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Corresponding Banach spaces on time scales☆

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Abstract

We will provide a short introduction to the calculus on a time scale $\mathbb{T}$, in order to make the reader familiar with the basics. Then we intend to have a closer look at the so-called "cylinder transform" ξ_μ which maps a positively regressive function $p : \mathbb{T} \to \mathbb{R}$ to another function $\tilde{p} : \mathbb{T} \to \mathbb{R}$. It will turn out that, under certain conditions, this cylinder transform acts as an isometry between two normed spaces. Therefore, we obtain a two-fold generalization of the well-known Banach and Hilbert spaces of functions in continuum analysis. Finally, we shall give some examples concerning this structure of corresponding spaces—for instance an example of orthogonal polynomials on equidistant lattices. In order to achieve this, we shall state a theorem on how to take orthogonality theory over from a Hilbert space to its corresponding Hilbert space.

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1. Motivation of the subject

In order to solve analytic Schrödinger difference equations, the concept of unitary linear lattices was introduced in [7]. Let us briefly refer to the results which we had prepared there: In [7], we focussed first on *regular lattices* by which we understand sets $\mathbb{T} \subset \mathbb{R}$ such that there exists a bijective map $\gamma : \mathbb{Z} \to \mathbb{T}$. For abbreviation, we have defined $t_n := \gamma(n)$, $n \in \mathbb{Z}$.

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Let moreover $(e_n)_{n\in\mathbb{Z}}$ be an orthonormal basis for $l^2(\mathbb{Z})$ and let

$$l^2_{\mathbb{T}}(\mathbb{Z}) := \left\{ F \in l^2(\mathbb{Z}) \,\middle|\, F = \sum_{n=-\infty}^{\infty} |t_{n+1} - t_n|^{1/2} f(t_n) e_n,\ f : \mathbb{T} \to \mathbb{C} \right\},$$

where the scalar product in $l^2_{\mathbb{T}}(\mathbb{Z})$ is induced by the standard one in $l^2(\mathbb{Z})$. There is a one-to-one correspondence between $l^2_{\mathbb{T}}(\mathbb{Z})$ and the space of all square integrable functions on the grid $\mathbb{T}$,

$$L^2(\mathbb{T}) = \left\{ f : \mathbb{T} \to \mathbb{C} \,\middle|\, \sum_{n=-\infty}^{\infty} |t_{n+1} - t_n| f(t_n) \overline{f(t_n)} < \infty \right\}.$$

The induced scalar product for any two elements in $L^2(\mathbb{T})$ is

$$\langle f, g \rangle = \sum_{n=-\infty}^{\infty} |t_{n+1} - t_n|\, f(t_n) \overline{g(t_n)}.$$

As for the corresponding L^1-space, we had chosen

$$L^1(\mathbb{T}) := \left\{ f : \mathbb{T} \to \mathbb{C} \,\middle|\, \sum_{n=-\infty}^{\infty} |t_{n+1} - t_n|\, |f(t_n)| < \infty \right\}.$$

As for $L^2(\mathbb{T})$, we define shift operators V resp. W on their maximal definition ranges $D(V) \subset L^2(\mathbb{T})$ and $D(W) \subseteq L^2(\mathbb{T})$ by

$$(Vf)(t_n) := f(t_{n-1}), \quad (Wf)(t_n) := f(t_{n+1}) \quad \forall n \in \mathbb{Z}.$$

Let now $\mathbb{T}$ be a regular lattice. The uniquely defined function $u : \mathbb{T} \to \mathbb{T}$, fixed by

$$u(t_n) = t_{n+1}, \quad u^{-1}(t_{n+1}) = t_n \quad \forall n \in \mathbb{Z},$$

was referred to in [7] as generating function of the lattice $\mathbb{T}$. Moreover, we call $\mathbb{T}$ a unitary lattice if the adjoint V^* of V and the operator W fulfill

$$V^* \varphi = \rho W \varphi = \rho V^{-1} \varphi$$

for any function $\varphi : \mathbb{T} \to \mathbb{C}$ with compact support, $\rho > 0$ denoting a universal constant.

In [7], a characterization property of unitary lattices could be stated: Let $u : \mathbb{T} \to \mathbb{T}$ be the generating function of a regular lattice $\mathbb{T}$. The lattice $\mathbb{T}$ is unitary if and only if there exists a constant $\rho > 0$ such that

$$|u(x) - x| = \rho\, |x - u^{-1}(x)| \quad \forall x \in \mathbb{T}.$$

The structure of unitary linear lattices turns out to be quite sophisticated. However, these lattices and the spectral theory of the linear difference operators behind can still be generalized. They are based on the q-linear grids and equidistant lattices from the related "quantum calculus".

Investigations on discrete Schrödinger equations in context of the *Undergraduate Research Program AbiTUMath* have led to the question of how to extend Schrödinger theory to discrete structures which are even more general than those of unitary linear lattices. From the level of difference operators, this means moving the classical Askey–Wilson divided difference operator setting to a more general type of difference operators. An essential key to this understanding lies in the so-called time scale analysis. In the sequel, we will deal with the concept of corresponding Banach spaces on time scales.

The organization of this article shall be as follows: a short introduction to time scale calculus is provided in Section 2. The concept of exponential functions and positive regressivity is referred to by Section 3. In Section 4, we interpret the so-called cylinder transform as an isomorphism. A two-fold Banach structure is elucidated in Section 5. Some concrete examples are presented in Section 6.

2. A short introduction to time scales calculus

A time scale $\mathbb{T}$ is any closed (nonempty) subset of the real numbers. Frequently considered examples of time scales are the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$ or the q-linear grid $\overline{q^{\mathbb{Z}}}$ where $0 < q < 1$. Those time scales lead to different types of "calculus", for instance continuum calculus, difference calculus and the already mentioned quantum calculus. (An introduction to quantum calculus can for instance be found in [6].) However, all those different calculi can be summarized under the general *time scale calculus*. For a thorough introduction to this special field of analysis, the reader is referred to the books [2,3]. Stefan Hilger, the author of the fundamental article [5], initiated the research on time scales in 1988. A major advantage of the time scales point of view is that, no matter what strange lattice or structure on $\mathbb{R}$ one is given, one can apply quite general concepts. Therefore, let us now revise the basic ideas behind the time scale approach.

Given a point x on a time scale $\mathbb{T}$, we state the following fundamental

Definition. The "right-shift" $\sigma(x)$ and the "forward graininess" $\mu(x)$ are given by

$$\sigma(x) = \inf\{t \in \mathbb{T} \mid t > x\}, \quad \mu(x) = \sigma(x) - x.$$

Similarly, the "left-shift" $\rho(x)$ and the "backward graininess" $v(x)$ are defined by

$$\rho(x) = \sup\{t \in \mathbb{T} \mid t < x\}, \quad v(x) = x - \rho(x).$$

Points fulfilling $\mu(x) = 0$ are called *right-dense*. If $\mu(x) > 0$, x is said to be *right-scattered*. If moreover $v(x) = 0$, x is called *left-dense*, otherwise *left-scattered*.

According to this definition, we can meet the upcoming four essentially different scenarios:

- x is dense. $\iff \rho(x) = x = \sigma(x)$.
- x is left-dense and right-scattered. $\iff \rho(x) = x < \sigma(x)$.
- x is left-scattered and right-dense. $\iff \rho(x) < x = \sigma(x)$.
- x is isolated. $\iff \rho(x) < x < \sigma(x)$.

Remark. All the above expressions are well defined because the time scale $\mathbb{T}$ is defined to be a *closed* subset of the real numbers.

Now, given a function $f : \mathbb{T} \to \mathbb{R}$, we want to define a *derivative*—generalizing the continuum derivative $f'(x)$ on $\mathbb{T} = \mathbb{R}$ and the difference operator $\Delta f(x) = f(x+1) - f(x)$ on $\mathbb{T} = \mathbb{Z}$. This is achieved by the so-called *delta-derivative*

$$f^{\Delta}(x) = \begin{cases} f'(x) \equiv & \text{if } \mu(x) = 0, \\ \dfrac{f(\sigma(x)) - f(x)}{\mu(x)} & \text{if } \mu(x) > 0. \end{cases} \tag{1}$$

In an analogous way, one can define a *nabla-derivative* by

$$f^{\nabla}(x) \equiv \begin{cases} f'(x) & \text{if } \nu(x) = 0, \\ \dfrac{f(x) - f(\rho(x))}{\nu(x)} & \text{if } \nu(x) > 0, \end{cases}$$

generalizing the "backward" difference operator $\nabla f(x) = f(x) - f(x-1)$ on $\mathbb{T} = \mathbb{Z}$. To become more familiar with these derivatives, let us consider a *q*-linear grid.

Example 1. Let $\mathbb{T} \equiv \mathbb{R}_q = \overline{\{\pm q^n \mid n \in \mathbb{Z}\}}$, where $0 < q < 1$. Then the nabla-derivative of a function $f : \mathbb{R}_q \to \mathbb{R}$ is computed via

$$f^{\nabla}(x) = \begin{cases} \dfrac{f(qx) - f(x)}{qx - x} & \text{if } x > 0, \\ f'(x) & \text{if } x = 0, \\ \dfrac{f(q^{-1}x) - f(x)}{q^{-1}x - x} & \text{if } x < 0, \end{cases}$$

hence equal to the usual *q*-difference operator on $q^{\mathbb{Z}}$. On the negative part of this time scale, the *q*-difference operator coincides with the delta-derivative.

Remark. In the above, we understand by

$$f'(x) \equiv \lim_{y \to x} \frac{f(x) - f(y)}{x - y},$$

a generalized continuum derivative. Hence, the delta-derivative exists at a right-dense point provided the function $f : \mathbb{T} \to \mathbb{R}$ is "differentiable" at this point, i.e., the above limit exists. For the existence at a right-scattered point we must guarantee *f* is continuous at that point. (And analogous conditions hold for the existence of f^{∇}.)

To see where the differences to the continuum calculus arise, let us, e.g., formulate the *product rule* for derivatives.

Proposition 2. *Let* $f, g : \mathbb{T} \to \mathbb{R}$ *be differentiable functions* (*in the time scale sense*). *Then* $fg : \mathbb{T} \to \mathbb{R}$ *is differentiable, satisfying*

$$(fg)^{\Delta}(x) = f^{\Delta}(x)g(x) + f(\sigma(x))g^{\Delta}(x),$$

$$(fg)^{\nabla}(x) = f^{\nabla}(x)g(x) + f(\rho(x))g^{\nabla}(x).$$

Note that some shifts in the argument appear, like in the classical equidistant case of $\mathbb{T} = \mathbb{Z}$, being of course a special case of our general considerations.

From now on we will concentrate on the calculus involving the delta-derivative, the so-called *delta-calculus*.

Having defined derivatives, the next question is how to define *integrals*. The easiest way of doing this is just to "invert" the derivative operation. Therefore we define the (Cauchy) *delta-integral* of $f : \mathbb{T} \to \mathbb{R}$ via

$$\left(\int_{x_0}^{x} f(t)\Delta t\right)^{\Delta} \equiv f(x) \quad \forall x \in \mathbb{T}. \tag{2}$$

But when does the integral exist? This "primitive" integral can be shown to exist provided the function *f* is *rd-continuous*, i.e., *f* is a regulated function and continuous at right-dense points. ("Regulated" means that, at each point $x \in \mathbb{T}$, right-hand and left-hand limits exist and are finite.) However, there have already been some deeper investigations concerning the theory of integration on time scales. In fact, one can define both *Riemann* and *Lebesgue* integrals on general time scales, see e.g. [3], in particular the whole corresponding continuum theory is preserved.

3. Exponential functions and positive regressivity

Let us consider the *dynamic initial value problem*

$$f^{\Delta}(x) = p(x)f(x) \quad \forall x \in \mathbb{T}, \quad f(x_0) = 1, \tag{3}$$

where $p : \mathbb{T} \to \mathbb{R}$ is a given function. In [2] it is shown that (3) has a unique solution $f(x) \equiv e_p(x, x_0)$ if the function *p* is rd-continuous and *regressive*, i.e.,

$$1 + \mu(x)p(x) \neq 0 \quad \forall x \in \mathbb{T}. \tag{4}$$

Under these conditions the solution is explicitly given by

$$e_p(x, x_0) = \mathrm{e}^{\int_{x_0}^{x} \tilde{p}(t)\Delta t} \quad \forall x \in \mathbb{T},$$

where the *cylinder transform* $\xi_\mu(p) \equiv \tilde{p}$ is defined as follows:

$$\xi_\mu(p)(x) \equiv \begin{cases} p(x) & \text{if } \mu(x) = 0, \\ \dfrac{\log(1 + \mu(x)p(x))}{\mu(x)} & \text{if } \mu(x) > 0. \end{cases} \tag{5}$$

(Here $\log(z)$ denotes the principal logarithm of $z \neq 0$.) Since $e_p(x, x_0)$ generalizes the continuum solution

$$f(x) = \mathrm{e}^{\int_{x_0}^{x} p(x)\Delta x}$$

of (3), the function $e_p(x, x_0)$ is called *exponential function*. In the special case $p \equiv 1$ we obtain the generalization of the "ordinary" exponential function e^{x-x_0}.

Having a close look at the definition of the cylinder transform, we recognize that the regressivity condition (4) is not only sufficient, but also necessary. Concerning the oscillation behavior of $e_p(x, x_0)$, one can easily prove the following:

Proposition 3. *If we have $1+\mu p>0$ on $\mathbb{T}$, then the exponential function is positive, i.e., $e_p(x,x_0)>0\ \forall x\in\mathbb{T}$. If contrarily $1+\mu p<0$ on $\mathbb{T}$, then the time scale $\mathbb{T}$ is isolated and the exponential function changes sign at every point.* (*In general, we have $e_p(x,x_0)\neq 0$ on $\mathbb{T}$.*)

Although the proof for this statement is quite simple, e.g., given in the book [2], let us rather illustrate it.

Example 4. Let $\mathbb{T}\equiv\mathbb{Z}$. Then the constant function $p(x)\equiv\lambda$ is regressive iff $\lambda\neq -1$. The corresponding exponential function with initial point $x_0=0$ can be computed easily, since the dynamic condition in (3) can be rewritten as

$$f(\sigma(x))=(1+\mu(x)p(x))f(x)\quad \forall x\in\mathbb{T}$$

in this case simplifying to

$$f(x+1)=(1+\lambda)f(x)\quad \forall x\in\mathbb{Z}.$$

The solution of this last equation is just

$$f(x)=(1+\lambda)^x f(0)\equiv(1+\lambda)^x.$$

Here the exponential function $e_\lambda(x,0)=(1+\lambda)^x$ clearly satisfies the assertions of the proposition.

Functions $p:\mathbb{T}\to\mathbb{R}$ fulfilling

$$1+\mu(x)p(x)>0\quad \forall x\in\mathbb{T} \tag{6}$$

are called *positively regressive*. Considering those functions as "comfortable" is quite straightforward since their exponential function $e_p(x,x_0)$ is positive—preserving the continuum property.

Definition. We denote by $\Gamma(\mathbb{T})$ the set of all positively regressive functions $p:\mathbb{T}\to\mathbb{R}$, furthermore by $\mathscr{R}^+(\mathbb{T})$ the set of all functions $p\in\Gamma(\mathbb{T})$ which are *in addition* rd-continuous. Here we should remark that we understand by a (positively) regressive function p just one satisfying property (4) resp. (6), not necessarily rd-continuous. (In this way, we differ from the original definition given in [2].) Now, given functions $p,q\in\Gamma(\mathbb{T})$ and an arbitrary scalar $\alpha\in\mathbb{R}$, we define the so-called "circle-plus" addition via

$$(p\oplus q)(x)=p(x)+q(x)+\mu(x)p(x)q(x)\quad \forall x\in\mathbb{T} \tag{7}$$

and the so-called "circle-dot" multiplication via

$$\alpha\odot p(x)=\begin{cases}\alpha p(x) & \text{if } \mu(x)=0,\\ \dfrac{(1+\mu(x)p(x))^\alpha-1}{\mu(x)} & \text{if } \mu(x)>0.\end{cases} \tag{8}$$

Remark. The $\odot$-multiplication can still be generalized to a $\odot$-product of a (general) function $f:\mathbb{T}\to\mathbb{R}$ and a function $p\in\Gamma(\mathbb{T})$ in a pointwise sense:

$$f\odot p(x)\equiv f(x)\odot p(x)=\begin{cases}\dfrac{(1+\mu(x)p(x))^{f(x)}-1}{\mu(x)} & \text{if } \mu(x)>0,\\ f(x)p(x) & \text{if } \mu(x)=0.\end{cases}$$

Notice that the product $f \odot p$ is *noncommutative*. For the question of the usefulness of this definition we refer the reader to article [4] about Gaussian bells on time scales. Anyway later on, when considering orthogonal polynomials on time scales, we will make use of this more general definition.

Akın-Bohner and Bohner have discovered in [1] that the set $\mathscr{R}^+(\mathbb{T})$, supplied with the addition $\oplus$ and the scalar multiplication $\odot$, constitutes a real vector space $(\mathscr{R}^+, \oplus, \odot)$. That is the starting point of our investigations. We want to generalize this result in view of taking over many spaces of functions we know from the continuum setting to the time scales setting—involving positive regressivity. A first step towards this goal is the upcoming result.

Theorem 5. *The set $\Gamma(\mathbb{T})$, supplied with $\oplus$ and $\odot$, is also a real linear space $(\Gamma, \oplus, \odot)$—generalizing the space $\mathscr{F}(\mathbb{R}) \equiv \{f \mid f : \mathbb{R} \to \mathbb{R}\}$ from the continuum scenario.*

Proof. We have to show that $(\Gamma, \oplus)$ is a commutative group, furthermore that Γ is closed with respect to the scalar multiplication $\odot$. In addition, we must prove the distributive laws to be true, as well as the properties $1 \odot p = p\ \forall p \in \Gamma(\mathbb{T})$ and

$$\alpha \odot (\beta \odot p) = (\alpha\beta) \odot p \quad \forall p \in \Gamma(\mathbb{T}). \tag{9}$$

Let us just show (9), which is done in a straightforward way—same as all the other properties follow directly from the definitions.

At right-dense points $x \in \mathbb{T}$, assertion (9) is clearly fulfilled. Therefore, assume $\mu(x) > 0$. Then the left-hand side yields

$$\alpha \odot (\beta \odot p)(x) = \frac{(1 + \mu(x)\beta \odot p(x))^{\alpha} - 1}{\mu(x)} = \frac{((1 + \mu(x)p(x))^{\beta})^{\alpha} - 1}{\mu(x)},$$

whereas the right-hand side gives us

$$(\alpha\beta) \odot p(x) = \frac{(1 + \mu(x)p(x))^{\alpha\beta} - 1}{\mu(x)}.$$

But those expressions coincide by a power law, proving (9). □

In the next section, we are going to pay some attention on properties of the cylinder transform and to see how they can be exploited.

4. The cylinder transform as an isomorphism

The first result we prove next about the cylinder transform concerns the fact that this operator is one-to-one.

Lemma 6. *Assume $\mathscr{S}$ is some set of regressive functions $p : \mathbb{T} \to \mathbb{R}$. Then the cylinder transform $\xi_\mu : \mathscr{S} \to \xi_\mu(\mathscr{S})$ is bijective.*

Proof. Let $p, q \in \mathscr{S}$ fulfill $p \neq q$. Then there is at least one point $x \in \mathbb{T}$, where $p(x) \neq q(x)$. If this point is right-dense, the statement will follow at once as the cylinder transform coincides with the identity mapping at right-dense points. So let us suppose x is right-scattered, i.e., $\mu(x) > 0$. Now we have

$$\xi_\mu(p)(x) = \frac{\log(1+\mu(x)p(x))}{\mu(x)}, \quad \xi_\mu(q)(x) = \frac{\log(1+\mu(x)q(x))}{\mu(x)}.$$

But since the principal logarithm is one-to-one, we must have

$$\log(1+\mu(x)p(x)) \neq \log(1+\mu(x)q(x))$$

and therefore $\tilde{p}(x) \neq \tilde{q}(x)$—proving the statement. □

After this simple proof, let us assume we have some (real) vector space $(\tilde{V}, +, \cdot)$ of functions on a time scale. The next theorem shows that this space corresponds to another linear space $(V, \oplus, \odot)$ of positively regressive functions—the cylinder transform acting as an *isomorphism* between those spaces.

Theorem 7. *Let $(\tilde{V}, +, \cdot)$ be a vector space of functions $f : \mathbb{T} \to \mathbb{R}$ on a time scale. Then, defining the set V of positively regressive functions by*

$$p \in V \Longleftrightarrow \tilde{p} \in \tilde{V}, \tag{10}$$

we obtain another vector space $(V, \oplus, \odot)$. The cylinder transform $\xi_\mu : (V, \oplus, \odot) \to (\tilde{V}, +, \cdot)$ is an isomorphism between these spaces.

Proof. To see that V really consists of positively regressive functions, we must first show that $\xi_\mu : \Gamma(\mathbb{T}) \to \mathscr{F}(\mathbb{T})$ is onto. In order to do this, let $f : \mathbb{T} \to \mathbb{R}$ be arbitrary. Then define $p : \mathbb{T} \to \mathbb{R}$ by

$$p(x) = \frac{e^{\mu(x)f(x)} - 1}{\mu(x)} \equiv \xi_\mu^{-1}(f)(x)$$

at right-scattered points, otherwise by $p(x) = f(x)$. This function p is well defined and positively regressive by construction. Furthermore we have $\xi_\mu(p) = f$, which had to be shown. Now Lemma 6 tells us that $\xi_\mu : V \to \tilde{V}$ is bijective.

It remains to show that ξ_μ is linear. But this is again straightforward by the definition of the cylinder transform; let us just consider a right-scattered point $x \in \mathbb{T}$, where ξ_μ can be rewritten in a "discrete" sense:

$$\begin{aligned}
\xi_\mu((\alpha \odot p) \oplus (\beta \odot q))(x) &= \frac{\log[(1+\mu(x)\alpha \odot p(x))(1+\mu(x)\beta \odot q(x))]}{\mu(x)} \\
&= \frac{\log[(1+\mu(x)p(x))^\alpha] + \log[(1+\mu(x)q(x))^\beta]}{\mu(x)} \\
&= \alpha\xi_\mu(p)(x) + \beta\xi_\mu(q)(x).
\end{aligned}$$

Since this holds for whatever $p, q \in V$ and $\alpha, \beta \in \mathbb{R}$, we are done. □

Remark. The statements in Theorem 7 could have been formulated the other way round, i.e., for every vector space $(V, \oplus, \odot)$ of positively regressive functions there is a corresponding one $(\tilde{V}, +, \cdot)$. The inverse cylinder transform $\xi_\mu^{-1} : \tilde{V} \to V$ is an isomorphism. That shows that there is really a one-to-one correspondence between the spaces V and $\tilde{V}$.

Finally, let us give three examples illustrating this correspondence, two of them giving a hint on what still remains to be worked out.

Example 8. Let $\tilde{V}$ be the vector space of functions on a time scale $\mathbb{T}$, i.e., $V \equiv \mathscr{F}(\mathbb{T})$. Theorem 7 shows that this space $(\mathscr{F}(\mathbb{T}), +, \cdot)$ corresponds to the space $(\Gamma(\mathbb{T}), \oplus, \odot)$ of positively regressive functions. They both generalize the continuum space $\mathscr{F}(\mathbb{R})$.

The next example concerns the generalization of the Banach space $(\mathrm{C}[a, b], \|.\|_\infty)$ of continuous functions on a compact interval $[a, b] \subset \mathbb{R}$.

Example 9. Let $\tilde{V} \equiv \mathrm{C}_{\mathrm{rd}}[a, b]$ be the vector space of rd-continuous functions on a finite interval $[a, b] \subset \mathbb{T}$. It corresponds to the linear space $(\mathscr{R}^+([a, b]), \oplus, \odot)$ which we defined before. (The cylinder transform preserves rd-continuity.)

The final example treats the two-fold generalization of the Hilbert space $L^2(\mathbb{R})$ of square integrable functions (resp. equivalence classes of functions).

Example 10. Consider $\tilde{V} \equiv L^2(\mathbb{T})$, where $L^2(\mathbb{T})$ denotes the set of all square integrable functions with respect to the (Lebesgue) delta-integral. Then define $V \equiv \{p : \mathbb{T} \to \mathbb{R} \mid \tilde{p} \in L^2(\mathbb{T})\}$. Again we have a one-to-one correspondence.

The space $\tilde{V} = \mathrm{C}_{\mathrm{rd}}[a, b]$ in Example 9 is a normed space $(\tilde{V}, \|.\|_\infty)$. Using standard arguments of functional analysis, one can show that it is *complete*, i.e., a Banach space. Analogously, the space $\tilde{V} = L^2(\mathbb{R})$ in Example 10 is an inner product space, supplied with the inner product

$$\langle f, g\rangle = \int_{\mathbb{T}} f(x)g(x)\Delta x.$$

Indeed it is a Hilbert space $(\tilde{V}, \langle ., .\rangle)$. Are the corresponding spaces also Banach resp. Hilbert spaces? We shall give a positive answer to this question.

Remark. One of us (Moritz Simon) gave a direct analytical proof of the fact that $(\mathscr{R}^+([a, b]), \oplus, \odot)$, supplied with the norm $\|p\|_\mu \equiv \|\tilde{p}\|_\infty$, is a Banach space, see [8]. However, this proof turns out to be rather tedious. In the next section we will establish a more general proof.

5. A two-fold Banach structure

The following theorem provides the main result of this paper:

Theorem 11. *Let* $(\tilde{V}, +, \cdot, \|.\|)$ *be a normed space of functions on a time scale* $\mathbb{T}$. *Then the space* $(V, \oplus, \odot)$ *of positively regressive functions, defined by*

$$p \in V \Longleftrightarrow \tilde{p} \in \tilde{V},$$

is also a normed space $(V, \|.\|_\mu)$, *where the norm is given by*

$$\|p\|_\mu \equiv \|\tilde{p}\| \quad \forall p \in V. \tag{11}$$

Therefore, the cylinder transform yields an isometry $\xi_\mu : (V, \|.\|_\mu) \to (\tilde{V}, \|.\|)$. *Moreover, if the space* $(\tilde{V}, \|.\|)$ *is complete, i.e., a Banach space, then* $(V, \|.\|_\mu)$ *is also a Banach space.* (*A similar statement holds when interchanging V and* $\tilde{V}$, *again pointing out the bijective correspondence.*)

Proof. Let us first show that $\|.\|_\mu$ really constitutes a norm on $(V, \oplus, \odot)$. By Theorem 7, the cylinder transform satisfies the following properties:

$$\begin{aligned}
&\xi_\mu(p) \equiv 0 \Longleftrightarrow p \equiv 0,\\
&\xi_\mu(\alpha \odot p) = \alpha\xi_\mu(p) \quad \forall p \in V, \quad \alpha \in \mathbb{R},\\
&\xi_\mu(p \oplus q) = \xi_\mu(p) + \xi_\mu(q) \quad \forall p, q \in V.
\end{aligned}$$

Taking furthermore into account that $\|.\|$ is a norm on $\tilde{V}$, we directly obtain the three axioms for $\|.\|_\mu$—guaranteeing it is a norm. That shows the first part of the theorem. (The "isometry" statement is simply fulfilled because of the construction of $\|.\|_\mu$.)

Now suppose $(\tilde{V}, \|.\|)$ is complete. Let $(p_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in V. Since ξ_μ is an isometry, this is equivalent to the fact that $(\tilde{p}_n)_{n\in\mathbb{N}}$, i.e., the sequence of the cylinder transforms of the p_n, is a Cauchy sequence in $\tilde{V}$. By the completeness of $\tilde{V}$, that sequence must converge to some element $f \in \tilde{V}$—with respect to $\|.\|$. Now, as $\xi_\mu : V \to \tilde{V}$ is bijective, we can find $p \in V$ such that $f \equiv \tilde{p}$. Again using the isometric argument, the convergence $\tilde{p}_n \to \tilde{p}$ is equivalent to

$$\|p_n \ominus p\|_\mu \longrightarrow 0 \tag{12}$$

as $n \to \infty$. Finally (12) shows that $(V, \|.\|_\mu)$ is complete, i.e., a Banach space. □

Let us consider some examples one more time, in order to take benefit from the results of this theorem.

Example 12. Again consider the Banach space $(\mathrm{C_{rd}}[a, b], \|.\|_\infty)$. Now Theorem 11 guarantees $(\mathscr{R}^+([a, b]), \|.\|_\mu)$ is also a Banach space, where $\|.\|_\mu$ is given by

$$\|p\|_\mu = \|\tilde{p}\|_\infty \quad \forall p \in \mathscr{R}^+([a, b]). \tag{13}$$

As was indicated, we do not need a direct proof of the completeness.

Example 13. Consider $\tilde{V} = L^2(\mathbb{T})$ from Example 10. The Banach space of positively regressive functions we obtain here is indeed a Hilbert space $(V \equiv T^2(\mathbb{T}), \langle ., . \rangle_\mu)$, where the scalar product is given by

$$\langle p, q\rangle_\mu = \langle \tilde{p}, \tilde{q}\rangle = \int_{\mathbb{T}} \tilde{p}(x)\tilde{q}(x)\Delta x. \tag{14}$$

Even more generally, for $\varphi \in [1, \infty)$, we know from Lebesgue theory on time scales that the vector space $L^\varphi(\mathbb{T}) = \{f \in \mathbb{T} \mid |f|^\varphi \text{ integrable}\}$, supplied with the norm

$$\|f\|_\varphi \equiv \left(\int_{\mathbb{T}} |f(x)|^\varphi \Delta x\right)^{1/\varphi} \tag{15}$$

is a Banach space (which has to be read in an "almost everywhere" sense). Each of these $L^\varphi-$spaces corresponds to a Banach space of positively regressive functions—denoted by $T^\varphi(\mathbb{T})$.

After all, we may conclude that a generalization of continuum vector spaces of functions to the time scales setting cannot be achieved in a “unique” sense. Rather one obtains a *two-fold* structure of spaces: For every continuum space there seems to be, on the one hand, an “ordinary” vector space $(\tilde{V}, +, \cdot)$ of functions on a time scale, on the other hand, a “strange” linear space $(V, \oplus, \odot)$ of positively regressive functions on the same time scale. Theorem 11 guarantees that V conserves all the topological properties of $\tilde{V}$—and vice versa. Indeed, they can be *identified* in some way, being (isometrically) isomorphic. Quite an advantage of this two-fold structure lies in the following rationale:

Having established analytic results in the space $\tilde{V}$, *one can “translate” the results to* V*—and vice versa*.

This is of particular interest when it comes to looking at *orthogonal polynomials* on time scales. One has to consider two Hilbert spaces, $L^2(\mathbb{T})$ and $T^2(\mathbb{T})$. In the following section, we will give an example on the “translation process” between those two spaces.

6. Some concrete examples

An example concerning the difference between the spaces $C_{\mathrm{rd}}[a, b]$ and $\mathscr{R}^+([a, b])$, which is due to the cylinder transform, is given in [8]. There it is worked out in detail that the function $f(x) \equiv e_\lambda(x, 0)$ on the time scale $\mathbb{T} \equiv \mathbb{Z} \cap [0, N]$ where $N \geqslant 2$—refer to Example 4—is positively regressive iff $\lambda > -2$ and $\lambda \neq 1$. Now for $\lambda \in (-2, -1)$, the supremum norm of f is equal to 1 all the time. On the contrary, the μ-norm of f diverges as $\lambda \to -2$, i.e.,

$$\|e_\lambda(., 0)\|_\mu \overset{\lambda \to -2}{\longrightarrow} \infty.$$

That already reminds us to be careful comparing the spaces V and $\tilde{V}$. Let us consider a situation involving integrals.

Example 14. Let $\mathbb{T} \equiv \mathbb{N}_0$. Then a “Gaussian bell” on this time scale can be defined via $\mathbf{E}(0) = 1$ and $\mathbf{E}^\Delta(x) = \ominus(x \odot 1)\mathbf{E}(x)$, i.e.,—after some algebraic transformations—

$$\mathbf{E}(x + 1) = (1 + 1)^{-x}\mathbf{E}(x) \quad \forall x \in \mathbb{N}_0.$$

(For a thorough treatment of these generalized Gaussian bells, we recommend to consider the recent article [4], in which several properties of those functions are worked out. In fact, sufficient and necessary conditions for the square integrability of a Gaussian bell on a time scale are provided.) A solution to this recurrence relation is explicitly given by

$$\mathbf{E}(x) = 2^{x(1-x)} \quad \forall x \in \mathbb{T}. \tag{16}$$

In [4] it is shown that $\mathbf{E} \in L^2(\mathbb{T})$, which can also be seen directly. But what about the corresponding space $T^2(\mathbb{T})$?

By construction we have $\hat{\mathbf{E}} \in T^2(\mathbb{T})$, where $\hat{f} \equiv \xi_\mu^{-1}(f)$ simply denotes the inverse cylinder transform. Can we also prove $\mathbf{E} \in T^2(\mathbb{T})$, being equivalent to $\tilde{\mathbf{E}} \in L^2(\mathbb{T})$? The cylinder transform of the function $\mathbf{E}$ is

$$\xi_\mu(\mathbf{E})(x) = \log(1 + \mathbf{E}(x)) = \log(1 + 2^{x(1-x)}).$$

For large x, we have $\mathbf{E}(x) \approx 0$ and thus obtain $\tilde{\mathbf{E}}(x) \approx \mathbf{E}(x)$ by the Taylor expansion of the logarithm. In fact, this asymptotic behavior guarantees $\mathbf{E} \in T^2(\mathbb{T})$, however, not going into the ε-δ-details at this point. Note that considering a more general time scale, we will usually not result in a similarity like that.

The following treatment is going to be more challenging. We shall consider orthogonal polynomials on an equidistant lattice. The question is how to take over results on $L^2(\mathbb{T})$ to the corresponding space of positively regressive functions.

To prepare the next theorem, assume $(P_n)_{n\in\mathbb{N}_0}$ are orthogonal polynomials (each of exact degree n) with respect to some weight function $\varphi : \mathbb{T} \to \mathbb{R}_0^+$, i.e.,

$$\int_{\mathbb{T}} P_n(x)P_m(x)\varphi(x)\Delta x = 0 \quad \forall n \neq m. \tag{17}$$

The idea is to take the inverse cylinder transform of $P_n\psi$, where $\psi \equiv \sqrt{\varphi}$. If we denote

$$P_n(x) = \sum_{k=0}^{n} \alpha_{nk}x^k,$$

then we obtain

$$\xi_\mu^{-1}(P_n\psi)(x) = \left(\bigoplus_{k=0}^{n} \alpha_{nk} \odot x^k\right) \odot \hat{\psi}(x)$$

because of the linearity of the inverse cylinder transform. (The definition of $\bigoplus$ should be clear—we made use of the general $\odot$-product.) The corresponding weight function is indeed $\hat{\psi} \equiv \xi_\mu^{-1}(\psi)$. However, the "polynomials" in the sense of $T^2(\mathbb{T})$ are no more polynomials as we know them; they rather have to be understood in the sense of *operators* working on the ground state $\hat{\psi}$. Let us denote those "operator polynomials" by $\mathscr{P}_n$. Now define the operator $\mathscr{X}$, having a domain which is dense in $T^2(\mathbb{T})$, via the relation

$$\mathscr{X}p(x) \equiv x \odot p(x) \quad \forall x \in \mathbb{T}.$$

Using this relation, the operator polynomial $\mathscr{P}_n$ can be put into another form, namely

$$\mathscr{P}_n(\mathscr{X}) \equiv \bigoplus_{k=0}^{n} \alpha_{nk} \odot \mathscr{X}^k. \tag{18}$$

Representation (18) of the "polynomials" in the positively regressive sense finally shows us the link to the original scenario. To round this up, let us remark that one might also think of the polynomials $P_n(x)$ as operators $P_n(X)$ working on the ground state ψ, in detail

$$P_n(X) \equiv \sum_{k=0}^{n} \alpha_{nk} \cdot X^k, \tag{19}$$

where the operator $Xf(x) \equiv xf(x)$ $\forall x \in \mathbb{T}$ is densely defined in $L^2(\mathbb{T})$. Representations (18) and (19) make the correspondence even more understandable. However, notice that (18) is a necessary operator representation, whereas (19) is not necessary.

Theorem 15. *Assume $(P_n)_{n\in\mathbb{N}_0}$ are orthogonal polynomials with respect to some weight function $\varphi \geqslant 0$ in $L^2(\mathbb{T})$. Then, denoting $P_n(x) = \sum_{k=0}^{n} \alpha_{nk} x^k$ $\forall n \in \mathbb{N}_0$, we obtain "orthogonal operator polynomials" $\mathscr{P}_n(\mathscr{X})$ with respect to $T^2(\mathbb{T})$, working on the ground state $\Psi \equiv \xi_\mu^{-1}(\sqrt{\varphi})$. They are explicitly given by* (18).

Proof. In principle, everything has been shown above. The main reason for the possibility of calculations like those lies again in the fundamental theorem on the Banach resp. Hilbert correspondence. □

Remark. The "orthogonality" of the operator polynomials $\mathscr{P}_n$ should be understood by their action on the ground state:

$$\mathscr{P}_n \perp \mathscr{P}_m \quad \forall n \neq m \Longleftrightarrow \langle \mathscr{P}_n(\mathscr{X})\Psi, \mathscr{P}_m(\mathscr{X})\Psi\rangle_\mu = 0 \quad \forall n \neq m.$$

An interesting question might be whether one could introduce a scalar product on some space of operators via this relation. Anyway, one cannot consider a "weight function" $\Phi \equiv \Psi^2$. Hence this observation encourages us to consider for instance *ladder formalisms* generated by some ground states rather than Gram–Schmidt orthonormalization with respect to some weight function.

To sum up, we give an example on how to apply Theorem 15, which was originally considered in context of [7]. It concerns equidistant lattices, as already mentioned.

Example 16. Let $\mathbb{T} = y - h\mathbb{N}_0$, where $h > 0$ and $y = 1/h\alpha$. Here $\alpha > 0$ has to be chosen such that $\alpha^{-1} \in h\mathbb{N}$. Then the polynomials $H_n(x) = (-h\alpha)^n C_n^{(a)}(a - x/h)$, where the $C_n^{(a)}$ denote the well-known Charlier polynomials, are orthogonal on $\mathbb{T}$ with respect to the weight function φ given by

$$\varphi(y - nh) = \prod_{k=1}^{n} [1 - \alpha h(y - k)]^{-1} \quad \forall n \in \mathbb{N}_0.$$

Now the theorem yields that the operator polynomials $\mathscr{H}_n(\mathscr{X})$ are "orthogonal" in $T^2(\mathbb{T})$ with respect to some ground state Ψ. This ground state is explicitly computed as

$$\Psi(y - hn) = \frac{\mathrm{e}^{h\sqrt{\varphi(y-hn)}} - 1}{h} \quad \forall n \in \mathbb{N}_0.$$

Theorem 15provides a general correspondence statement, but the computation in special cases is not as clear as the correspondence itself—one of the rare cases where the abstract theorem is easier to understand than concrete examples.

In any case, it is still an open question, which of the two Hilbert spaces is more likely to deliver a sufficient theory of orthogonal function systems.

7. Future perspectives

A main perspective will be the extension of the considered structures to the theory of orthogonal polynomials. A main branch of these future investigations shall deal with the question of how to classify different orthogonal function systems in Hilbert spaces of time scale analysis. Fundamental results in this direction will also shed some more light on the question of how to generalize Schrödinger's equation within the framework of time scale analysis.

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References

[1] E. Akın-Bohner, M. Bohner, Miscellaneous Dynamic Equations, Methods and Applications of Analysis, vol. 10, no. 1, 2003, pp. 11–30.
[2] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.
[3] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.
[4] L. Erbe, A. Peterson, M. Simon, Square integrability of Gaussian bells on time scales, Math. Comput. Modell., preprint, 2003 (accepted).
[5] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, Results Math. 18 (1990) 18–56.
[6] V. Kac, P. Cheung, Quantum Calculus, Springer, New York, 2001.
[7] A. Ruffing, J. Lorenz, K. Ziegler, Difference ladder operators for a harmonic Schrödinger oscillator using unitary linear lattices, J. Comput. Appl. Math. 153 (2003) 395–410.
[8] M. Simon, The Banach Space of Positively Regressive Functions on a Time Scale, Talk at ICDEA, vol. 8, 2003.