JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

ELSEVIER Journal of Computational and Applied Mathematics 68 (1996) 115–149

# Contiguous relations, basic hypergeometric functions, and orthogonal polynomials. III. Associated continuous dual $q$-Hahn polynomials [1]

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Received 27 September 1994; revised 10 July 1995

**Abstract**

Explicit solutions for the three-term recurrence satisfied by associated continuous dual $q$-Hahn polynomials are obtained. A minimal solution is identified and an explicit expression for the related continued fraction is derived. The absolutely continuous component of the spectral measure is obtained. Eleven limit cases are discussed in some detail. These include associated big $q$-Laguerre, associated Wall, associated Al-Salam–Chihara, associated Al-Salam–Carlitz I, and associated continuous $q$-Hermite polynomials.

*Keywords:* Basic hypergeometric series; Contiguous relations; Continued fractions; Generating functions; Weight functions; Associated orthogonal polynomials

*AMS classification:* 33D15; 33D45; 30B70; 39A10

## 1. Introduction

If $\{p_n(x)\}$ is a monic polynomial sequence given by the three-term recurrence

$$p_{n+1}(x) - (x - a_n)p_n(x) + b_n^2 p_{n-1}(x) = 0, \quad n \geqslant 0,\ p_{-1} = 0,\ p_0 = 1,$$

then the associated monic polynomial sequence $\{p_n^{(\alpha)}(x)\}$, $\alpha = 1, 2, \ldots$, is given by

$$p_{n+1}^{(\alpha)}(x) - (x - a_{n+\alpha})p_n^{(\alpha)}(x) + b_{n+\alpha}^2 p_{n-1}^{(\alpha)}(x) = 0, \quad n \geqslant 0,\ p_{-1}^{(\alpha)} = 0,\ p_0^{(\alpha)} = 1.$$

The particular case $\alpha = 1$ yields the polynomials of the second kind. If $a_n$ and $b_n$ have an explicit $n$ dependence, then one has more general associated polynomials with $\alpha \in \mathbb{R}$ or $\mathbb{C}$. In this paper

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[1] Research partially supported by NSF grant DMS 9203659 and NSERC grant A5384.



*SSDI* 0377-0427(95)00264-2

we consider associated continuous dual $q$-Hahn polynomials with general complex parameters and $\alpha \in \mathbb{C}$.

The discrete $q$-Hahn and the discrete dual $q$-Hahn polynomials were introduced by Hahn in [10] and [11], respectively. These are particular cases of polynomials introduced by Askey and Wilson and called $q$-Racah polynomials [3]. Askey and Wilson [3] have also considered continuous dual $q$-Hahn polynomials as a particular case of the Askey–Wilson polynomials. The objective of the present study is to generalize the continuous dual $q$-Hahn polynomials to the associated continuous dual $q$-Hahn polynomials. The $q \to 1$ limit gives the case of associated continuous Hahn polynomials which have been studied by Ismail et al. [13]. It may be mentioned that in two earlier papers [7, 8], we have discussed the associated continuous Hahn (for continuous Hahn polynomials see [1]) and the associated big $q$-Jacobi polynomials, respectively. In both of these associated cases we made extensive use of contiguous relations for hypergeometric and $q$-hypergeometric functions. Although the use of contiguous relations in connection with continued fractions goes back to Gauss [23], the importance of contiguous relations in relation to the theory of orthogonal polynomials was first stressed by Wilson [25].

In Section 2, we obtain six solutions to the three-term recurrence relation satisfied by associated continuous dual $q$-Hahn polynomials. This is done with the help of three-term contiguous relations satisfied by balanced ${}_3\phi_2$'s. It is also demonstrated how an existing three-term transformation formula for balanced ${}_3\phi_2$'s connects any three of these solutions. By examining the large $n$ asymptotics of the solutions and the associated second-order difference equation we show that one of the solutions is a minimal [6, 16] solution. When one of the four parameters is equal to '$q$', another of our solutions reduces to the continuous dual $q$-Hahn polynomial solution [3].

In Section 3, the related infinite continued fraction is obtained. Following the procedure employed in several other cases (see [21–23]) we then derive the explicit weight function for the absolutely continuous component of the spectrum.

Section 4 is devoted to obtaining a generating function and hence an explicit expression for the associated continuous dual $q$-Hahn polynomials. The method employed is the same as by Ismail and Libis [14] for big $q$-Laguerre polynomials.

In Section 5, we examine four limiting cases of the original recurrence relation together with their solutions, related continued fractions and explicit polynomials. The first two limits are associated big $q$-Laguerre and associated Wall polynomials. These are at the ${}_2\phi_1$ and ${}_1\phi_1$ levels, respectively. Two further limits are found at the ${}_0\phi_1$ level.

In Section 6, we consider seven additional limiting cases. These include the associated cases for Al-Salam–Chihara, Al-Salam–Carlitz I, and continuous $q$-Hermite polynomials.

In Section 7, we give the connection between solutions to the associated Askey–Wilson [15, 9] and the associated continuous dual $q$-Hahn polynomial recurrence relations.

The results in this paper are intended to complement the survey by Koekoek and Swarttouw [17] which gives the limiting cases for Askey–Wilson polynomials including those for the continuous dual $q$-Hahn polynomials. However they did not include results for the minimal solutions to the three-term recurrences, the corresponding continued fractions or the associated polynomial cases. By providing these additional results we have enlarged the class of explicitly solvable models. We believe this is more than just an interesting exercise. Explicit solutions are important. They have a habit of reoccurring in seemingly unrelated problems.

## 2. Three-term contiguous relations and solutions

The recurrence relation satisfied by associated continuous dual $q$-Hahn polynomials can be expressed as

$$X_{n+1} - (z - a_n)X_n + b_n^2 X_{n-1} = 0,$$

$$\begin{aligned} a_n &:= a_n(z;A,B,C,D),\\ &= \left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}+\frac{1}{D}\right)q^n - (1+q)q^{2n-1},\\ b_n^2 &:= b_n^2(z;A,B,C,D)\\ &= \frac{q}{ABCD}(1-Aq^{n-1})(1-Bq^{n-1})(1-Cq^{n-1})(1-Dq^{n-1}). \end{aligned} \tag{2.1}$$

The symmetry with respect to the parameters $A,B,C,D$ is obvious. With this form of the recurrence, it is easy to take successive limits $A,B,C,D \to \infty$. The limit $D \to \infty$ gives associated big $q$-Laguerre polynomials (see [14]) and a subsequent limit $C \to \infty$ gives associated Wall polynomials (see [4, p. 198]). Finally, $B \to \infty$ and then $A \to \infty$ give additional cases.

Note that in (2.1) the $a_n$ can also be expressed as

$$\begin{aligned} a_n &= -(\lambda_n + \mu_n) + \frac{1}{AB} + \frac{q}{CD},\\ \lambda_n &= (1-Aq^n)(1-Bq^n)/AB,\\ \mu_n &= q(1-Cq^{n-1})(1-Dq^{n-1})/CD. \end{aligned}$$

This means that with a renormalization and a translation of the coordinate $z$ we may re-express (2.1) and the aforementioned limits as birth and death processes with birth and death rates $\lambda_n$ and $\mu_n$, respectively [12]. A second family with $\mu_0 :=0$ should also be investigated. This has already been done in the more general case of associated Askey–Wilson polynomials [15].

The solutions to (2.1) and its limit cases will be expressed in terms of the basic hypergeometric functions

$$_r\phi_s\left(\begin{matrix} a_1,a_2,\ldots,a_r\\ b_1,b_2,\ldots,b_s\end{matrix};z\right) = \sum_{k=0}^{\infty} \frac{(a_1,a_2,\ldots,a_r)_k}{(b_1,b_2,\ldots,b_s,q)_k}[(-1)^k q^{k(k-1)/2}]^{1+s-r} z^k, \quad |z|<1,$$

where

$$(a)_\infty = \prod_{j=1}^{\infty}(1-aq^{j-1}), \qquad (a)_n = (a)_\infty/(aq^n)_\infty, \quad n \text{ integer},$$

$$(a_1,a_2,\ldots,a_m)_n = \prod_{k=1}^{m}(a_k)_n, \quad n \text{ integer or } \infty.$$

We will use the notation

$$\phi := \phi(a,b,c,d,e) = {}_3\phi_2\left(\begin{matrix} a,b,c\\ d,e\end{matrix};\frac{de}{abc}\right), \quad \left|\frac{de}{abc}\right| < 1, \tag{2.2a}$$

and its analytic continuation for $|de/abc| \geqslant 1$ given by the following transformations [5, (III.9), (III.10), p. 241]:

$$ {}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}; \frac{de}{abc}\right) = \frac{(e/a, de/bc)_\infty}{(e, de/abc)_\infty}\, {}_3\phi_2\left(\begin{matrix} a, d/b, d/c \\ d, de/bc \end{matrix}; \frac{e}{a}\right), \tag{2.2b} $$

$$ {}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix}; \frac{de}{abc}\right) = \frac{(b, de/ab, de/bc)_\infty}{(d, e, de/abc)_\infty}\, {}_3\phi_2\left(\begin{matrix} d/b, e/b, de/abc \\ de/ab, de/bc \end{matrix}; b\right). \tag{2.2c} $$

To obtain solutions to (2.1) we use ${}_3\phi_2$ contiguous relations and the usual notation

$$ \begin{aligned} &\phi(a\pm) = \phi(aq^{\pm 1}, b, c, d, e), \\ &\phi_\pm = \phi(aq^{\pm 1}, bq^{\pm 1}, cq^{\pm 1}, dq^{\pm 1}, eq^{\pm 1}). \end{aligned} $$

Two such contiguous relations are [18]:

$$ \phi - \phi(a+) + \frac{(1-b)(1-c)}{(1-d)(1-e)} \frac{de}{abcq} \phi_+ = 0, \tag{2.3} $$

$$ (1-d)(1-e)\phi + (d-a)\left(1 - \frac{e}{a}\right)\phi_+(a-) - (1-a)\left(1 - \frac{de}{abcq}\right)\phi_+ = 0. \tag{2.4} $$

Changing $(a,b,c,d,e) \longrightarrow (a, b/q, c/q, d/q, e/q)$ in (2.4) and then eliminating $\phi(a+)$ and $\phi_+$ from the resulting equation together with (2.3) and (2.4), we obtain

$$ \begin{aligned} &\frac{(1-b)(1-c)(1-d/a)(1-e/a)}{(1-d)(1-e)} \frac{de}{bcq} \phi_+(a-) \\ &\quad - \left[(1-a)\left(1 - \frac{de}{abcq}\right) + a\left(1 - \frac{d}{aq}\right)\left(1 - \frac{e}{aq}\right) + \frac{de}{abcq}(1-b)(1-c)\right]\phi \\ &\quad + \left(1 - \frac{d}{q}\right)\left(1 - \frac{e}{q}\right)\phi_-(a+) = 0. \end{aligned} \tag{2.5} $$

With the replacements $b = Bq^n$, $c = Cq^n$, $d/a = Dq^n$, $e/a = Aq^n$, this becomes

$$ \begin{aligned} &\frac{(1-Bq^n)(1-Cq^n)(1-Dq^n)(1-Aq^n)}{(1-aDq^n)(1-aAq^n)} \frac{DAa^2}{BCq} \phi_{n+1} \\ &\quad - \left[(1-a)\left(1 - \frac{DAa}{BCq}\right) + a(1-Dq^{n-1})(1-Aq^{n-1}) + \frac{DAa}{BCq}(1-Bq^n)(1-Cq^n)\right]\phi_n \\ &\quad + (1-Daq^{n-1})(1-Aaq^{n-1})\phi_{n-1} = 0, \end{aligned} \tag{2.6} $$

where

$$ \phi_n = {}_3\phi_2\left(\begin{matrix} a, Bq^n, Cq^n \\ aDq^n, aAq^n \end{matrix}; \frac{DAa}{BC}\right). $$

We write $z = q/aDA + a/BC$ so that

$$\begin{aligned} a &= \frac{BC}{2}\left(z \pm \sqrt{z^2 - \frac{4q}{ABCD}}\right) \\ &= BC\lambda_\pm, \quad \text{say.} \end{aligned} \tag{2.7}$$

After renormalization, (2.6) becomes (2.1) with a solution

$$\begin{aligned} X_n^{(1),\pm} &= X_n^{(1),\pm}(z;A,B,C,D) \\ &= \frac{(A,B,C,D)_n}{(BCD\lambda_\pm, ABC\lambda_\pm)_n}(\lambda_\pm)^n\ {}_3\phi_2\left(\begin{matrix} BC\lambda_\pm, Bq^n, Cq^n \\ BCD\lambda_\pm q^n, ABC\lambda_\pm q^n \end{matrix}; AD\lambda_\pm\right). \end{aligned} \tag{2.8}$$

We shall show later that, with a suitable choice of square root branch, $X_n^{(1),-}$ is a minimal solution of (2.1). Because of the symmetry in (2.1) and the fact that the minimal solution is unique up to a constant multiple, the parameter interchanges $A \leftrightarrow C$ or $B \leftrightarrow D$ in the above solution must yield only an $n$ independent multiple of $X_n^{(1),-}$. Let us verify this. If we make the interchange $A \leftrightarrow C$, $X_n^{(1),\pm}$ changes to

$$\xi_n^{(1),\pm} = \frac{(A,B,C,D)_n}{(BAD\lambda_\pm, ABC\lambda_\pm)_n}(\lambda_\pm)^n\ {}_3\phi_2\left(\begin{matrix} BA\lambda_\pm, Bq^n, Aq^n \\ BAD\lambda_\pm q^n, ABC\lambda_\pm q^n \end{matrix}; CD\lambda_\pm\right). \tag{2.9}$$

If we apply the transformation (2.2b) to (2.9) we obtain

$$\begin{aligned} \xi_n^{(1),\pm} &= \frac{(A,B,C,D)_n}{(BAD\lambda_\pm, ABC\lambda_\pm)_n}(\lambda_\pm)^n \frac{(AD\lambda_\pm, BCD\lambda_\pm q^n)_\infty}{(BAD\lambda_\pm q^n, CD\lambda_\pm)_\infty} \\ &\quad \times {}_3\phi_2\left(\begin{matrix} Bq^n, Cq^n, BC\lambda_\pm \\ ABC\lambda_\pm q^n, BCD\lambda_\pm q^n \end{matrix}; AD\lambda_\pm\right), \end{aligned}$$

which is clearly just a constant multiple of $X_n^{(1),\pm}$.

In order to obtain a second solution of (2.1) we start from a different three-term contiguous relation [18];

$$\begin{aligned} &[de(a-b-c) + abc(d+e+q-a-aq)]\phi \\ &\quad + (1-a)(de-abcq)\phi(a+) + bc(d-a)(e-a)\phi(a-) = 0. \end{aligned} \tag{2.10}$$

Changing $a \to aq^n$, we can write the above as

$$\begin{aligned} &[de(aq^n-b-c) + abcq^n(d+e+q-aq^n-aq^{n+1})]Y_n \\ &\quad + (1-aq^n)(de-abcq^{n+1})Y_{n+1} + bc(d-aq^n)(e-aq^n)Y_{n-1} = 0, \end{aligned}$$

where

$$Y_n = {}_3\phi_2\left(\begin{matrix} aq^n, b, c \\ d, e \end{matrix}; \frac{de}{abc}q^{-n}\right).$$

Writing

$$a = A, \quad b = AB\lambda_+, \quad c = AB\lambda_-, \quad d = \frac{Aq}{C}, \quad e = \frac{Aq}{D} \quad \text{and} \quad \frac{q}{CD}\left(\frac{1}{b} + \frac{1}{c}\right) = z,$$

and renormalizing we again arrive at Eq. (2.1) with a new solution

$$X_n^{(2)} = \frac{(A,B)_n}{(AB)^n}\,{}_3\phi_2\left(\begin{matrix}Aq^n, AB\lambda_+, AB\lambda_-\\ Aq/C, Aq/D\end{matrix}; \frac{1}{B}q^{-n+1}\right). \tag{2.11}$$

From (2.11) we obtain additional solutions by parameter interchanges due to the symmetry of (2.1). However, we find that this is not true for the interchanges $B \leftrightarrow C$ or $B \leftrightarrow D$. This can be seen by applying transformation (2.2b) to $X_n^{(2)}$. We get

$$\begin{aligned} X_n^{(2)} &= \frac{(A,B)_n}{(AB)^n}\frac{(q^{-n+1}/D, Aq/B)_\infty}{(Aq/D, q^{-n+1}/B)_\infty}\,{}_3\phi_2\left(\begin{matrix}Aq^n, AD\lambda_-, AD\lambda_+\\ Aq/C, Aq/B\end{matrix}; \frac{q^{-n+1}}{D}\right) \\ &= \frac{(A,D)_n}{(AD)^n}\frac{(Aq/B, q/D)_\infty}{(Aq/D, q/B)_\infty}\,{}_3\phi_2\left(\begin{matrix}Aq^n, AD\lambda_-, AD\lambda_+\\ Aq/C, Aq/B\end{matrix}; \frac{q^{-n+1}}{D}\right), \end{aligned} \tag{2.12}$$

where the right-hand side is a constant multiple of the $B \leftrightarrow D$ interchange applied to $X_n^{(2)}$. It is similarly seen that $B \leftrightarrow C$ does not yield a new solution. However, the interchanges $A \leftrightarrow B$, $A \leftrightarrow C$ and $A \leftrightarrow D$ do yield the following new solutions:

$$X_n^{(3)} = \frac{(A,B)_n}{(AB)^n}\,{}_3\phi_2\left(\begin{matrix}Bq^n, AB\lambda_+, AB\lambda_-\\ Bq/C, Bq/D\end{matrix}; \frac{q^{-n+1}}{A}\right), \tag{2.13}$$

$$X_n^{(4)} = \frac{(B,C)_n}{(BC)^n}\,{}_3\phi_2\left(\begin{matrix}Cq^n, BC\lambda_+, BC\lambda_-\\ Cq/A, Cq/D\end{matrix}; \frac{q^{-n+1}}{B}\right), \tag{2.14}$$

$$X_n^{(5)} = \frac{(B,D)_n}{(BD)^n}\,{}_3\phi_2\left(\begin{matrix}Dq^n, BD\lambda_+, BD\lambda_-\\ Dq/C, Dq/A\end{matrix}; \frac{q^{-n+1}}{B}\right). \tag{2.15}$$

It can be shown that the three-term transformation formula [5, (III.33), p. 245] connects $X_n^{(1)}$ with any two of the solutions $X_n^{(2)}, X_n^{(3)}, X_n^{(4)}$ and $X_n^{(5)}$. One such relation works out to be

$$\begin{aligned} &\left(ABC\lambda_+, AC\lambda_-, \frac{q}{D}, \frac{A}{C}\right)_\infty X_n^{(1),+} - \left(A, A\lambda_-, AB\lambda_+, \frac{Cq}{D}\right)_\infty X_n^{(4)} \\ &\quad = \frac{\left(C, C\lambda_-, \frac{A}{C}, \frac{Aq}{D}, BC\lambda_+, CD\lambda_+, ABD\lambda_+\right)_\infty}{\left(\frac{C}{A}, AD\lambda_+, BCD\lambda_+\right)_\infty} X_n^{(2)}. \end{aligned} \tag{2.16}$$

Another three-term contiguous relation satisfied by balanced ${}_3\phi_2$'s also yields solutions to (2.1). The required contiguous relation, which can be deduced from (2.3), (2.4) and (2.10), is

$$\begin{aligned} &\frac{(1-a)(1-b)(1-c)}{(1-d)(1-e)}\frac{de}{abcq}(de - abcq)\phi \\ &\quad + [abc(d+e-q) + de(1+q-a-b-c)]\,\phi + abcq\left(1-\frac{d}{q}\right)\left(1-\frac{e}{q}\right)\phi_- = 0. \end{aligned} \tag{2.17}$$

Replacing $(a,b,c,d,e)$ by $(aq^{-n},bq^{-n},cq^{-n},dq^{-n},eq^{-n})$ and writing

$$Z_n = {}_3\phi_2\left(\begin{matrix} aq^{-n},bq^{-n},cq^{-n} \\ dq^{-n},eq^{-n} \end{matrix};\frac{de}{abc}q^n\right),$$

we have from (2.17)

$$\begin{aligned} &q^{-n-1}\left(1-\frac{q^{n+1}}{d}\right)\left(1-\frac{q^{n+1}}{e}\right)Z_{n+1}+\left\{\frac{1}{d}+\frac{1}{e}-\frac{q^{n+1}}{de}+\frac{q^n}{abc}(q^n+q^{n+1}-a-b-c)\right\}Z_n \\ &\quad+\frac{q^n}{de}\frac{\left(1-\frac{1}{a}q^n\right)\left(1-\frac{1}{b}q^n\right)\left(1-\frac{1}{c}q^n\right)\left(1-\frac{de}{abcq}q^n\right)}{\left(1-\frac{1}{d}q^n\right)\left(1-\frac{1}{e}q^n\right)}Z_{n-1}=0. \end{aligned} \tag{2.18}$$

Choosing the parameters $a=q/B$, $b=q/A$, $c=q/D$, $d=Cq\lambda_+$ and $e=Cq\lambda_-$ and renormalizing we again obtain Eq. (2.1) with a solution

$$\begin{aligned} X_n^{(6)} &= (-1)^n\left(\frac{q}{ABD}\right)^n q^{-n(n-1)/2}\left(\frac{ABD}{q}\lambda_+,\frac{ABD}{q}\lambda_-\right)_n \\ &\quad\times{}_3\phi_2\left(\begin{matrix} q^{-n+1}/B,q^{-n+1}/A,q^{-n+1}/D \\ C\lambda_+q^{-n+1},C\lambda_-q^{-n+1} \end{matrix};Cq^n\right) \end{aligned} \tag{2.19}$$

and three similar solutions obtained by parameter interchanges $C\leftrightarrow A$ or $C\leftrightarrow B$ or $C\leftrightarrow D$. However, the solution $X_n^{(6)}$ and its $C\leftrightarrow A$, $C\leftrightarrow B$ or $C\leftrightarrow D$ interchanges do not give new solutions. They are related to the solutions $X_n^{(2)},X_n^{(3)},X_n^{(4)}$ and $X_n^{(5)}$ by the transformation formula (2.2c). For example, with the help of this formula we find that

$$X_n^{(6)}=\frac{(Cq/A,Cq/D,Cq/B)_\infty}{(C,C\lambda_+q,C\lambda_-q)_\infty}X_n^{(4)}$$

and thus $X_n^{(6)}$ is the same solution as $X_n^{(4)}$ except for a constant factor.

We next show that continuous dual $q$-Hahn polynomials ([3, pp. 3, 28]) are obtained as a particular case of the above solutions. This is true for the solution $X_n^{(2)}$ if $C=q$ or $X_n^{(6)}$ if $A,B$ or $D=q$. That is why our general case represents associated continuous dual $q$-Hahn polynomials. In order to show this for $X_n^{(2)}$ we first apply the transformation formula [5, (III.34), p. 245], connecting ${}_3\phi_2$'s of types I and II

$$\begin{aligned} {}_3\phi_2\left(\begin{matrix} a,b,c \\ d,e \end{matrix};\frac{de}{abc}\right) &= \frac{(e/b,e/c)_\infty}{(e,e/bc)_\infty}{}_3\phi_2\left(\begin{matrix} d/a,b,c \\ d,bcq/e \end{matrix};q\right) \\ &\quad+\frac{(d/a,b,c,de/bc)_\infty}{(d,e,bc/e,de/abc)_\infty}{}_3\phi_2\left(\begin{matrix} e/b,e/c,de/abc \\ de/bc,eq/bc \end{matrix};q\right). \end{aligned}$$

We have from (2.11)

$$X_n^{(2)} = \frac{(A,B)_n}{(AB)^n}\left[\frac{(AC\lambda_-,AC\lambda_+)_\infty}{(Aq/D,C/B)_\infty}\,{}_3\phi_2\left(\begin{matrix} q^{-n+1}/C,AB\lambda_+,AB\lambda_- \\ Aq/C,Bq/C \end{matrix};q\right)\right.$$
$$\left.+\frac{(q^{-n+1}/C,AB\lambda_+,AB\lambda_-,Aq/B)_\infty}{(Aq/C,Aq/D,B/C,q^{-n+1}/B)_\infty}\,{}_3\phi_2\left(\begin{matrix} q^{-n+1}/B,AC\lambda_-,AC\lambda_+ \\ Aq/B,Cq/B \end{matrix};q\right)\right]. \tag{2.20}$$

When we write $C=q$ in (2.20), the right-hand side becomes

$$\frac{(A,B)_n}{(AB)^n}\frac{(Aq\lambda_-,Aq\lambda_+)_\infty}{(Aq/D,q/B)_\infty}\,{}_3\phi_2\left(\begin{matrix} q^{-n},AB\lambda_+,AB\lambda_- \\ A,B \end{matrix};q\right). \tag{2.21}$$

In order to compare this with continuous dual $q$-Hahn polynomials we apply a transformation which will ultimately change the interval of orthogonality from $[-1/\alpha,1/\alpha]$ when $\alpha=\frac{1}{2}\sqrt{ABCD/q}$ is real, to $[-1,1]$. Take $x=\alpha z=\cos\theta$, $u=\mathrm{e}^{\mathrm{i}\theta}$, which means $u=2\alpha\lambda_+=1/2\alpha\lambda_-$. Thus, omitting constant factors we can write (2.21) as

$$\frac{(A,B)_n}{(AB)^n}\,{}_3\phi_2\left(\begin{matrix} q^{-n},ABu/2\alpha,AB/2\alpha u \\ A,B \end{matrix};q\right), \tag{2.22}$$

which except for a normalization factor of $(2\alpha)^{-n}$, is the same as the continuous dual $q$-Hahn polynomials of Askey and Wilson [3].

We now proceed to show that $X_n^{(1),-}$ is a minimal solution of (2.1) for a particular branch in the complex plane. This is done by evaluating the large-$n$ asymptotics of $X_n^{(1),\pm}$ and Eq. (2.1). Applying the transformation (2.2c) to (2.8) i.e. to $X_n^{(1),\pm}$ we obtain for $n\to\infty$

$$\begin{aligned} X_n^{(1),\pm} &= \frac{(A,B,C,D)_n}{(BCD\lambda_\pm,ABC\lambda_\pm)_n}(\lambda_\pm)^n\frac{(Bq^n,ACD\lambda_\pm q^n,ABCD\lambda_\pm^2)_\infty}{(BCD\lambda_\pm q^n,ABC\lambda_\pm q^n,AD\lambda_\pm)_\infty} \\ &\quad\times{}_3\phi_2\left(\begin{matrix} CD\lambda_\pm,AC\lambda_\pm,AD\lambda_\pm \\ ACD\lambda_\pm q^n,ABCD\lambda_\pm^2 \end{matrix};Bq^n\right) \\ &\approx \text{const. }(\lambda_\pm)^n. \end{aligned} \tag{2.23}$$

On the other hand, asymptotics of the second-order difference equation (2.1) is given by

$$X_{n+1}-zX_n+\frac{q}{ABCD}X_{n-1}=0, \tag{2.24}$$

from which we have for large $n$

$$X_n\approx \text{const. }(\lambda_\pm)^n. \tag{2.25}$$

Choosing the square root branch for which $|\lambda_-/\lambda_+|<1$, the minimal (or the subdominant) solution in terms of the variable $x=\alpha z$ is given by

$$\begin{aligned} X_n^{(\min)}(x) &= X_n^{(1),-} \\ &= (\lambda_-)^n\frac{(A,B,C,D)_n}{(BCD\lambda_-,ABC\lambda_-)_n}\,{}_3\phi_2\left(\begin{matrix} Bq^n,Cq^n,BC\lambda_- \\ BCD\lambda_-q^n,ABC\lambda_-q^n \end{matrix};AD\lambda_-\right) \end{aligned} \tag{2.26}$$

valid for $z\alpha = x \in \mathbb{C} \setminus [-1,1]$, $\alpha = \frac{1}{2}\sqrt{ABCD/q}$, $\lambda_\pm = (1/2\alpha)(x \pm \sqrt{x^2-1})$. Summarizing what we have done so far, we have:

**Theorem 1.** *The functions* $X_n^{(1),\pm}$, $X_n^{(k)}$, $k = 2,3,4,5$ *of* (2.8), (2.11), (2.13)–(2.15), *respectively, are solutions to the recurrence relation* (2.1) *for associated continuous dual q-Hahn polynomials. These solutions are pairwise linearly independent. The minimal solution of* (2.1) *is given, up to a multiplicative factor, by* (2.26) *with the square root branch chosen so that* $(\lambda_-/\lambda_+)^n \to 0$ *as* $n \to \infty$ *with* $z\alpha = x \in \mathbb{C} \setminus [-1,1]$, $\alpha = \frac{1}{2}\sqrt{ABCD/q}$.

## 3. The continued fraction and measure

The infinite continued fraction associated with (2.1) is

$$CF(z) = z - a_0 - \frac{b_1^2}{z - a_1 -} \frac{b_2^2}{z - a_2 - \cdots}, \quad b_n^2 \neq 0, \ n > 0. \tag{3.1}$$

Pincherle's theorem [6, 16] connects the minimal solution of (2.1) with the continued fraction (3.1) by the formula

$$\frac{1}{CF(z)} = \frac{X_0^{(\min)}(x)}{b_0^2 X_{-1}^{(\min)}(x)}, \quad z\alpha = x, \quad \alpha = \frac{1}{2}\sqrt{ABCD/q}. \tag{3.2}$$

Therefore from Theorem 1, we obtain the continued fraction representation

$$\frac{1}{CF(z)} = \frac{ABCD\lambda_-}{q\left(1 - \dfrac{BCD\lambda_-}{q}\right)\left(1 - \dfrac{ABC\lambda_-}{q}\right)} \frac{{}_3\phi_2\left(\begin{matrix} BC\lambda_-, B, C \\ BCD\lambda_-, ABC\lambda_- \end{matrix}; AD\lambda_-\right)}{{}_3\phi_2\left(\begin{matrix} BC\lambda_-, B/q, C/q \\ BCD\lambda_-/q, ABC\lambda_-/q \end{matrix}; AD\lambda_-\right)}. \tag{3.3a}$$

We can also write (3.3a) as

$$\frac{1}{CF(z)} = \frac{1}{\lambda_+\left(1 - \dfrac{1}{A\lambda_+}\right)\left(1 - \dfrac{1}{D\lambda_+}\right)} \frac{{}_3\phi_2\left(\begin{matrix} BC\lambda_-, B, C \\ q/A\lambda_+, q/D\lambda_+ \end{matrix}; AD\lambda_-\right)}{{}_3\phi_2\left(\begin{matrix} BC\lambda_-, B/q, C/q \\ 1/A\lambda_+, 1/D\lambda_+ \end{matrix}; AD\lambda_-\right)}. \tag{3.3b}$$

(3.3a) or the alternative form (3.3b) are valid for

$$z\alpha = x \in C \setminus [-1,1], \quad |\lambda_-/\lambda_+| < 1, \quad \alpha = \tfrac{1}{2}\sqrt{ABCD/q}.$$

In the particular case $C = q$ (the case of continuous dual $q$-Hahn polynomials), (3.3a) reduces to

$$\frac{1}{CF(z)} = \frac{ABD\lambda_-}{(1 - BD\lambda_-)(1 - AB\lambda_-)} {}_3\phi_2\left(\begin{matrix} Bq\lambda_-, B, q \\ DBq\lambda_-, ABq\lambda_- \end{matrix}; AD\lambda_-\right) \tag{3.4}$$

which can also be written with the help of (2.2c) in the form

$$\frac{1}{CF(z)} = \frac{ABD\lambda_-(q, ABD\lambda_-, ABDq\lambda_-^2)_\infty}{(BD\lambda_-, AB\lambda_-, AD\lambda_-)_\infty} {}_3\phi_2\left(\begin{matrix} BD\lambda_-, AB\lambda_-, AD\lambda_- \\ ABD\lambda_-, ABDq\lambda_-^2 \end{matrix}; q\right), \tag{3.5}$$

with explicit pole terms given by the zeros of the denominator $(BD\lambda_-, AB\lambda_-, AD\lambda_-)_\infty$. These pole singularities and their residues determine the discrete component of the spectral measure of orthogonality for continuous dual $q$-Hahn polynomials.

We now determine the absolutely continuous part of the spectrum for the general associated case. If $x = z\alpha \in (-1,1)$, then (2.1) has linearly independent solutions given by the boundary values of (2.26) as $x$ approaches $(-1,1)$ from above and below. With now $\lambda_\pm = (1/2\alpha)[x \pm \mathrm{i}\sqrt{1-x^2}]$, $x = z\alpha$, we have the large $n$ asymptotics

$$X_n^{(\min)}(x+\mathrm{i}0) \approx (\lambda_-)^n \frac{(A,B,C,D)_n (ABCD\lambda_-^2)_\infty}{(BCD\lambda_-, ABC\lambda_-, AD\lambda_-)_\infty}, \tag{3.6}$$

and

$$X_n^{(\min)}(x-\mathrm{i}0) \approx (\lambda_+)^n \frac{(A,B,C,D)_n (ABCD\lambda_+^2)_\infty}{(BCD\lambda_+, ABC\lambda_+, AD\lambda_+)_\infty}. \tag{3.7}$$

Since the minimal solution changes as we cross the line segment $z\alpha = x \in (-1,1)$, we have the representation

$$\frac{1}{CF(z)} = \int_{\mathbb{R}} \frac{\omega(t)\,\mathrm{d}t}{z - t/\alpha} + \text{possible pole terms}, \tag{3.8}$$

Also $\omega(x)$, $x \in (-1,1)$ can be obtained by using the formula [21]

$$\begin{aligned}\omega(x) &= \frac{1}{2\pi\mathrm{i}\alpha} \frac{W(X_{-1}^{(\min)}(x+\mathrm{i}0), X_{-1}^{(\min)}(x-\mathrm{i}0))}{b_0^2 X_{-1}^{(\min)}(x+\mathrm{i}0) X_{-1}^{(\min)}(x-\mathrm{i}0)} \\ &= \frac{1}{2\pi\mathrm{i}\alpha} \lim_{n\to\infty} \frac{W(X_n^{(\min)}(x+\mathrm{i}0), X_n^{(\min)}(x-\mathrm{i}0))}{b_1^2 b_2^2 \cdots b_n^2 b_0^4 X_{-1}^{(\min)}(x+\mathrm{i}0) X_{-1}^{(\min)}(x-\mathrm{i}0)},\end{aligned} \tag{3.9}$$

where

$$W(X_n, Y_n) = X_n Y_{n+1} - X_{n+1} Y_n.$$

Using (2.1), (2.26), (3.6) and (3.7) and simplifying we have

$$\begin{aligned}\omega(x) = \frac{1}{2\pi\sqrt{1-x^2}} &\frac{(A,B,C,D)_\infty \left(\dfrac{1}{u^2}, u^2\right)_\infty}{\left(\dfrac{2\alpha}{Au}, \dfrac{2\alpha}{A}u, \dfrac{2\alpha}{Du}, \dfrac{2\alpha}{D}u, \dfrac{2\alpha q}{BCu}, \dfrac{2\alpha q}{BC}u\right)_\infty} \\ &\times \left[ {}_3\phi_2\left(\begin{matrix} BC\lambda_-, \dfrac{B}{q}, \dfrac{C}{q} \\ \dfrac{BCD\lambda_-}{q}, \dfrac{ABC\lambda_-}{q} \end{matrix}; AD\lambda_-\right) {}_3\phi_2\left(\begin{matrix} BC\lambda_+, \dfrac{B}{q}, \dfrac{C}{q} \\ \dfrac{BCD\lambda_+}{q}, \dfrac{ABC\lambda_+}{q} \end{matrix}; AD\lambda_+\right)\right]^{-1}.\end{aligned} \tag{3.10}$$

In the particular case $C = q$ this reduces to

$$\omega(x) = \frac{1}{2\pi\sqrt{1-x^2}} \frac{(A,B,q,D)_\infty \left(\frac{1}{u^2}, u^2\right)_\infty}{\left(\sqrt{\frac{BD}{A}}\Big/u, \sqrt{\frac{BD}{A}}u, \sqrt{\frac{AB}{D}}\Big/u, \sqrt{\frac{AB}{D}}u, \sqrt{\frac{AD}{B}}\Big/u, \sqrt{\frac{AD}{B}}u\right)_\infty}. \tag{3.11}$$

Taking the appropriate value of the parameters from (2.22) we find that this weight function is the same as the one obtained by Askey and Wilson ([3, p.11, Theorem 2.2 in the special case $d = 0$]) for continuous dual $q$-Hahn polynomials. Summarizing the above we have

**Theorem 2.** *The associated continuous dual q-Hahn polynomials $P_n(x/\alpha)$ given by* (4.12) *of the next section are orthogonal with respect to a measure with absolutely continuous component given by the weight function* (3.10) *on* $(-1,1)$. *In the particular case C=q this absolutely continuous component reduces to* (3.11) *and the discrete spectrum is given by the zeros of* $(BD\lambda_-, AB\lambda_-, AD\lambda_-)_\infty$ *where* $\lambda_- = (x \mp \sqrt{x^2-1})/2\alpha$ *for* $x > 1$ *and* $x < -1$, *respectively.*

The $q \to 1$ limit of continuous dual $q$-Hahn polynomials yields the case of continuous dual Hahn polynomials. The corresponding results of this Section for associated continuous dual Hahn polynomials with $q = 1$ are given by Ismail et al. [13].

## 4. Generating function

The associated continuous dual $q$-Hahn polynomials $P_n(z; A, B, C, D)$ satisfy the second-order difference equation (2.1) i.e. the equation

$$X_{n+1} - \left[z - \left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D}\right) q^n + (1+q)q^{2n-1}\right] X_n + \frac{1}{4\alpha^2}(1-Aq^{n-1})(1-Bq^{n-1})(1-Cq^{n-1})(1-Dq^{n-1})X_{n-1} = 0. \tag{4.1}$$

A renormalized form of (4.1) is, with $x = \alpha z$, $\alpha = \frac{1}{2}\sqrt{ABCD/q}$,

$$(1-Aq^n)(1-Dq^n)\zeta_{n+1} - \left[2x - 2\alpha\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C} + \frac{1}{D}\right) q^n + 2\alpha(1+q)q^{2n-1}\right] \zeta_n + (1-Bq^{n-1})(1-Cq^{n-1})\zeta_{n-1} = 0. \tag{4.2}$$

This is satisfied by the polynomials

$$\zeta_n(x; A, B, C, D) = \frac{(2\alpha)^n P_n(z; A, B, C, D)}{(A)_n (D)_n}.$$

Let the generating function of the polynomials $\zeta_n$ be

$$G(x,t)=\sum_{n=0}^{\infty}\frac{(2\alpha)^n P_n(z;A,B,C,D)}{(A)_n(D)_n}t^n$$
$$=\sum_{n=0}^{\infty}\zeta_n(x;A,B,C,D)t^n. \tag{4.3}$$

An explicit form for the generating function may be obtained by employing the procedure given in [14]. First multiply (4.2) by $t^n$ and sum up the resulting equations from $n=0$ through $n=\infty$. Using the initial conditions for the polynomial solutions of the first kind, we obtain

$$(1-ut)\left(1-\frac{t}{u}\right)G(x,t)-\left(1-\frac{2\alpha tq}{AD}\right)\left(\frac{A}{q}+\frac{D}{q}-\frac{2\alpha t}{B}-\frac{2\alpha t}{C}\right)G(x,tq)$$
$$+\left(1-\frac{2\alpha tq}{AD}\right)\left(1-\frac{2\alpha tq^2}{AD}\right)\frac{AD}{q^2}G(x,tq^2)=\left(1-\frac{A}{q}\right)\left(1-\frac{D}{q}\right). \tag{4.4}$$

In (4.4) we put

$$G(x,t)=\frac{\left(\frac{2\alpha tq}{AD}\right)_\infty}{(tu)_\infty}f(t) \tag{4.5}$$

to obtain

$$\left(1-\frac{t}{u}\right)f(t)-\left(\frac{A}{q}+\frac{D}{q}-\frac{2\alpha t}{B}-\frac{2\alpha t}{C}\right)f(tq)$$
$$+\frac{AD}{q^2}(1-tuq)f(tq^2)=\left(1-\frac{A}{q}\right)\left(1-\frac{D}{q}\right)\frac{(tuq)_\infty}{\left(\frac{2\alpha tq}{AD}\right)_\infty}. \tag{4.6}$$

In the left-hand side of (4.6) we write $f(t)=\sum_{n=0}^{\infty}f_nt^n$ and on the right-hand side we use the $q$-binomial theorem to replace $(tuq)_\infty/(2\alpha tq/AD)_\infty$ by

$$\sum_{n=0}^{\infty}\frac{\left(\frac{ADu}{2\alpha}\right)_n}{(q)_n}\left(\frac{2\alpha tq}{AD}\right)^n.$$

If we now equate coefficients of $t^n$ on both sides, we obtain the first-order difference equation

$$(1-Aq^{n-1})(1-Dq^{n-1})f_n=\frac{1}{u}\left(1-\frac{2\alpha u}{B}q^{n-1}\right)\left(1-\frac{2\alpha u}{C}q^{n-1}\right)f_{n-1}$$
$$+\left(\frac{2\alpha q}{AD}\right)^n\left(1-\frac{A}{q}\right)\left(1-\frac{D}{q}\right)\frac{\left(\frac{ADu}{2\alpha}\right)_n}{(q)_n}. \tag{4.7}$$

Rewriting (4.7) as

$$u^n \frac{(A)_n(D)_n}{\left(\frac{2\alpha u}{B}\right)_n\left(\frac{2\alpha u}{C}\right)_n} f_n = u^{n-1}\frac{(A)_{n-1}(D)_{n-1}}{\left(\frac{2\alpha u}{B}\right)_{n-1}\left(\frac{2\alpha u}{C}\right)_{n-1}} f_{n-1} + \left(\frac{2\alpha q}{AD}\right)^n u^n \frac{\left(\frac{A}{q}\right)_n\left(\frac{D}{q}\right)_n\left(\frac{ADu}{2\alpha}\right)_n}{\left(\frac{2\alpha u}{B}\right)_n\left(\frac{2\alpha u}{C}\right)_n (q)_n}, \tag{4.8}$$

the general solution of (4.7) is

$$f_n = \frac{\left(\frac{2\alpha u}{B}\right)_n\left(\frac{2\alpha u}{C}\right)_n}{u^n(A)_n(D)_n}\left[E + \sum_{j=0}^{n}\frac{\left(\frac{A}{q}\right)_j\left(\frac{D}{q}\right)_j\left(\frac{ADu}{2\alpha}\right)_j}{\left(\frac{2\alpha u}{B}\right)_j\left(\frac{2\alpha u}{C}\right)_j (q)_j}\left(\frac{2\alpha q}{AD}\right)^j u^j\right], \tag{4.9}$$

where $E$ is a constant which by the boundary conditions may be taken as 0. Consequently, we have the generating function

$$\sum_{n=0}^{\infty}\frac{(2\alpha)^n P_n(z;A,B,C,D)}{(A)_n(D)_n}t^n = \frac{\left(\frac{2\alpha tq}{AD}\right)_\infty}{(tu)_\infty}\sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{\left(\frac{2\alpha u}{B}\right)_n\left(\frac{2\alpha u}{C}\right)_n}{(A)_n(D)_n}\frac{\left(\frac{A}{q}\right)_j\left(\frac{D}{q}\right)_j\left(\frac{ADu}{2\alpha}\right)_j}{\left(\frac{2\alpha u}{B}\right)_j\left(\frac{2\alpha u}{C}\right)_j (q)_j}\left(\frac{2\alpha q}{AD}\right)^j u^{-n+j}t^n, \tag{4.10}$$

where $z = x/\alpha$ and $x = \cos\theta$, $u = \mathrm{e}^{\mathrm{i}\theta}$. If we interchange $C \leftrightarrow D$ in (4.10) and write $C = q$ we obtain, with $a = 2\alpha/B, b = 2\alpha/D, c = 2\alpha/A$,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(2\alpha)^n P_n(z;A,B,C)}{(A)_n(q)_n}t^n &= \frac{(ct)_\infty}{(tu)_\infty}\sum_{n=0}^{\infty}\frac{(au)_n(bu)_n}{(q)_n(ab)_n}\left(\frac{t}{u}\right)^n \\ &= \frac{(ct)_\infty}{(tu)_\infty}\,{}_2\phi_1\left(\begin{matrix}au, bu\\ ab\end{matrix};\frac{t}{u}\right).\end{aligned} \tag{4.11}$$

This gives the generating function result for the continuous dual $q$-Hahn polynomials (see [13, (3.3.7), p. 55]).

By comparing coefficients of $t^n$ on the left- and right-hand sides of (4.10) we obtain an explicit expression for our monic associated continuous dual $q$-Hahn polynomials.

$$\begin{aligned} P_n(z) &= P_n(z;A,B,C,D) \\ &= \left(\frac{u}{2\alpha}\right)^n \frac{\left(A,D,\dfrac{2\alpha q}{ADu}\right)_n}{(q)_n} \sum_{\ell=0}^{n} \left\{ \frac{\left(q^{-n},\dfrac{2\alpha u}{B},\dfrac{2\alpha u}{C}\right)_\ell}{\left(\dfrac{ADu}{2\alpha}q^{-n},A,D\right)_\ell} \left(\frac{AD}{2\alpha u}\right)^\ell \right. \\ &\qquad \left. \times \sum_{j=0}^{\ell} \frac{\left(\dfrac{A}{q},\dfrac{D}{q},\dfrac{ADu}{2\alpha}\right)_j}{\left(q,\dfrac{2\alpha u}{B},\dfrac{2\alpha u}{C}\right)_j} \left(\frac{2\alpha uq}{AD}\right)^j \right\}, \end{aligned} \tag{4.12}$$

$$\alpha = \tfrac{1}{2}\sqrt{ABCD/q}, \quad x = \tfrac{1}{2}(u+1/u), \quad z = x/\alpha.$$

Note that $P_n(z)$ is of course symmetric in the parameters $A,B,C,D$. The symmetry under the interchanges $A \leftrightarrow D$ or $B \leftrightarrow C$ is obvious from (4.12). However, the symmetry under the interchanges $A \leftrightarrow B, A \leftrightarrow C, D \leftrightarrow B$ or $D \leftrightarrow C$ is hidden. $P_n(z)$ is also symmetric under the interchange $u \leftrightarrow u^{-1}$. Again this is not apparent from (4.12). Applying any one of these hidden symmetry interchanges to (4.12) gives us a type of transformation formula.

A different expression for $P_n(z)$ is obtained in Section 7. It is derived from the associated Askey–Wilson polynomial formula of Ismail and Rahman [15]. In order to contrast (7.3) with (4.12) we repeat it here as

$$\begin{aligned} P_n(z;A,B,C,D) &= \frac{(B,C)_n}{(BC)^n} \sum_{k=0}^{n} \left\{ \frac{\left(q^{-n},\sqrt{\dfrac{BCq}{AD}}u,\sqrt{\dfrac{BCq}{AD}}\dfrac{1}{u}\right)_k q^k}{(q,B,C)_k} \right. \\ &\qquad \left. \times \sum_{j=0}^{n-k} \frac{\left(\dfrac{A}{q},\dfrac{D}{q},q^{k+1},q^{k-n}\right)_j}{(q,Cq^k,Bq^k,q^{-n})_j} \left(\frac{BCq}{AD}\right)^j \right\}. \end{aligned} \tag{4.13}$$

This formula also does not reveal the full symmetry with respect to $A,B,C,D$. However, it does make explicit the $u \leftrightarrow u^{-1}$ symmetry.

## 5. Four limiting cases

We now take successive limits $D \to \infty$, $C \to \infty$, $B \to \infty$, $A \to \infty$.

*5.1. Associated big q-Laguerre ($D \to \infty$)*

The recurrence relation (2.1) becomes

$$L_{n+1} - \left[z - \left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) q^n + (1+q)q^{2n-1}\right] L_n - \frac{q^n}{ABC}(1 - Aq^{n-1})(1 - Bq^{n-1})(1 - Cq^{n-1})L_{n-1} = 0. \tag{5.1}$$

If we write $A = aq$, $B = bq$, $C = -abq/s$ and $z = t/abq$, and renormalize with a factor $(abq)^n$, (5.1) gives Eq. (4.3) of Ismail and Libis [14] for associated big $q$-Laguerre polynomials in monic form.

The solutions of (5.1) are obtained as $D \to \infty$ limits of solutions of (2.1). We have $\lambda_+ \approx z$ and $\lambda_- \approx q/ABCDz$ and consequently from (2.8)

$$\begin{aligned} L_n^{(1)}(z;A,B,C) &= \lim_{D\to\infty} X_n^{(1),-}(z;A,B,C,D) \\ &= (-1)^n \frac{q^{n(n+1)/2}}{(ABCz)^n} \frac{(A,B,C)_n}{\left(\frac{q}{Az}\right)_n} {}_2\phi_1\left(\begin{matrix} Bq^n, Cq^n \\ q^{n+1}/Az \end{matrix}; \frac{q}{BCz}\right), \end{aligned} \tag{5.2}$$

and from (2.12)

$$\begin{aligned} L_n^{(2)}(z;A,B,C) &= \frac{(q/B)_\infty}{(Aq/B)_\infty} \lim_{D\to\infty} X_n^{(2)}(z;A,B,C,D) \\ &= (-1)^n q^{n(n-1)/2} (A)_n A^{-n} {}_2\phi_2\left(\begin{matrix} Aq^n, \dfrac{q}{BCz} \\ \dfrac{Aq}{B}, \dfrac{Aq}{C} \end{matrix}; Azq^{-n+1}\right). \end{aligned} \tag{5.3}$$

Interchanging $A \leftrightarrow B$ and $A \leftrightarrow C$ in (5.3) yields the limits of solutions $X_n^{(3)}$, $X_n^{(4)}$. We have

$$L_n^{(3)}(z;A,B,C) = (-1)^n q^{n(n-1)/2} (B)_n B^{-n} {}_2\phi_2\left(\begin{matrix} Bq^n, \dfrac{q}{ACz} \\ \dfrac{Bq}{C}, \dfrac{Bq}{A} \end{matrix}; Bzq^{-n+1}\right), \tag{5.4}$$

and

$$L_n^{(4)}(z;A,B,C) = (-1)^n q^{n(n-1)/2} (C)_n C^{-n} {}_2\phi_2\left(\begin{matrix} Cq^n, \dfrac{q}{ABz} \\ \dfrac{Cq}{A}, \dfrac{Cq}{B} \end{matrix}; Czq^{-n+1}\right). \tag{5.5}$$

Limits of $X_n^{(6)}$ given by (2.19) and its parameter interchanges will be in terms of ${}_2\phi_1$ and these three limits will simply be transforms of $L_n^{(2)}, L_n^{(3)}$ and $L_n^{(4)}$. We write below one of these limits

$$\begin{aligned} L_n^{(5)}(z;A,B,C) &= \lim_{D\to\infty} X_n^{(6)}(z;A,B,C,D) \\ &= z^n (1/Cz)_n \, {}_2\phi_1\left(\begin{matrix} q^{-n+1}/A, q^{-n+1}/B \\ Czq^{-n+1} \end{matrix}; Cq^n\right). \end{aligned} \tag{5.6}$$

Next we take the $D \to \infty$ limit of the explicit expression (4.12) for the polynomial solution to get

$$P_n^{(1)}(z;A,B,C) = \lim_{D\to\infty} P_n(z;A,D,C,B)$$
$$= z^n \frac{\left(A,B,\frac{q}{ABz}\right)_n}{(q)_n} \sum_{\ell=0}^{n} \left\{ \frac{\left(q^{-n},\frac{ABCz}{q}\right)_\ell (-1)^\ell q^{\ell(\ell-1)/2}}{(ABzq^{-n},A,B)_\ell} \left(\frac{AB}{C}\right)^\ell \right.$$
$$\left. \times \sum_{j=0}^{\ell} \frac{\left(\frac{A}{q},\frac{B}{q},ABz\right)_j}{\left(\frac{ABCz}{q},q\right)_j} (-1)^j q^{-j(j-1)/2} \left(\frac{Cq}{AB}\right)^j \right\}. \tag{5.7}$$

Note that before taking limit $D \to \infty$ of (4.12) we have made the parameter interchange $B \leftrightarrow D$ in (4.12).

The minimal solution of (5.1) is $L_n^{(1)}$ and we therefore have the continued fraction representation

$$\frac{1}{z-a_0}-\frac{b_1^2}{z-a_1}-\frac{b_2^2}{z-a_2}-\cdots = \frac{{}_2\phi_1\left(\begin{matrix}B,C\\ q/Az\end{matrix};\frac{q}{BCz}\right)}{z(1-1/Az)\,{}_2\phi_1\left(\begin{matrix}B/q,C/q\\ 1/Az\end{matrix};\frac{q}{BCz}\right)},$$

$$a_n = \left(\frac{1}{A}+\frac{1}{B}+\frac{1}{C}\right)q^n - (1+q)q^{2n-1}, \tag{5.8}$$
$$b_n^2 = -\frac{q^n}{ABC}(1-Aq^{n-1})(1-Bq^{n-1})(1-Cq^{n-1}).$$

The associated orthogonality is discrete and only explicit in the case $A,B$ or $C=q$.

### 5.2. *Associated wall* $(C,D \to \infty)$

With $C \to \infty$ the recurrence relation (5.1) changes to

$$W_{n+1} - \left[z - \left(\frac{1}{A}+\frac{1}{B}\right)q^n + (1+q)q^{2n-1}\right] W_n + \frac{q^{2n-1}}{AB}(1-Aq^{n-1})(1-Bq^{n-1})W_{n-1} = 0. \tag{5.9}$$

Solutions are

$$W_n^{(1)}(z;A,B) = \lim_{C\to\infty} L_n^{(1)}(z;A,B,C)$$
$$= \left(\frac{q}{ABz}\right)^n q^{n(n-1)} \frac{(A,B)_n}{(q/Az)_n}\, {}_1\phi_1\left(\begin{matrix}Bq^n\\ q^{n+1}/Az\end{matrix};\frac{q^{n+1}}{Bz}\right), \tag{5.10}$$

$$W_n^{(2)}(z;A,B) = \lim_{C\to\infty} L_n^{(2)}(z;A,B,C)$$
$$= (-1)^n q^{n(n-1)/2}(A)_n A^{-n}\, {}_1\phi_1\left(\begin{matrix}Aq^n\\ Aq/B\end{matrix};Azq^{-n+1}\right), \tag{5.11}$$

and

$$\begin{aligned} W_n^{(3)}(z;A,B) &= \lim_{C\to\infty} L_n^{(3)}(z;A,B,C) \\ &= (-1)^n q^{n(n-1)/2}(B)_n B^{-n}\ {}_1\phi_1\left(\begin{matrix} Bq^n \\ Bq/A \end{matrix}; Bzq^{-n+1}\right), \end{aligned} \tag{5.12}$$

which is just $A \leftrightarrow B$ interchange of (5.11).

Also from (5.6)

$$\begin{aligned} W_n^{(4)}(z;A,B) &= \lim_{C\to\infty} L_n^{(5)}(z;A,B,C) \\ &= z^n {}_2\phi_0\left(\begin{matrix} q^{-n+1}/A, q^{-n+1}/B \\ - \end{matrix}; \frac{q^{2n-1}}{z}\right). \end{aligned} \tag{5.13}$$

The series representing the above ${}_2\phi_0$ converges only when it terminates. The relevant terminating cases are when $A = q$ or $B = q$.

A limit $C \to \infty$ of (5.7) gives the explicit expression for associated Wall polynomials

$$\begin{aligned} P_n^{(2)}(z;A,B) &= \lim_{C\to\infty} P_n^{(1)}(z;A,B,C) \\ &= z^n \frac{(q/ABz,A,B)_n}{(q)_n} \sum_{\ell=0}^{n} \left\{ \frac{(q^{-n})_\ell q^{\ell(\ell-1)}}{(q^{-n}ABz,A,B)_\ell} \left(\frac{A^2B^2z}{q}\right)^\ell \right. \\ &\qquad \left. \times \sum_{j=0}^{\ell} \frac{(A/q,B/q,ABz)_j}{(q)_j} \left(\frac{q}{AB}\right)^{2j} z^{-j} q^{-j(j-1)} \right\}. \end{aligned} \tag{5.14}$$

The minimal solution of (5.9) is given by (5.10) and therefore we have the associated continued fraction representation

$$\frac{1}{z-a_0} \, \frac{b_1^2}{-z-a_1} \, \frac{b_2^2}{-z-a_2} - \cdots = \frac{{}_1\phi_1\left(\begin{matrix} B \\ q/Az \end{matrix}; \frac{q}{Bz}\right)}{z(1-1/Az)_1\phi_1\left(\begin{matrix} B/q \\ 1/Az \end{matrix}; \frac{1}{Bz}\right)}, \tag{5.15}$$

where

$$a_n = \left(\frac{1}{A} + \frac{1}{B}\right) q^n - (1+q) q^{2n-1},$$

$$b_n^2 = \frac{q^{2n-1}}{AB}(1 - Aq^{n-1})(1 - Bq^{n-1}).$$

When $A$ or $B = q$, (5.15) has associated with it an explicit discrete orthogonality for $P_n^{(2)}(z)$. We note here that our associated Wall polynomials given by (5.9) reduce to the Wall polynomials when $B = q$. In fact, if we make the substitutions $A = aq$, $B = q$, $z = x/aq$ in (5.9) and renormalize, the equation can be written in the form (see [4])

$$q^n(1 - aq^{n+1})p_{n+1}(x) - \left[q^n(1 - aq^{n+1}) + aq^n(1 - q^n) - x\right] p_n(x) + aq^n(1 - q^n)p_{n-1}(x) = 0$$

with

$$p_n(x) = p_n(x;a) = (-1)^n q^{-n(n-1)/2} \frac{(aq)^n}{(aq)_n} W_n^{(2)}\left(\frac{x}{aq}; aq, q\right)$$

where we use solution $W_n^{(2)}$ from (5.11). From (5.28) we obtain the standard expression for Wall polynomials (see [17, (3.20.1), p. 83] and [4, p. 198])

$$p_n(x;a) = {}_2\phi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix}; qx\right).$$

*5.3. Limit wall $(B, C, D \to \infty)$*

The three-term recurrence now becomes

$$U_{n+1} - \left[z - \frac{q^n}{A} + (1+q)q^{2n-1}\right] U_n - \frac{q^{3n-2}}{A}(1 - Aq^{n-1})U_{n-1} = 0. \tag{5.16}$$

Using (5.10), (5.11) and (5.13), we have the solutions

$$\begin{aligned} U_n^{(1)}(z;A) &= \lim_{B\to\infty} W_n^{(1)}(z;A,B) \\ &= (-1)^n \left(\frac{q}{Az}\right)^n q^{3n(n-1)/2} \frac{(A)_n}{\left(\frac{q}{Az}\right)_n} {}_0\phi_1\left(\begin{matrix} - \\ q^{n+1}/Az \end{matrix}; \frac{q^{2n+1}}{z}\right), \end{aligned} \tag{5.17}$$

$$\begin{aligned} U_n^{(2)}(z;A) &= \lim_{B\to\infty} W_n^{(2)}(z;A,B) \\ &= (-1)^n q^{n(n-1)/2} \frac{(A)_n}{A^n} {}_1\phi_1\left(\begin{matrix} Aq^n \\ 0 \end{matrix}; Azq^{-n+1}\right), \end{aligned} \tag{5.18}$$

and

$$U_n^{(3)}(z;A) = \lim_{B\to\infty} W_n^{(4)}(z;A,B) = z^n {}_2\phi_0\left(\begin{matrix} q^{-n+1}/A, 0 \\ - \end{matrix}; \frac{q^{2n-1}}{z}\right), \tag{5.19}$$

where $U_n^{(3)}$ converges when it terminates with say $A = q$.

A direct limit of (5.14) i.e. $P_n^{(2)}(z;A,B)$ as $B \to \infty$ leads to an indeterminate form. However, we can obtain the explicit form of the polynomials by applying the method of Section 4 ab initio to Eq. (5.16). The result is

$$\begin{aligned} P_n^{(3)}(z;A) = \frac{q^{n^2}}{A^n} \frac{(A)_n}{(q)_n} \sum_{\ell=0}^{n} \Bigg\{ & \frac{(q^{-n})_\ell}{(A)_\ell} (-1)^\ell q^{-\ell(\ell-1)/2} (Az)^\ell \\ & \times \sum_{j=0}^{\ell} \frac{(A/q)_j}{(q)_j} q^{j(j-1)/2} (-Az)^{-j} \Bigg\}. \end{aligned} \tag{5.20}$$

The minimal solution of (5.16) being (5.17) we have the related continued fraction

$$\frac{1}{z-a_0}\,\frac{b_1^2}{-z-a_1}\,\frac{b_2^2}{-z-a_2}\,-\cdots = \frac{1}{z}\,\frac{{}_0\phi_1\left(\begin{matrix}-\\ q/Az\end{matrix};\frac{q}{z}\right)}{(1-1/Az)_0\phi_1\left(\begin{matrix}-\\ 1/Az\end{matrix};\frac{1}{qz}\right)},$$

$$= \frac{1}{z}\,\frac{{}_1\phi_1\left(\begin{matrix}A\\ 0\end{matrix};\frac{q}{Az}\right)}{{}_1\phi_1\left(\begin{matrix}A/q\\ 0\end{matrix};\frac{1}{Az}\right)}, \tag{5.21}$$

where

$$a_n = \frac{q^n}{A} - (1+q)q^{2n-1}$$

$$b_n^2 = -\frac{q^{3n-2}}{A}(1-Aq^{n-1}).$$

The second expression on the right-hand side of (5.21) comes from the transformation

$${}_0\phi_1\left(\begin{matrix}-\\ c\end{matrix};cz\right) = \frac{1}{(c)_\infty}\,{}_1\phi_1\left(\begin{matrix}z\\ 0\end{matrix};c\right),$$

which can be derived from (5.30) by letting $a=0$. Note that when $A \neq q$, the right-hand side of (5.21) is a meromorphic function of $z$. When $A \to q$ the singularities coalesce at $z=0$ to produce an essential singularity.

### 5.4. A fourth limit $(A,B,C,D \to \infty)$

The three-term recurrence is now

$$V_{n+1} - [z+(1+q)q^{2n-1}]V_n + q^{4n-3}V_{n-1} = 0. \tag{5.22}$$

Using (5.17) and (5.19) we have the solutions

$$V_n^{(1)}(z) = \lim_{A\to\infty} U_n^{(1)}(z;A) = q^{2n(n-1)}\left(\frac{q}{z}\right)^n {}_0\phi_1\left(\begin{matrix}-\\ 0\end{matrix};\frac{q^{2n+1}}{z}\right), \tag{5.23}$$

$$V_n^{(2)}(z) = \lim_{A\to\infty} U_n^{(3)}(z;A) = z^n{}_2\phi_0\left(\begin{matrix}0,0\\ -\end{matrix};\frac{q^{2n-1}}{z}\right). \tag{5.24}$$

The solution $V_n^{(2)}$ is divergent and is thus only a formal solution. The associated polynomials are given by

$$P_n^{(4)}(z) = \lim_{A\to\infty} P_n^{(3)}(z;A)$$

$$= \frac{(-1)^n q^{n^2} q^{n(n-1)/2}}{(q)_n}\sum_{\ell=0}^{n}\left\{(q^{-n})_\ell q^{-\ell(\ell-1)} z^\ell \sum_{j=0}^{\ell}\frac{q^{j(j-1)}}{(q)_j}(qz)^{-j}\right\}. \tag{5.25}$$

The minimal solution of (5.22), being given by (5.23), yields the associated continued fraction

$$\frac{1}{z-a_0}\,\underset{-}{}\,\frac{b_1^2}{z-a_1}\,\underset{-}{}\,\frac{b_2^2}{z-a_2}\,\underset{-}{}\cdots=\frac{1}{z}\frac{{}_0\phi_1\left(\begin{matrix}-\\0\end{matrix};q/z\right)}{{}_0\phi_1\left(\begin{matrix}-\\0\end{matrix};1/qz\right)},\tag{5.26}$$

where

$$a_n=-(1+q)q^{2n-1},$$
$$b_n^2=q^{4n-3}.$$

We can also write (5.26) more explicitly as

$$\frac{1}{z+(1+q)q^{-1}}\,\underset{-}{}\,\frac{q}{z+(1+q)q}\,\underset{-}{}\,\frac{q^5}{z+(1+q)q^3}\,\underset{-}{}\cdots=\frac{1}{z}\frac{\sum_{n=0}^{\infty}\dfrac{q^{n^2}}{(q)_n}z^{-n}}{\sum_{n=0}^{\infty}\dfrac{q^{n^2-2n}}{(q)_n}z^{-n}}.\tag{5.27}$$

Using (5.22), (5.23) and (5.26) we have the following:

**Corollary.** *If* $0<q<1$ *and* $n$ *is an integer, then*

$$f_n(z)={}_0\phi_1\left(\begin{matrix}-\\0\end{matrix};\frac{q^{2n+1}}{z}\right)$$

*has only real simple negative zeros which interlace those of* $f_{n+1}(z)$.

**Proof.** If $0<q<1$ then (5.26) is a completely convergent positive definite $J$-fraction which can be represented as a Stieltjes transform of a unique positive discrete measure [21]. This means that (5.26) can have only simple pole sigularities on the real axis with positive residues. Hence we must have simple real intertwining zeros for $f_0(z)$ and $f_{-1}(z)$. From the series representation $f_n(z)=\sum_{k=0}^{\infty}(q^{k(k-1)}/(q)_k)(q^{2n+1}/z)^k$ we see that the zeros of $f_{-1}(z)$ and $f_0(z)$ must be negative. This establishes the result for $n=-1$. The proof for other values of $n$ is the same if one starts from the continued fraction

$$\frac{1}{z-a_{n+1}}\,\underset{-}{}\,\frac{b_{n+2}^2}{z-a_{n+2}}\,\underset{-}{}\cdots=\frac{1}{z}\frac{f_{n+1}(z)}{f_n(z)}.\qquad\square$$

*Note* 1. There are similar corollaries associated with the positive-definite cases for the continued fractions (5.8), (5.15) and (5.21). These require special parameter conditions. See also the remarks after (6.18), (6.28), (6.33) and (6.63).

*Note* 2. The identities

$${}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};z\right)=\frac{(b,az)_\infty}{(c,z)_\infty}\,{}_2\phi_1\left(\begin{matrix}c/b,z\\az\end{matrix};b\right),\tag{5.28}$$

$${}_2\phi_1\left(\begin{matrix}a,b\\c\end{matrix};z\right)=\frac{(az)_\infty}{(z)_\infty}\,{}_2\phi_2\left(\begin{matrix}a,c/b\\c,az\end{matrix};bz\right),\tag{5.29}$$

$$\frac{(z)_\infty}{(az)_\infty}\,{}_2\phi_1\left(\begin{matrix}a,0\\c\end{matrix};z\right) = {}_1\phi_2\left(\begin{matrix}a\\c,az\end{matrix};cz\right)$$
$$= \frac{1}{(c)_\infty}\,{}_1\phi_1\left(\begin{matrix}z\\az\end{matrix};c\right) \tag{5.30}$$

can be used to relate some of the above solutions. For (5.28) and (5.29) see [5, (III.1), (III.4), p. 241]. (5.30) follows from $b \to 0$ in (5.28) and (5.29).

## 6. Additional limits

There are other less obvious limiting cases which we can obtain from (2.1) and its solutions. These may also be re-expressed as birth and death processes. In the cases of Sections 6.4 and 6.5 there are processes based seperately on the even and odd approximants with $z^2$ replaced by $z$ [7, Section 4]. We begin with

### *6.1. Associated Al-Salam–Chihara*

In (2.1) we put $D = \delta C$, multiply by $C$, replace $zC$ by $z$ and renormalize and let $C \to 0$ to get

$$Q_{n+1} - (z - (1+\delta^{-1})q^n)Q_n + \frac{q}{AB\delta}(1 - Aq^{n-1})(1 - Bq^{n-1})Q_{n-1} = 0 \tag{6.1}$$

with

$$Q_n = \lim_{C\to 0} C^n X_n(z/C; A, B, C, C\delta). \tag{6.2}$$

This can be recognized as the recurrence relation for associated Al-Salam–Chihara polynomials given in [2, (3.54) with $a = (1+\gamma^2 b)/\gamma$] if we make the replacements

$$(A, B, \delta^{-1}, z) \to (b\gamma/c, \gamma q, b\gamma^2, x) \tag{6.3}$$

and renormalize (see also [20]).

If $A = q$ or $B = q$ then (6.1) becomes the recurrence for monic Al-Salam–Chihara polynomials [2]. For other references to this case see [17, p. 63].

We record here the solutions to (6.1) based on (6.2) and the solutions we obtained for $X_n$ in Sections 2 and 4.

Using (2.8) we have

$$Q_n^{(1),\pm}(z; A, B, \delta) = \lim_{C\to 0} C^n X_n^{(1),\pm}\left(\frac{z}{C}; A, B, C, \delta C\right)$$
$$= \frac{(A,B)_n}{(AB\Lambda_\pm)_n}(\Lambda_\pm)^n{}_2\phi_1\left(\begin{matrix}B\Lambda_\pm, Bq^n\\AB\Lambda_\pm q^n\end{matrix}; A\delta\Lambda_\pm\right), \tag{6.4}$$

$$\Lambda_\pm = \tfrac{1}{2}\left(z \pm \sqrt{z^2 - \gamma^2}\right),\qquad \gamma = 2(q/AB\delta)^{1/2}. \tag{6.5}$$

Also from (2.14), (2.15) and (2.19), respectively, we similarly obtain

$$Q_n^{(2)}(z;A,B,\delta) = (B)_n B^{-n} {}_2\phi_1\left(\begin{matrix} BA_+, BA_- \\ q/\delta \end{matrix}; q^{-n+1}/B\right), \tag{6.6}$$

$$Q_n^{(3)}(z;A,B,\delta) = (B)_n(\delta B)^{-n} {}_2\phi_1\left(\begin{matrix} B\delta A_+, B\delta A_- \\ q\delta \end{matrix}; q^{-n+1}/B\right), \tag{6.7}$$

$$\begin{aligned} Q_n^{(4)}(z;A,B,\delta) &= (-q/AB\delta)^n q^{-n(n-1)/2} \\ &\quad \times (AB\delta A_+/q, AB\delta A_-/q)_n {}_2\phi_2\left(\begin{matrix} q^{-n+1}/A, q^{-n+1}/B \\ A_+ q^{-n+1}, A_- q^{-n+1} \end{matrix}; q/\delta\right), \end{aligned} \tag{6.8}$$

with $Q_n^{(4)}$ proportional to $Q_n^{(2)}$ via (5.28) and (5.29).

Using (4.12) we obtain the explicit polynomial formula (first make the interchange $B \leftrightarrow D$)

$$\begin{aligned} Q_n(z;A,B,\delta) &= \lim_{C\to 0} C^n P_n(z/C; A, \delta C, C, B) \\ &= (\tfrac{1}{2}\gamma u)^n \frac{(A,B)_n}{(q)_n} \\ &\quad \times \left\{ \sum_{\ell=0}^{n} \frac{(q^{-n}, 2u/\gamma\delta, 2u/\gamma)_\ell}{(A,B)_\ell} (-1)^\ell u^{-2\ell} q^{n\ell} q^{-\ell(\ell-1)/2} \right. \\ &\quad \left. \times \sum_{j=0}^{\ell} \frac{(A/q, B/q)_j (-1)^j u^{2j} q^{j(j+1)/2}}{(q, 2u/\gamma\delta, 2u/\gamma)_j} \right\}, \end{aligned} \tag{6.9}$$

$$z = \tfrac{1}{2}\gamma(u + u^{-1}), \qquad \gamma = 2(q/AB\delta)^{1/2}.$$

With the choice of square root branch chosen so that $|A_-/A_+| < 1$ for $x = z/\gamma \in \mathbb{C}\setminus[-1,1]$, we have the minimal solution to (6.1) given by (6.5). As a consequence we may give an explicit expression for the corresponding continued fraction and the absolutely continuous component of the measure which gives its representation as a Stieltjes transform. The calculations proceed as in Section 3 and yield the following. For $z/\gamma = x \in \mathbb{C}\setminus[-1,1]$ and $|A_-/A_+| < 1$,

$$\begin{aligned} \frac{1}{z - a_0 -}\,\frac{b_1^2}{z - a_1 -}\,\frac{b_2^2}{z - a_2 -}\cdots &= \frac{AB\delta A_-}{q(1 - ABA_-/q)} \frac{{}_2\phi_1\left(\begin{matrix} BA_-, B \\ ABA_- \end{matrix}; A\delta A_-\right)}{{}_2\phi_1\left(\begin{matrix} BA_-, B/q \\ ABA_-/q \end{matrix}; A\delta A_-\right)} \\ &= \int_{-1}^{1} \frac{\omega(t)\,\mathrm{d}t}{z - \gamma t} + \text{possible pole terms}, \end{aligned} \tag{6.10}$$

with

$$\begin{aligned} \omega(x) &= \frac{1}{2\pi\sqrt{1-x^2}} \frac{(A, B, 1/u^2, u^2)_\infty}{(A\delta\gamma u/2, A\delta\gamma/2u, AB\gamma u/2q, AB\gamma/2qu)_\infty} \\ &\quad \times \left[ {}_2\phi_1\left(\begin{matrix} BA_-, B/q \\ ABA_-/q \end{matrix}; A\delta A_-\right) {}_2\phi_1\left(\begin{matrix} BA_+, B/q \\ ABA_+/q \end{matrix}; A\delta A_+\right) \right]^{-1}. \end{aligned} \tag{6.11}$$

Note that (6.11) agrees with the weight function derived in [2,(3.64) with $a = q(1+\delta^{-1})/B, b = q^2/B^2\delta, c = q/AB\delta, \gamma = B/q$ and $x^2$ replaced by $x^2/4c$].

### 6.2. Associated Al-Salam–Carlitz I

We take the $B \to \infty$ limit of (6.1) and its corresponding solutions to obtain the case of associated Al-Salam–Carlitz I. The recurrence becomes

$$R_{n+1} - [z - (1+\delta^{-1})q^n]R_n - \frac{q^n}{A\delta}(1 - Aq^{n-1})R_{n-1} = 0, \tag{6.12}$$

with solutions from (6.5)–(6.8) given by

$$\begin{aligned} R_n^{(1)}(z;A,\delta) &= \lim_{B\to\infty} Q_n^{(1),-}(z;A,B,\delta) \\ &= \frac{(A)_n}{(q/\delta z)_n}(-q/A\delta z)^n q^{n(n-1)/2}{}_1\phi_1\left(\begin{matrix} q/Az\delta \\ q^{n+1}/z\delta \end{matrix};q^{n+1}/z\right), \end{aligned} \tag{6.13}$$

$$\begin{aligned} R_n^{(2)}(z;A,\delta) &= \lim_{B\to\infty} Q_n^{(2)} \\ &= (-1)^n q^{n(n-1)/2}{}_1\phi_1\left(\begin{matrix} q/Az\delta \\ q/\delta \end{matrix};zq^{-n+1}\right), \end{aligned} \tag{6.14}$$

$$\begin{aligned} R_n^{(3)}(z;A,\delta) &= \lim_{B\to\infty} Q_n^{(3)} \\ &= (-\delta)^{-n} q^{n(n-1)/2}{}_1\phi_1\left(\begin{matrix} q/Az \\ q\delta \end{matrix};\delta zq^{-n+1}\right), \end{aligned} \tag{6.15}$$

$$\begin{aligned} R_n^{(4)}(z;A,\delta) &= \lim_{B\to\infty} Q_n^{(4)} \\ &= (1/z)_n\, z^n{}_1\phi_1\left(\begin{matrix} q^{-n+1}/A \\ zq^{-n+1} \end{matrix};q/\delta\right), \end{aligned} \tag{6.16}$$

with $R_n^{(4)}$ proportional to $R_n^{(2)}$ via (5.29) and (5.30); namely

$${}_1\phi_1\left(\begin{matrix} c/b \\ c \end{matrix};bz\right) = \frac{(bz)_\infty}{(c)_\infty}{}_1\phi_1\left(\begin{matrix} z \\ bz \end{matrix};c\right). \tag{6.17}$$

The minimal solution to (6.12) is given by (6.13). Using Pincherle's theorem we obtain the continued fraction representation

$$\begin{aligned} &\frac{1}{z-(1+\delta^{-1})}\,\frac{}{+}\,\frac{q(1-A)/A\delta}{z-(1+\delta^{-1})q}\,\frac{}{+}\,\frac{q^2(1-Aq)/A\delta}{z-(1+\delta^{-1})q^2}\,\frac{}{+}\cdots \\ &= \frac{1}{z(1-1/\delta z)}\frac{{}_1\phi_1\left(\begin{matrix} q/Az\delta \\ q/z\delta \end{matrix};q/z\right)}{{}_1\phi_1\left(\begin{matrix} q/Az\delta \\ 1/z\delta \end{matrix};1/z\right)}. \end{aligned} \tag{6.18}$$

This is a positive definite $J$-fraction in the case when $0 < q < 1$ and $A < 1$, $A\delta < 0$. We may then deduce that the zeros of the ${}_1\phi_1$'s in the numerator and denominator on the right-hand side of (6.18) are real and simple and interlace. See for example the Corollary in Section 5.4.

When $A=q$, the pole singularities in (6.18) become explicit, since we may then use (6.17) to obtain

$$\frac{1}{z-(1+\delta^{-1})}+\frac{(1-q)/\delta}{z-(1+\delta^{-1})q}+\frac{q(1-q^2)/\delta}{z-(1+\delta^{-1})q^2}+\cdots$$
$$=\frac{(q/\delta z)_\infty}{z(1/z)_\infty(1/z\delta)_\infty}\,{}_1\phi_1\left(\begin{matrix}1/z\delta\\ q/z\delta\end{matrix};q/z\right)$$
$$=\sum_{n=0}^{\infty}\left\{\frac{q^n}{(z-q^n)(q)_n(\delta q)_n(\delta^{-1})_\infty}+\frac{q^n}{(z-q^n/\delta)(q)_n(q/\delta)_n(\delta)_\infty}\right\}. \tag{6.18'}$$

In the last equality we have assumed that $\delta\neq q^{-m}, m$ an integer. The explicit polynomial solution to (6.12) can be obtained from the $B\to\infty$ limit of (6.9). We get

$$R_n(z;A,\delta)=\lim_{B\to\infty}Q_n(z;A,B,\delta)$$
$$=(-q/A\delta z)^n\frac{(A)_n}{(q)_n}q^{n(n-1)/2}$$
$$\times\left\{\sum_{\ell=0}^{n}\frac{(q^{-n},1/z\delta,1/z)_\ell}{(A)_\ell}q^{-\ell(\ell-1)}(A\delta z^2/q)^\ell q^{n\ell}\sum_{j=0}^{\ell}\frac{(A/q)_j q^{j^2}(A\delta z^2)^{-j}}{(q,1/z\delta,1/z)_j}\right\}. \tag{6.19}$$

When $A=q$, the expression for $R_n(z;q,\delta)$ must be equal to $R_n^{(4)}(z;q,\delta)$ given by (6.16), since they are both monic polynomial solutions. Equating these two expressions we obtain

$$R_n(z;q,\delta)=(-\delta z)^{-n}q^{n(n-1)/2}{}_3\phi_0\left(\begin{matrix}q^{-n},1/z\delta,1/z\\ -\end{matrix};\delta z^2q^n\right)$$
$$=z^n(1/z)_n\,{}_1\phi_1\left(\begin{matrix}q^{-n}\\ zq^{-n+1}\end{matrix};q/\delta\right). \tag{6.20}$$

The above connection between a terminating ${}_3\phi_0$ and a terminating ${}_1\phi_1$ appears to be new. Note that (6.20) differs also from the standard expression in [17,(3.24.1) with $a=\delta^{-1}$ and $x$ replaced by $z$] which gives

$$R_n(z;q,\delta)=(-\delta)^{-n}q^{n(n-1)/2}{}_2\phi_1\left(\begin{matrix}q^{-n},z^{-1}\\ 0\end{matrix};qz\delta\right)$$
$$=z^n(1/z\delta)_n\,{}_1\phi_1\left(\begin{matrix}q^{-n}\\ q^{-n+1}z\delta\end{matrix};q\delta\right). \tag{6.21}$$

For the last equality we have used (5.29) with $c=0$.

### 6.3. Limit Al-Salam–Carlitz I

We now take the $A\to\infty$ limit of (6.12) and its corresponding solutions. The recurrence becomes

$$S_{n+1}-[z-(1+\delta^{-1})q^n]S_n+\frac{q^{2n-1}}{\delta}S_{n-1}=0 \tag{6.22}$$

with solutions

$$S_n^{(1)}(z;\delta) = \lim_{A\to\infty} R_n^{(1)}(z;A,\delta)$$
$$= \frac{q^{n^2}}{(q/\delta z)_n(\delta z)^n}\,{}_1\phi_1\left(\begin{matrix}0\\ q^{n+1}/z\delta\end{matrix};q^{n+1}/z\right), \tag{6.23}$$
$$S_n^{(2)}(z;\delta) = \lim_{A\to\infty} R_n^{(2)}(z;A,\delta)$$
$$= (-1)^n q^{n(n-1)/2}\,{}_1\phi_1\left(\begin{matrix}0\\ q/\delta\end{matrix};zq^{-n+1}\right), \tag{6.24}$$
$$S_n^{(3)}(z;\delta) = \lim_{A\to\infty} R_n^{(3)}(z;A,\delta)$$
$$= (-\delta)^{-n} q^{n(n-1)/2}\,{}_1\phi_1\left(\begin{matrix}0\\ q\delta\end{matrix};\delta z q^{-n+1}\right), \tag{6.25}$$
$$S_n^{(4)}(z;\delta) = \lim_{A\to\infty} R_n^{(4)}(z;A,\delta)$$
$$= (1/z)_n z^n\,{}_1\phi_1\left(\begin{matrix}0\\ zq^{-n+1}\end{matrix};q/\delta\right). \tag{6.26}$$

Note that $S_n^{(4)}$ is proportional to $S_n^{(2)}$ via the transformation (6.17) which yields the identity

$${}_1\phi_1\left(\begin{matrix}0\\ c\end{matrix};z\right) = \frac{(z)_\infty}{(c)_\infty}\,{}_1\phi_1\left(\begin{matrix}0\\ z\end{matrix};c\right). \tag{6.27}$$

The minimal solution to (6.22) is $S_n^{(1)}(z;\delta)$. Using Pincherle's theorem we then obtain the evaluation of the continued fraction associated with (6.22). Namely,

$$\frac{1}{z-(1+\delta^{-1})-}\;\frac{q/\delta}{z-(1+\delta^{-1})q}\;-\;\frac{q^3/\delta}{z-(1+\delta^{-1})q^2}\;-\cdots$$
$$= \frac{1}{z(1-1/\delta z)}\,\frac{{}_1\phi_1\left(\begin{matrix}0\\ q/z\delta\end{matrix};q/z\right)}{{}_1\phi_1\left(\begin{matrix}0\\ 1/z\delta\end{matrix};q/z\right)}. \tag{6.28}$$

This is a positive definite $J$-fraction if $0<q<1$ and $\delta>0$ and we may then deduce that the zeros of the ${}_1\phi_1$'s on the right of (6.28) are real, simple and interlacing. See the Corollary in Section 5.4.

The monic polynomials are given explicitly by

$$S_n(z;\delta) = \lim_{A\to\infty} R_n(z;A,\delta)$$
$$= \frac{(z\delta)^{-n}q^{n^2}}{(q)_n}\left\{\sum_{\ell=0}^{n}(q^{-n},1/z\delta,1/z)_\ell(-\delta)^\ell q^{-3\ell(\ell-1)/2}\right.$$
$$\left.\times z^{2\ell}q^{\ell(n-1)}\sum_{j=0}^{\ell}\frac{z^{-2j}(-\delta)^{-j}q^{3j(j-1)/2}}{(q,1/z\delta,1/z)_j}\right\}. \tag{6.29}$$

A simpler expression is obtained if one applies the generating function method of Section 4 directly to (6.22). This results in

$$S_n(z;\delta) = \frac{\delta^{-n}q^{n(n+1)/2}}{(q)_n}\left\{\sum_{\ell=0}^{n}(q^{-n})_\ell(1/z)_\ell(-\delta z)^\ell q^{-\ell(\ell-1)/2}\right.$$
$$\left.\times\sum_{j=0}^{\ell}\frac{(-z\delta)^{-j}q^{j(j-1)/2}}{(1/z)_j(q)_j}\right\}. \tag{6.30}$$

### 6.4. Associated continuous q-Hermite

If we multiply (6.1) by $B^{1/2}$, replace $zB^{1/2}$ by $z$, renormalize and let $B\to 0$, we get the recurrence

$$H_{n+1} - zH_n + \frac{q}{A\delta}(1 - Aq^{n-1})H_{n-1} = 0. \tag{6.31}$$

This will become the continuous $q$-Hermite case if we set $A=q$. The solutions to (6.31) are obtained as limits of the solutions to (6.1). They are

$$H_n^{(1),\pm}(z;A,\delta) = \lim_{B\to 0} B^{n/2}Q_n^{(1),\pm}(z/B^{1/2};A,B,\delta)$$
$$= (A)_n(\mu_\pm)^n{}_1\phi_1\left(\begin{matrix}Aq^n\\0\end{matrix};A\delta\mu_\pm^2\right), \tag{6.32}$$

$$H_n^{(2)}(z;A,\delta) = (\mu_-)^n{}_2\phi_0\left(\begin{matrix}q^{-n+1}/A,0\\-\end{matrix};q^n/\delta\mu_-^2\right), \tag{6.33}$$

where

$$\mu_\pm = \tfrac{1}{2}\left(z \pm \sqrt{z^2-\gamma'^2}\right),\quad \gamma' = 2\sqrt{q/A\delta}. \tag{6.34}$$

To derive (6.32) we first made the transformation (5.28) before taking the limit and the transformation (5.30) in the form

$$ {}_0\phi_1\left(\begin{matrix}-\\c\end{matrix};cz\right) = \frac{1}{(c)_\infty}{}_1\phi_1\left(\begin{matrix}z\\0\end{matrix};c\right) \tag{6.35}$$

after the limit was taken.

To derive (6.33) we transformed either $Q_n^{(2)}$ or $Q_n^{(3)}$ using an iterate of (5.28), discarded factors which were $n$-independent, multiplied by $B^{n/2}$ and then let $B\to 0$. Note that (6.33) is only a formal solution unless it terminates by having 'say' $A=q$.

For the general polynomial solution we take the limit of (6.9) to obtain

$$H_n(z;A,\delta) = \lim_{B\to 0} B^{n/2} Q_n(z;A,B,\delta)$$
$$= (\gamma' u/2)^n \frac{(A)_n}{(q)_n} \left\{ \sum_{\ell=0}^{n} \frac{(q^{-n})_\ell}{(A)_\ell} (-1)^\ell u^{-2\ell} q^{n\ell} \right.$$
$$\left. \times q^{-\ell(\ell-1)/2} \sum_{j=0}^{\ell} \frac{(A/q)_j (-1)^j u^{2j} q^{j(j+1)/2}}{(q)_j} \right\}, \tag{6.36}$$

$$z = \tfrac{1}{2}\gamma'(u+u^{-1}), \quad \gamma' = 2(q/A\delta)^{1/2}.$$

Note that when $A = q$, the expressions (6.33) and (6.36) become equal.

For $z = \gamma' x$, $x \in \mathbb{C} \setminus [-1,1]$ and with the square root branch chosen so that $|\mu_-/\mu_+| < 1$, we have $H_n^{(1),-}(z;A,\delta)$ as a minimal solution to (6.31) and hence the continued fraction representation

$$\frac{1}{z-}\;\frac{q(1-A)/A\delta}{z\;-}\;\frac{q(1-Aq)/A\delta}{z\;-}\;\frac{q(1-Aq^2)/A\delta}{z\;-}\cdots$$
$$= \frac{A\delta\mu_-}{q} \frac{{}_1\phi_1\left(\begin{matrix} A \\ 0 \end{matrix}; A\delta\mu_-^2\right)}{{}_1\phi_1\left(\begin{matrix} A/q \\ 0 \end{matrix}; A\delta\mu_-^2\right)}$$
$$= \int_{-1}^{1} \frac{\omega(t)\,\mathrm{d}t}{z-\gamma' t} + \text{possible pole terms.} \tag{6.37}$$

Repeating the method of Section 3, we find that

$$\omega(x) = \frac{1}{2\pi\sqrt{1-x^2}} \frac{(A, u^2, u^{-2})_\infty}{{}_1\phi_1\left(\begin{matrix} A/q \\ 0 \end{matrix}; u^2/q\right) {}_1\phi_1\left(\begin{matrix} A/q \\ 0 \end{matrix}; 1/u^2 q\right)}, \tag{6.38}$$
$$z/\gamma' = x = \tfrac{1}{2}(u+u^{-1}).$$

When $A = q$ the denominator ${}_1\phi_1$'s in (6.37) and (6.38) become equal to 1 and there are no pole terms. This is then the continuous $q$-Hermite case. See [17, p. 88], for a list of references.

### 6.5. Limit q-Hermite

We take the $A \to \infty$ limit of the results in the previous Section 6.4. This gives the recurrence relation

$$T_{n+1} - zT_n - \frac{q^n}{\delta} T_{n-1} = 0, \tag{6.39}$$

with minimal solution

$$T_n^{(1)}(z;\delta) = \lim_{A\to\infty} H_n^{(1),-}(z;A,\delta)$$
$$= (-1)^n q^{n(n-1)/2}(q/\delta z)^n {}_0\phi_1\left(\begin{matrix} - \\ 0 \end{matrix}; q^{n+2}/\delta z^2\right) \tag{6.40}$$

and the polynomial solution

$$T_n(z;\delta) = \lim_{A\to\infty} H_n(z;A,\delta)$$
$$= \frac{(-z)^{-n}q^{n(n-1)/2}}{(q)_n}(q/\delta)^n\left\{\sum_{\ell=0}^{n}(q^{-n})_\ell \right.$$
$$\left. \times z^{2\ell}q^{n\ell}(\delta/q)^\ell q^{-\ell(\ell-1)}\sum_{j=0}^{\ell}\frac{z^{-2j}\delta^{-j}q^{j^2}}{(q)_j}\right\}. \tag{6.41}$$

Using the minimal solution we get the continued fraction result

$$\frac{1}{z+}\,\frac{q/\delta}{z+}\,\frac{q^2/\delta}{z+}\cdots = \frac{1}{z}\frac{{}_0\phi_1\left(\begin{matrix} - \\ 0 \end{matrix}; q^2/\delta z^2\right)}{{}_0\phi_1\left(\begin{matrix} - \\ 0 \end{matrix}; q/\delta z^2\right)}. \tag{6.42}$$

This is a positive definite $J$-fraction if $0 < q < 1$ and $\delta < 0$ and one then has simple real interlacing zeros for the ${}_0\phi_1$'s on the right of (6.32) (see the Corollary in Section 5.4).

### 6.6. Associated continuous big q-Hermite

In (6.1) let $\delta = 1/aB$ and then let $B \to 0$ to get the recurrence

$$C_{n+1} - (z - q^n)C_n + \frac{aq}{A}(1 - Aq^{n-1})C_{n-1} = 0. \tag{6.43}$$

The solutions to (6.43) may be obtained from the solutions of (6.1) by using

$$C_n(z;A,a) = \lim_{B\to 0} Q_n(z;A,B,1/aB). \tag{6.44}$$

First making the parameter interchange $A \leftrightarrow B$ in (6.5), then writing $\delta = 1/aB$ and taking limit $B \to 0$ we obtain the solution

$$C_n^{(1),\pm}(z;A,a) = \lim_{B\to 0}\frac{(A,B)_n}{(ABA'_\pm)_n}(A'_\pm)^n {}_2\phi_1\left(\begin{matrix} AA'_\pm, Aq^n \\ ABA'_\pm q^n \end{matrix}; \frac{A'_\pm}{a}\right)$$
$$= (A)_n(A'_\pm)^n {}_2\phi_1\left(\begin{matrix} AA'_\pm, Aq^n \\ 0 \end{matrix}; \frac{A'_\pm}{a}\right),$$

$$A'_\pm = \tfrac{1}{2}\left(z \pm \sqrt{z^2 - \gamma_1^2}\right), \tag{6.45}$$

$$\gamma_1 = 2\sqrt{aq/A}.$$

Also from (6.2) we similarly obtain

$$C_n^{(2)}(z;A,a) = \lim_{B\to 0} Q_n^{(2)}(z;B,A,1/aB)$$
$$= \text{const. } (A'_+)^n (AA'_-/aq)_n \; {}_2\phi_1\left(\begin{matrix} q^{-n+1}/A, 0 \\ A'_+ q^{-n+1} \end{matrix}; AA'_-\right), \tag{6.46}$$

which is a polynomial solution when $A = q$.

In (6.7) we first apply the transformation [5, (III.2), p. 241], write $\delta = 1/aB$, $B = q^m$ and let $m \to \infty$. We obtain the solution

$$C_n^{(3)}(z;A,a) = \text{const. } (A'_+)^n {}_2\phi_0\left(\begin{matrix} q^{-n+1}/A, A'_+/a \\ - \end{matrix}; \frac{A^2 A'^2_- q^{n-2}}{a}\right), \tag{6.47}$$

which is again a polynomial solution for $A = q$. Using (6.8), transformation (5.30) and taking limits we have the solution

$$C_n^{(4)}(z;A,a) = \text{const. } (A'_+)^n (AA'_-/aq)_n \times {}_2\phi_1\left(\begin{matrix} q^{-n+1}/A, 0 \\ A'_+ q^{-n+1} \end{matrix}; AA'_-\right). \tag{6.48}$$

Also from (6.9) we obtain the explicit polynomial solution

$$C_n(z;A,a) = \lim_{B\to 0} Q_n\left(z;A,B,\frac{1}{aB}\right)$$
$$= (\gamma_1 u/2)^n \frac{(A)_n}{(q)_n} \left\{ \sum_{\ell=0}^{n} \frac{(q^n, 2u/\gamma_1)_\ell}{(A)_\ell} (-1)^\ell u^{-2\ell} q^{n\ell} q^{-\ell(\ell-1)/2} \right.$$
$$\left. \times \sum_{j=0}^{\ell} \frac{(A/q)_j (-1)^j u^{2j} q^{j(j+1)/2}}{(q, 2u/\gamma_1)_j} \right\}, \tag{6.49}$$

$$\gamma_1 = 2\sqrt{aq/A}.$$

If in (6.43) we write $z = x\gamma_1 = 2(aq/A)^{1/2}x$, and renormalize, we obtain the recurrence for the associated continuous big $q$-Hermite polynomials (see [17, (3.18.4)]) viz., the relation

$$\xi_{n+1} - (2x - bq^n)\xi_n + (1 - Aq^{n-1})\xi_{n-1} = 0, \tag{6.50}$$

with $b = (A/aq)^{1/2}$. Writing $A = q$ in the renormalized solution given by (6.47) yields continuous big $q$-Hermite polynomials [17, (3.18.1)].

Choosing, for $x = z/\gamma_1 \in \mathbb{C} \setminus [-1,1]$, the square root branch for which $|A'_-/A'_+| < 1$, the minimal solution of (6.43) is given by (6.45). Consequently, we have the continued fraction representation

$$\frac{1}{z-a_0-}\,\frac{b_1^2}{z-a_1-}\,\frac{b_2^2}{z-a_2-}\cdots = \frac{AA'_-}{aq} \frac{{}_2\phi_1\left(\begin{matrix} A, AA'_- \\ 0 \end{matrix}; A'_-/a\right)}{{}_2\phi_1\left(\begin{matrix} A, AA'_- \\ 0 \end{matrix}; A'_-/a\right)}$$
$$= \int_{-1}^{1} \frac{\omega(t)\,\mathrm{d}t}{z - \gamma_1 t} + \text{possible pole terms}, \tag{6.51}$$

with

$$\omega(x) = \frac{1}{2\pi\sqrt{1-x^2}} \frac{(A)_\infty(1/u^2,u^2)_\infty}{(\gamma_1 u/2a, \gamma_1/2au)_\infty} \left[ {}_2\phi_1\left(\begin{matrix} A/q, AA'_- \\ 0 \end{matrix}; \frac{A'_-}{a}\right) {}_2\phi_1\left(\begin{matrix} A/q, AA'_+ \\ 0 \end{matrix}; \frac{A'_+}{a}\right)\right]^{-1}, \tag{6.52}$$

where

$$x = \tfrac{1}{2}(u+u^{-1}),$$
$$\gamma_1 = 2(aq/A)^{1/2},$$
$$a_n = q^n,$$
$$b_n^2 = \frac{qa}{A}(1-Aq^{n-1}).$$

In the particular case $A = q$,

$$\omega(x) = \frac{1}{2\pi\sqrt{1-x^2}} \frac{(q)_\infty(1/u^2,u^2)_\infty}{(u/\sqrt{a}, 1/u\sqrt{a})_\infty}. \tag{6.53}$$

This weight function agrees with the form given in [17, (3.18.2)] with $b$ in (6.50) replaced by $1/\sqrt{a}$ and taking into consideration the normalization factor $(A/aq)^{n/2} = a^{-n/2} = b^n$ used in (6.50) and also that $x$ is replaced by $bx$.

### 6.7. q-Bessel order

The limit $A \to \infty$ of the recurrence (6.43) gives

$$B_{n+1} - (z-q^n)B_n - aq^nB_{n-1} = 0. \tag{6.54}$$

It is clear that (6.54) will give real orthogonal polynomials only for $0 < q < 1$, $a < 0$. The solutions of (6.54) can be obtained by taking $A \to \infty$ limits of (6.45)–(6.48). We have

$$B_n^{(1)}(z;a) = \lim_{A\to\infty} C_n^{(1),-}(z;A,a) = (-aq/z)^n q^{n(n-1)/2} {}_1\phi_1\left(\begin{matrix} aq/z \\ 0 \end{matrix}; q^{n+1}/z\right) \tag{6.55}$$

$$= (-aq/z)^n q^{n(n-1)/2} (q^{n+1}/z)_\infty \, {}_0\phi_1\left(\begin{matrix} - \\ q^{n+1}/z \end{matrix}; aq^{n+2}/z^2\right), \tag{6.56}$$

using the transformation (6.35). Also

$$B_n^{(2)}(z;a) = \lim_{A\to\infty} C_n^{(2)}(z;A,a) = \text{const. } z^n(1/z)_n {}_2\phi_1\left(\begin{matrix} 0,0 \\ zq^{-n+1} \end{matrix}; aq/z\right), \tag{6.57}$$

$$B_n^{(3)}(z;a) = \lim_{A\to\infty} C_n^{(3)}(z;A,a) = \text{const. } z^n {}_2\phi_0\left(\begin{matrix} 0, z/a \\ - \end{matrix}; aq^n/z^2\right), \tag{6.58}$$

$$\lim_{A\to\infty} C_n^{(4)}(z;A,a) = \text{const. } B_n^{(2)}(z;a). \tag{6.59}$$

Using (6.49) we obtain the explicit polynomial

$$\begin{aligned} B_n(z;a) &= \lim_{A\to\infty} C_n(z;A,a) \\ &= \frac{(-a/z)^n}{(q)_n} q^{n(n+1)/2} \left\{ \sum_{\ell=0}^{n} (q^{-n}, 1/z)_\ell \right. \\ &\qquad \left. \times\, q^{-\ell^2} q^{n\ell} \left(\frac{z^2}{a}\right)^\ell \sum_{j=0}^{\ell} \frac{q^{j^2}(a/z^2)^j}{(q)_j (1/z)_j} \right\}. \end{aligned} \tag{6.60}$$

We now demonstrate the relationship of the above solutions with Jackson's $q$-analogues of Bessel functions (see [5, Exercise 1.24, p. 25]). Using the notation of [5] and writing

$$q^\nu = z^{-1}, \quad -\tfrac{1}{4}x^2 = aq/z,$$

we find from (6.57) and (6.56) that

$$J^{(1)}_{-\nu-n}(2\mathrm{i}(aq/z)^{1/2};q) = \text{const.}\ (-1)^n q^{-n(n+4)/2} \sqrt{z}\left(-\mathrm{i}\sqrt{z/a}\right)^{n+\nu} \frac{(q^2/z)_n}{(1/z)_n} B^{(2)}_n(z;a) \tag{6.61}$$

and

$$J^{(2)}_{\nu+n}(2\mathrm{i}(aq/z)^{1/2};q) = \text{const.}\ (-1)^n \frac{q^{-n(n-1)/2}}{\sqrt{z}} (z/a\sqrt{q})^n \left(\mathrm{i}\sqrt{a/z}\right)^{n+\nu} B^{(1)}_n(z;a). \tag{6.62}$$

This shows that (6.54) is connected with the recurrence for $q$-Bessel functions with $z$ appearing both in the arguement and the order of the $q$-Bessel function. Thus we choose to call this the $q$-Bessel order case. The continued fraction representation obtained with the help of the minimal solution of (6.54) given by (6.55) is

$$\frac{1}{z-1+}\,\frac{aq}{z-q+}\,\frac{aq^2}{z-q^2+}\,\cdots = \frac{1}{(z-1)} \frac{{}_0\phi_1\left(\begin{matrix}-\\ q/z\end{matrix}; aq^2/z^2\right)}{{}_0\phi_1\left(\begin{matrix}-\\ 1/z\end{matrix}; aq/z^2\right)}. \tag{6.63}$$

This is a positive definite $J$-fraction if $0 < q < 1$ and $a < 0$ and the zeros of the ${}_0\phi_1$'s on the right are then real, simple and interlacing.

A $q \to 1$ limiting case is due to Maki [19] (see also [21]).

## 7. Limit Askey–Wilson

In this section we give the connection between solutions to the associated Askey–Wilson and the associated continuous dual $q$-Hahn polynomial recurrence relations. The associated Askey–Wilson polynomial recurrence relation in monic form is given by

$$\begin{aligned} &p_{n+1}(x) - (x-a_n)p_n(x) + b_n^2 p_{n-1}(x) = 0, \\ &a_n = -A_n - B_n + \frac{a}{2} + \frac{1}{2a}, \end{aligned}$$

$$
\begin{aligned}
b_n^2 &= A_{n-1}B_n, \\
A_n &= \frac{(1-abcdq^{n+\alpha-1})(1-abq^{n+\alpha})(1-acq^{n+\alpha})(1-adq^{n+\alpha})}{2a(1-abcdq^{2n+2\alpha-1})(1-abcdq^{2n+2\alpha})}, \\
B_n &= \frac{a(1-q^{n+\alpha})(1-bcq^{n+\alpha-1})(1-bdq^{n+\alpha-1})(1-cdq^{n+\alpha-1})}{2(1-abcdq^{2n+2\alpha-2})(1-abcdq^{2n+2\alpha-1})}.
\end{aligned} \tag{7.1}
$$

When $\alpha = 0$, (7.1) reduces to the nonassociated monic Askey–Wilson case [3]. If one further puts $d = 0$, one has the monic continuous dual $q$-Hahn case [3, pp. 3, 28]. Our associated dual $q$-Hahn case may be obtained from (7.1) by multiplying (7.1) by $k^{-1} = 2\sqrt{q/ABCD}$, replacing $(a,b,c,d,q^\alpha)$ by $(2kq/AD, 2kq/AC, 2kq/AB, 0, A/q)$ and renormalizing to monic form. Note that '$k$' is now used to denote the '$\alpha$' of previous sections so as to avoid confusion with the '$\alpha$' in (7.1).

There are two papers [15, 9], which deal with the associated Askey–Wilson polynomial case. We now indicate how our associated dual $q$-Hahn recurrence solutions in Section 2 are connected with the solutions in [15] and [9].

Using the solutions to (7.1) which Ismail and Rahman have given in [15], we have the following limiting cases:

$$
P_n(z; A,B,C,D) = \lim_{d\to 0} \frac{(B,C)_n}{k^n} p_n\left(x; \frac{2kq}{AD}, \frac{2kq}{AC}, \frac{2kq}{AB}, d, A/q\right), \tag{7.2}
$$

where

$$
p_n(x) = p_n(x; a,b,c,d,q^\alpha) = \frac{1}{(2a)^n} p_n^{(\alpha)}(x)
$$

and $p_n^{(\alpha)}(x)$ is given explicitly in [15, (4.15)]. The calculation yields

$$
\begin{aligned}
P_n(z; A,B,C,D) = \frac{(B,C)_n}{(BC)^n} \Bigg\{ \sum_{k=0}^{n} & \frac{(q^{-n}, \sqrt{BCq/AD}\,u, \sqrt{BCq/AD}\,u^{-1})_k \, q^k}{(q,B,C)_k} \\
& \times \sum_{j=0}^{n-k} \frac{(A/q, D/q, q^{k+1}, q^{k-n})_j}{(q, Cq^k, Bq^k, q^{-n})_j} \left(\frac{BCq}{AD}\right)^j \Bigg\}.
\end{aligned} \tag{7.3}
$$

Ismail and Rahman have also obtained nonpolynomial solutions $r_{n+\alpha}(u)$ and $s_{n+\alpha}(u)$ which correspond to $p_n^{(\alpha)}(x)$. Using [15, (1.12), (1.13)] we obtain

$$
X_n^{(1),\pm}(z; A,B,C,D) = \lim_{d\to 0} C_1^{\pm} \frac{(B,C)_n}{(BC)^n} \left((BC\lambda_\pm)^{-\alpha} s_{n+\alpha}(u^{\pm 1})\right) \tag{7.4}
$$

where

$$
C_1^{\pm} = \frac{(A, D, BC\lambda_\pm)_\infty}{(BCD\lambda_\pm, ABC\lambda_\pm, AD\lambda_\pm)_\infty},
$$

and

$$
X_n^{(5)}(z; A,B,C,D) = \lim_{d\to 0} C_5 \frac{(B,C)_n}{(BC)^n} \left((BC\lambda_-)^{-\alpha} r_{n+\alpha}(u)\right), \tag{7.5}
$$

where

$$C_5 = \frac{(C, AD\lambda_+, Dq\lambda_-, ABC\lambda_+/q)_\infty}{(B, C, q/B, Dq/C)_\infty}.$$

In [9], the parameters $(a, b, c, d, q^\alpha)$ of [15] are replaced by $(\alpha, \beta, \gamma, \delta, \varepsilon)$. The solutions $X_n^{(r)}$ to (7.1) in [9] will now be denoted by $\widetilde{X}_n^{(r)}$. The $\delta \to 0$ limits of some of these solutions are given below in terms of the solutions $X_n^{(1)}, X_n^{(2)}, X_n^{(4)}, X_n^{(5)}$ and $X_n^{(6)}$ of Section 2. We have

$$\begin{aligned}
&\lim_{\delta\to 0} \widetilde{X}_n^{(6)}(u^{\pm 1}) = D_6^{\pm} k^n X_n^{(1),\pm}(z; A, B, C, D), \\
&D_6^{\pm} = \frac{(BCD\lambda_\pm, AD\lambda_\pm, ABC\lambda_\pm)_\infty}{(A, B, C, D, q/u^{\pm 2})_\infty},
\end{aligned} \tag{7.6}$$

$$\begin{aligned}
&\lim_{\delta\to 0} \widetilde{X}_n^{(5)}(u) = D_5 k^n X_n^{(5)}(z; A, B, C, D) + D_5' k^n X_n^{(2)}(z; A, B, C, D), \\
&D_5 = \frac{\left(\frac{A}{q}u^2, \frac{q^2}{Au^2}, \frac{q}{D}, AC\lambda_-, AB\lambda_-, \frac{Dq}{C}, \frac{q}{B}\right)_\infty}{\left(\frac{ABC}{q}\lambda_+, \frac{ABD}{q}\lambda_+, \frac{ACD}{q}\lambda_+, Cq\lambda_-, Bq\lambda_-, \frac{q}{u^2}\right)_\infty} \\
&\qquad \times \frac{1}{\left(\frac{A}{D}, Dq\lambda_-, AD\lambda_+\right)_\infty}, \\
&D_5' = \frac{\left(\frac{A}{q}u^2, \frac{q^2}{Au^2}, \frac{q}{A}, Aq\lambda_-, BD\lambda_-, CD\lambda_-, \frac{Dq}{2k}\right)_\infty}{\left(\frac{ABC}{q}\lambda_+, \frac{ABD}{q}\lambda_+, \frac{ACD}{q}\lambda_+, Cq\lambda_-, Bq\lambda_-, \frac{q}{u^2}\right)_\infty} \\
&\qquad \times \frac{\left(\frac{q^2}{Du}, \frac{q}{B}, \frac{Aq}{C}, \frac{2k}{D}, \frac{Du}{q}\right)_\infty}{\left(Dq\lambda_-, \frac{D}{A}, \frac{q^2}{Au}, \frac{Aq}{2k}, \frac{Au}{q}, \frac{2k}{A}, Aq\lambda_-, AD\lambda_+\right)_\infty},
\end{aligned} \tag{7.7}$$

$$\begin{aligned}
&\lim_{\delta\to 0} \widetilde{X}_n^{(2)}(u) = D_2 k^n X_n^{(6)}(z; C, B, A, D), \\
&D_2 = \frac{2}{u} \frac{(A^2 q\lambda_+, Aq\lambda_-)_\infty}{\left(\frac{Aq}{B}, \frac{Aq}{C}, \frac{Aq}{D}, \frac{BCD\lambda_-}{q}\right)_\infty},
\end{aligned} \tag{7.8}$$

$$\begin{aligned}
&\lim_{\delta\to 0}\widetilde{X}_n^{(1)}(u) = D_1 k^n X_n^{(5)}(z;A,B,C,D) + D_1' k^n X_n^{(1)}(z;A,B,C,D),\\
&D_1 = \frac{(q/B, BCDq\lambda_+/A, BC\lambda_-, Dq/C)_\infty}{(D, Cq/A, Bq/A, Dq\lambda_+, ABC\lambda_-/q, AD\lambda_-)_\infty},\\
&D_1' = \frac{\left(\frac{q}{A}, \frac{BCq}{2k}, \frac{2k}{BC}, \frac{Dq}{2k}, BD\lambda_+, CD\lambda_+, \frac{BCDq}{A}\lambda_+, BCD\lambda_-, \frac{2k}{D}\right)_\infty}{\left(B, C, D, \frac{Dq}{A}, \frac{Cq}{A}, \frac{Bq}{A}, u, \frac{q}{u}, \frac{uq^2}{A}, \frac{A}{uq}, D\lambda_+\right)_\infty}.
\end{aligned} \tag{7.9}$$

Parameter interchanges in [9] in some of the cases yield new solutions of (7.1). For example an interchange of $\delta \leftrightarrow \beta$ in solution $\widetilde{X}_n^{(1)}(u)$ gives a new solution $\approx X_n^{(1)}(u)$ for which we have

$$\begin{aligned}
&\lim_{\delta\to 0}\overset{\approx}{X}{}_n^{(1)}(u) = D_1'' k^n X_n^{(4)}(z;A,B,C,D),\\
&D_1'' = \frac{(q/B, Cq/D)_\infty}{(C, Cq\lambda_+, AC\lambda_-, ABD\lambda_-/q)_\infty}.
\end{aligned} \tag{7.10}$$

If we instead make the interchange $\alpha \leftrightarrow \delta$ in solution $\widetilde{X}_n^{(1)}(u)$, then the $\delta \to 0$ limit would be proportional to $k^n X_n^{(5)}(z;A,B,C,D)$.

## References


[1] R. Askey, Continuous Hahn polynomials, *J. Phys. A.* **18** (1985) L1017–L1019.

[2] R. Askey and M.E.H. Ismail, Recurrence relations, continued fractions and orthogonal polynomials, *Mem. Amer. Math. Soc.* **300** (1984).

[3] R. Askey and J. Wilson, Some basic hypergeometric orthogonal polynomials, *Mem. Amer. Math. Soc.* **319** (1985).

[4] T.S. Chihara, *An Introduction to Orthogonal Polynomials* (Gordon and Breach, New York, 1978).

[5] G. Gasper and M. Rahman, *Basic Hypergeometric Series* (Cambridge Univ. Press, Cambridge, 1990).

[6] W. Gautschi, Computational aspects of three-term recurrence relations, *SIAM Rev.* **9** (1967) 24–82.

[7] D.P. Gupta, M.E.H. Ismail and D.R. Masson, Associated continuous Hahn Polynomials, *Can. J. Math.* **43** (1991) 1263–1280.

[8] D.P. Gupta, M.E.H. Ismail and D.R. Masson, Contiguous relations, basic hypergeometric functions and orthogonal polynomials II. Associated big $q$-Jacobi polynomials, *J. Math. Anal. Appl.* **171** (1992) 477–497.

[9] D.P. Gupta and D.R. Masson, Solutions to the associated $q$-Askey–Wilson polynomial recurrence relation, in: *Approximation and Computation*, Vol. 158 (Birkhäuser, Cambridge, 1994) 273–284.

[10] W. Hahn, Über Orthogonalpolynome, die $q$-Differenzengleichungen genügen, *Math. Nachr.* **2** (1949) 4–34.

[11] W. Hahn, Über Polynome, die gleichzeitig zwei verschiedenen Orthogonalsystemen angehören, *Math. Nachr.* **2** (1949) 263–278.

[12] M.E.H. Ismail, J. Letessier, D.R. Masson and G. Valent, Birth and death processes and orthogonal polynomials, in: P. Nevai, Ed., *Orthogonal Polynomials* (Kluwer Academic Publishers, Dordrecht, 1990) 229–255.

[13] M.E.H. Ismail, J. Letessier and G. Valent, Quadratic birth and death models and associated continuous dual Hahn polynomials, *SIAM J. Math. Anal.* **20** (1989) 727–737.

[14] M.E.H. Ismail and C.A. Libis, Contiguous relations, basic hypergeometric functions and orthogonal polynomials I, *J. Math. Anal. Appl.* **141** (1989) 349–372.

[15] M.E.H. Ismail and M. Rahman, The associated Askey–Wilson polynomials, *Trans. Amer. Math. Soc.* **328** (1991) 201–237.

[16] W.B. Jones and W.J. Thron, *Continued Fractions: Analytic Theory and Applications* (Addison-Wesley, Reading, MA, 1980).
[17] R. Koekoek and R.F. Swarttouw, The Askey-scheme of hypergeometric orthogonal polynomials and its $q$-analogue, Reports of the Faculty of Technical Mathematics and Informatics No. 94-05, Delft, 1994.
[18] C.A. Libis, private communication.
[19] D. Maki, On constructing distribution functions with applications to Lommel polynomials and Bessel functions, *Trans. Amer. Math. Soc.* **130** (1968) 281–297.
[20] D.R. Masson, Difference equations revisited, in: J.S. Feldman and L.M. Rosen, Eds., *Canadian Mathematics Society Conference Proceedings*, Vol. 9 (AMS, Providence, RI, 1988) 73–82.
[21] D.R. Masson, Difference equations, continued fractions, Jacobi matrices and orthogonal polynomials, in: A Cuyt, Ed., *Non-linear Numerical Methods and Rational Approximation* (D. Reidel, Dordrecht, 1988) 239–257.
[22] D.R. Masson, Associated Wilson polynomials, *Constructive Approximation* **7** (1991) 521–534.
[23] D.R. Masson, Wilson polynomials and some continued fractions of Ramanujan, *Rocky Mountain J. Math.* **21** (1991) 489–499.
[24] H.S. Wall, *Analytic Theory of Continued Fractions* (D. Van Nostrand, Princeton, NJ, 1948).
[25] J. Wilson, Hypergeometric series, recurrence relations and some orthogonal functions, Doctoral Dissertation, University of Wisconsin, Madison, 1978.