Convergence of the family of the deformed Euler–Halley iterations under the Hölder condition of the second derivative[☆]

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Abstract

The convergence problem of the family of the deformed Euler–Halley iterations with parameters for solving nonlinear operator equations in Banach spaces is studied. Under the assumption that the second derivative of the operator satisfies the Hölder condition, a convergence criterion of the order $2 + p$ of the iteration family is established. An application to a nonlinear Hammerstein integral equation of the second kind is provided.
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1. Introduction

Let X and Y be (real or complex) Banach spaces, $\Omega \subseteq X$ an open subset and let $F : \Omega \subseteq X \to Y$ be a nonlinear operator with a second continuous derivative on Ω. Solving the operator equation

$$F(x) = 0 \tag{1.1}$$

is a basic and important problem in applied and computational mathematics. The most well-known method to solve (1.1) is the Newton method, which is quadratically convergent under the proper conditions, showed

by the well-known Kantorovich theorem; see e.g. [17]. Recent progress in the Newton method is referred to [21,20] and [22]. The important cubical generalizations of the Newton method are the well-known Chebyshev method (or Euler method) [1,3,15,16] and the Halley method [4,2,12,15]. More generally, Gutiérrez and Hernández proposed the family of the Euler–Halley iterations in [8], which is defined as follows:

$$x_{\alpha,n+1} = x_{\alpha,n} - [I + \tfrac{1}{2} L_F(x_{\alpha,n})[I - \alpha L_F(x_{\alpha,n})]^{-1}]F'(x_{\alpha,n})^{-1}F(x_{\alpha,n}), \ \ n = 0, 1, \ldots, \tag{1.2}$$

where $\alpha \in [0, 1]$ and

$$L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x), \quad x \in X. \tag{1.3}$$

The family of the Euler–Halley iterations includes, as particular cases, not only the Chebyshev method ($\alpha = 0$) and the Halley method ($\alpha = \tfrac{1}{2}$) but also the convex acceleration of the Newton method (or the super-Halley method) ($\alpha = 1$) (cf. [5,10,9]). Hence, this family has recently been studied extensively; see for example [11,23]. However, the disadvantage of this family is that evaluation of the second derivative of the operator F is required at every step, the operation cost of which may be very large in fact. To reduce the operation cost but also retain the cubical convergence, in [6] Ezquerro and Hernández gave a variant of avoiding computation of the second derivative for the convex acceleration of the Newton method. More precisely, the second derivative of F at x_n in this method is replaced by the difference of the first derivatives at x_n and z_n:

$$F''(x_n)(z_n - x_n) \approx F'(z_n) - F'(x_n), \tag{1.4}$$

where for some $\lambda \in [0, 1]$,

$$y_n = x_n - F'(x_n)^{-1}F(x_n) \quad \text{and} \quad z_n = x_n + \lambda(y_n - x_n). \tag{1.5}$$

Thus under the assumption that the second derivative of F satisfies the Lipschitz condition on Ω, a cubical convergence criterion based on the Lipschitz constant and boundary of the second derivative was established for this method. The same variant was given in [13] for the Chebyshev method, and cubical convergence criteria for the variant of the Chebyshev method were studied in [13,14]. Convergence criteria for these methods under the assumption that the first derivative satisfies the Lipschitz condition were discussed in [7,13].

Motivated by the idea of Ezquerro and Hernández in [6] and [13], we define, for arbitrary $\alpha \in [0, 1]$, the variant of the family of the Euler–Halley iterations with parameters as follows: Let $\lambda \in (0, 1]$ and $\alpha \in [0, 1]$. Define

$$H(x, y) = \frac{1}{\lambda} F'(x)^{-1}[F'(x + \lambda(y - x)) - F'(x)] \tag{1.6}$$

and

$$Q(x, y) = -\tfrac{1}{2} H(x, y)[\mathbf{I} + \alpha H(x, y)]^{-1}, \tag{1.7}$$

where $\mathbf{I}$ is the identity. Then the family of the deformed Euler–Halley iterations with parameters is defined by

$$y_n = x_n - F'(x_n)^{-1}F(x_n), \quad n = 0, 1, 2, \ldots, \tag{1.8}$$

$$x_{n+1} = y_n + Q(x_n, y_n)(y_n - x_n), \quad n = 0, 1, 2, \ldots . \tag{1.9}$$

Under the assumption that the second derivative of F satisfies the Hölder condition on the ball $\mathbf{B}(x_0, r) \subseteq \Omega$ for some proper $r > 0$, we will establish a unified convergence criterion based on the values of the operator and its second derivative at the initial point as well as the Hölder constant for the family of the deformed Euler–Halley iterations with parameters. The main theorem is stated in Section 3, which extends and improves the corresponding results for the deformed convex acceleration of the Newton method and the Chebyshev method obtained in [6,13,14]. An application to a nonlinear Hammerstein integral equation of the second kind (cf. [18]) is given in the final section.

Throughout the paper, we always assume that $\alpha \in [0, 1]$, $\lambda \in (0, 1]$, $p \in [0, 1]$ and that $x_0 \in X$ such that an inverse $F'(x_0)^{-1}$ of F' at x_0 exists. For $r > 0$, we use $\mathbf{B}(x_0, r)$ to denote the open ball with radius r and center x_0 while $\overline{\mathbf{B}(x_0, r)}$ stands for the corresponding closed ball.

2. Iteration sequences $\{a_n\}$ and $\{b_n\}$

We begin with two real-valued functions defined below, which will play an important role in analyzing the convergence order of the family of the Euler–Halley iterations. Write

$$s_\alpha = \frac{2}{1 + \alpha + \sqrt{(1 - \alpha)^2 + 2}}, \quad \alpha \in [0, 1]. \tag{2.1}$$

Then

$$s\left(1 + \frac{s}{2(1 - \alpha s)}\right) < 1, \quad \forall s \in [0, s_\alpha). \tag{2.2}$$

Now we define continuous functions g on $[0, s_\alpha)$ and h on $[0, s_\alpha) \times [0, +\infty)$, respectively, by

$$g(s) = \frac{1}{1 - s(1 + \frac{s}{2(1-\alpha s)})} = \frac{2(1 - \alpha s)}{2(1 - \alpha s)(1 - s) - s^2}, \quad \forall s \in [0, s_\alpha) \tag{2.3}$$

and

$$h(s, t) = \frac{(2 + (p + 2)\lambda^p)t}{2(p + 1)(p + 2)} + \frac{\alpha(1 - \lambda)^p s t}{2(p + 1)(1 - \alpha s)} + \frac{(1 - \alpha)s^2}{2(1 - \alpha s)} + \frac{s^3}{8(1 - \alpha s)^2},$$
$$\forall (s, t) \in [0, s_\alpha) \times [0, +\infty), \tag{2.4}$$

where we adopt the convention that $0^0 = 0$. Then the following lemma on the properties of the functions is trivial and so the proof is omitted.

Lemma 2.1. *Let g and h be defined by* (2.3) *and* (2.4), *respectively. Then*

(1) *$g(\cdot)$ is strictly monotonically increasing on $[0, s_\alpha)$ and $g(0) = 1$;*
(2) *for each $(s, t) \in [0, s_\alpha) \times [0, +\infty)$, $h(s, \cdot)$ and $h(\cdot, t)$ are strictly monotonically increasing, respectively, on $[0, +\infty)$ and $[0, s_\alpha)$; and*
(3) *for each $\gamma \in (0, 1)$,*

$$h(\gamma s, \gamma^{1+p} t) \leqslant \gamma^{1+p} h(s, t), \quad \forall (s, t) \in [0, s_\alpha) \times [0, +\infty). \tag{2.5}$$

Let $a_0 > 0$ and $b_0 > 0$. Define real sequences $\{a_n\}$ and $\{b_n\}$, respectively, by

$$a_n = a_{n-1} g(a_{n-1})^2 h(a_{n-1}, b_{n-1}), \quad \forall n \geqslant 1 \tag{2.6}$$

and

$$b_n = b_{n-1} g(a_{n-1})^{p+2} h(a_{n-1}, b_{n-1})^{p+1}, \quad \forall n \geqslant 1. \tag{2.7}$$

Then we have the following lemma.

Lemma 2.2. *Let $a_0 \in (0, s_\alpha)$ and $b_0 > 0$ such that $g(a_0)^2 h(a_0, b_0) < 1$. Set $\gamma = g(a_0)^2 h(a_0, b_0)$ (hence $\gamma = a_1/a_0 < 1$) and $\Delta = g(a_0)^{-1}$. Then*

$$a_n \leqslant \gamma^{(p+2)^{n-1}} a_{n-1} \leqslant \gamma^{\frac{(p+2)^n-1}{p+1}} a_0, \quad \forall n \geqslant 1, \tag{2.8}$$

$$b_n \leqslant (\gamma^{(p+2)^{n-1}})^{(p+1)} b_{n-1} \leqslant \gamma^{(p+2)^n-1} b_0, \quad \forall n \geqslant 1, \tag{2.9}$$

$$\prod_{j=0}^{n-1} g(a_j) h(a_j, b_j) \leqslant \gamma^{\frac{(p+2)^n-1}{p+1}} \Delta^n, \quad \forall n \geqslant 1. \tag{2.10}$$

Proof. To show (2.8), it suffices to show that

$$a_n \leqslant \gamma^{(p+2)^{n-1}} a_{n-1} \tag{2.11}$$

holds for each $n \geqslant 1$. We proceed by mathematical induction. Clearly, (2.11) holds for $n = 1$. Now assume that (2.11) holds for $n = k$. Then, by (2.6), we have that

$$a_{k+1} = a_k g(a_k)^2 h(a_k, b_k) \leqslant \gamma^{(p+2)^{k-1}} a_{k-1} g(\gamma^{(p+2)^{k-1}} a_{k-1})^2 h(\gamma^{(p+2)^{k-1}} a_{k-1}, (\gamma^{(p+2)^{k-1}})^{p+1} b_{k-1}).$$

Consequently, by (2.5), we obtain that

$$a_{k+1} \leqslant \gamma^{(p+2)^{k-1}} a_{k-1} g(a_{k-1})^2 (\gamma^{(p+2)^{k-1}})^{p+1} h(a_{k-1}, b_{k-1}) = \gamma^{(p+2)^k} a_k. \tag{2.12}$$

This shows that (2.11) holds for $n = k + 1$. Therefore, (2.11) holds for all $n \geqslant 1$.

Next, by (2.7) and (2.11), we obtain that, for each n,

$$b_{n+1} = b_n g(a_n)^{p+2} h(a_n, b_n)^{p+1} < b_n [g(a_n)^2 h(a_n, b_n)]^{p+1} \leqslant \left(\frac{a_{n+1}}{a_n} \right)^{p+1} b_n \leqslant (\gamma^{(p+2)^n})^{p+1} b_n \tag{2.13}$$

and so (2.9) is proved.

Finally, from (2.8) and (2.9), one has that, for each $j \geqslant 0$,

$$g(a_j) h(a_j, b_j) \leqslant g(\gamma^{\frac{(p+2)^j-1}{p+1}} a_0) h(\gamma^{\frac{(p+2)^j-1}{p+1}} a_0, \gamma^{(p+2)^j-1} b_0) \leqslant \gamma^{(p+2)^j-1} g(a_0) h(a_0, b_0) = \gamma^{(p+2)^j} \Delta.$$

Hence,

$$\prod_{j=0}^{n-1} g(a_j) h(a_j, b_j) \leqslant \prod_{j=0}^{n-1} \gamma^{(p+2)^j} \Delta = \gamma^{\frac{(p+2)^n-1}{p+1}} \Delta^n.$$

The proof is complete. $\quad \square$

We still need two simple lemmas. The first one is clear while the second one is a direct consequence of Lemma 2.1. Recall that g and h are given by (2.3) and (2.4), respectively.

Lemma 2.3. *Let $K \geqslant 0$ and $\beta > 0$. Then the equation*

$$K t^{p+1} + \beta t - 1 = 0 \tag{2.14}$$

has a unique positive solution r satisfying $r \leqslant \frac{1}{\beta}$.

Lemma 2.4. *Let $K \geqslant 0$, $r > 0$ and let s_α be defined by (2.1). Define the function q on $[0, r s_\alpha)$ by*

$$q(s) = g(s/r)^2 h(s/r, K s^{p+1}), \quad \forall s \in [0, r s_\alpha). \tag{2.15}$$

Then q is strictly monotonically increasing and continuous on $[0, r s_\alpha)$ with $q(0) = 0$ and $q(r s_\alpha -) = +\infty$. Consequently, there exists a unique $r_\alpha \in (0, r s_\alpha)$ such that $q(r_\alpha) = 1$.

3. Convergence of the iterations

Let r be the unique positive solution of equation (2.14). Then

$$\beta + K r^p = \frac{1}{r}. \tag{3.1}$$

Throughout this section, we assume that the nonlinear operator $F : \mathbf{B}(x_0, r) \subseteq X \to Y$ satisfies the following conditions:

$$\| F'(x_0)^{-1} F(x_0) \| \leqslant \eta, \tag{3.2}$$

$$\| F'(x_0)^{-1} F''(x_0) \| \leqslant \beta \tag{3.3}$$

and

$$\| F'(x_0)^{-1} (F''(x) - F''(y)) \| \leqslant K \| x - y \|^p, \quad \forall x, y \in \mathbf{B}(x_0, r). \tag{3.4}$$

Below we will establish the convergence of order $p + 2$ for the family of the Euler–Halley iterations. For this purpose, we first give several lemmas.

Lemma 3.1. *Let $x \in \mathbf{B}(x_0, r)$. Then $F'(x)^{-1}$ exists and satisfies that*

$$\| F'(x)^{-1} F'(x_0) \| \leqslant \frac{1}{1 - \frac{1}{r} \| x - x_0 \|}. \tag{3.5}$$

Proof. Since by (3.1)

$$\beta + K \| x - x_0 \|^p \leqslant \beta + K r^p = \frac{1}{r},$$

it follows from (3.4) that

$$
\begin{aligned}
\|I - F'(x_0)^{-1}F'(x)\| \\
&= \|F'(x_0)^{-1}F''(x_0)(x - x_0) + \int_0^1 F'(x_0)^{-1}[F''(x_0 + \tau(x - x_0)) - F''(x_0)](x - x_0)\,d\tau\| \\
&\leqslant \left(\|F'(x_0)^{-1}F''(x_0)\| + \int_0^1 \|F'(x_0)^{-1}[F''(x_0 + \tau(x - x_0)) - F''(x_0)]\|\,d\tau\right)\|x - x_0\| \\
&\leqslant (\beta + K\|x - x_0\|^p)\|x - x_0\| \leqslant \tfrac{1}{r}\|x - x_0\| < 1.
\end{aligned}
$$

Consequently, the well-known Banach Lemma yields that $F'(x)^{-1}$ exists and (3.5) holds.   □

Let $\{y_n\}$ and $\{x_n\}$ denote the sequences generated by (1.8) and (1.9), respectively. Then by (1.6) and (1.7)

$$
H(x_n, y_n) = \frac{1}{\lambda}\, F'(x_n)^{-1}[F'(x_n + \lambda(y_n - x_n)) - F'(x_n)] \tag{3.6}
$$

and

$$
Q(x_n, y_n) = -\tfrac{1}{2}\, H(x_n, y_n)[I + \alpha H(x_n, y_n)]^{-1}. \tag{3.7}
$$

Lemma 3.2. *Suppose that $y_n, x_n \in \mathbf{B}(x_0, r)$ and that*

$$
\tfrac{1}{r}\|F'(x_n)^{-1}F'(x_0)\|\,\|y_n - x_n\| < 1. \tag{3.8}
$$

Then

$$
\|Q(x_n, y_n)\| \leqslant \frac{\tfrac{1}{r}\|F'(x_n)^{-1}F'(x_0)\|\,\|y_n - x_n\|}{2(1 - \tfrac{\alpha}{r}\|F'(x_n)^{-1}F'(x_0)\|\,\|y_n - x_n\|)}, \tag{3.9}
$$

$$
\|x_{n+1} - x_n\| \leqslant (1 + \|Q(x_n, y_n)\|)\|y_n - x_n\| \tag{3.10}
$$

and

$$
\begin{aligned}
\|F'(x_0)^{-1}F(x_{n+1})\| \leqslant {} & \left(\frac{2 + (p + 2)\lambda^p}{2(p + 1)(p + 2)} + \frac{\alpha(1 - \lambda)^p}{(p + 1)}\|Q(x_n, y_n)\|\right) K\|y_n - x_n\|^{p+2} \\
& + \frac{1}{r}\left((1 - \alpha)\|Q(x_n, y_n)\| + \frac{\|Q(x_n, y_n)\|^2}{2}\right)\|y_n - x_n\|^2.
\end{aligned} \tag{3.11}
$$

Proof. Note that

$$
H(x_n, y_n) = \int_0^1 [F'(x_n)^{-1}F''(x_0) + F'(x_n)^{-1}(F''(x_n + \tau\lambda(y_n - x_n)) - F''(x_0))](y_n - x_n)\,d\tau.
$$

Hence, by (3.4) and (3.1),

$$
\begin{aligned}
\|H(x_n, y_n)\| &\leqslant \|F'(x_n)^{-1}F'(x_0)\| \int_0^1 [\beta + K((1-\tau\lambda)\|x_n - x_0\| + \tau\lambda\|y_n - x_0\|)^p]\,\mathrm{d}\tau \|y_n - x_n\| \\
&\leqslant \|F'(x_n)^{-1}F'(x_0)\|(\beta + Kr^p)\|y_n - x_n\| \\
&= \tfrac{1}{r}\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\|.
\end{aligned} \tag{3.12}
$$

It follows from (3.8) and the Banach Lemma that

$$
\|Q(x_n, y_n)\| \leqslant \frac{\|H(x_n, y_n)\|}{2(1 - \alpha\|H(x_n, y_n)\|)} \leqslant \frac{\tfrac{1}{r}\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\|}{2(1 - \tfrac{\alpha}{r}\|F'(x_n)^{-1}F'(x_0)\|\|y_n - x_n\|)},
$$

hence, (3.9) is proved. (3.10) follows from

$$
x_{n+1} - x_n = y_n - x_n + Q(x_n, y_n)(y_n - x_n). \qquad \square
$$

Finally, to verify (3.11), noting that $x_{n+1} - y_n = Q(x_n, y_n)(y_n - x_n)$ and applying Taylor's expression, we attain that

$$
\begin{aligned}
F(x_{n+1}) = &\int_0^1 [F''(x_n + \tau(y_n - x_n)) - F''(x_n)](1 - \tau)(y_n - x_n)^2\,\mathrm{d}\tau \\
&+ \frac{1}{2}\int_0^1 [F''(x_n) - F''(x_n + \tau\lambda(y_n - x_n))](y_n - x_n)^2\,\mathrm{d}\tau \\
&- \int_0^1 [\alpha F''(x_n + \tau\lambda(y_n - x_n)) - \alpha F''(x_n + \tau(y_n - x_n))]\,\mathrm{d}\tau \\
&\times (y_n - x_n)Q(x_n, y_n)(y_n - x_n) \\
&+ (1 - \alpha)\int_0^1 F''(x_n + \tau(y_n - x_n))]\,\mathrm{d}\tau(y_n - x_n)Q(x_n, y_n)(y_n - x_n) \\
&+ \int_0^1 F''(y_n + \tau(x_{n+1} - y_n))(1 - \tau)(Q(x_n, y_n)(y_n - x_n))^2\,\mathrm{d}\tau.
\end{aligned}
$$

Consequently, by (3.4),

$$
\begin{aligned}
\|F'(x_0)^{-1}F(x_{n+1})\| \leqslant &\int_0^1 K\tau^p\|y_n - x_n\|^{p+2}(1 - \tau)\,\mathrm{d}\tau + \frac{1}{2}\int_0^1 K\lambda^p\tau^p\|y_n - x_n\|^{p+2}\,\mathrm{d}\tau \\
&+ \alpha\int_0^1 K(1 - \lambda)^p\tau^p\|y_n - x_n\|^{p+2}\,\mathrm{d}\tau\|Q(x_n, y_n)\| \\
&+ (1 - \alpha)\int_0^1 (\beta + K\|(1 - \tau)(x_n - x_0) + \tau(y_n - x_0)\|^p)\,\mathrm{d}\tau\|Q(x_n, y_n)\| \\
&+ \int_0^1 (\beta + K\|(1 - \tau)(y_n - x_0) + \tau(x_n - x_0)\|^p)(1 - \tau)\mathrm{d}\tau\|x_{n+1} - y_n\|^2.
\end{aligned}
$$

Note that

$$\beta + K\|(1-\tau)(x_n - x_0) + \tau(y_n - x_0)\|^p \leqslant \frac{1}{r},$$

$$\beta + K\|(1-\tau)(y_n - x_0) + \tau(x_n - x_0)\|^p \leqslant \frac{1}{r}$$

and

$$\|x_{n+1} - y_n\| = \|Q(x_n, y_n)\| \|y_n - x_n\|.$$

(3.11) follows. The proof is complete. $\square$

Recall that q is defined by

$$q(s) = g(s/r)^2 h(s/r, Ks^{p+1}), \quad \forall s \in [0, rs_\alpha)$$

and $r_\alpha \in (0, rs_\alpha)$ is a unique solution of the equation $q(s) = 1$.

Lemma 3.3. *Let $a_0 = \eta/r$, $b_0 = K\eta^{p+1}$ and $\Delta = g(a_0)^{-1}$. Let $\{a_n\}$ and $\{b_n\}$ be the sequences generated by (2.6) and (2.7), respectively. Suppose that $\eta < r_\alpha$. Then the following inequalities hold for all $n = 0, 1, \ldots$.*

$$K\|F'(x_n)^{-1}F'(x_0)\| \|y_n - x_n\|^{p+1} \leqslant b_n, \tag{3.13}$$

$$\|y_n - x_n\| \leqslant \gamma^{\frac{(p+2)^n - 1}{p+1}} \Delta^n \|y_0 - x_0\|, \tag{3.14}$$

$$\frac{1}{r}\|F'(x_n)^{-1}F'(x_0)\| \|y_n - x_n\| \leqslant a_n, \tag{3.15}$$

$$\|y_n - x_0\| \leqslant (1 - (\gamma\Delta)^{n+1})r, \tag{3.16}$$

$$\|x_{n+1} - x_n\| \leqslant \left(1 + \frac{a_n}{2(1 - \alpha a_n)}\right)\|y_n - x_n\|, \tag{3.17}$$

$$\|x_{n+1} - x_0\| \leqslant (1 - (\gamma\Delta)^{n+1})r \tag{3.18}$$

and

$$\|F'(x_{n+1})^{-1}F'(x_0)\| \leqslant g(a_n)\|F'(x_n)^{-1}F'(x_0)\|. \tag{3.19}$$

Proof. Since $\eta < r_\alpha$, we have that $g(a_0)^2 h(a_0, b_0) < 1$ by Lemma 2.4. Hence, by Lemma 2.2, $\{a_n\}$ and $\{b_n\}$ are strictly monotonically decreasing. In addition, as $\eta = a_0 r$,

$$\left(1 + \frac{a_n}{2(1 - \alpha a_n)}\right)\eta \leqslant \left(1 + \frac{a_0}{2(1 - \alpha a_0)}\right)\eta = (1 - \Delta)r \leqslant (1 - \gamma\Delta)r, \quad \forall n \geqslant 1. \tag{3.20}$$

Clearly, (3.13), (3.14) and (3.15) hold for $n = 0$. Since $\frac{1}{r}\|y_0 - x_0\| \leqslant a_0 < s_\alpha < 1$, it follows from (3.9) and (3.10) that

$$\|x_1 - x_0\| \leqslant a_0\left(1 + \frac{a_0}{2(1 - \alpha a_0)}\right)r. \tag{3.21}$$

Therefore $\|x_1 - x_0\| < r$ thanks to (2.2). This means that (3.17) and (3.18) hold for $n = 0$. Furthermore, by Lemma 3.1 and (3.21),

$$\|F'(x_1)^{-1}F'(x_0)\| \leqslant \frac{1}{1 - \frac{1}{r}\|x_1 - x_0\|} \leqslant \frac{1}{1 - a_0(1 + \frac{a_0}{2(1-\alpha a_0)})} = g(a_0). \tag{3.22}$$

That is, (3.19) holds for $n = 0$. We will proceed by mathematical induction. For this purpose, assume that (3.13)–(3.19) hold for all $n \leqslant k$. Then it suffices to show that they hold for $n = k + 1$. We first show that

$$\|y_n - x_n\| \leqslant g(a_{n-1})h(a_{n-1}, b_{n-1})\|y_{n-1} - x_{n-1}\|, \quad \forall n \leqslant k + 1. \tag{3.23}$$

Indeed, for each $n \leqslant k + 1$, since $n - 1 \leqslant k$, we obtain that, by inductional assumption (3.15),

$$\frac{1}{r}\|F'(x_{n-1})^{-1}F'(x_0)\|\|y_{n-1} - x_{n-1}\| \leqslant a_{n-1} \leqslant s_\alpha < 1. \tag{3.24}$$

By inductional assumptions (3.18) and (3.16), $x_{n-1}, y_{n-1} \in \mathbf{B}(x_0, r)$. It follows from (3.11) that

$$\begin{aligned}
&\|F'(x_{n-1})^{-1}F(x_n)\| \\
&\quad \leqslant \|F'(x_{n-1})^{-1}F'(x_0)\|\|F'(x_0)^{-1}F(x_n)\| \\
&\quad \leqslant \left(\frac{2 + (p+2)\lambda^p}{2(p+1)(p+2)} + \frac{\alpha(1-\lambda)^p}{(p+1)}\|Q(x_{n-1}, y_{n-1})\|\right) K\|F'(x_{n-1})^{-1}F'(x_0)\|\|y_{n-1} - x_{n-1}\|^{p+2} \\
&\qquad + \frac{1}{r}\left((1-\alpha)\|Q(x_{n-1}, y_{n-1})\| + \frac{\|Q(x_{n-1}, y_{n-1})\|^2}{2}\right)\|F'(x_{n-1})^{-1}F'(x_0)\|\|y_{n-1} - x_{n-1}\|^2.
\end{aligned} \tag{3.25}$$

Using (3.9) and (3.24), we have that

$$\|Q(x_{n-1}, y_{n-1})\| \leqslant \frac{a_{n-1}}{2(1 - \alpha a_{n-1})}. \tag{3.26}$$

Note that, by inductional assumption (3.13),

$$K\|F'(x_{n-1})^{-1}F'(x_0)\|\|y_{n-1} - x_{n-1}\|^{p+1} \leqslant b_{n-1}. \tag{3.27}$$

Therefore, by (3.24)–(3.27), we obtain that

$$\begin{aligned}
&\|F'(x_{n-1})^{-1}F(x_n)\| \\
&\quad \leqslant \left\{\frac{1}{2(p+1)}\left[\frac{2 + (p+2)\lambda^p}{(p+2)} + \frac{\alpha(1-\lambda)^p a_{n-1}}{(1 - \alpha a_{n-1})}\right]b_{n-1} + \frac{(1-\alpha)a_{n-1}^2}{2(1 - \alpha a_{n-1})}\right. \\
&\qquad \left. + \frac{a_{n-1}^3}{8(1 - \alpha a_{n-1})^2}\right\}\|y_{n-1} - x_{n-1}\| = h(a_{n-1}, b_{n-1})\|y_{n-1} - x_{n-1}\|.
\end{aligned} \tag{3.28}$$

Since

$$\|y_n - x_n\| \leqslant \|F'(x_n)^{-1}F'(x_{n-1})\|\|F'(x_{n-1})^{-1}F(x_n)\|,$$

(3.23) follows from (3.28) and inductional assumption (3.19); hence (3.23) is proved. $\quad\square$

Thus, by (3.23) and inductional assumptions (3.19) and (3.13), we have that

$$
\begin{aligned}
K\|F'(x_{k+1})^{-1}F'(x_0)\| \|y_{k+1} - x_{k+1}\|^{p+1} \\
\leqslant g(a_k)K\|F'(x_k)^{-1}F'(x_0)\| \|y_k - x_k\|^{p+1} g(a_k)^{p+1} h(a_k, b_k)^{p+1} \\
\leqslant b_k g(a_k)^{p+2} h(a_k, b_k)^{p+1} = b_{k+1}.
\end{aligned}
\tag{3.29}
$$

This shows that (3.13) holds for $n = k + 1$. As for (3.14), it can be shown easily by (2.10) and (3.23), while (3.15) follows from (3.23) and inductional assumptions (3.19) and (3.15). Hence (3.14) and (3.15) hold for $n = k + 1$. By (3.14) and inductional assumption (3.18), one has that

$$
\begin{aligned}
\|y_{k+1} - x_0\| \leqslant \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_0\| \leqslant \gamma^{\frac{(p+2)^{k+1}-1}{p+1}} \Delta^{k+1}\eta + (1 - (\gamma\Delta)^{k+1})r \\
\leqslant (\gamma\Delta)^{k+1})\eta + (1 - (\gamma\Delta)^{k+1})r \leqslant (1 - (\gamma\Delta)^{k+2})r
\end{aligned}
\tag{3.30}
$$

because $\eta \leqslant (1 + \frac{a_0}{2(1-\alpha a_0)})\eta \leqslant (1 - \gamma\Delta)r$. Therefore (3.16) holds for $n = k + 1$. Consequently, $x_{k+1}, y_{k+1} \in B(x_0, r)$ by (3.30) and inductional assumption (3.18). Hence, by (3.9) and (3.10), we have that

$$
\|x_{k+2} - x_{k+1}\| \leqslant \left(1 + \frac{\frac{1}{r}\|F'(x_{k+1})^{-1}F'(x_0)\| \|y_{k+1} - x_{k+1}\|}{2(1 - \frac{\alpha}{r}\|F'(x_{k+1})^{-1}F'(x_0)\| \|y_{k+1} - x_{k+1}\|)}\right) \|y_{k+1} - x_{k+1}\|.
$$

Since (3.15) holds for $n = k + 1$, we obtain that

$$
\|x_{k+2} - x_{k+1}\| \leqslant (1 + \frac{a_{k+1}}{2(1 - \alpha a_{k+1})})\|y_{k+1} - x_{k+1}\|.
\tag{3.31}
$$

This shows that (3.17) holds for $n = k + 1$. Now let us show that (3.18) holds for $n = k + 1$. In fact, by (3.31) and (3.14),

$$
\|x_{k+2} - x_{k+1}\| \leqslant \left(1 + \frac{a_{k+1}}{2(1 - \alpha a_{k+1})}\right)\gamma^{\frac{(p+2)^{k+1}-1}{p+1}}\Delta^{k+1}\eta \leqslant \left(1 + \frac{a_{k+1}}{2(1 - \alpha a_{k+1})}\right)\eta(\gamma\Delta)^{k+1}.
\tag{3.32}
$$

Combining this with (3.20) and inductional assumption (3.18) gives that

$$
\begin{aligned}
\|x_{k+2} - x_0\| \leqslant \|x_{k+2} - x_{k+1}\| + \|x_{k+1} - x_0\| \\
\leqslant (1 - \gamma\Delta)r(\gamma\Delta)^{k+1} + (1 - (\gamma\Delta)^{k+1})r = (1 - (\gamma\Delta)^{k+2})r
\end{aligned}
\tag{3.33}
$$

and so (3.18) holds for $n = k + 1$. Thus, it remains to show that (3.19) holds for $n = k + 1$. Since $x_{k+1}, x_{k+2} \in \mathbf{B}(x_0, r)$, similar to the proof of (3.12), we obtain that

$$
\|I - F'(x_{k+1})^{-1}F'(x_{k+2})\| \leqslant \frac{1}{r}\|F'(x_{k+1})^{-1}F'(x_0)\| \|x_{k+2} - x_{k+1}\|.
\tag{3.34}
$$

Using (3.15) and (3.17), we obtain that

$$
\|I - F'(x_{k+1})^{-1}F'(x_{k+2})\| \leqslant a_{k+1}\left(1 + \frac{a_{k+1}}{2(1 - \alpha a_{k+1})}\right) < 1,
$$

where the last inequality is because of (2.2). Then, by the Banach Lemma,

$$\|F'(x_{k+2})^{-1}F'(x_0)\| \leqslant \frac{\|F'(x_{k+1})^{-1}F'(x_0)\|}{1 - a_{k+1}\left(1 + \frac{a_{k+1}}{2(1 - \alpha a_{k+1})}\right)} = g(a_{k+1})\|F'(x_{k+1})^{-1}F'(x_0)\|. \tag{3.35}$$

That is, (3.19) holds for $n = k + 1$. The proof is complete. $\quad\square$

Now we are ready to state the main theorem. Recall that q is defined by

$$q(s) = g(s/r)^2 h(s/r, Ks^{p+1}), \quad \forall s \in [0, rs_\alpha),$$

while g and h are, respectively, defined by

$$g(s) = \frac{2(1 - \alpha s)}{2(1 - \alpha s)(1 - s) - s^2}, \quad \forall s \in [0, s_\alpha)$$

and

$$h(s, t) = \frac{(2 + (p + 2)\lambda^p)t}{2(p + 1)(p + 2)} + \frac{\alpha(1 - \lambda)^p st}{2(p + 1)(1 - \alpha s)} + \frac{(1 - \alpha)s^2}{2(1 - \alpha s)} + \frac{s^3}{8(1 - \alpha s)^2},$$
$$\forall (s, t) \in [0, s_\alpha) \times [0, +\infty).$$

Theorem 3.1. *Let $F : \Omega \subseteq X \to Y$ be a nonlinear operator with a continuous second derivative F'' and let $x_0 \in \Omega$ be such that $F'(x_0)^{-1}$ exists. Let r be a unique positive solution of the equation $Kt^{p+1} + \beta t - 1 = 0$. Suppose that $\mathbf{B}(x_0, r) \subseteq \Omega$ and that conditions (3.2)–(3.4) are satisfied. Let r_α be a unique solution of the equation $q(s) = 1$ on $(0, rs_\alpha)$. If $\eta < r_\alpha$, then the sequence $\{x_n\}$ generated by (1.8) and (1.9) with initial point x_0 converges at a rate of order $2 + p$ to a unique solution x^* of the equation $F(x) = 0$ on $\mathbf{B}(x_0, r)$.*

Proof. Recall that $a_0 = \eta/r$, $b_0 = K\eta^{p+1}$ and $\Delta = g(a_0)^{-1}$. Since $\eta < r_\alpha$, it follows from (3.14) and (3.17) that, for each $n \geqslant 0$,

$$\|x_{n+1} - x_n\| \leqslant \left(1 + \frac{a_n}{2(1 - \alpha a_n)}\right)\eta\gamma^{\frac{(p+2)^n - 1}{p+1}}\Delta^n \leqslant (1 - \Delta)r\gamma^{\frac{(p+2)^n - 1}{p+1}}\Delta^n. \tag{3.36}$$

As

$$\frac{(p + 2)^i - 1}{p + 1} \geqslant \frac{(p + 2)^n - 1}{p + 1} + (i - n), \quad \forall i \geqslant n,$$

one has that, for all $m, n \geqslant 0$,

$$\|x_{m+n} - x_n\| \leqslant (1 - \Delta)r\gamma^{\frac{(p+2)^n - 1}{p+1}}\Delta^n \sum_{i=n}^{n+m-1}(\gamma\Delta)^{i-n} \leqslant \frac{(1 - \Delta)r}{(1 - \gamma\Delta)}\gamma^{\frac{(p+2)^n - 1}{p+1}}\Delta^n. \tag{3.37}$$

This means that $\{x_n\}$ is a Cauchy sequence and hence $\{x_n\}$ converges, say, to x^*. Letting $m \to \infty$ in (3.37) yields

$$\|x^* - x_n\| \leqslant \frac{(1 - \Delta)r}{(1 - \gamma\Delta)}\gamma^{\frac{(p+2)^n - 1}{p+1}}\Delta^n. \tag{3.38}$$

That is, $\{x_n\}$ converges to x^* at a rate of order $2 + p$. In particular, letting $n = 0$, we have that $\|x^* - x_0\| < r$. Since $F'(x_n)^{-1} F(x_n) \to 0$ and $F'(x)^{-1} F(x)$ is continuous, $F'(x^*)^{-1} F(x^*) = 0$; hence, $F(x^*) = 0$.

Thus, to complete the proof, it remains to show that the solution of the equation $F(x) = 0$ is unique on $\mathbf{B}(x_0, r)$. To this end, let $y^* \in \mathbf{B}(x_0, r)$ be such that $F(y^*) = 0$. Then

$$\int_0^1 F'(x_0)^{-1} F'(x^* + t(y^* - x^*)) \, dt \, (y^* - x^*) = F'(x_0)^{-1} [F(y^*) - F(x^*)] = 0. \tag{3.39}$$

As

$$\left\| \int_0^1 F'(x_0)^{-1} [F'(x^* + t(y^* - x^*)) - F'(x_0)] \, dt \right\|$$

$$\leqslant \int_0^1 \int_0^1 \|F'(x_0)^{-1} F''(x_0 + \tau(x^* + t(y^* - x^*) - x_0))\| \, d\tau [(1 - t)\|x^* - x_0\| + t\|y^* - x_0\|] dt$$

$$\leqslant r \int_0^1 \int_0^1 (\beta + K\tau^p \|(1 - t)(x^* - x_0) + t(y^* - x_0)\|^p) \, d\tau \, dt < 1,$$

by the Banach Lemma, $\int_0^1 F'(x^* + t(y^* - x^*)) \, dt$ is invertible. Hence, $y^* = x^*$ by (3.39). The proof is complete. $\square$

In particular, taking $\alpha = 0$, $\frac{1}{2}$ and 1 in Theorem 3.1, respectively, we immediately obtain the semi-local convergence results of the Euler method, the Halley method and the super-Halley method, which are not necessary to restate here. The first and the third were, respectively, investigated in [13,14] and [6].

4. Application to a nonlinear integral equation of Hammerstein type

In this section, we provide an application of the main result to a special nonlinear Hammerstein integral equation of the second kind (cf. [18]). Letting $\mu \in \mathbf{R}$ and $p \in [0, 1]$, we consider

$$x(s) = l(s) + \int_a^b G(s, t)[x(t)^{2+p} + \mu x(t)^2] \, dt, \quad s \in [a, b], \tag{4.1}$$

where l is a continuous function such that $l(s) > 0$ for all $s \in [a, b]$ and the kernel G is a non-negative continuous function on $[a, b] \times [a, b]$.

Note that if G is the Green function defined by

$$G(s, t) = \begin{cases} \dfrac{(b - s)(t - a)}{b - a}, & t \leqslant s, \\ \dfrac{(s - a)(b - t)}{b - a}, & s \leqslant t, \end{cases} \tag{4.2}$$

Eq. (4.2) is equivalent to the following boundary value problem (cf. [19]):

$$\begin{cases} x'' = -x^{2+p} - \mu x^2, \\ x(a) = v(a), \quad x(b) = v(b). \end{cases}$$

To apply Theorem 3.1, let $X = Y = C[a, b]$, the Banach space of real-valued continuous functions on $[a, b]$ with the uniform norm, and let

$$\Omega_p = \begin{cases} \{x \in C[a, b] : x(s) > 0, \; s \in [a, b]\}, & p \in (0, 1), \\ C[a, b], & p = 0, \; 1. \end{cases}$$

Define $F : \Omega_p \to C[a, b]$ by

$$[F(x)](s) = x(s) - l(s) - \int_a^b G(s, t)[x(t)^{2+p} + \mu x(t)^2] \, dt, \quad s \in [a, b]. \tag{4.3}$$

Then solving Eq. (4.1) is equivalent to solving Eq. (1.1) with F being defined by (4.3).

We start by calculating the parameters β and η in the study. Firstly, we have

$$[F'(x)u](s) = u(s) - \int_a^b G(s, t)[(2 + p)x(t)^{1+p} + 2\mu x(t)]u(t) \, dt, \quad s \in [a, b]$$

and

$$[F''(x)u\,z](s) = -\int_a^b G(s, t)[(2 + p)(1 + p)x(t)^p + 2\mu]u(t)z(t) \, dt, \quad s \in [a, b].$$

Let $x_0 \in \Omega_p$ be fixed. Then

$$\|I - F'(x_0)\| \leqslant M((2 + p)\|x_0\|^{1+p} + 2\mu\|x_0\|),$$

where

$$M = \max_{s \in [a,b]} \int_a^b |G(s, t)| \, dt.$$

By the Banach Lemma, if $M((2 + p)\|x_0\|^{1+p} + 2\mu\|x_0\|) < 1$, one has

$$\|F'(x_0)^{-1}\| \leqslant \frac{1}{1 - M((2 + p)\|x_0\|^{1+p} + 2\mu\|x_0\|)}.$$

Since

$$\|F(x_0)\| \leqslant \|x_0 - l\| + M(\|x_0\|^{2+p} + \mu\|x_0\|^2)$$

and

$$\|F''(x)\| = M((2 + p)(1 + p)\|x(t)\|^p + 2\mu),$$

it follows that

$$\|F'(x_0)^{-1}F(x_0)\| \leqslant \frac{\|x_0 - l\| + M(\|x_0\|^{2+p} + \mu\|x_0\|^2)}{1 - M((2 + p)\|x_0\|^{1+p} + 2\mu\|x_0\|)}$$

and

$$\|F'(x_0)^{-1}F''(x)\| \leqslant \frac{M((2 + p)(1 + p)\|x\|^p + 2\mu)}{1 - M((2 + p)\|x_0\|^{1+p} + 2\mu\|x_0\|)}.$$

Therefore, β and η are estimated.

On the other hand, for x, $y \in \Omega_p$,

$$[(F''(x) - F''(y))uz](s) = - \int_a^b G(s, t)(2 + p)(1 + p)(x(t)^p - y(t)^p)u(t)z(t)\,\mathrm{d}t, \quad s \in [a, b]$$

and consequently,

$$\|F''(x) - F''(y)\| \leqslant M(2 + p)(1 + p)\|x - y\|^p, \quad x, y \in \Omega_p.$$

This means that $K = M(2 + p)(1 + p)$. Thus, we can establish the following result from Theorem 3.1.

Theorem 4.1. *Let F be the nonlinear operator defined in (4.3) and $x_0 \in \Omega_p$ be a point such that $M((2 + p)\|x_0\|^{1+p} + 2\mu\|x_0\|) < 1$. Suppose that $\mathbf{B}(x_0, r) \subseteq \Omega_p$, where r is a unique positive solution of the equation $Kt^{p+1} + \beta t - 1 = 0$. Let r_α be a unique solution of the Eq. $q(s) = 1$ on $(0, rs_\alpha)$. If $\eta < r_\alpha$, then the sequence $\{x_n\}$ generated by (1.8) and (1.9) with initial point x_0 converges at a rate of order $2 + p$ to a unique solution x^* of Eq. (4.1) on $\mathbf{B}(x_0, r)$.*

Example 4.1. Let l_0 be defined by

$$l_0(s) = s, \quad s \in [0, 1]$$

and G Green's function on $[0, 1] \times [0, 1]$ defined by (4.2). Consider the following particular case of (4.1):

$$x(s) = l_0(s) + \int_0^1 G(s, t)[2x(t)^2]\,\mathrm{d}t, \quad s \in [a, b]. \tag{4.4}$$

Choose $x_0 = l_0$ for Theorem 4.1. Clearly, $K = 0$ and $p = 0$. Since $M = 1/8$, we have $\beta = 1$ and so $r = 1$. Hence $\mathbf{B}(x_0, 1) \subseteq \Omega_0$. Note that

$$\|F'(x_0)^{-1}F(x_0)\| < \frac{2M\|x_0\|^2}{1 - 4M\|x_0\|} = \frac{1}{2}.$$

Then $\eta = \|F'(x_0)^{-1}F(x_0)\| < \frac{1}{2}$. Furthermore, we have $r_\alpha = \frac{1}{2}$. Thus, assumptions of Theorem 4.1 are satisfied. Consequently, the sequence $\{x_n\}$ generated by (1.8) and (1.9) with initial point x_0 converges quadratically to a unique solution x^* of Eq. (4.4) on $\mathbf{B}(x_0, 1)$.

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