Journal of Computational and Applied Mathematics 39 (1992) 1–6
North-Holland

Some generalized projection methods for solving operator equations

Ioannis K. Argyros
Department of Mathematics, Cameron University, Lawton, OK 73505-6377, United States

Received 9 November 1989
Revised 8 November 1990

Abstract

Argyros, I.K., Some generalized projection methods for solving operator equations, Journal of Computational and Applied Mathematics 39 (1992) 1–6.

We provide sufficient conditions for the convergence of a certain Newton-like method to the solution of an equation with a nondifferentiable term.

Keywords: Banach space, Newton-like method, majorant theory.

1. Introduction

Consider the fixed-point problem

$$T(x)=x, \quad \text{with } T(x)=F(x)+G(x), \tag{1}$$

where F, G are nonlinear operators defined on some convex subset D of a Banach space E with values in a Banach space $\hat{E}$. We assume that F is Fréchet differentiable on D, whereas G is not. Let $x^0 \in D$ and choose $R>0$ such that the closed ball with center x^0 and radius R, denoted by $\overline{U}(x_0, R)$ is included in D. Chen and Yamamoto [3] and others [10,11] proposed the Newton-like iteration

$$\bar{y}_{n+1}=\bar{y}_n-(I-A(\bar{y}_n))^{-1}(\bar{y}_n-T(\bar{y}_n)), \quad n \geqslant 0,\ \bar{y}_0 \in \overline{U}(x^0, R), \tag{2}$$

for approximating a fixed point x^* of (1). Here $A(x)$ denotes a linear operator which approximates the Fréchet derivative $F'(x)$ of F at $x \in \overline{U}(x^0, R)$. The above authors showed that under certain conditions iteration (2) generates a sequence which converges to x^*. For $G=0$, iteration (2) reduces to the classical Newton-like method which has been studied extensively by several authors [1–9]. Whereas for $G \neq 0$ and $A(x)=F'(x)$, it has been studied in [4,9–12]. Let $\bar{x}_0=x^0$ and define the sequence

$$\bar{x}_{n+1}=\bar{x}_n-(I-A(\bar{x}_n))^{-1}(\bar{x}_n-T(\bar{x}_n)), \quad n \geqslant 0. \tag{3}$$

In either case the iterates $\{\bar{y}_n\}$ and $\{\bar{x}_n\}$, $n \geqslant 0$, can rarely be computed in infinite-dimensional spaces, since it may be difficult or even impossible to compute the inverses of the linear operators $A(\bar{y}_n)$ or $A(\bar{x}_n)$, $n \geqslant 0$.

In this paper we will make practical use of iterations (2) and (3), by considering the iterations

$$y_{n+1} = y_n - (I - PA(y_n))^{-1}(y_n - T(y_n)), \quad y_0 \in \bar{U}(x^0, R), \ n \geqslant 0, \tag{4}$$

and

$$x_{n+1} = x_n - (I - PA(x_n))^{-1}(x_n - T(x_n)), \quad x_0 = x^0, \ n \geqslant 0, \tag{5}$$

where P is a projection operator ($P^2 = P$) on D.

Let us assume that the inverse of the linear operator $I - PA(x^0)$ exists and

$$\|(I - PA(x^0))^{-1}[PA(x) - PA(x^0)]\| \leqslant v_0(\|x - x^0\|) + b, \tag{6}$$

$$\|(I - PA(x^0))^{-1}[PF'(x + t(y-x)) - PA(x)]\| \leqslant v(\|x - x^0\| + t\|y - x\|) - v_0(\|x - x^0\|) + c, \tag{7}$$

and

$$\|(I - PA(x^0))^{-1}[(QF(x) + G(x)) - (QF(y) + G(y))]\| \leqslant v_1(r)\|x - y\|, \tag{8}$$

for any $x, y \in \bar{U}(x^0, r) \subseteq \bar{U}(x^0, R)$, with $Q = I - P$. Here $v(r+t) - v_0(r)$, $t \geqslant 0$, and $v_1(r)$ are nondecreasing nonnegative functions with $v(0) = v_0(0) = v_1(0) = 0$, $v_0(r)$ is differentiable, $v_0'(r) > 0$ for all $r \in [0, R]$, and the constants b, c satisfy $b \geqslant 0$, $c \geqslant 0$ and $b + c < 1$.

We note that for $P = I$ the conditions (6)–(8) reduce to the Zabrejko–Nguen type conditions, considered in [3].

It is easy to see that the solutions of iterations (4) and (5) reduce to solving certain operator equations in the space E_P. If, moreover, E_P is a finite-dimensional space of dimension N, we obtain a system of linear algebraic equations of order at most N.

We will provide sufficient conditions for the convergence of iterations (4) and (5) to x^* as well as error bounds on the distances $\|x_{n+1} - x_n\|$ and $\|x_n - x^*\|$, $n \geqslant 0$.

Finally, we illustrate our results by considering a nondifferentiable nonlinear integral equation.

2. Convergence results

We introduce the constant

$$a = \left\|(I - PA(x^0))^{-1}(x_0 - T(x_0))\right\|,$$

the functions

$$\phi(r) = a - r + \int_0^r v(t)\,dt,$$

$$\psi(r) = \int_0^r v_1(t)\,dt,$$

$$\chi(r) = \phi(r) + \psi(r) + (b + c)r,$$

and the sequences

$$r_{n+1} = r_n + \frac{u(r_n)}{w(r_n)}, \quad r_0 \in [0, R], \; n \geqslant 0, \tag{9}$$

$$v_{n+1} = v_n + \frac{u(v_n)}{w(v_n)}, \quad v_0 = 0, \; n \geqslant 0, \tag{10}$$

where

$$u(r) = \chi(r) - \alpha^* \quad \text{and} \quad w(r) = 1 - v_0(r) - b.$$

Here α^* denotes the minimal value in [0, R]; let r^* be the minimal point. As in [3, p.39] we can easily show that if $\chi(R) \leqslant 0$, then $\chi(r)$ has a unique zero t^* in $(0, r^*]$, since $\chi(r)$ is strictly convex. Moreover, r^* can be obtained as the limit of the monotonically increasing sequences $\{s_n\}$ and $\{v_n\}$, $n \geqslant 0$. Furthermore, $w(r) > 0$ for all $r \in [0, r^*)$.

If $\chi(R) \leqslant 0$, let us define the sets

$$\tilde{U} = \begin{cases} \bar{U}(x^0, R), & \text{if } \chi(R) < 0 \quad \text{or} \quad \chi(R) = 0 \text{ and } t^* = R, \\ U(x^0, R), & \text{if } \chi(R) = 0 \text{ and } t^* < R, \end{cases}$$

$$H = \bigcup_{r \in [0, r^*)} \left\{ y \in \bar{U}(x^0 r) \mid \| (I - PA(y))^{-1}(y - T(y)) \| \leqslant \frac{u(r)}{w(r)} \right\},$$

and

$$R_y = \left\{ r \in [0, r^*) \mid \| (I - PA(y))^{-1}(y - T(y)) \| \leqslant \frac{u(r)}{w(r)}, \; \| y - x^0 \| \leqslant r \right\}.$$

We set $f(x) = P(F(x) - x)$ and $g(x) = PG(x)$, where f and g are as defined in [3]. Then by slightly modifying the proof of [3, Theorem 1] we can show the following.

Theorem 1. *Suppose that* $\chi(R) \leqslant 0$. *Then*

(a) *equation* (1) *has a fixed point* x^* *in* $\bar{U}(x^0, t^*)$, *which is unique in* $\tilde{U}$;

(b) *for any* $y_0 \in H$, *the iteration* (4) *is well defined, remains in* $\bar{U}(x^0, r^*)$, $n \geqslant 0$, *and satisfies*

$$\| y_{n+1} - y_n \| \leqslant r_{n+1} - r_n, \quad n \geqslant 0, \tag{11}$$

and

$$\| y_n - x^* \| \leqslant r^* - r_n, \quad n \geqslant 0, \tag{12}$$

provided that r_0 *is chosen in* (9) *such that* $r_0 \in R_{y_0}$.

For completion we wiil now generalize [3, Proposition 1 and Theorem 2]. For any $y \in H$, we choose a number $r_y \in R_y$, which we fix and set

$$a_y = \| (I - PA(y))^{-1}(y - T(y)) \|,$$

$$d_y = \begin{cases} 1, & \text{if } y = x^0 \text{ and } r_y = 0, \\ w(r_y)^{-1}, & \text{otherwise,} \end{cases}$$

and

$$\chi_y(r) = a_y + d_y\left(\int_0^r \left(v(r_y+t) + v_1(r_y+t)\right) \mathrm{d}t + (b+c-1)r\right).$$

Moreover, we define the sequence

$$q_{n+1} = q_n + \frac{\chi_y(q_n)}{d_y w(r_y+q_n)}, \quad n \geqslant 0, \; q_0 = 0.$$

Then we can show the next theorem.

Theorem 2. *Suppose that the hypotheses of Theorem* 1 *are true. Then*

(a) $\bar{U}\left(x^0, \dfrac{|\alpha^*|}{2-b}\right) \subset H$;

(b) *the estimate* $\chi_y(r^*-r_y) \leqslant 0$ *is true and the function* $\chi_y(r)$ *has a unique zero* q^* *in* $[0, r^*-r_y]$.

Moreover, the sequence $\{y_n\}$, $n \geqslant 0$, *with* $y_0 = y$ *satisfies*

$$\| y_{n+1} - y_n \| \leqslant q_{n+1} - q_n,$$

and

$$\| y_n - x^* \| \leqslant q^* - q_n \leqslant r^* - r_y, \quad n \geqslant 0.$$

We now complete this paper with an example. For simplicity we will assume that $A(x) = F'(x)$ on D.

3. Applications

Consider the integral equation

$$x(t) = \int_0^1 K(t, s, x(s)) \, \mathrm{d}s, \tag{13}$$

where the kernel $K(t, s, x(s))$ is nondifferentiable on some convex subset $D \subset E = C[0, 1]$ equipped with the sup-norm. We set

$$T(x) = \int_0^1 K(t, s, x(s)) \, \mathrm{d}s \quad \text{and} \quad F(x) = \int_0^1 \bar{K}(t, s, x(s)) \, \mathrm{d}s,$$

where $\bar{K}(t, s, x(s))$ is differentiable on D. Then

$$PF'(x) = \int_0^1 \bar{\bar{K}}'_x(t, s, x(s)) \, \mathrm{d}s,$$

where

$$\bar{\bar{K}}(t, s, x(s)) = \sum_{i=1}^{\infty} A_i(t) B_i(s, x(s))$$

is a degenerate kernel approximating the functions $\bar{K}(t, s, x)$, e.g., a portion of the Taylor or Fourier series for the function $\bar{K}(t, s, x)$ if we consider it as a function of t. The modified Newton–Kantorovich iteration (3) can now be written as

$$x_{n+1}(t) = \int_0^1 K(t, s, x_n(s))\, ds - \int_0^1 \bar{\bar{K}}'_x(t, s, x_n(s)) x_n(s)\, ds + \int_0^1 \bar{\bar{K}}'_x(t, s, x_n(s)) x_{n+1}(s)\, ds \tag{14}$$

Let

$$f_n(t) = \int_0^1 K(t, s, x_n(s))\, ds - \int_0^1 \bar{\bar{K}}'_x(t, s, x_n(s)) x_n(s)\, ds;$$

then iteration (14) can be written as

$$x_{n+1}(t) = f_r(t) + \sum_{i=1}^{m} A_i(t) \int_0^1 B'_i(x, x_n(s)) x_{n+1}(s)\, ds,$$

which can be solved to give a system of linear algebraic equations

$$\int_0^1 B'_i(s, x_n(s)) x_{n+1}(s)\, ds - \sum_{i=1}^{m} \int_0^1 B'_i(s, x_n(s)) A_j(s)\, ds \int_0^1 B'_i(s, x_n(s)) x_{n+1}(s)\, ds = \int_0^1 B'_i(s, x_n(s)) f_n(s)\, ds.$$

Denote by $D(x_n)$ the determinant of the above system and assume $D(x_n) \neq 0$, $n \geqslant 0$. Then,

$$\int_0^1 B'_i(s, x_n(s)) x_{n+1}(s)\, ds = \frac{1}{D(x_n)} \int_0^1 \sum_{k=1}^{n} D_{ki}(x_n) B'_k(s, x_n(s)) f_n(s)\, ds,$$

and

$$x_{n+1}(t) = f_n(t) + \int_0^1 \sum_{i=1}^{m} \sum_{k=1}^{m} \frac{A_i(t) D_{ki}(x_n) B'_k(s, x_n(s))}{D(x_n)} f_n(s)\, ds,$$

where $D_{ki}(x_n)$ is the cofactor of the element in the ith row and kth column of the determinant $D(x_n)$.

Define the operators $\bar{\bar{K}}_x(t, s, x)$, $Q(t, s, x)$, $G(t, s, x)$ and $L(t, s, x)$ by $Q(t, s, x) = \bar{K}(t, s, x) - \bar{\bar{K}}(t, s, x)$, $G(t, s, x) = K(t, s, x) - \bar{K}(t, s, x)$ and

$$L(t, s, x) = \frac{1}{D(x)} \sum_{i=1}^{m} \sum_{k=1}^{m} A_i(t) D_{ki}(x) B'_k(s, x).$$

Let us now consider a ball $U(x_0, R) \subset D$ for some $R > 0$ fixed such that the inverse $I - PF'(x_0)$ exists on $U(x_0, R)$. Assume that for each r with $r \in [0, R]$ the functions defined above satisfy the conditions

$$\left| \bar{\bar{K}}'_x(t, s, x) - \bar{\bar{K}}'_x(t, s, y) \right| \leqslant M_r(t, s) |x - y|,$$

$$|Q(t, s, x) + G(t, s, x) - (Q(t, s, y) + G(t, s, y))| \leqslant N_r(t, s) |x - y|,$$

and

$$|L(t, s, x_0)| \leqslant J_r(t, s),$$

for all $t, s \in [0, 1]$ and $x, y \in U(x_0, R)$. If equation (13) is considered in E, then choose real nonnegative functions v_0, v and v_1 to satisfy the conditions listed after (8), and

$$v_0(r) \leqslant 2r\left(1 + \sup_{t\in[0,1]} \int_0^1 J_r(t, s)\, \mathrm{d}s\right) \sup_{t\in[0,1]} \int_0^1 M_r(t, s)\, \mathrm{d}s,$$

$$v_1(r) \leqslant 2r\left(1 + \sup_{t\in[0,1]} \int_0^1 J_r(t, s)\, \mathrm{d}s\right) \sup_{t\in[0,1]} \int_0^1 N_r(t, s)\, \mathrm{d}s,$$

$v = v_0$ and linear. Moreover, choose $b = c = 0$. It can now easily be seen that conditions (6)–(8) are satisfied. The conclusions of Theorems 1 and 2 can now apply provided that $\chi(R) \leqslant 0$.

References

[1] I.K. Argyros, On Newton's method and nondiscrete mathematical induction, *Bull. Austral. Math. Soc.* **38** (1988) 131–140.

[2] M. Balazs and G. Goldner, On the method of the cord and on a modification of it for the solution of nonlinear operator equations, *Stud. Cerc. Mat.* **20** (1968) 981–990.

[3] X. Chen and T. Yamamoto, Convergence domains of certain iterative methods for solving nonlinear equations, *Numer. Funct. Anal. Optim.* **10** (1 & 2) (1989) 37–48.

[4] W.B. Gragg and R.A. Tapia, Optimal error bounds for the Newton–Kantorovich Theorem, *SIAM J. Numer. Anal.* **11** (1974) 10–13.

[5] F.M. Migovich, On the convergence of projection-iterative methods for solving nonlinear operator equations, *Dopovidi Akad. Nauk. Ukraïn RSR Ser. A* **1** (1970) 20–23.

[6] F.A. Potra and V. Ptâk, Sharp error bounds for Newton's process, *Numer. Math.* **34** (1980) 63–72.

[7] W.C. Rheinboldt, A unified convergence theory for a class of iterative processes, *SIAM J. Numer. Anal.* **5** (1969) 42–63.

[8] J.W. Schmidt, Unter Fehrerschranker für regular-falsi-verfahren, *Period. Math. Hungar.* **9** (1978) 241–247.

[9] T. Yamamoto, A method for finding sharp error bounds for Newton's method under the Kantorovich assumptions, *Numer. Math.* **44** (1986) 203–220.

[10] T. Yamamoto, A note on a posteriori error bound of Zabrejko and Nguen for Zincenko's iteration, *Numer. Funct. Anal. Optim.* **9** (9 & 10) (1987) 987–994

[11] P.P. Zabrejko and D.F. Nguen, The majorant method in the theory of Newton–Kantorovich approximations and the Ptâk error estimates, *Numer. Funct. Anal. Optim.* **9** (1987) 671–684.

[12] A.I. Zincenko, Some approximate methods of solving equations with nondifferentiable operators, *Dopovidi Akad. Nauk. Ukraïn. RSR* **115** (1963) 156–161 (in Ukrainian).