}^c(\xi_T^2\phi_\infty)\big)\big|_{\Omega\setminus\omega_{T,L-1}} = Q_H\phi_\infty\big|_{\Omega\setminus\omega_{T,L-1}} = 0$, we have $\mathrm{supp}\big(Q_H\big(I_{h,T}^c(\xi_T^2\phi_\infty)\big)\big) \subseteq \omega_{T,L-1}\setminus\omega_{T,L-2}$, so is $\mathrm{supp}(b)$ (see Remark 4.4). Therefore $I_{h,T}^c(\xi_T^2\phi_\infty) - b \in V^f$ vanishes in $\omega_{T,L-2}$. It follows from (2.12) that

$$
I_2 = a_h\big(\lambda_T^j, I_{h,T}^c(\xi_T^2\phi_\infty) - b\big) = 0.
$$

Moreover, from (3.25), the trace inequalities, Lemma 4.2, (4.51), we conclude that

$$
\begin{aligned}
I_3 \lesssim & \|\phi_\infty\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}\|b\|_h + k\big|\langle\phi_\infty, b\rangle\big| \\
\lesssim & \|\phi_\infty\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}\|b\|_h + kH^{\frac{1}{2}}\|\nabla_h\phi_\infty\|_{0,\omega_{T,L-1}\setminus\omega_{T,L-2}}\|b\|_0^{\frac{1}{2}}\|b\|_1^{\frac{1}{2}} \\
\lesssim & \|\phi_\infty\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}\|b\|_h + (kH)^{\frac{1}{2}}\|\nabla_h\phi_\infty\|_{0,\omega_{T,L-1}\setminus\omega_{T,L-2}}\big(k\|b\|_0 + \|b\|_1\big)^{\frac{1}{2}} \\
\lesssim & \|\phi_\infty\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}\big(\|b\|_h + k\|b - Q_Hb\|_0 + k\|Q_Hb\|_0\big) \\
\lesssim & \|\phi_\infty\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}\big(\|b\|_h + k\|Q_Hb\|_0\big) \\
\lesssim & \|\phi_\infty\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}\big\|Q_H\big(I_{h,T}^c(\xi_T^2\phi_\infty)\big)\big\|_{H,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}} \\
= & \|\phi_\infty\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}\big\|Q_H\big(I_{h,T}^c(\xi_T^2\phi_\infty) - \phi_\infty\big)\big\|_{H,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}} \\
\lesssim & \|\phi_\infty\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}\big\|I_{h,T}^c(\xi_T^2\phi_\infty) - \phi_\infty\big\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}} \\
\lesssim & \|\phi_\infty\|^2_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}.
\end{aligned}
$$

It is clear that

$$
I_4 \lesssim k^2\|\xi_T\phi_\infty\|^2_{0,\Omega\setminus\omega_{T,L-2}} \lesssim (kH)^2\|\phi_\infty\|^2_{h,\Omega\setminus\omega_{T,L-2}}.
$$

By combing the above estimates for I_1-I_4, we obtain

$$
I \lesssim (kH)^2\|\phi_\infty\|^2_{h,\Omega\setminus\omega_{T,L-2}} + \|\phi_\infty\|_{h,\overline{\omega_{T,L-1}\setminus\omega_{T,L-2}}}. \tag{4.52}
$$

Then we estimate the second term II. Clearly, $[\xi_T]$ vanishes at e if $e \not\subseteq \big(\omega_{T,L-1}\setminus\omega_{T,L-2}\big)^\circ$. It follows from the local trace inequality, the inverse inequality, and Lemma 4.2 that

$$
\begin{aligned}
II &\lesssim \left(\frac{h}{H}\right)^2 \sum_{E\in(\omega_{T,L-1}\backslash\omega_{T,L-2})^\circ} \left(\left\| \left\{ \frac{\partial \phi_\infty}{\partial n} \right\} \right\|_{0,E} \| [\phi_\infty] \|_{0,E} + \left\| \left[\frac{\partial \phi_\infty}{\partial n} \right] \right\|_{0,E} \| \{\phi_\infty\} \|_{0,E} \right) \\
&\lesssim \frac{h^{\frac{5}{2}}}{H^2} \left(\sum_{t\subset\overline{\omega_{T,L-1}\backslash\omega_{T,L-2}}} h^{-1} \|\nabla\phi_\infty\|_{0,t}^2 \right)^{\frac12} \left(\sum_{E\in(\omega_{T,L-1}\backslash\omega_{T,L-2})^\circ} \frac{1}{h} \|[\phi_\infty]\|_{0,E}^2 \right)^{\frac12} \\
&\quad + \frac{h^{\frac{3}{2}}}{H^2} \left(\sum_{E\in(\omega_{T,L-1}\backslash\omega_{T,L-2})^\circ} h \left\| \left[\frac{\partial \phi_\infty}{\partial n} \right] \right\|_{0,E}^2 \right)^{\frac12} \left(\sum_{t\subset\overline{\omega_{T,L-1}\backslash\omega_{T,L-2}}} h^{-1} \|\phi_\infty\|_{0,t}^2 \right)^{\frac12} \\
&\lesssim \frac{h}{H} \|\phi_\infty\|_{h,\omega_{T,L-1}\backslash\omega_{T,L-2}}^2 .
\end{aligned} \tag{4.53}
$$

Similarly,

$$
\begin{aligned}
III &\lesssim \left(\frac{h}{H}\right)^2 \sum_{E\in(\omega_{T,L-1}\backslash\omega_{T,L-2})^\circ} \frac{1}{h} \left(\| \{\phi_\infty\} \|_{0,E}^2 + \| [\phi_\infty] \|_{0,E}^2 \right) \\
&\lesssim \frac{1}{H^2} \|\phi_\infty\|_{0,\omega_{T,L-1}\backslash\omega_{T,L-2}}^2 \lesssim \|\phi_\infty\|_{h,\omega_{T,L-1}\backslash\omega_{T,L-2}}^2 ,
\end{aligned} \tag{4.54}
$$

and

$$
\begin{aligned}
IV &\lesssim \left(\frac{h}{H}\right)^2 \sum_{e\in(\omega_{T,L-1}\backslash\omega_{T,L-2})^\circ} h \left(\left\| \left\{ \frac{\partial \phi_\infty}{\partial n} \right\} \right\|_e^2 + \left\| \left[\frac{\partial \phi_\infty}{\partial n} \right] \right\|_e^2 \right) \\
&\lesssim \left(\frac{h}{H}\right)^2 \|\nabla_h \phi_\infty\|_{0,\omega_{T,L-1}\backslash\omega_{T,L-2}}^2 \lesssim \left(\frac{h}{H}\right)^2 \|\phi_\infty\|_{h,\omega_{T,L-1}\backslash\omega_{T,L-2}}^2 .
\end{aligned} \tag{4.55}
$$

By plugging the estimates (4.52)–(4.55) into (4.48) we obtain

$$
\begin{aligned}
C_2^2 \|\phi_\infty\|_{h,\Omega\backslash\omega_{T,L-1}}^2 &\le (kH)^2 \|\phi_\infty\|_{h,\Omega\backslash\omega_{T,L-2}}^2 + \|\phi_\infty\|_{h,\overline{\omega_{T,L-1}\backslash\omega_{T,L-2}}} \\
&\le (kH)^2 \|\phi_\infty\|_{h,\Omega\backslash\omega_{T,L-3}}^2 + \left(\|\phi_\infty\|_{h,\Omega\backslash\omega_{T,L-3}}^2 - \|\phi_\infty\|_{h,\Omega\backslash\omega_{T,L-1}}^2 \right)
\end{aligned}
$$

for some constant C_2 independent of $k, h, H,$ and f. As a consequence,

$$
\|\phi_\infty\|_{h,\Omega\backslash\omega_{T,L-1}}^2 \le \beta^4 \|\phi_\infty\|_{h,\Omega\backslash\omega_{T,L-3}}^2
$$

where $\beta := \left(\frac{(kH)^2+1}{C_2^2+1}\right)^{\frac14} < 1$ if $KH < C_2$. Recursively, we have

$$
\begin{aligned}
\|\phi_\infty\|_{h,\Omega\backslash\omega_{T,L-1}} \le \beta^2 \|\phi_\infty\|_{h,\Omega\backslash\omega_{T,L-3}} \le \cdots \le &\begin{cases} \beta^{L-2} \|\phi_\infty\|_{h,\Omega\backslash\omega_{T,1}} & \text{if } L \text{ even} \\ \beta^{L-1} \|\phi_\infty\|_{h,\Omega\backslash\omega_{T,0}} & \text{if } L \text{ odd} \end{cases} \\
\lesssim \beta^L \|\phi_\infty\|_{h,\Omega\backslash\omega_{T,0}} = \beta^L \|\phi_\infty - \lambda_T^j\|_{h,\Omega\backslash\omega_{T,0}} &\lesssim \beta^L \|\!|\lambda_T^j|\!\|_H,
\end{aligned}
$$

where we have used Lemma 4.8 to prove the last inequality. The proof of the theorem follows by combining the above estimate and (4.46). $\square$

Theorem 4.10. *Under the conditions of Theorem 4.9, there holds*

$$\begin{aligned}&\big\|(E_\infty - E_L)v_H\big\|_h \lesssim \beta^L (kH)^{-1} |||v_H|||_H,\\&\big\|(E^*_\infty - E^*_L)v_H\big\|_h \lesssim \beta^L (kH)^{-1} |||v_H|||_H, \quad \forall v_H \in V_H.\end{aligned}$$

Proof. From (2.19), it suffice to prove the first inequality. Denote by

$$Z = (E_\infty - E_L)v_H = \sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} v_H(x_T^j)(E_\infty - E_{T,L})\lambda_T^j \in V^f.$$

Let ζ_T be a piecewise constant cut-off function defined as ξ_T in (4.47) but with L replaced by $L+2$. That is

$$\zeta_T = \begin{cases} 0 & \text{in } \omega_{T,L},\\ 1 & \text{in } \Omega\setminus\omega_{T,L+1},\end{cases} \quad \text{and} \quad \|[\zeta_T]\|_{L^\infty(e)} \begin{cases} \lesssim \frac{h}{H}, & \text{if} e\in\mathcal{E}_h, e\subset(\omega_{T,L+1}\setminus\omega_{T,L})^\circ,\\ =0, & \text{otherwise}.\end{cases} \tag{4.56}$$

Similar to Lemma 4.5, there exists $I^c_{h,T}(\zeta_T Z)\in V_h$ satisfying the following estimate:

$$H^{-1}\big\|Z - I^c_{h,T}(\zeta_T Z)\big\|_0 + \big\|Z - I^c_{h,T}(\zeta_T Z)\big\|_h \lesssim \|Z\|_{h,\,\omega_{T,L+1}}. \tag{4.57}$$

Lemma 4.3 implies that there is a function $b\in V_h\cap H^1(\Omega)$ which satisfies that $Q_H b = Q_H\big(I^c_{h,T}(\zeta_T Z)\big)$ and $\|b\|_h \lesssim \big\|Q_H\big(I^c_{h,T}(\zeta_T Z)\big)\big\|_H = \big\|Q_H\big(I^c_{h,T}(\zeta_T Z) - Z\big)\big\|_H$. Since $I^c_{h,T}(\zeta_T Z) - b\in V^f$ and $\operatorname{supp}\big(I^c_{h,T}(\zeta_T Z) - b\big)\subseteq\Omega\setminus\omega_{T,L}$, we have

$$a_h\big((E_\infty - E_{T,L})\lambda_T^j, I^c_{h,T}(\zeta_T Z) - b\big) = 0.$$

Therefore, from Lemmas 4.2 and 4.6, (4.57), the fact that $\operatorname{supp}\big(Q_H(I^c_{h,T}(\zeta_T Z))\big)\subseteq\overline{\omega_{T,L+1}\setminus\omega_{T,L}}$, and $|||b|||_h \lesssim \|b\|_h + k\|Q_H b\|_0 + k\|b - Q_H b\|_0 \lesssim \|b\|_h + k\|Q_H b\|_0 \lesssim \big|\big|\big|Q_H\big(I^c_{h,T}(\zeta_T Z) - Z\big)\big|\big|\big|_H$, we conclude that

$$\begin{aligned}\|Z\|_h^2 \lesssim &\operatorname{Re} a_h(Z,Z) = \operatorname{Re}\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} v_H(x_T^j)\, a_h\big((E_\infty - E_{T,L})\lambda_T^j, Z\big)\\
=&\operatorname{Re}\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} v_H(x_T^j)\, a_h\big((E_\infty - E_{T,L})\lambda_T^j, Z - I^c_{h,T}(\zeta_T Z)\big)\\
&+\operatorname{Re}\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} v_H(x_T^j)\, a_h\big((E_\infty - E_{T,L})\lambda_T^j, I^c_{h,T}(\zeta_T Z) - b\big)\\
&+\operatorname{Re}\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} v_H(x_T^j)\, a_h\big((E_\infty - E_{T,L})\lambda_T^j, b\big)\\
=&\operatorname{Re}\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} v_H(x_T^j)\, a_h\big((E_\infty - E_{T,L})\lambda_T^j, Z - I^c_{h,T}(\zeta_T Z)\big)\\
&+\operatorname{Re}\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} v_H(x_T^j)\, a_h\big((E_\infty - E_{T,L})\lambda_T^j, b\big)\\
\lesssim&\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} |v_H(x_T^j)|\big\|(E_\infty - E_{T,L})\lambda_T^j\big\|_h\Big(\big|\big|\big|Z - I^c_{h,T}(\zeta_T Z)\big|\big|\big|_h + |||b|||_h\Big)\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} |v_H(x_T^j)| \|(E_\infty - E_{T,L})\lambda_T^j\|_h \Big(\| Z - I_{h,T}^c(\zeta_T Z) \|_h + \| Q_H\big(I_{h,T}^c(\zeta_T Z) - Z\big)\|_H \Big) \\
&\lesssim \sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} |v_H(x_T^j)| \|(E_\infty - E_{T,L})\lambda_T^j\|_h \| Z - I_{h,T}^c(\zeta_T Z) \|_h \\
&\lesssim \sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} |v_H(x_T^j)| \|(E_\infty - E_{T,L})\lambda_T^j\|_h \|Z\|_{h,\omega_{T,L+1}} \\
&\lesssim \Big(\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} |v_H(x_T^j)|^2 \|(E_\infty - E_{T,L})\lambda_T^j\|_h^2 \Big)^{\frac12} \Big(\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} \|Z\|_{h,\omega_{T,L+1}}^2 \Big)^{\frac12} \\
&\lesssim \Big(\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} |v_H(x_T^j)|^2 \|(E_\infty - E_{T,L})\lambda_T^j\|_h^2 \Big)^{\frac12} \|Z\|_h.
\end{aligned}
$$

Therefore it follows from Theorem 4.9 and $\|\lambda_T^j\|_H \lesssim H^{\frac{d}{2}-1}$ that

$$
\begin{aligned}
\|Z\|_h &\lesssim \beta^L \Big(\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} |v_H(x_T^j)|^2 \|\lambda_T^j\|_H^2 \Big)^{\frac12} \\
&\lesssim \beta^L H^{-1} \Big(\sum_{\substack{T\in\mathcal{T}_H\\ j=1\cdots r}} |v_H(x_T^j)|^2 |T| \Big)^{\frac12} \\
&\lesssim \beta^L H^{-1} \|v_H\|_0 \lesssim \beta^L (kH)^{-1} \|v_H\|_H.
\end{aligned}
$$

This complete the proof of the theorem. □

4.3 Preasymptotic error analysis of Multiscale IPCDGM

In this subsection, we present the pre-asymptotic error analysis of Multiscale IPCDGM.

First, we prove the inf-sup condition of $a_h(\cdot,\cdot)$ with respect to the pair of spaces $V_{H,L}^{ms}$ and $V_{H,L}^{ms*}$.

Lemma 4.11. *Under the conditions of Theorems 3.6 and 4.9, there exist a constant $C_3 > 0$ such that if $L \ge |\log(C_3 H)/\log\beta|$, then*

$$
\inf_{u_{H,L}^{ms*}\in V_{H,L}^{ms*}\setminus\{0\}} \ \sup_{v_{H,L}^{ms}\in V_{H,L}^{ms}\setminus\{0\}} \frac{\mathrm{Re}\, a_h(v_{H,L}^{ms}, u_{H,L}^{ms*})}{\|v_{H,L}^{ms}\|_h \|u_{H,L}^{ms*}\|_h} \gtrsim \frac{1}{k}. \tag{4.58}
$$

Proof. For any $u_{H,L}^{ms*} \in V_{H,L}^{ms*}$, let $u_H^{ms*} = (1-E^{\infty*})Q_H u_{H,L}^{ms*}$. Lemma 3.9 implies that there exists $v_h \in V_h$ such that

$$
\mathrm{Re}\, a_h(v_h, u_H^{ms*}) \gtrsim \frac{1}{k} \|v_h\|_h \|u_H^{ms*}\|_h.
$$

Set $v_H^{ms} = (1-E_\infty)Q_H v_h$ and $v_{H,L}^{ms} = (1-E_L)Q_H v_h \in V_{H,L}^{ms}$. Clearly $Q_H v_{H,L}^{ms} = Q_H v_H^{ms} = Q_H v_h$. Therefore, from Lemmas 4.8 and 4.2 and Theorem 4.10, we obtain

$$
\begin{aligned}
\|v_{H,L}^{ms}\|_h &\lesssim \|v_H^{ms}\|_h + \|(E_\infty - E_L)Q_H v_h\|_h \\
&\lesssim \|Q_H v_h\|_H + \beta^L (kH)^{-1} \|Q_H v_h\|_H \lesssim \|Q_H v_h\|_H \lesssim \|v_h\|_h.
\end{aligned}
$$

Similarly, noting that $u^{ms*}_{H,L}=(1-E^*_L)Q_H u^{ms*}_{H,L}$ and $Q_H u^{ms*}_{H,L}=Q_H u^{ms*}_H$, we have

$$\|u^{ms*}_{H,L}\|_h \lesssim \|Q_H u^{ms*}_{H,L}\|_H = \|Q_H u^{ms*}_H\|_H \lesssim \|u^{ms*}_H\|_h.$$

By the above three estimates, (2.18), Lemma 4.6, Theorem 4.10, and Lemma 4.2, we have

$$\begin{aligned}
\frac{1}{k}\|v^{ms}_{H,L}\|_h\|u^{ms*}_{H,L}\|_h \lesssim & \frac{1}{k}\|v_h\|_h\|u^{ms*}_H\|_h \lesssim \mathrm{Re}\,a_h(v_h,u^{ms*}_H)\\
=&\mathrm{Re}\,a_h(v^{ms}_H,u^{ms*}_H)=\mathrm{Re}\,a_h(v^{ms}_H,u^{ms*}_{H,L})\\
=&\mathrm{Re}\,a_h(v^{ms}_{H,L},u^{ms*}_{H,L})+\mathrm{Re}\,a_h((E_L-E_\infty)Q_H v_h,u^{ms*}_{H,L})\\
\le&\mathrm{Re}\,a_h(v^{ms}_{H,L},u^{ms*}_{H,L})+C\beta^L(kH)^{-1}\|Q_H v_h\|_H\|u^{ms*}_{H,L}\|_h\\
=&\mathrm{Re}\,a_h(v^{ms}_{H,L},u^{ms*}_{H,L})+C\beta^L(kH)^{-1}\|Q_H v^{ms}_{H,L}\|_H\|u^{ms*}_{H,L}\|_h\\
\le&\mathrm{Re}\,a_h(v^{ms}_{H,L},u^{ms*}_{H,L})+C\beta^L(kH)^{-1}\|v^{ms}_{H,L}\|_h\|u^{ms*}_{H,L}\|_h,
\end{aligned}$$

which implies (4.58) if $\beta^L H^{-1}$ is sufficiently small, or equivalently, $L\ge|\log(C_3H)/\log\beta|$ for some constant $C_3>0$. This completes the proof of the lemma. $\square$

The following theorem gives the error estimate between the MsIPDPG solution and the IPCDG solution, which is the main result of this paper.

Theorem 4.12. *Assume that $f\in H^j(\Omega)$ $(j=0,1,2)$, $k^3h^2\le C_0$, $kH<\min(C_1,C_2)$, and $L\ge|\log(C_3H)/\log\beta|$. Let $u^{ms}_{H,L}$ and u_h be the MsIPDPG solution to (2.20) and the IPCDG solution to (2.7), respectively. Under the conditions of Lemma 3.3, there holds*

$$\|u^{ms}_{H,L}-u_h\|_h \lesssim H^{j+1}|f|_j+\beta^L(kH)^{-1}\|f\|_0.$$

Here $C_l(l=0,1,2,3)$ are from Theorem 3.6, Lemma 4.6, Theorem 4.9, and Lemma 4.11, respectively, and $\beta\in(0,1)$ is from Theorem 4.9.

Proof. Denote by $\tau=u^{ms}_{H,L}-u_h\in V_h$, $\tau_{H,L}=(1-E_L)Q_H\tau\in v^{ms}_{H,L}$. Introduce the following adjoint problem: find $w^*_{H,L}=(1-E^*_L)w_H\in V^{ms*}_{H,L}$ with $w_H\in V_H$ such that

$$\begin{aligned}
a_h(v_{H,L},w^*_{H,L})=&(\nabla_h v_{H,L},\nabla_h\tau_{H,L})+k^2(v_{H,L},\tau_{H,L})+\sum_{E\in\mathscr{E}_H}\frac{1}{h}([v_{H,L}],[\tau_{H,L}])_E\\
&+\sum_{e\in\mathscr{E}_h}h\left(\left[\frac{\partial v_{H,L}}{\partial n}\right],\left[\frac{\partial \tau_{H,L}}{\partial n}\right]\right)_e,\qquad \forall v_{H,L}\in V^{ms}_{H,L}.
\end{aligned}\tag{4.59}$$

Clearly, $w_H=Q_H w^*_{H,L}$. From Lemma 4.11, the adjoint problem satisfies the stability $\|w^*_{H,L}\|_h\lesssim k\|\tau_{H,L}\|_h$.

Setting $v_{H,L}=\tau_{H,L}$ in (4.59) and denoting by $w^*_{H,\infty}=(1-E^*_\infty)w_H$, we use the fact $\tau-\tau_{H,L}\in V^f$, the orthogonality $a_h(\tau, w^*_{H,L}) = 0$, Theorem 4.10, and Lemmas 4.2 and 4.6 to get

$$\begin{aligned}
\|\!|\tau_{H,L}|\!\|_h^2 =& a_h(\tau_{H,L}, w^*_{H,L}) = a_h(\tau_{H,L}, w^*_{H,L} - w^*_{H,\infty}) + a_h(\tau_{H,L}, w^*_{H,\infty})\\
=& a_h(\tau_{H,L}, w^*_{H,L} - w^*_{H,\infty}) + a_h(\tau, w^*_{H,\infty})\\
=& a_h(\tau_{H,L}, w^*_{H,L} - w^*_{H,\infty}) + a_h(\tau, w^*_{H,\infty} - w^*_{H,L})\\
=& a_h(\tau - \tau_{H,L}, w^*_{H,\infty} - w^*_{H,L}) = a_h(\tau - \tau_{H,L}, (E^*_L - E^*_\infty) w_H)\\
\lesssim& \|\!|\tau - \tau_{H,L}|\!\|_h \left\|\!\left| (E^*_L - E^*_\infty) Q_H w^*_{H,L} \right|\!\right\|_h\\
\lesssim& \|\!|\tau - \tau_{H,L}|\!\|_h \beta^L (kH)^{-1} \|\!| w^*_{H,L} |\!\|_h\\
\lesssim& \beta^L H^{-1} \|\!|\tau - \tau_{H,L}|\!\|_h \|\!|\tau_{H,L}|\!\|_h.
\end{aligned}$$

Noting that $\beta^L H^{-1} \lesssim 1$, we have

$$\|\!|\tau|\!\|_h \lesssim \|\!|\tau - \tau_{H,L}|\!\|_h + \|\!|\tau_{H,L}|\!\|_h \lesssim \|\!|\tau - \tau_{H,L}|\!\|_h. \tag{4.60}$$

On the other hand, from the fact $\tau - \tau_{H,L} \in V^f$ and Lemma 4.2, we obtain

$$\begin{aligned}
(f, \tau - \tau_{H,L}) =& \sum_{T\in\mathcal{T}_H} (f, \tau - \tau_{H,L})_T = \sum_{T\in\mathcal{T}_H} \big(f - Q_H f, (\tau - \tau_{H,L}) - Q_H(\tau - \tau_{H,L})\big)_T\\
\lesssim& H^{j+1} |f|_j \|\nabla_h (\tau - \tau_{H,L})\|_0 \lesssim H^{j+1} |f|_j \|\!|\tau - \tau_{H,L}|\!\|_h.
\end{aligned} \tag{4.61}$$

Since $(1 - E_L) Q_H v^{ms}_{H,L} = v^{ms}_{H,L}, \forall v^{ms}_{H,L} \in V^{ms}_{H,L}$, we have $\tau - \tau_{H,L} = u^{ms}_{H,L} - u_h - (1 - E_L) Q_H (u^{ms}_{H,L} - u_h) = (1 - E_L) Q_H u_h - u_h$. It follows from Lemma 4.6, (4.61), Theorem 4.10 , Lemma 4.2, and Corollary 3.7 that

$$\begin{aligned}
\|\!|\tau - \tau_{H,L}|\!\|_h^2 \lesssim& \mathrm{Re}\, a_h(\tau - \tau_{H,L}, \tau - \tau_{H,L})\\
=& \mathrm{Re}\, a_h(u_h, \tau_{H,L} - \tau) + \mathrm{Re}\, a_h\big((E_\infty - E_L) Q_H u_h, \tau - \tau_{H,L}\big)\\
=& \mathrm{Re}\big(f, \tau_{H,L} - \tau\big) + \mathrm{Re}\, a_h\big((E_\infty - E_L) Q_H u_h, \tau - \tau_{H,L}\big)\\
\lesssim& H^{j+1} |f|_j \|\!|\tau - \tau_{H,L}|\!\|_h + \beta^L (kH)^{-1} \|\!| u_h |\!\|_h \|\!|\tau - \tau_{H,L}|\!\|_h\\
\lesssim& \left(H^{j+1} |f|_j + \beta^L (kH)^{-1} \|f\|_0\right) \|\!|\tau - \tau_{H,L}|\!\|_h,
\end{aligned}$$

that is,

$$\|\!|\tau - \tau_{H,L}|\!\|_h \lesssim H^{j+1} |f|_j + \beta^L (kH)^{-1} \|f\|_0. \tag{4.62}$$

At last the proof of the theorem follows by plugging (4.62) into (4.60). □

Remark 4.13. *Theorem 4.12 shows that there is no pollution error between the MsIPDPG solution and the IPCDG solution and the MsIPDPG solution is a good approximation to the IPCDG solution even in the case of large wave number.*

As a consequence of Theorems 3.6 and 4.12, we have the following corollary which gives the error between the MsIPDPG solution and the exact solution.

Corollary 4.14. *Let $u_{H,L}^{ms}$ and u be the MsIPDPG solution to* (2.20) *and the exact solution to* (1.1)*, respectively. Under the conditions of Theorem 4.12, there holds*

$$|||u-u_{H,L}^{ms}|||_h \lesssim H^{j+1}|f|_j + \left(kh+k^3h^2+\beta^L(kH)^{-1}\right)\|f\|_0.$$

Remark 4.15. *The error bound in Corollary 4.14 consists of four parts: the interpolation error $O(kh)$, the pollution error $O(k^3h^2)$, the multiscale approximation error $O(H^{j+1})$, and the multiscale truncation error $O(\beta^L(kH)^{-1})$. In the next section we will investigate numerically the influence of each part. In particular, we will show that the pollution error may be greatly reduced by tuning the penalty parameters.*

5 Numerical example

In this section, we will simulate the Helmholtz problem (5.63) in two-dimension domain Ω by IPCDGM and MsIPDPGM.

$$\begin{cases} -\triangle u - k^2 u = f & \text{in } \Omega, \\ \frac{\partial u}{\partial n} + \mathrm{i}ku = g & \text{on } \Gamma := \partial\Omega. \end{cases} \tag{5.63}$$

f is so chosen that the exact solution is

$$u = \frac{1}{4k^2}\left(e^{\mathrm{i}k(x-1)} + e^{-\mathrm{i}kx} - 2\right)\left(y^2 - y + \frac{i}{k}\right).$$

The computational domain Ω is the unit square $[0,1]\times[0,1]$. $\mathcal{T}_H$ and $\mathcal{T}_h$ are the two sets of uniform isosceles right-angled triangular meshes of Ω. The mesh $\mathcal{T}_h$ is derived by uniform mesh refinement of $\mathcal{T}_H$. In the example, we will give out the H^1 relative errors between the exact solutions and IPCDGM, MsIPDPGM solutions respectively. The proper penalty parameters $\gamma_{1,e}$ of (2.7) and (2.20) will be chosen as $\gamma_{1,e} = [-1/12, -1/24]$ on two different interior edge sets which are obtained by a dispersion analysis on the two meshes such that the phase errors can be reduced efficiently. In [6], the parameter $\gamma_{1,e}$ is derived for one dimensional problem by dispersion analysis, we use them in our computational for the two dimensional problem.

In the mesh sets $\mathcal{T}_H$ and $\mathcal{T}_h$, the corrector of an IPCDG basis of one coarse mesh point is depicted in Figure 1 which is obtained by solved the local problem in the whole domain Ω.

In Figure 2, fixing the parameters relations $kH = 2$ and $kh = 1/2$, we show the H1-relative errors between the exact solution and IPCDGM, MsIPDPGM solutions with $\gamma_{1,e} = O(h)$ and $\gamma_{1,e} = [-1/12, -1/24]$ respectively. It shows that the proper parameter $\gamma_{1,e}$ can reduce the pollution error effectively for both IPCDGM and MsIPDPGM.

In Table 1, fixing $k = 32$, $h = 512$ and changing $H = 1/16, 1/32, 1/64, 1/128$, we show the changing of H1-relative errors between MsIPDPGM and IPCDG solutions with the sufficiently large sampling number L (Layer). The results are in accord with the following super convergence.

$$|||u_h - u_{H,L}^{ms}|||_h \approx |||u_h - u_h^{ms}|||_h \lesssim H^3\|f\|_2.$$

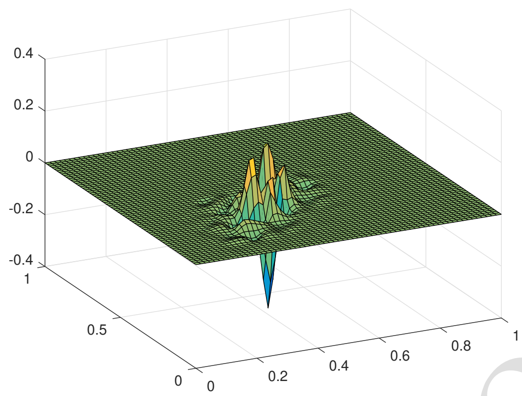

Figure 1: The corrector of a coarse point IPCDG-basis.

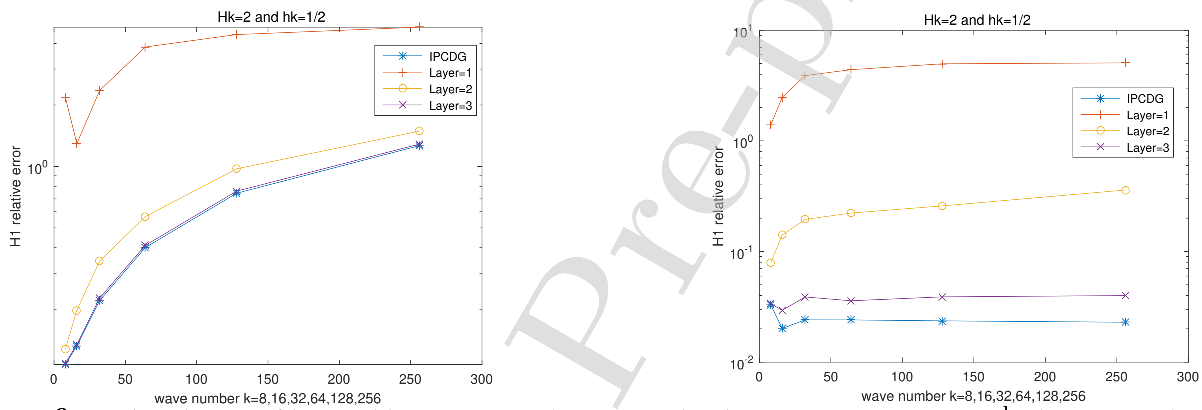

Figure 2: Set the relations of $kH = 2, kh = 1/2$. For each wave number $k = 8, 16, 32, 64, 128, 256$, H^1 relative errors between the exact solutions and IPCDGM, MsIPDPGM solutions are shown with $\gamma_{1,e} = O(h)$ (left) and $\gamma_{1,e} = [-1/12, -1/24]$ (right).

Conclusion

The proposed IPCDGM and MsIPDPGM can greatly reduce the pollution error by choosing a proper penalty parameter $\gamma_{1,e}$. Preasymptotic error estimates are proved for both methods. In particular, it is shown that the error between the IPCDG solution and the MsIPDPG solution in the broken H^1-norm is $O(H^3)$ under proper assumptions. For Helmholtz equations with large wave number, the IPCDGM on a fine mesh involves an enormous algebraic system that need to be solved, while the MsIPDPGM can be assembled on a coarse mesh and the local basis functions on patches of macro elements can be computed in parallel. The layers of the local patches can be chosen according to

H=1/16	H=1/32	H=1/64	H=1/128
5.124867e-04	6.138274e-05	7.353098e-06	9.057307e-07

Table 1: Fix $k = 32$, $h = 512$; Changing $H = 1/16, 1/32, 1/64, 1/128$ and choosing the sampling parameter sufficient large, H^1 relative errors between MsIPDPGM solutions and IPCDGM solutions.

$O(\log k)$ without affecting the accuracy. Numerical tests are provided to verify the present analysis.

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