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Further results on oscillation of a class of second-order neutral equations☆

Peiguang Wang[a,*], Xuewen Li[b]

[a] Department of Mathematics, College of Electronic and Information Engineering, Hebei University, Baoding 071002, People's Republic of China

[b] School of Mathematical Sciences, Peking University, Beijing 100871, People's Republic of China

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Abstract

This paper establishes several results on oscillation of a class of second neutral differential equations with distributed deviating arguments.

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1. Introduction

The study of the oscillatory and asymptotic behavior of the solutions of neutral differential equations, besides its theoretical interest, is important from the viewpoint of applications. For example, neutral differential equations arise frequently in many applications such as population growth models, distributed networks with lossless transmission lines, control problem (see [4]). There have been many results on the oscillatory and asymptotic behavior of second-order neutral differential equations, refer to the monographs of Bainov and Mishev [1] and Erbe et al. [3]. Several papers concerning neutral equations with distributed deviating arguments have appeared recently, refer to Yu and Fu [7], Liu and Fu [5], Bainov and Petrov [2] and Wang and Yu [6] and their references cited therein.

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* Corresponding author.

E-mail address: pgwang@mail.hbu.edu.cn (P. Wang).

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In this paper, we consider the following second-order neutral differential equation with distributed deviating arguments:

$$[x(t)+c(t)x(t-\tau)]''+\int_a^b p(t,\xi)x[g(t,\xi)]\,\mathrm{d}\sigma(\xi)=0, \tag{1}$$

where $\tau>0$ is a constant; $c(t)\in C([t_0,\infty),[0,1])$; $p(t,\xi)\in C([t_0,\infty)\times[a,b],R_+)$, and $p(t,\xi)$ is not eventually zero on any $[t_\mu,\infty)\times[a,b]$, $t_\mu\geqslant t_0$, $R_+=[0,\infty)$; $g(t,\xi)\in C([t_0,\infty)\times[a,b],R)$, $g(t,\xi)\leqslant t$, $\xi\in[a,b]$; $g(t,\xi)$ is nondecreasing with respect to t and ξ, respectively, and $\lim_{t\to\infty}$, $\inf_{\xi\in[a,b]}\{g(t,\xi)\}=\infty$; $\sigma(\xi)\in([a,b],R)$ is nondecreasing, the integral of Eq. (1) is a Stieltjes one.

We restrict our attention to proper solutions of Eq. (1), that is, to nonconstant solutions existing on $[T,\infty)$ for $T\geqslant t_0$ and satisfying $\sup_{t\geqslant T}|x(t)|>0$. A proper solution $x(t)$ of Eq. (1) is called oscillatory if it does not have the largest zero, otherwise, it is called nonoscillatory.

The objective of this paper is to obtain some general oscillatory criteria of solutions of Eq. (1) by introducing parameter function and using integral averaging technique. By choosing different $H(t,s)$, we can obtain various corollaries, namely various conditions under each of which Eq. (1) has oscillatory solution. The results generalize and improve some known results. For example, in [7], the following oscillatory criteria of Eq. (1) were obtained.

Theorem A. *If*

$$\int_{t_0}^{\infty}\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s=\infty, \tag{2}$$

then every solution of Eq. (1) *is oscillatory.*

Theorem B. *If there exists* $a(\mathrm{d}/\mathrm{d}t)g(t,a)$ *and* $\varphi(t)\in C'([t_0,\infty),(0,\infty))$ *such that*

$$\int_{t_0}^{\infty}\left[\varphi(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)-\frac{\varphi'^2(s)}{4\varphi(s)g'(s,a)}\right]\mathrm{d}s=\infty, \tag{3}$$

then every solution of Eq. (1) *is oscillatory.*

2. Main results

The following theorems provide sufficient conditions on oscillation of solutions of Eq. (1).

Theorem 1. *Assume that there exist* $(\mathrm{d}/\mathrm{d}t)g(t,a)$ *and function* $H(t,s)\in C'(D;R)$, $h(t,s)\in C(D;R)$, *in which* $D=\{(t,s)|t\geqslant s\geqslant t_0\}$ *satisfying*

(H_1) $H(t,t)=0$, $t\geqslant t_0$; $H(t,s)>0$, $t>s\geqslant t_0$;
(H_2) $H_t'(t,s)\geqslant 0$, $H_s'(t,s)\leqslant 0$, *and* $-H_s'(t,s)=h(t,s)\sqrt{H(t,s)}$, $(t,s)\in D$.

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)-\frac{h^2(t,s)}{4g'(s,a)}\right]\mathrm{d}s=\infty, \tag{4}$$

then every solution of Eq. (1) *is oscillatory.*

Proof. Assume the contrary, then there exists a nonoscillatory solution $x(t)$ of Eq. (1) on $[t_0,\infty)$, such that $x(t)\neq 0$ on $[t_0,\infty)$. Without loss of generality, assume that $x(t)>0,\ t\geqslant t_0$. Then, from $\lim_{t\to\infty}, \inf_{\xi\in[a,b]}\{g(t,\xi)\}=\infty$, there exists a $T_0\geqslant t_0$ such that

$$x(t)>0,\quad x(t-\tau)>0\quad\text{and}\quad x[g(t,\xi)]>0,\quad t\geqslant T_0,\ \xi\in[a,b].$$

Set

$$y(t)=x(t)+c(t)x(t-\tau), \tag{5}$$

then, we have that $y(t)\geqslant x(t)>0,\ y''(t)\leqslant 0,\ t\geqslant T_0$, and we can claim that $y'(t)\geqslant 0,\ t\geqslant T_0$. In fact, assume that it is not true, then there exists a $t_1\geqslant T_0$ such that $y'(t_1)<0$. According to the fact of $y'(t)$ is decreasing, there exists a $t_2\geqslant t_1$ such that $y'(t_2)<0$ and $y'(t)\leqslant y'(t_2)<0,\ t\geqslant t_2$. Integrating from t_2 to t, we have $y(t)\leqslant y(t_2)+y'(t_2)(t-t_2)$. Thus, we conclude that $\lim_{t\to\infty}y(t)=-\infty$, this contradicts $y(t)>0$. From (1) and (5), we obtain

$$\begin{aligned}0&=y''(t)+\int_a^b p(t,\xi)x[g(t,\xi)]\,\mathrm{d}\sigma(\xi)\\&=y''(t)+\int_a^b p(t,\xi)\{y[g(t,\xi)]-c[g(t,\xi)]x[g(t,\xi)-\tau]\}\,\mathrm{d}\sigma(\xi).\end{aligned}\tag{6}$$

Using $y'(t)\geqslant 0$, and $y(t)\geqslant x(t),\ t\geqslant t_1$, we have $y[g(t,\xi)]\geqslant y[g(t,\xi)-\tau]\geqslant x[g(t,\xi)-\tau]$, thus

$$y''(t)+\int_a^b p(t,\xi)\{1-c[g(t,\xi)]\}y[g(t,\xi)]\,\mathrm{d}\sigma(\xi)\leqslant 0,\quad t\geqslant t_1. \tag{7}$$

Furthermore, using $g(t,\xi)$ is nondecreasing with respect to ξ, we have $y[g(t,a)]\leqslant y[g(t,\xi)]$, thus

$$y''(t)+y[g(t,a)]\int_a^b p(t,\xi)\{1-c[g(t,\xi)]\}\,\mathrm{d}\sigma(\xi)\leqslant 0,\quad t\geqslant t_1. \tag{8}$$

Let

$$z(t)=\frac{y'(t)}{y[g(t,a)]}. \tag{9}$$

Then $z(t)\geqslant 0$. According to the fact that there exists a $(\mathrm{d}/\mathrm{dt})g(t,a)$, we obtain $y'[g(t,a)]=(\mathrm{d}y/\mathrm{d}g)(\mathrm{d}/\mathrm{d}t)g(t,a)$, and noting that $g(t,\xi)$ is nondecreasing with respect to $\xi, g(t,\xi)\leqslant t$ for $\xi\in[a,b]$, we obtain $y'(t)\leqslant y'[g(t,a)]$. Thus

$$\begin{aligned}z'(t)&=\frac{y''(t)}{y[g(t,a)]}-\frac{y'(t)y'[g(t,a)]g'(t,a)}{y^2[g(t,a)]}\\&\leqslant-\int_a^b p(t,\xi)\{1-c[g(t,\xi)]\}\,\mathrm{d}\sigma(\xi)-g'(t,a)z^2(t),\quad t\geqslant t_1.\end{aligned}\tag{10}$$

Integrating by parts for any $t > T \geqslant t_1$, and using properties (H_1) and (H_2), we have

$$\begin{aligned}
&\int_T^t H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s\\
&\quad\leqslant -\int_T^t H(t,s)z'(s)\,\mathrm{d}s-\int_T^t H(t,s)g'(s,a)z^2(s)\,\mathrm{d}s\\
&\quad= -\int_T^t H(t,s)\,\mathrm{d}z(s)-\int_T^t H(t,s)g'(s,a)z^2(s)\,\mathrm{d}s\\
&\quad= H(t,T)z(T)-\int_T^t h(t,s)\sqrt{H(t,s)}z(s)\,\mathrm{d}s-\int_T^t H(t,s)g'(s,a)z^2(s)\,\mathrm{d}s\\
&\quad= H(t,T)z(T)-\int_T^t\left[\sqrt{H(t,s)g'(s,a)}z(s)+\frac{h(t,s)}{2\sqrt{g'(s,a)}}\right]^2\mathrm{d}s+\int_T^t\frac{h^2(t,s)}{4g'(s,a)}\,\mathrm{d}s,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_T^t\left[H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)-\frac{h^2(t,s)}{4g'(s,a)}\right]\mathrm{d}s\\
&\quad\leqslant H(t,T)z(T)-\int_T^t\left[\sqrt{H(t,s)g'(s,a)}z(s)+\frac{h(t,s)}{2\sqrt{g'(s,a)}}\right]^2\mathrm{d}s. \qquad (11)
\end{aligned}$$

Furthermore, according to (H_2), for $t_1 \geqslant t_0$, we have $H(t,t_1)\leqslant H(t,t_0)$, thus

$$\begin{aligned}
&\int_{t_1}^t\left[H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)-\frac{h^2(t,s)}{4g'(s,a)}\right]\mathrm{d}s\\
&\quad\leqslant H(t,t_1)z(t_1)-\int_{t_1}^t\left[\sqrt{H(t,s)g'(s,a)}z(s)+\frac{h(t,s)}{2\sqrt{g'(s,a)}}\right]^2\mathrm{d}s\\
&\quad\leqslant H(t,t_1)z(t_1)\leqslant H(t,t_0)z(t_1),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{H(t,t_0)}\int_{t_0}^t\left[H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)-\frac{h^2(t,s)}{4g'(s,a)}\right]\mathrm{d}s\\
&\quad=\frac{1}{H(t,t_0)}\left[\int_{t_0}^{t_1}+\int_{t_1}^{t}\right]\left[H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)-\frac{h^2(t,s)}{4g'(s,a)}\right]\mathrm{d}s\\
&\quad\leqslant z(t_1)+\int_{t_0}^{t_1}\frac{H(t,s)}{H(t,t_0)}\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s\\
&\quad\leqslant z(t_1)+\int_{t_0}^{t_1}\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s, \qquad (12)
\end{aligned}$$

which implies that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s) \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi) - \frac{h^2(t,s)}{4g'(s,a)} \right] \mathrm{d}s$$

$$\leqslant z(t_1) + \int_{t_0}^{t_1} \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s < \infty,$$

this contradicts (4). Therefore, the proof of Theorem 1 is completed. □

According to the proved process of Theorem 1, we have the following corollary.

Corollary 1. *If condition* (4) *of Theorem* 1 *is replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s = \infty, \tag{13}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{h^2(t,s)}{g'(s,a)}\,\mathrm{d}s < \infty, \tag{14}$$

then every solution of Eq. (1) *is oscillatory.*

Remark 1. From Theorem 1 and Corollary 1, we can obtain various oscillatory criteria by means of the choices of weighted function $H(t,s)$. For example, choosing $H(t,s) = (t-s)^{m-1}$, $t \geqslant s \geqslant t_0$, in which $m > 2$ is an integer, then $h(t,s) = (m-1)(t-s)^{(m-3)/2}$, $t \geqslant s \geqslant t_0$. According to Corollary 1, we have

Corollary 2. *If there exists* $a(\mathrm{d}/\mathrm{d}t)g(t,a)$ *and an integer* $m > 2$ *such that*

$$\limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s = \infty, \tag{15}$$

$$\limsup_{t\to\infty} \frac{1}{t^{m-1}} \int_{t_0}^{t} \frac{(t-s)^{m-3}}{g'(s,a)}\,\mathrm{d}s < \infty, \tag{16}$$

then every solution of Eq. (1) *is oscillatory.*

Example. Consider the neutral differential equation

$$[x(t) + (1-\mathrm{e}^{-t})x(t-1)]'' + \int_1^2 \mathrm{e}^{-2t+\xi} x(t+\xi)\,\mathrm{d}\xi = 0, \quad t \geqslant 1, \tag{17}$$

where $\tau = 1$, $a = 1$, $b = 2$, $c(t) = 1 - \mathrm{e}^{-t}$, $g(t,\xi) = t + \xi$, $p(t,\xi) = \mathrm{e}^{2t+\xi}$. By choosing $m = 3$, then

$$\limsup_{t\to\infty} \frac{t}{t^{m-1}} \int_{t_0}^{t} (t-s)^{m-1} \int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s$$

$$= \limsup_{t\to\infty} \frac{1}{t^2} \int_1^t \mathrm{e}^s (t-s)^2\,\mathrm{d}s = \infty,$$

then, from Corollary 2, every solution of Eq. (17) is oscillatory.

Theorem 2. *Assume that there exist* $(\mathrm{d}/\mathrm{d}t)g(t,a)$ *and function* $H(t,s) \in C'(D;R)$, $h(t,s) \in C(D;R)$ *such that* (H_1) *and* (H_2) *hold. If there exists a function* $\rho(t) \in C'([t_0,\infty),(0,\infty))$ *satisfying*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)]^2}{\rho(s)g'(s,a)}\, \mathrm{d}s < \infty, \tag{18}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\, \mathrm{d}\sigma(\xi)\, \mathrm{d}s = \infty, \tag{19}$$

then every solution of Eq. (1) *is oscillatory.*

Proof. Assume the contrary, then there exists a nonoscillatory solution $x(t)$ of Eq. (1) on $[t_0,\infty)$. Without loss of generality, assume that $x(t) > 0,\ t \geqslant t_0$. Then from proof of Theorem 1, there exists a $t_1 \geqslant t_0$ such that

$$z'(t) \leqslant -\int_a^b p(t,\xi)\{1 - c[g(t,\xi)]\}\, \mathrm{d}\sigma(\xi) - g'(t,a)z^2(t), \quad t \geqslant t_1. \tag{10$'$}$$

Thus

$$\begin{aligned}
&\int_{t_1}^{t} H(t,s)\rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\, \mathrm{d}\sigma(\xi)\, \mathrm{d}s \\
&\leqslant -\int_{t_1}^{t} H(t,s)\rho(s)z'(s)\, \mathrm{d}s - \int_{t_1}^{t} H(t,s)\rho(s)g'(s,a)z^2(s)\, \mathrm{d}s \\
&= H(t,t_1)\rho(t_1)z(t_1) - \int_{t_1}^{t} \sqrt{H(t,s)}[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)]z(s)\, \mathrm{d}s \\
&\quad - \int_{t_1}^{t} H(t,s)\rho(s)g'(s,a)z^2(s)\, \mathrm{d}s.
\end{aligned} \tag{20}$$

Furthermore, we conclude that

$$\begin{aligned}
&\int_{t_1}^{t} H(t,s)\rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\, \mathrm{d}\sigma(\xi)\, \mathrm{d}s \\
&\leqslant H(t,t_1)\rho(t_1)z(t_1) \\
&\quad - \int_{t_1}^{t} \left\{ \sqrt{\rho(s)g'(s,a)H(t,s)}z(s) + \frac{[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)]^2}{2\sqrt{\rho(s)g'(s,a)}} \right\} \mathrm{d}s \\
&\quad + \frac{1}{4} \int_{t_1}^{t} \frac{[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)]^2}{\rho(s)g'(s,a)}\, \mathrm{d}s \\
&\leqslant H(t,t_1)\rho(t_1)z(t_1) + \frac{1}{4} \int_{t_1}^{t} \frac{[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)]^2}{\rho(s)g'(s,a)}\, \mathrm{d}s.
\end{aligned} \tag{21}$$

From (21), for $t > t_1 \geqslant t_0$, we obtain that

$$\begin{aligned}
&\frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s \\
&\quad = \frac{1}{H(t,t_0)} \left[\int_{t_0}^{t_1} + \int_{t_1}^{t}\right] \left[H(t,s)\rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\right] \mathrm{d}\sigma(\xi)\,\mathrm{d}s \\
&\quad \leqslant \frac{1}{H(t,t_0)} \int_{t_0}^{t_1} H(t,s)\rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s \\
&\qquad + \frac{H(t,t_1)}{H(t,t_0)}\rho(t_1)z(t_1) + \frac{1}{4H(t,t_0)} \int_{t_1}^{t} \frac{[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)]^2}{\rho(s)g'(s,a)}\,\mathrm{d}s,
\end{aligned}$$

noting that $H_s'(t,s) \leqslant 0$, which implies that

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s \\
&\quad \leqslant L + \frac{1}{4}\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)]^2}{\rho(s)g'(s,a)}\,\mathrm{d}s,
\end{aligned} \tag{22}$$

where $L = \rho(t_1)z(t_1) + \int_{t_0}^{t_1} \rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s$. Thus, according to condition (18), we conclude that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s < \infty,$$

which contradicts (19). Therefore, the proof of Theorem 2 is completed. □

Remark 2. In Theorem 2, by choosing $\rho(s) \equiv 1$, we can obtain Corollary 1.

In the case of

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} (t-s)^m \rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s < \infty, \tag{23}$$

we have the following result.

Theorem 3. *Assume that there exist* $(\mathrm{d}/\mathrm{d}t)g(t,a)$ *and function* $H(t,s) \in C'(D;R)$, $h(t,s) \in C(D;R)$ *such that* (H_1) *and* (H_2) *hold. If* $H_t'(t,s)$ *is nondecreasing, and there exists a function*

$\varphi(t)\in C([t_0,\infty),R)$ satisfying

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_u^t\left[H(t,s)\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\right.$$
$$\left.-\frac{[h(t,s)\rho(s)-\sqrt{H(t,s)}\rho'(s)]^2}{4\rho(s)g'(s,a)}\right]\mathrm{d}s\geqslant\varphi(u),\quad u\geqslant t_0, \tag{24}$$

$$\lim_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^t\frac{H(t,s)g'(s,a)\varphi_+^2(s)}{\rho(s)}\,\mathrm{d}s=\infty,\quad \varphi_+(s)=\max_{s\geqslant t_0}\{\varphi(s),0\}, \tag{25}$$

then every solution of Eq. (1) *is oscillatory.*

Proof. Assume the contrary, then there exists a nonoscillatory solution $x(t)$ of Eq. (1) on $[t_0,\infty)$. Without loss of generality, assume that $x(t)>0,\ t\geqslant t_0$. Then proceeding as Theorem 2, there exists a $t_1>u\geqslant t_0$ such that

$$\int_u^t H(t,s)\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s$$
$$\leqslant H(t,u)\rho(u)z(u)+\frac{1}{4}\int_u^t\frac{[h(t,s)\rho(s)-\sqrt{H(t,s)}\rho'(s)]^2}{\rho(s)g'(s,a)}\,\mathrm{d}s. \tag{21$'$}$$

Furthermore, for $t>u\geqslant t_0$, we have

$$\frac{1}{H(t,t_0)}\int_u^t\left[H(t,s)\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\right.$$
$$\left.-\frac{[h(t,s)\rho(s)-\sqrt{H(t,s)}\rho'(s)]^2}{4\rho(s)g'(s,a)}\right]\mathrm{d}s$$
$$\leqslant\frac{H(t,u)}{H(t,t_0)}\rho(u)z(u). \tag{26}$$

According to (24) and (H_2), we conclude that

$$\varphi(u)\leqslant\frac{1}{H(t,t_0)}\int_u^t\left[H(t,s)\rho(s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\right.$$
$$\left.-\frac{[h(t,s)\rho(s)-\sqrt{H(t,s)}\rho'(s)]^2}{4\rho(s)g'(s,a)}\right]\mathrm{d}s$$
$$\leqslant\frac{H(t,u)}{H(t,t_0)}\rho(u)z(u)\leqslant\rho(u)z(u), \tag{27}$$

which implies that

$$\varphi_+^2(u) \leqslant \rho^2(u)z^2(u). \tag{28}$$

Let

$$v(t) = \frac{1}{H(t,t_0)} \int_{t_1}^{t} \sqrt{H(t,s)}[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)]z(s)\,\mathrm{d}s,$$

$$w(t) = \frac{1}{H(t,t_0)} \int_{t_1}^{t} H(t,s)\rho(s)g'(s,a)z^2(s)\,\mathrm{d}s,$$

then, according to (20), we have

$$\begin{aligned} v(t) + w(t) \leqslant{}& \frac{H(t,t_1)}{H(t,t_0)}\rho(t_1)z(t_1) \\ &- \frac{1}{H(t,t_0)} \int_{t_1}^{t} H(t,s)\rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s, \end{aligned} \tag{29}$$

from (24), we have

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_u^t H(t,s)\rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s \geqslant \varphi(u),$$

furthermore, we obtain

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^t H(t,s)\rho(s) \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s \\ &\quad -\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^t \frac{[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)]^2}{4\rho(s)g'(s,a)}\,\mathrm{d}s \geqslant \varphi(t_1). \end{aligned} \tag{30}$$

According to (30) and (23), we conclude that

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^t \frac{[h(t,s)\rho(s) - \sqrt{H(t,s)}\rho'(s)]^2}{4\rho(s)g'(s,a)}\,\mathrm{d}s < \infty.$$

Thus, there exists a sequence $\{t_n\}_1^\infty$ in $[t_1,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and satisfying

$$\lim_{n\to\infty} \frac{1}{H(t_n,t_0)} \int_{t_1}^{t_n} \frac{[h(t_n,s)\rho(s) - \sqrt{H(t_n,s)}\rho'(s)]^2}{4\rho(s)g'(s,a)}\,\mathrm{d}s < \infty, \tag{31}$$

which implies that

$$\begin{aligned} \limsup_{t\to\infty}\{v(t) + w(t)\} &\leqslant \rho(t_1)z(t_1) - \liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_1}^t H(t,s)\rho(s) \\ &\quad \times \int_a^b p(s,\xi)\{1 - c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)\,\mathrm{d}s \\ &\leqslant \rho(t_1)z(t_1) - \varphi(t_1) \triangleq M. \end{aligned} \tag{32}$$

Then, for any sufficiently large n, we have

$$u(t_n) + v(t_n) < M_1, \tag{33}$$

where $M_1 > M$, M and M_1 are constant. According to definition of $w(t)$, we have

$$w'(t) = \int_{t_1}^{t} \frac{(H_t'(t,s)H(t,t_0) - H_t'(t,t_0)H(t,s))}{H^2(t,t_0)} \rho(s)g'(s,a)z^2(s)\,\mathrm{d}s,$$

from $H_t'(t,s)$ is nondecreasing, and (H_2), we have $w'(t) \geqslant 0$, thus, $w(t)$ is increasing, and $\lim_{t\to\infty} w(t) = l$ exists, where l is finite or infinite. In the case of $l = \infty$, then $\lim_{n\to\infty} w(t_n) = \infty$, which implies that from (33)

$$\lim_{n\to\infty} v(t_n) = -\infty \tag{34}$$

and

$$\frac{v(t_n)}{w(t_n)} + 1 < \frac{M_1}{w(t_n)},$$

thus, for any $0 < \varepsilon < 1$ and sufficiently large n, we have

$$\frac{v(t_n)}{w(t_n)} < \varepsilon - 1 < 0. \tag{35}$$

On the other hand, by using the Schwartz inequality for $t \geqslant t_1$, we obtain

$$\begin{aligned} 0 \leqslant v^2(t_n) &= \frac{1}{H^2(t_n,t_0)} \left\{ \int_{t_1}^{t_n} \sqrt{H(t_n,s)} \left[h(t_n,s)\rho(s) - \sqrt{H(t_n,s)}\rho'(s) \right] z(s)\,\mathrm{d}s \right\}^2 \\ &\leqslant \left\{ \frac{1}{H(t_n,t_0)} \int_{t_1}^{t_n} H(t_n,s)\rho(s)g'(s,a)z^2(s)\,\mathrm{d}s \right\} \\ &\quad \times \left\{ \frac{1}{H(t_n,t_0)} \int_{t_1}^{t_n} \frac{[h(t_n,s)\rho(s) - \sqrt{H(t_n,s)}\rho'(s)]^2}{\rho(s)g'(s,a)}\,\mathrm{d}s \right\} \\ &= w(t_n)\frac{1}{H(t_n,t_0)} \int_{t_1}^{t_n} \frac{[h(t_n,s)\rho(s) - \sqrt{H(t_n,s)}\rho'(s)]^2}{\rho(s)g'(s,a)}\,\mathrm{d}s. \end{aligned}$$

Then

$$0 \leqslant \frac{v^2(t_n)}{w(t_n)} \leqslant \frac{1}{H(t_n,t_0)} \int_{t_1}^{t_n} \frac{[h(t_n,s)\rho(s) - \sqrt{H(t_n,s)}\rho'(s)]^2}{\rho(s)g'(s,a)}\,\mathrm{d}s. \tag{36}$$

It follows that from (31)

$$0 \leqslant \lim_{n\to\infty} \frac{v^2(t_n)}{w(t_n)} < \infty. \tag{37}$$

According to (35), we have

$$\lim_{n\to\infty} \frac{v(t_n)}{w(t_n)} = \lim_{n\to\infty} \frac{v'(t_n)}{w'(t_n)} \leqslant \varepsilon - 1 < 0,$$

then

$$\lim_{n\to\infty}\frac{v^2(t_n)}{w(t_n)}=\lim_{n\to\infty}\frac{2v(t_n)v'(t_n)}{w'(t_n)}\geqslant 2\lim_{n\to\infty}v(t_n)(\varepsilon-1)=\infty,$$

which contradicts (37). Thus, we have $\lim_{t\to\infty}w(t)=l<\infty$. Furthermore, according to (28), we conclude that

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_1}^{t}\frac{H(t,s)g'(s,a)\varphi_+^2(s)}{\rho(s)}\,\mathrm{d}s\\&\quad\leqslant\lim_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_1}^{t}H(t,s)\rho(s)g'(s,a)z^2(s)\,\mathrm{d}s=\lim_{t\to\infty}w(t)<\infty,\end{aligned}\tag{38}$$

which implies that

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{H(t,s)g'(s,a)\varphi_+^2(s)}{\rho(s)}\,\mathrm{d}s\\&\quad=\lim_{t\to\infty}\frac{1}{H(t,t_0)}\left[\int_{t_0}^{t_1}+\int_{t_1}^{t}\right]\frac{H(t,s)g'(s,a)\varphi_+^2(s)}{\rho(s)}\,\mathrm{d}s\\&\quad\leqslant\int_{t_0}^{t_1}\frac{H(t,s)g'(s,a)\varphi_+^2(s)}{\rho(s)}\,\mathrm{d}s+\lim_{t\to\infty}w(t)<\infty,\end{aligned}$$

which contradicts (25). Therefore, the proof of Theorem 3 is completed. □

Remark 3. In Theorem 3, by choosing $\rho(t)\equiv 1$, we have the following result.

Corollary 3. *Assume that there exist* $(\mathrm{d}/\mathrm{d}t)g(t,a)$ *and function* $H(t,s)\in C'(D;R)$, $h(t,s)\in C(D;R)$ *such that* (H_1) *and* (H_2) *hold. If* $H_t'(t,s)$ *is nondecreasing, and there exists a function* $\varphi(t)\in C([t_0,\infty),R)$ *satisfying*

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_u^t\left[H(t,s)\int_a^b p(s,\xi)\{1-c[g(s,\xi)]\}\,\mathrm{d}\sigma(\xi)-\frac{h^2(t,s)}{4g'(s,a)}\right]\mathrm{d}s\geqslant\varphi(u),\tag{39}$$

$$\lim_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)g'(s,a)\varphi_+^2(s)\,\mathrm{d}s=\infty,\qquad \varphi_+(s)=\max_{s\geqslant t_0}\{\varphi(s),0\},\tag{40}$$

then every solution of Eq. (1) is *oscillatory*.

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