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The Bessel differential equation and the Hankel transform

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Dedicated to Professor W.D. Evans on the occasion of his 65th birthday

Abstract

This paper studies the classical second-order Bessel differential equation in Liouville form:

$$-y''(x) + (\nu^2 - \tfrac{1}{4})x^{-2}y(x) = \lambda y(x) \quad \text{for all } x \in (0, \infty).$$

Here, the parameter ν represents the order of the associated Bessel functions and λ is the complex spectral parameter involved in considering properties of the equation in the Hilbert function space $L^2(0, \infty)$.

Properties of the equation are considered when the order $\nu \in [0, 1)$; in this case the singular end-point 0 is in the limit-circle non-oscillatory classification in the space $L^2(0, \infty)$; the equation is in the strong limit-point and Dirichlet condition at the end-point $+\infty$.

Applying the generalised initial value theorem at the singular end-point 0 allows of the definition of a single Titchmarsh–Weyl m-coefficient for the whole interval $(0, \infty)$. In turn this information yields a proof of the Hankel transform as an eigenfunction expansion for the case when $\nu \in [0, 1)$, a result which is not available in the existing literature.

The application of the principal solution, from the end-point 0 of the Bessel equation, as a boundary condition function yields the Friedrichs self-adjoint extension in $L^2(0, \infty)$; the domain of this extension has many special known properties, of which new proofs are presented.

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Keywords: Bessel differential equation; Titchmarsh–Weyl m-coefficient; Hankel transform; Friedrichs extension

1. Introduction

In this paper we consider the Bessel differential equation in the classical form

$$-y''(x) + (\nu^2 - 1/4)x^{-2}y(x) = \lambda y(x) \quad \text{for all } x \in (0, \infty). \tag{1.1}$$

Here, the parameter $\nu \in [0, \infty) \subset \mathbb{R}$ is the order of the Bessel functions involved, and the parameter $\lambda \in \mathbb{C}$ is the spectral parameter. Properties of this equation are considered in the Hilbert function space $L^2(0, \infty)$.

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We restrict attention to the situation when $\nu \in [0, 1)$; in this case the endpoint 0 of Eq. (1.1) is in the singular limit-circle case, with respect to $L^2(0, \infty)$, except for the regular case when $\nu = \frac{1}{2}$.

For a regular endpoint a Sturm–Liouville equation, such as (1.1) when $\nu = \frac{1}{2}$, there is a classical solution to the initial value problem which yields the analytic dependence of the solutions on the complex spectral parameter λ; see for example [19, Chapter I, Section 1.5]. Recent studies have shown that for a limit-circle endpoint there is a generalised solution to the initial value problem, which reduces to the classical solution when the endpoint is regular; see [7, Sections 1–5, in particular Theorem 2; 1, Section 5, Theorem 5.1].

This generalised solution, to the initial value problem, allows for the definition of the Titchmarsh–Weyl m-coefficient associated with a singular boundary condition at the limit-circle endpoint; see details in the recent paper [1, Section 8]. From the Nevanlinna representation of this m-coefficient the spectral function ρ can be obtained to describe the spectrum of the associated self-adjoint operator in $L^2(0, \infty)$.

By choosing the self-adjoint operator to be the Friedrichs extension, see [15,16,8,13,17], it then proves possible to obtain the Hankel transform formula, see [18, Chapter VIII, Section 8.18; 19, Chapter IV, Section 4.11], as an eigenfunction expansion, even in the case when $\nu \in [0, 1)$.

Additional analysis then yields the limit behaviour of the functions in the domain of the Friedrichs extension, as previously discussed by a number of authors [11,17]. These results allow for discussion of a possible HELP-type integral inequality as previously considered for regular endpoints in [5,6].

We have made reference in the text to the earlier work of other authors whose papers are listed in the References.

2. Bessel differential equation

For the differential equation (1.1) on the interval $(0, \infty)$ we restrict attention to the case when the order parameter $\nu \in [0, 1)$; this restriction has the implications (a) and (b) below for the endpoint classifications at 0 and ∞ in the space $L^2(0, \infty)$; for general details of these classifications see [5, Section 3; 6, Section 5].

We use the following named Bessel functions, see [20, Chapter III], as solutions of (1.1):

(i) For $\nu = 0$

$$x^{1/2} J_0(x\sqrt{\lambda}) \quad \text{and} \quad x^{1/2} Y_0(x\sqrt{\lambda}) \ \text{ for all } x \in (0, \infty). \tag{2.1}$$

(ii) For $\nu \in (0, 1)$

$$x^{1/2} J_\nu(x\sqrt{\lambda}) \quad \text{and} \quad x^{1/2} J_{-\nu}(x\sqrt{\lambda}) \ \text{ for all } x \in (0, \infty). \tag{2.2}$$

Here and after the analytic function $\sqrt{\cdot} : \mathbb{C} \to \mathbb{C}$ is defined as follows:

$$\text{given } \lambda = \rho \exp(\mathrm{i}\eta) \quad \text{define } \sqrt{\lambda} := \rho^{1/2} \exp(\tfrac{1}{2}\,\mathrm{i}\eta) \ \text{ for } \rho \in [0, \infty) \ \text{ and } \ \eta \in [0, 2\pi). \tag{2.3}$$

In the solution $x^{1/2} Y_0(x\sqrt{\lambda})$ there is a term $\log(\frac{1}{2}\,x\sqrt{\lambda})$ which here is defined by, using again $\lambda = \rho \exp(\mathrm{i}\eta)$,

$$\log(\tfrac{1}{2}\,x\sqrt{\lambda}) := \ln(\tfrac{1}{2}\,x\rho^{1/2}) + \mathrm{i}\,\tfrac{1}{2}\,\eta. \tag{2.4}$$

It is clear from the properties of solutions (2.1) and (2.2), see [5, Section 3; 20, Chapter III], that:

(a) The endpoint $+\infty$ is strong limit-point and Dirichlet in the space $L^2(0, \infty)$; see [6, Section 5] for details of these properties.
(b) The endpoint 0^+ is limit-circle non-oscillatory (for $\nu = \frac{1}{2}$ this endpoint is regular but this classification may be regarded as limit-circle non-oscillatory; we make only infrequent special mention of this exception).

3. Hankel transform

Formally the Hankel inversion formula (here also called the Hankel transform) can be written in symmetrical form as

$$f(x) = \int_0^\infty (xs)^{1/2} J_\nu(xs)\,\mathrm{d}s \int_0^\infty (s\xi)^{1/2} J_\nu(s\xi) f(\xi)\,\mathrm{d}\xi \quad \text{for all } x \in (0, \infty). \tag{3.1}$$

A systematic account of the various forms in which this transform can be considered is given in [18, Chapter VIII].

There are three forms which command attention:

(1) Theory of direct convergence as given in [18, Chapter VIII, Section 18.8]; this method assumes $f \in L^1(0, \infty)$ and requires local bounded variation of f, as defined on $(0, \infty)$.
(2) Theory of general symmetric integral transforms as given in [18, Chapter VIII, Sections 18.1–18.4]; this method requires only that $f \in L^2(0, \infty)$ but convergence of the integrals in (3.1) is mean convergence in $L^2(0, \infty)$.
(3) Theory of Sturm–Liouville eigenfunction expansions as given in [19, Chapter IV, Section 4.11]; this method applies the general expansion theory to the Bessel differential equation (1.1) and yields both direct and mean convergence results; however, these methods yield the Hankel transform as an eigenfunction expansion (3.1) only for the case when $v \in [1, \infty)$ and this restriction seems to be due to the limit-circle classification of Eq. (1.1) in $L^2(0, \infty)$ when $v \in [0, 1)$.

One of the results of this present paper is to show that the Hankel transform for $v \in [0, 1)$ can be obtained as an eigenfunction expansion when the generalised initial value conditions are applied to Eq. (1.1) at the singular limit-circle endpoint 0.

4. Differential expression M_v

For all $v \in [0, 1)$ the differential expression $M_v : D(M_v) \to \mathbb{C}$ is defined by

(i) $D(M_v) := \{f : (0, \infty) \to \mathbb{C} : f, f' \in AC_{\mathrm{loc}}(0, \infty)\}$; and
(ii) $M_v[f](x) := -f''(x) + (v^2 - \frac{1}{4})x^{-2} f(x)$ for all $f \in D(M_v)$ and all $x \in (0, \infty)$.

The Green's formula for M_v is, for all compact intervals $[\alpha, \beta] \subset (0, \infty)$ and for all $f, g \in D(M_v)$,

$$\int_\alpha^\beta \{\overline{g}(x) M_v[f](x) - f(x)\overline{M_v[g]}(x)\}\, \mathrm{d}x = [f, g](\beta) - [f, g](\alpha). \tag{4.1}$$

Here the symplectic form $[f, g](\cdot)$ is defined by

$$[f, g](x) := (f(x)\overline{g}'(x) - f'(x)\overline{g}(x)) \quad \text{for all } x \in (0, \infty). \tag{4.2}$$

5. Differential operators

In the space $L^2(0, \infty)$ the maximal operator $T_{v,1}$ is defined by

$$\begin{aligned} &D(T_{v,1}) := \{f \in D(M_v) : f, M_v[f] \in L^2(0, \infty)\}, \\ &T_{v,1} f := M_v[f] \quad \text{for all } f \in D(T_{v,1}). \end{aligned} \tag{5.1}$$

From Green's formula (4.1) both the limits

$$[f, g](0) := \lim_{x \to 0+} [f, g](x) \quad \text{and} \quad [f, g](\infty) := \lim_{x \to +\infty} [f, g](x), \tag{5.2}$$

exist and are finite in $\mathbb{C}$, for all $f, g \in D(T_{v,1})$. Since M_v is in the limit-point case at $+\infty$ it follows that

$$[f, g](\infty) = 0 \quad \text{for all } f, g \in D(T_{v,1}). \tag{5.3}$$

The minimal operator $T_{v,0}$ is defined by

$$\begin{aligned} &D(T_{v,0}) := \{f \in D(T_{v,1}) : [f, g](0) = 0 \quad \text{for all } g \in D(T_{v,1})\}, \\ &T_{v,0} f := M_v[f] \quad \text{for all } f \in D(T_{v,0}). \end{aligned} \tag{5.4}$$

The operator $T_{v,0}$ is closed and symmetric in the Hilbert function space, and has equal deficiency indices $d^-(T_{v,0}) = d^+(T_{v,0}) = 1$; the self-adjoint extensions $\{T\}$ of $T_{v,0}$ satisfy $T_{v,0} \subset T \subset T_{v,1}$. The domain of any such operator T has the definition

$$D(T) := \{f \in D(T_{v,1}) : [f, \omega](0) = 0\}, \tag{5.5}$$

where ω is a real-valued, non-null element of $D(T_{v,1})$ such that $\omega \notin D(T_{v,0})$; for this result see [14, Chapter V, Section 18.1, Theorem 4] and recall that the differential expression M_v is in the limit-point case at $+\infty$.

Remark 5.1. From an application of the well known Hardy inequality to the domain of the closure of the maximal operator but restricted to functions of compact support, and then using results from [14, Chapter V, Section 17.4] it follows that the minimal operators $T_{v,0}$, for all $v \in [0, 1)$, are bounded below by zero in the space $L^2(0, \infty)$.

6. The generalised initial value problem

The general theory of the solution of this problem in the case of a limit-circle endpoint is given in [7, Sections 1–5, in particular Theorem 2; 1, Section 5, Theorem 5.1].

In this Bessel equation case we have

Lemma 6.1. *Given the Bessel differential expression M_v as above in Section 4, for all $v \in [0, 1)$, and the differential operators as in Section 5, then there exists a pair $\{\gamma_v, \delta_v : v \in [0, 1)\}$ with the properties*

$$\begin{cases} \text{(i)} & \gamma_v, \delta_v : (0, \infty) \to \mathbb{R}, \\ \text{(ii)} & \gamma_v, \delta_v \in D(T_{v,1}), \\ \text{(iii)} & [\gamma_v, \delta_v](0) = 1. \end{cases} \tag{6.1}$$

Proof. See [1, Section 5]. □

Remark 6.1. We call the pair $\{\gamma_v, \delta_v\}$ boundary condition functions for the limit-circle singular endpoint 0, for the Bessel equation (1.1).

We now have the existence theorem for the basic solutions θ_v and φ_v of the Bessel differential equation that generalise the fundamental solutions θ and φ introduced by Titchmarsh in his text [19, Chapters I and II].

Theorem 6.1. *Given the Bessel differential equation* (1.1) *then for all $v \in [0, 1)$ there exist linearly independent solutions θ_v and φ_v with the properties*:

(1) $\theta_v, \varphi_v : (0, \infty) \times \mathbb{C} \to \mathbb{C}$;
(2) $\theta_v(x, \cdot), \theta'_v(x, \cdot), \varphi_v(x, \cdot), \varphi'_v(x, \cdot) : \mathbb{C} \to \mathbb{C}$ *are entire analytic functions, for all $x \in (0, \infty)$*;
(3) $\theta_v(\cdot, \lambda), \varphi_v(\cdot, \lambda) \in L^2(0, X)$ *for all $X \in (0, \infty)$ and for all $\lambda \in \mathbb{C}$; and*
(4) *Given γ_v, δ_v as in Lemma* 6.1 *then θ_v, φ_v satisfy the following generalised initial conditions*:

$$\begin{cases} [\theta_v(\cdot, \lambda), \gamma_v(\cdot)](0) = 1 & [\theta_v(\cdot, \lambda), \delta_v(\cdot)](0) = 0, \\ [\varphi_v(\cdot, \lambda), \gamma_v(\cdot)](0) = 0 & [\varphi_v(\cdot, \lambda), \delta_v(\cdot)](0) = 1 \end{cases} \tag{6.2}$$

for all $\lambda \in \mathbb{C}$.

Proof. See [7, Section 5; 1, Section 6]. □

7. The case $v = 0$

It proves necessary to distinguish between the case for $v = 0$ and the cases when $v \in (0, 1)$.

7.1. The boundary condition functions

For the case $v = 0$ a calculation shows a pair $\{\gamma_0, \delta_0\}$ may be defined by

$$\gamma_0(x) := x^{1/2} \quad \text{and} \quad \delta_0(x) := x^{1/2}\ln(x) \ \text{ for all } x \in (0, 1], \tag{7.1}$$

and then further defined on $[1, \infty)$ so that both γ_0, δ_0 are zero in some neighbourhood of the endpoint $+\infty$; this can be done using the Naimark patching lemma, see [14, Chapter V, Section 17.3, Lemma 2]; then (6.1) is satisfied. Note that both γ_0 and δ_0 satisfy the differential equation (1.1) with $\lambda = \nu = 0$, i.e. for all $x \in (0, 1]$,

$$\gamma_0''(x) + \tfrac{1}{4}x^{-2}\gamma_0(x) = 0 \quad \text{and} \quad \delta_0''(x) + \tfrac{1}{4}x^{-2}\delta_0(x) = 0. \tag{7.2}$$

7.2. The basis solutions θ_0 and φ_0

Further define the solutions θ_0 and φ_0, for all $x \in (0, \infty)$ and all $\lambda \in \mathbb{C}$, by

$$\varphi_0(x, \lambda) := x^{1/2}J({}_0x\sqrt{\lambda}), \tag{7.3}$$

$$\theta_0(x, \lambda) := -\frac{\pi}{2}x^{1/2}Y_0(x\sqrt{\lambda}) + \left[\log\left(\frac{1}{2}\sqrt{\lambda}\right) + \gamma\right]x^{1/2}J_0(x\sqrt{\lambda}), \tag{7.4}$$

where γ is Euler's constant.

A calculation, details omitted, then shows that the basis $\{\theta_0, \varphi_0\}$ satisfies the four properties given in Theorem 6.1; note in particular that the generalised initial conditions (6.2) are satisfied.

7.3. The Friedrichs extension F_0

For any abstract operator in Hilbert space that is closed, symmetric and bounded below there is a distinguished self-adjoint extension named for Friedrichs, see [9]; a later characterisation of this extension was given in [8]; at that time and in following years there has been an extensive study of the Friedrichs extension for Sturm–Liouville differential operators, see [16,4,11,13,17].

It is possible to show that the closed, symmetric operator $T_{0,0}$, see Remark 5.1 in Section 5 above, is bounded below in the space $L^2(0, \infty)$, and so has a Friedrichs extension which we denote here by F_0.

From the established properties of Sturm–Liouville differential expressions, see in particular paper [17], the operator F_0 is determined by, compare with (5.5), for any $\lambda \in \mathbb{R}$,

$$D(F_0) := \{f \in D(T_{0,1}) : [f, \gamma_0](0) = 0\} = \{f \in D(T_{0,1}) : [f, \varphi_0(\cdot, \lambda)](0) = 0\}, \tag{7.5}$$

$$F_0 f := M_0[f] \quad \text{for all } f \in D(F_0). \tag{7.6}$$

Note that we are using result (5.5), with $\omega = \gamma_0$, to determine the domain $D(F_0)$.

For an alternative definition of the Friedrichs operator, for a similar but more complicated differential expression, see [2, Section III].

Remark 7.1. Following the argument similar to that used in Remark 5.1, an application of the Hardy inequality implies that the Friedrichs operator F_0 is also bounded below by zero in the space $L^2(0, \infty)$.

7.4. The Titchmarsh–Weyl m-coefficient m_0

As is shown in paper [1, Section 8] the Friedrichs boundary condition (7.5) at the singular endpoint 0 and the limit-point condition at the endpoint $+\infty$, serve to determine a unique m-coefficient m_0 such that the solution $\psi_0(\cdot, \lambda)$ of Eq. (1.1) defined by, recall [19, Chapter II, Section 2.1],

$$\psi_0(x, \lambda) := \theta_0(x, \lambda) + m_0(\lambda)\varphi_0(x, \lambda) \quad \text{for all } x \in (0, \infty) \text{ and all } \lambda \in \mathbb{C}\backslash\mathbb{R}, \tag{7.7}$$

has the property

$$\psi_0(\cdot, \lambda) \in L^2(0, \infty) \quad \text{for all } \lambda \in \mathbb{C}\backslash\mathbb{R}. \tag{7.8}$$

Remark 7.2. Note that in choosing to represent the solution ψ_0 in form (7.7), with the solution φ_0 in the product with the coefficient m_0, we are connecting with the boundary value problem associated with the Friedrichs extension F_0; see the boundary condition in (7.5).

To compute this m-coefficient note that, see [19, Chapter IV, Section 4.10], the Bessel function

$$x \longmapsto x^{1/2} H_0^{(1)}(x\sqrt{\lambda}) = x^{1/2} J_0(x\sqrt{\lambda}) + \mathrm{i} x^{1/2} Y_0(x\sqrt{\lambda}),$$

lies in the space $L^2(0,\infty)$ for all $\lambda \in \mathbb{C}\backslash[0,\infty)$; thus we have for some function $k : \mathbb{C}\backslash[0,\infty) \to \mathbb{C}\backslash\{0\}$

$$\psi_0(x,\lambda) = k(\lambda) x^{1/2} H_0^{(1)}(x\sqrt{\lambda}) \quad \text{for all } \lambda \in \mathbb{C}\backslash[0,\infty) \text{ and all } x \in (0,\infty). \tag{7.9}$$

Using the initial conditions (6.2) and representation (7.7) we obtain

$$[\psi_0, \delta_0](0) = m_0(\lambda) \quad \text{and} \quad [\psi_0, \gamma_0](0) = 1,$$

so that from (7.9), since $k(\lambda) \neq 0$,

$$m_0(\lambda) = \frac{[\psi_0, \delta_0](0)}{[\psi_0, \gamma_0](0)} = \frac{[x^{1/2} H_0^{(1)}(x\sqrt{\lambda}), \delta_0(x)](0)}{[x^{1/2} H_0^{(1)}(x\sqrt{\lambda}), \gamma_0(x)](0)}. \tag{7.10}$$

From the properties for Bessel functions, see [20, Chapter III], it may be shown that the following results hold:

$$[x^{1/2} J_0(x\sqrt{\lambda}), \gamma_0(x)](0) = 0 \quad \text{and} \quad [x^{1/2} J_0(x\sqrt{\lambda}), \delta_0(x)](0) = 1,$$

$$[x^{1/2} Y_0(x\sqrt{\lambda}), \gamma_0(x)](0) = -\frac{2}{\pi} \quad \text{and} \quad [x^{1/2} Y_0(x\sqrt{\lambda}), \delta_0(x)](0) = \frac{2}{\pi}\left(\gamma + \log\left(\frac{1}{2}\sqrt{\lambda}\right)\right),$$

which combine to give

$$[x^{1/2} H_0(x\sqrt{\lambda}), \gamma_0(x)](0) = -\frac{2\mathrm{i}}{\pi} \quad \text{and} \quad [x^{1/2} H_0(x\sqrt{\lambda}), \delta_0(x)](0) = 1 + \frac{2\mathrm{i}}{\pi}\left(\gamma + \log\left(\frac{1}{2}\sqrt{\lambda}\right)\right).$$

From these results and using (7.10) we obtain the following explicit form of this m-coefficient:

$$m_0(\lambda) = \frac{\mathrm{i}\pi}{2} - \gamma - \log\left(\frac{1}{2}\sqrt{\lambda}\right) \quad \text{for all } \lambda \in \mathbb{C}\backslash[0,\infty). \tag{7.11}$$

If, as before, we write $\lambda = \rho \exp(\mathrm{i}\eta)$ then

$$m_0(\lambda) = \frac{\mathrm{i}}{2}(\pi - \eta) - \left(\gamma + \ln(1/2) + \frac{1}{2}\ln(\rho)\right), \tag{7.12}$$

from which result it follows that

$$\begin{cases} \mathrm{Im}(m_0(\lambda)) & = \frac{1}{2}\pi & \text{for } \eta = 0, \\ & = \frac{1}{2}(\pi - \eta) > 0 & \text{for all } \eta \in (0,\pi), \\ & = 0 & \text{for } \eta = \pi, \\ & = \frac{1}{2}(\pi - \eta) < 0 & \text{for all } \eta \in (\pi, 2\pi), \\ & = -\frac{1}{2}\pi & \text{for } \eta = 2\pi. \end{cases} \tag{7.13}$$

Remark 7.3. (1) The results in (7.13) confirm the Nevanlinna properties, see [1, Section 10], of this m-coefficient.

(2) Form (7.11) of this m-coefficient shows that this Nevanlinna function has logarithmic growth as $|\lambda| \to +\infty$ in the open domain $\mathbb{C}\backslash\mathbb{R}$.

7.5. The spectral function ρ_0

The explicit form of the m-coefficient m_0 leads to the spectral function ρ_0 for this Nevanlinna function, and hence for the Friedrichs self-adjoint operator F_0; for the appropriate definitions see [14, Chapter VI, Section 20.2, and Appendix I; 1, Section 10].

Define the monotonic non-decreasing function $\rho_0 : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned}\rho_0(t) &:= \frac{1}{\pi} \lim_{v\to 0+} \int_0^t \operatorname{Im}(m_0(\mu + \mathrm{i}v))\,\mathrm{d}\mu \quad \text{for all } t \in \mathbb{R} \\ &= t/2 \quad \text{for all } t \in [0, \infty) \\ \rho_0(t) &:= 0 \quad \text{for all } t \in (-\infty, 0). \end{aligned} \tag{7.14}$$

Remark 7.4. (1) We note, and this point is significant for the Hankel transform to be discussed in the following section, that for the Friedrichs operator F_0 the corresponding spectral Hilbert function space $L^2(\mathbb{R}; \rho_0)$, see [1, Section 10], is essentially identical with the original space $L^2(0, \infty)$.

(2) The explicit information about the spectral function ρ_0 in 7.14 implies the following spectral properties of the Friedrichs extension F_0, defined in (7.5) and (7.6).

$$\begin{cases} \text{(i)} & F_0 \text{ has no eigenvalues,} \\ \text{(ii)} & \sigma(F_0) = \sigma_{\mathrm{ess}}(F_0) = \sigma_{\mathrm{cont}}(F_0) = \sigma_{\mathrm{abcont}}(F_0) = [0, \infty). \end{cases} \tag{7.15}$$

(3) The spectral properties (7.15) confirm the previous results, see Remarks 5.1 and 7.1, that the two symmetric operators $T_{0,0}$ and F_0 are bounded below by 0, i.e.,

$$(T_{0,0}f, f) \geqslant 0 \quad \text{for all } f \in D(T_{0,0}) \text{ and } (F_0 f, f) \geqslant 0 \text{ for all } f \in D(F_0); \tag{7.16}$$

both the Hardy inequality applications and these spectral results imply that these two inequalities are best possible, with only the null function as cases of equality.

7.6. The Hankel transform

With the spectral function ρ_0 for the operator F_0 now obtained in explicit form we can apply the general Sturm–Liouville expansion theorem as given in [1, Section 11, Theorem 11.1 and Corollary 11.1] to obtain the Hankel transform, as discussed in Section 3 above, for the case $v = 0$.

Formally, this Hankel transform result may be written in the form:

(1) Given $f \in L^2(0, \infty)$ it is shown that F defined on $(0, \infty)$ by

$$F(t) = \int_0^\infty \xi^{1/2} J_0(\xi\sqrt{t}) f(\xi)\,\mathrm{d}\xi, \tag{7.17}$$

satisfies $F \in L^2(\mathbb{R}; \rho_0)$.

(2) Then f is recovered by (this is the eigenfunction expansion for f which recognises that $\sigma(F_0)=\sigma_{\mathrm{cont}}(F_0)=[0, \infty)$)

$$f(x) = \int_0^\infty x^{1/2} J_0(x\sqrt{t}) F(t)\,\mathrm{d}\rho_0(t). \tag{7.18}$$

As in [19, Chapter IV, Section 4.11] we obtain, again formally, using the spectral result (7.14) and writing $t = s^2$

$$\begin{aligned} f(x) &= \int_0^\infty x^{1/2} J_0(x\sqrt{t})\,\mathrm{d}\rho_0(t) \int_0^\infty \xi^{1/2} J_0(\xi\sqrt{t}) f(\xi)\,\mathrm{d}\xi \\ &= \frac{1}{2} \int_0^\infty x^{1/2} J_0(x\sqrt{t})\,\mathrm{d} \int_0^\infty \xi^{1/2} J_0(\xi\sqrt{t}) f(\xi)\,\mathrm{d}\xi \\ &= \int_0^\infty x^{1/2} J_0(xs) s\,\mathrm{d}s \int_0^\infty \xi^{1/2} J_0(s\xi) f(\xi)\,\mathrm{d}\xi \\ &= \int_0^\infty (xs)^{1/2} J_0(xs)\,\mathrm{d}s \int_0^\infty (s\xi)^{1/2} J_0(s\xi) f(\xi)\,\mathrm{d}\xi. \end{aligned} \tag{7.19}$$

7.7. Properties of the Friedrichs domain

The asymptotic properties of f and f', where $f \in D(F_0)$, near the singular endpoint 0^+ have been the subject of study by a number of authors, see in particular [11,17]. Here we give a new proof of these results together with some additional explicit information of the value of certain limits.

Given $f \in D(T_{0,1})$, the maximal domain of the differential expression M_0, we note the integrability result, for $t \in (0,1]$,

$$t \mapsto t^{-1}\int_0^t \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi \in L^1(0,1). \tag{7.20}$$

This follows since for all $x \in (0,1]$:

$$\int_x^1 t^{-1}\left(\int_0^t \xi^{1/2}|M_0[f](\xi)|\,\mathrm{d}\xi\right)\mathrm{d}t \leqslant \int_0^1 t^{-1/2}\,\mathrm{d}t \int_0^1 |M_0[f](\xi)|\,\mathrm{d}\xi < +\infty,$$

since $M_0[f] \in L^2(0,1)$.

We define $K_0 : D(T_{0,1}) \to \mathbb{C}$ by

$$K_0(f) := f(1) + \int_0^1 t^{-1}\left(\int_0^t \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi\right)\mathrm{d}t. \tag{7.21}$$

Theorem 7.1. *Let $f \in D(F_0)$ and let $K_0(f) \in \mathbb{C}$ be defined by* (7.21); *then*:

(1) *The following limit properties hold*

$$\lim_{x\to 0+} x^{-1/2} f(x) = K_0(f), \quad \lim_{x\to 0+} x^{1/2} f'(x) = \frac{1}{2}K_0(f), \quad \lim_{x\to 0+} f(x)f'(x) = \frac{1}{2}K_0(f)^2. \tag{7.22}$$

(2) *In general $K_0(f) \neq 0$.*

(3) *The following properties of f hold*

$$f' \in L^1(0,1), \quad f \in AC_{\mathrm{loc}}[0,\infty), \quad f(0) = 0. \tag{7.23}$$

(4) *In general $f' \notin L^2(0,1)$.*

Proof. Since $f \in D(F_0)$ we have

$$[f, x^{1/2}](0) = \lim_{x\to 0+}[f, x^{1/2}](x) = \lim_{x\to 0+}\left(f(x)\frac{1}{2}x^{-1/2} - f'(x)x^{1/2}\right) = 0.$$

From Green's formula (4.1) on the interval $(0, x]$ and noting again that $M_0[x^{1/2}] = 0$ on $(0,1]$, we have for $x \in (0,1]$

$$[f, x^{1/2}](x) - [f, x^{1/2}](0) = [f, x^{1/2}](x) = \int_0^x \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi$$

to give

$$f'(x)x^{-1/2} - f(x)\frac{1}{2}x^{-3/2} = -x^{-1}\int_0^x \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi.$$

Thus, again for all $x \in (0,1]$,

$$(f(x)x^{-1/2})' = -x^{-1}\int_0^x \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi;$$

integrating over the interval $(x, 1]$ gives

$$f(x)x^{-1/2} = f(1) + \int_x^1 t^{-1}\left(\int_0^t \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi\right)\mathrm{d}t. \tag{7.24}$$

Hence, as required for the first part of (7.22), using (7.21),

$$\lim_{x\to 0+} f(x)x^{-1/2} = f(1) + \int_0^1 t^{-1}\left(\int_0^t \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi\right)\mathrm{d}t = K_0(f). \tag{7.25}$$

From (7.24) we obtain, on differentiation,

$$f'(x) = \frac{1}{2}x^{-1/2}f(1) + \frac{1}{2}x^{-1/2}\int_x^1 t^{-1}\left(\int_0^t \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi\right)\mathrm{d}t - x^{-1/2}\int_0^x \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi; \tag{7.26}$$

hence, for all $x \in (0, 1]$,

$$x^{1/2}f'(x) = \frac{1}{2}\left\{f(1) + \int_x^1 t^{-1}\left(\int_0^t \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi\right)\mathrm{d}t\right\} - \int_0^x \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi.$$

From this last result we obtain, as required for the second part of (7.22),

$$\lim_{x\to 0+} x^{1/2}f'(x) = \frac{1}{2}\left\{f(1) + \int_0^1 t^{-1}\left(\int_0^t \xi^{1/2} M_0[f](\xi)\,\mathrm{d}\xi\right)\mathrm{d}t\right\} = \frac{1}{2}K_0(f). \tag{7.27}$$

The third part of (7.22) now follows.

If $f = \gamma_0$, as defined by (7.1), then $\gamma_0 \in D(F_0)$ and it may be seen that $K_0(f) = f(1) = 1 \neq 0$.

From (7.27), for any $f \in D(F_0)$, if follows that $f' \in L^1(0, 1)$ and so $f \in AC_{\mathrm{loc}}[0, \infty)$; this now defines the value of f at the endpoint 0 and gives $f(0) = 0$, otherwise there is a contradiction to result (7.25).

For $f = \gamma_0$ we have $f' \in L^1(0, 1)$ but $f' \notin L^2(0, 1)$.

This completes the proof of the theorem. □

Corollary 7.1. *Let $f \in D(F_0)$; then the following limit result exists*

$$\lim_{x\to 0+}\{f'(x)^2 - \tfrac{1}{4}x^{-2}f(x)^2\} = 0. \tag{7.28}$$

Proof. This result follows from a direct calculation using (7.24) and (7.26) and taking the limit at the point 0. □

Remark 7.5. In general, the properties of the domain $D(F_0)$, as given in Theorem 7.1, do not hold on the domain of any other self-adjoint extension of the minimal operator $T_{0,0}$; in this respect, as in other respects, the Friedrichs extension is a distinguished self-adjoint operator in the Hilbert space $L^2(0, \infty)$.

7.8. The Dirichlet formula

The endpoint classification in the space $L^2(0, \infty)$ of the differential expression M_0 is given in Section 2 above; thus:

(1) M_0 is limit-circle non-oscillatory at 0^+; for any $\lambda \in [0, \infty)$ the principal solution is $x^{1/2}J_0(x\sqrt{\lambda})$.
(2) M_0 is strong limit-point and Dirichlet at $+\infty$, and this condition implies:
   (i) $\lim_{x\to\infty} f(x)\overline{g}'(x) = 0$ for all $f, g \in D(T_{0,1})$, see (5.3),
   (ii) $f' \in L^2(\beta, \infty)$ for all $\beta \in (0, \infty)$ and for all $f \in D(T_{0,1})$,
   (iii) $x^{-1}f \in L^2(\beta, \infty)$ for all $\beta \in (0, \infty)$ and for all $f \in D(T_{0,1})$.

Integration by parts and using item 2 above yields, for all $f, g \in D(T_{0,1})$ and for all $\kappa \in (0, \infty)$,

$$\begin{aligned}\int_\kappa^\infty \left\{ f'(x)\overline{g}'(x) - \frac{1}{4}x^{-2} f(x)\overline{g}(x) \right\} \mathrm{d}x &= [f(x)\overline{g}'(x)]_\kappa^\infty - \int_\kappa^\infty \left\{ f(x)\overline{g}''(x) + \frac{1}{4}x^{-2} f(x)\overline{g}(x) \right\} \mathrm{d}x \\ &= -f(\kappa)\overline{g}'(\kappa) + \int_\kappa^\infty f(x)\overline{M_0[g]}(x)\,\mathrm{d}x. \end{aligned} \tag{7.29}$$

This result is the general Dirichlet formula for M_0 on the domain $D(T_{0,1})$ of the maximal operator in $L^2(0, \infty)$.

Now take $f \in D(F_0)$ and to be real-valued on $(0, \infty)$; also let $g = f$; then from (7.29)

$$\lim_{\kappa \to 0+} \int_\kappa^\infty \left\{ f'(x)^2 - \frac{1}{4}x^{-2} f(x)^2 \right\} \mathrm{d}x = -\lim_{\kappa \to 0+} f(\kappa)f'(\kappa) + \int_0^\infty f(x)M_0[f](x)\,\mathrm{d}x.$$

From (7.22) of Theorem 7.1, using the notation in [18, Chapter I, Section 1.7] and letting $\kappa \to 0^+$ gives

$$\int_{\to 0}^\infty \left\{ f'(x)^2 - \frac{1}{4}x^{-2} f(x)^2 \right\} \mathrm{d}x = -\frac{1}{2}K_0(f)^2 + \int_0^\infty f(x)M_0[f](x)\,\mathrm{d}x, \tag{7.30}$$

which implies that the limit integral on the left-hand side exists and is finite, for all $f \in D(F_0)$.

From Corollary 7.1 if follows that the integral on the left of (7.30) is a Lebesgue integral and the Dirichlet formula for M_0 on the domain $D(F_0)$ of the Friedrichs operator in $L^2(0, \infty)$ is

$$\int_0^\infty \left\{ f'(x)^2 - \frac{1}{4}x^{-2} f(x)^2 \right\} \mathrm{d}x = -\frac{1}{2}K_0(f)^2 + \int_0^\infty f(x)M_0[f](x)\,\mathrm{d}x. \tag{7.31}$$

Remark 7.6. (1) In the Dirichlet formula (7.31) the out-integrated term involves $K_0(f)$, which depends on the behaviour of the function f in the neighbourhood of the singular endpoint 0. This is in contrast to the case when such an endpoint is regular when for the Dirichlet formula, see [5, Section 3, (3.13)], the out-integrated term involves the endpoint values of the function and its derivative.

(2) Any self-adjoint extension of the minimal operator $T_{0,0}$, other than the Friedrichs extension, involves a domain which essentially contains the function δ_0.For this function we have

$$\int_{\to 0}^1 \left\{ \delta_0'(x)^2 - \frac{1}{4}x^{-2}\delta_0(x)^2 \right\} \mathrm{d}x = +\infty.$$

This result implies that there cannot be a Dirichlet formula on the domains of such extensions; again this is an example of the special nature of the Friedrichs extension.

7.9. A HELP-type inequality

The Dirichlet formula on the domain $D(F_0)$ suggests, in the light of the HELP integral inequalities considered in [5,6], that there could be a HELP-type integral inequality of the form

$$\left(\int_0^\infty \left\{ |f'(x)|^2 - \frac{1}{4}x^{-2}|f(x)|^2 \right\} \mathrm{d}x \right)^2 \leqslant C \int_0^\infty |f(x)|^2\,\mathrm{d}x \int_0^\infty |M_0[f](x)|^2\,\mathrm{d}x \quad \text{for all } f \in D(F_0). \tag{7.32}$$

Here C would be a positive number independent of the elements of $D(F_0)$.

However the form of the out-integrated term $-\frac{1}{2}K_0(f)^2$ in equality (7.31) does not allow of an application of the methods used in [5]. Nevertheless, we conjecture:

Conjecture 7.1. *There exists a positive number C, which is independent of the elements of the domain $D(F_0)$, such that inequality* (7.32) *is valid.*

8. The case $\nu \in (0, 1)$

In this section we give the corresponding results from Section 7 but now for the case when $\nu \in (0, 1)$, making use of the previous formulae wherever possible.

8.1. The boundary condition functions

For the case $\nu \in (0, 1)$ a calculation shows a pair $\{\gamma_\nu, \delta_\nu)$ may be defined by

$$\gamma_\nu(x) := x^{1/2+\nu} \quad \text{and} \quad \delta_\nu(x) := -\frac{1}{2\nu} x^{1/2-\nu} \ \text{ for all } x \in (0, 1], \tag{8.1}$$

and then defined on $[1, \infty)$ so that both γ_ν, δ_ν are zero in some neighbourhood of the endpoint $+\infty$; this can be done using the Naimark patching lemma, see [14, Chapter V, Section 17.3, Lemma 2]; then (6.1) is satisfied. Note that both γ_ν and δ_ν satisfy the differential equation (1.1) with $\lambda = 0$, i.e. for all $x \in (0, 1]$,

$$-\gamma_\nu''(x) + (\nu^2 - \tfrac{1}{4}x^{-2})\gamma_\nu(x) = 0 \quad \text{and} \quad -\delta_\nu''(x) + (\nu^2 - \tfrac{1}{4}x^{-2})\delta_\nu(x) = 0. \tag{8.2}$$

8.2. The basic solutions θ_ν and φ_ν

Further define the solutions θ_ν and φ_ν for all $\nu \in (0, 1)$, for all $x \in (0, \infty)$ and all $\lambda \in \mathbb{C}$, by

$$\varphi_\nu(x, \lambda) := 2^\nu \Gamma(1 + \nu)\lambda^{-\nu/2} x^{1/2} J_\nu(x\sqrt{\lambda}), \tag{8.3}$$

$$\theta_\nu(x, \lambda) := \frac{2^{-\nu}\Gamma(1 - \nu)}{2\nu} \lambda^{\nu/2} x^{1/2} J_{-\nu}(x\sqrt{\lambda}). \tag{8.4}$$

A calculation, details omitted, then shows that for all $\nu \in (0, 1)$ the basis $\{\theta_\nu, \varphi_\nu\}$ satisfies the four properties given in Theorem 6.1; note in particular that the generalised initial conditions (6.2) are satisfied.

Note that for any $\nu \in (0, 1)$ and given $\lambda = \rho \exp(i\eta)$ as in (2.3), we define

$$\lambda^{\pm\nu} := \rho^{\pm\nu} \exp(\pm i\nu\eta) \quad \text{for all } \rho \in (0, \infty) \ \text{ and all } \eta \in [0, 2\pi).$$

Remark 8.1. A computation for the special case $\nu = \frac{1}{2}$ yields, compare with [19, Chapters II and III],

$$\varphi_{1/2}(x, \lambda) = \frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}} \quad \text{and} \quad \theta_{1/2}(x, \lambda) = \cos(x\sqrt{\lambda}). \tag{8.5}$$

8.3. The Friedrichs extension F_ν

Following the results stated in Section 7.3 we now define the corresponding Friedrichs operators F_ν, for all $\nu \in (0, 1)$.

From the established properties of Sturm–Liouville differential expressions, see in particular paper [17], the operator F_ν, for all $\nu \in (0, 1)$, is determined by, compare with (5.5), for any $\lambda \in \mathbb{R}$,

$$D(F_\nu) := \{f \in D(T_{\nu,1}) : [f, \gamma_\nu](0) = 0\} = \{f \in D(T_{\nu,1}) : [f, \varphi_\nu(\cdot, \lambda)](0) = 0\}, \tag{8.6}$$

$$F_\nu f := M_\nu[f] \quad \text{for all } f \in D(F_\nu). \tag{8.7}$$

8.4. The Titchmarsh–Weyl m-coefficient m_ν

As is shown in paper [1, Section 8] the Friedrichs boundary condition (8.6) at the singular endpoint 0 and the limit-point condition at the endpoint $+\infty$, serve to determine a unique m-coefficient m_ν such that the solution $\psi_\nu(\cdot, \lambda)$ of Eq. (1.1) defined by, recall [19, Chapter II, Section 2.1],

$$\psi_\nu(x, \lambda) := \theta_\nu(x, \lambda) + m_\nu(\lambda)\varphi_\nu(x, \lambda) \quad \text{for all } x \in (0, \infty) \ \text{ and all } \lambda \in \mathbb{C}\backslash\mathbb{R}, \tag{8.8}$$

has the property, for all $v \in (0, 1)$,

$$\psi_v(\cdot, \lambda) \in L^2(0, \infty) \quad \text{for all } \lambda \in \mathbb{C}\backslash\mathbb{R}. \tag{8.9}$$

The corresponding solution to $H_0^{(1)}(\cdot)$ given in Section 7.4 above is, for all $v \in (0, 1)$ and all $x \in (0, \infty)$,

$$\begin{aligned} H_v^{(1)}(x) &= J_v(x) + \mathrm{i}Y_v(x) \\ &= J_v(x) + \mathrm{i}\left[\frac{\cos(v\pi)J_v(x) - J_{-v}(x)}{\sin(v\pi)}\right]. \end{aligned} \tag{8.10}$$

Then the solution to the differential equation (1.1) that lies in the space $L^2(0, \infty)$ for all $\lambda \in \mathbb{C}\backslash[0, \infty)$, for all $x \in (0, \infty)$ and all $v \in (0, 1)$, is given by

$$x \mapsto x^{1/2} H_v^{(1)}(x\sqrt{\lambda}). \tag{8.11}$$

Using the properties of the basis solutions $\{\theta_v, \varphi_v\}$ and representation (8.10), we find

$$\begin{aligned} x^{1/2} H_v^{(1)}(x\sqrt{\lambda}) = {} & \frac{2^{1+v} v\mathrm{i}}{\Gamma(1 - v)\sin(v\pi)} \lambda^{-v/2} \theta(x, \lambda) \\ & - \frac{2^{-v}(1 + \mathrm{i}\cos(v\pi))}{\Gamma(1 + v)\sin(v\pi)} \lambda^{v/2} \varphi(x, \lambda). \end{aligned} \tag{8.12}$$

From definition (8.8) of the solution ψ_v it follows that:

$$m_v(\lambda) = \frac{\exp(\mathrm{i}(1 - v)\pi)}{2v4^v} \frac{\Gamma(1 - v)}{\Gamma(1 + v)} \lambda^v \quad \text{for all } \lambda \in \mathbb{C}\backslash[0, \infty), \tag{8.13}$$

where $\lambda = \rho \exp(\mathrm{i}\eta)$ and $\eta \in [0, 2\pi)$.

It follows that, writing:

$$K(v, \rho) = \frac{1}{2v4^v} \frac{\Gamma(1 - v)}{\Gamma(1 + v)} \rho^v > 0 \quad \text{for all } v \in (0, 1) \text{ and } \rho \in (0, \infty),$$

we obtain

$$\mathrm{Im}(m_v(\lambda)) = K(v, \rho)\sin(v(\pi - \eta));$$

hence, for $\rho \in (0, \infty)$,

$$\begin{cases} \mathrm{Im}(m_v(\lambda)) & = K(v, \rho)\sin(v\pi) > 0 \qquad \text{for } \eta = 0, \\ & = K(v, \rho)\sin(v(\pi - \eta)) > 0 \quad \text{for all } \eta \in (0, \pi), \\ & = 0 \qquad \text{for } \eta = \pi, \\ & = K(v, \rho)\sin(v(\pi - \eta)) < 0 \quad \text{for all } \eta \in (\pi, 2\pi), \\ & = -K(v, \rho)\sin(v\pi) < 0 \qquad \text{for } \eta = 2\pi. \end{cases} \tag{8.14}$$

Remark 8.2. (i) We note that table (8.14) confirms the required properties of $m_v(\cdot)$ as a Nevanlinna function, see [1, Section 10].

(ii) The asymptotic growth of $m_v(\lambda)$ covers the full range of powers λ^v as $|\lambda|$ tends to infinity, for all $v \in (0, 1)$; this result is to be compared with the logarithmic growth of $m_0(\lambda)$ when $v = 0$, see (2) of Remark 7.3.

(iii) If we take $v = \frac{1}{2}$ in formula (8.13) then we obtain $m_{1/2}(\lambda) = i\lambda^{1/2}$; this is the known result for the m-coefficient for the Fourier differential equation $-y'' = \lambda y$ on $[0, \infty)$ and the Dirichlet boundary condition $y(0) = 0$ at the now regular endpoint; this boundary condition is consistent with the definition of the domain of the Friedrichs extension $F_{1/2}$ as in (8.7) above, when now $\delta_{1/2}(x) = x$ for all $x \in (0, 1]$.

8.5. The spectral function ρ_ν

We can now define the spectral function $\rho_\nu(\cdot)$ for the Nevanlinna function $m_\nu(\cdot)$ from the formula, see [14], for all $\nu \in (0, 1)$,

$$\begin{aligned}\rho_\nu(t) &:= \frac{1}{\pi} \lim_{\delta \to 0+} \int_0^t \operatorname{Im}(m_\nu(\mu + \mathrm{i}\delta))\, \mathrm{d}\mu \quad \text{for all } t \in \mathbb{R} \\ &= \frac{1}{\pi} \int_0^t \frac{1}{2\nu 4^\nu} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \sin(\nu\pi)\tau^\nu\, \mathrm{d}\tau \quad \text{for all } t \in [0, \infty) \\ &= \frac{1}{\pi} \frac{1}{2\nu 4^\nu} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \frac{\sin(\nu\pi)}{\nu+1} t^{\nu+1} \quad \text{for all } t \in [0, \infty) \\ \rho_\nu(t) &= 0 \quad \text{for all } t \in (-\infty, 0). \end{aligned} \tag{8.15}$$

This explicit information about the spectral function ρ_ν in (8.15) implies that the following spectral properties hold for the Friedrichs extension F_ν defined in (8.7):

$$\begin{cases} \text{(i)} & F_\nu \text{ has no eigenvalues,} \\ \text{(ii)} & \sigma_{\text{ess}}(F_\nu) = \sigma_{\text{cont}}(F_\nu) = \sigma_{\text{abcont}}(F_\nu) = [0, \infty). \end{cases} \tag{8.16}$$

Remark 8.3. Note that for the case $\nu = \frac{1}{2}$,

$$\rho_{1/2}(t) = \frac{2}{3\pi} t^{3/2} \quad \text{for all } t \in [0, \infty).$$

Remark 8.4. The spectral properties (8.16) confirm the previous results, see Remarks 5.1 and 7.1, that, for all $\nu \in (0, 1)$, the symmetric operators $T_{\nu,0}$ and F_ν are bounded below by 0, i.e.

$$(T_{\nu,0} f, f) \geqslant 0 \quad \text{for all } f \in D(T_{\nu,0}) \text{ and } (F_\nu f, f) \geqslant 0 \text{ for all } f \in D(F_\nu); \tag{8.17}$$

both the Hardy inequality applications and these spectral results imply that these two inequalities are best possible, with only the null function as cases of equality.

8.6. The Hankel transform

Recall the general remarks made about the Hankel inversion formula at the beginning of Section 3.

From the general expansion theorem [1, Section 11, Theorem 11.1] applied to the Bessel equation on $(0, \infty)$ when $\nu \in (0, 1)$, in the case when the m-coefficient m_ν, equivalently the corresponding self-adjoint operator F_ν, is chosen to be the Friedrichs case, i.e. (8.8). Then we obtain the formula (recall the results that follow are formal but can be interpreted in rigorous form as in [19, Chapters II and III; 1, Section 11, Theorem 11.1]),

$$f(x) = \int_0^\infty \varphi_\nu(x, t)\, \mathrm{d}\rho_\nu(t) \int_0^\infty \varphi_\nu(\xi, t) f(\xi)\, \mathrm{d}\xi \quad \text{for all } x \in (0, \infty).$$

We now write this general result in the form specific to the Bessel equation, using the explicit form of φ_ν in (8.3) and the derivative of ρ_ν in (8.15),

$$\begin{aligned} f(x) &= \int_0^\infty 2^\nu \Gamma(1+\nu) t^{-\nu/2} x^{1/2} J_\nu(x\sqrt{t}) \frac{1}{\pi} \frac{1}{2\nu 4^\nu} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)} \sin(\nu\pi) t^\nu\, \mathrm{d}t \\ &\quad \times \int_0^\infty 2^\nu \Gamma(1+\nu) t^{-\nu/2} \xi^{1/2} J_\nu(\xi\sqrt{t}) f(\xi)\, \mathrm{d}\xi. \\ &= \frac{1}{\pi} \frac{\Gamma(1-\nu)\Gamma(1+\nu)}{2\nu} \sin(\nu\pi) \int_0^\infty x^{1/2} J_\nu(x\sqrt{t})\, \mathrm{d}t \int_0^\infty \xi^{1/2} J_\nu(\xi\sqrt{t}) f(\xi)\, \mathrm{d}\xi \\ &= \int_0^\infty (xs)^{1/2} J_\nu(xs)\, \mathrm{d}s \int_0^\infty (s\xi)^{1/2} J_\nu(s\xi) f(\xi)\, \mathrm{d}\xi, \end{aligned} \tag{8.18}$$

on using properties of the Γ-function and writing $t = s^2$.

This last result is the formal form of the Hankel transform, compare (3.1), but now for any $\nu \in (0, 1)$.

8.7. Properties of the Friedrichs domain

Lemma 8.1. *Given $f \in D(T_{\nu,1})$, the maximal domain of the differential expression M_ν, we have the integrability result, for $t \in (0, 1]$,*

$$t \mapsto t^{-2\nu-1} \int_0^t \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi \in L^1(0, 1). \tag{8.19}$$

Proof. We rewrite the integral term in (8.19) in the product form

$$t^{1/2-\nu} \times t^{-\nu-3/2} \int_0^t \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi. \tag{8.20}$$

The first factor $t \longmapsto t^{1/2-\nu} \in L^2(0, 1)$ for all $\nu \in (0, 1)$ since then $-1 < 2\nu - 1 < 1$.

To the second factor we can apply the criterion in [3, Section 1, Theorem 1] with $a = 0, b = 1, w(x) = 1$ for all $x \in (0, 1]$ and

$$\varphi(t) = t^{\nu+1/2} \quad \text{and} \quad \psi(t) = t^{-\nu-3/2} \text{ for all } t \in (0, 1].$$

With this choice we find, for all $x \in (0, 1]$,

$$\int_0^x \varphi(t)^2\,\mathrm{d}t = \int_0^x t^{2\nu+1}\,\mathrm{d}t = \frac{x^{2\nu+2}}{2\nu+2},$$
$$\int_x^1 \psi(t)^2\,\mathrm{d}t = \int_x^1 t^{-2\nu-3}\,\mathrm{d}t = \frac{x^{-2\nu-2}-1}{2\nu+2}.$$

Thus, see [3, Section 1, (1.12) with $p = q = 2$],

$$\int_0^x \varphi(t)^2\,\mathrm{d}t \times \int_x^1 \psi(t)^2\,\mathrm{d}t \leqslant \frac{1}{(2\nu+2)^2} \quad \text{for all } x \in (0, 1].$$

This result implies from [3, Theorem 1], noting that $M_\nu[f](\cdot) \in L^2(0, 1)$,

$$t \longmapsto t^{-\nu-3/2} \int_0^t \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi \in L^2(0, 1).$$

The Cauchy–Schwarz integral inequality then applied to (8.20) completes the proof of the Lemma. □

We now define, for all $\nu \in (0, 1)$, $K_\nu : D(T_{\nu,1}) \to \mathbb{C}$ by

$$K_\nu(f) := f(1) + \int_0^1 t^{-2\nu-1} \left(\int_0^t \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi \right) \mathrm{d}t. \tag{8.21}$$

Theorem 8.1. *The following results hold for all $\nu \in (0, 1)$.*

Let $f \in D(F_\nu)$ and let $K_\nu(f) \in \mathbb{C}$ be defined by (8.21); *then*:

(1) *The following limit properties hold*

$$\lim_{x\to 0+} x^{-\nu-1/2} f(x) = K_\nu(f), \quad \lim_{x\to 0+} x^{-\nu+1/2} f'(x) = \left(\nu + \frac{1}{2}\right) K_\nu(f), \quad \lim_{x\to 0+} f(x) f'(x) = 0. \tag{8.22}$$

(2) *In general $K_\nu(f) \neq 0$.*

(3) *The following properties of f hold*

$$f' \in L^2(0, 1), \quad f \in AC_{\mathrm{loc}}[0, \infty), \quad x^{-1} f \in L^2(0, 1), \quad f(0) = 0. \tag{8.23}$$

Proof. Since $f \in D(F_\nu)$ we have

$$[f, x^{\nu+1/2}](0) = \lim_{x\to 0+} [f, x^{\nu+1/2}](x) = \lim_{x\to 0+} \left(f(x)\left(\nu+\frac{1}{2}\right)x^{\nu-1/2} - f'(x)x^{\nu+1/2}\right) = 0.$$

From Green's formula (4.1) on the interval $(0, x]$ and noting again that $M_\nu[x^{\nu+1/2}] = 0$ on $(0, 1]$, we have for $x \in (0, 1]$

$$[f, x^{\nu+1/2}](x) - [f, x^{\nu+1/2}](0) = [f, x^{\nu+1/2}](x) = \int_0^x \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi$$

to give

$$f'(x)x^{-\nu-1/2} - f(x)\left(\nu+\frac{1}{2}\right)x^{-\nu-3/2} = -x^{-2\nu-1}\int_0^x \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi.$$

Thus, again for all $x \in (0, 1]$,

$$(f(x)x^{-\nu-1/2})' = -x^{-2\nu-1}\int_0^x \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi;$$

integrating over the interval $(x, 1]$ gives

$$f(x)x^{-\nu-1/2} = f(1) + \int_x^1 t^{-2\nu-1}\left(\int_0^t \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi\right)\mathrm{d}t \quad \text{for all } x \in (0, 1]. \tag{8.24}$$

From this last result, Lemma 8.1 and definition (8.21) we obtain

$$\lim_{x\to 0+} x^{-\nu-1/2} f(x) = f(1) + \int_0^1 t^{-2\nu-1}\left(\int_0^t \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi\right)\mathrm{d}t = K_\nu(f),$$

to give the first part of (8.22).

From (8.24) we obtain, on differentiating,

$$\begin{aligned} f'(x) = {} & \left(\nu+\frac{1}{2}\right)x^{\nu-1/2}\left(f(1) + \int_x^1 t^{-2\nu-1}\left(\int_0^t \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi\right)\mathrm{d}t\right) \\ & - x^{\nu+1/2}x^{-2\nu-1}\int_0^x \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi \end{aligned}$$

and so, for all $x \in (0, 1]$,

$$\begin{aligned} x^{-\nu+1/2} f'(x) = {} & \left(\nu+\frac{1}{2}\right)\left(f(1) + \int_x^1 t^{-2\nu-1}\left(\int_0^t \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi\right)\mathrm{d}t\right) \\ & - x^{-2\nu}\int_0^x \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi. \end{aligned} \tag{8.25}$$

Now, recall $\nu \in (0, 1)$,

$$\begin{aligned} \lim_{x\to 0+} x^{-2\nu}\int_0^x \xi^{\nu+1/2} M_\nu[f](\xi)\,\mathrm{d}\xi & \leqslant \lim_{x\to 0+} x^{-2\nu}\left\{\int_0^x \xi^{2\nu+1}\,\mathrm{d}\xi \int_0^x |M_\nu[f](\xi)|^2\,\mathrm{d}\xi\right\}^{1/2} \\ & \leqslant \lim_{x\to 0+} x^{-2\nu}x^{\nu+1}\left\{\int_0^1 |M_\nu[f](\xi)|^2\,\mathrm{d}\xi\right\}^{1/2} = 0. \end{aligned} \tag{8.26}$$

Taking the limit as $x \to 0^+$ in (8.25) and using result (8.26) gives the second part of (8.22).

For the third part of (8.22) we have, using the above limit results,

$$\lim_{x\to 0+} f(x)f'(x) = \lim_{x\to 0+} \frac{f(x)}{x^{\nu+1/2}} \frac{f'(x)}{x^{\nu-1/2}} \lim_{x\to 0+} x^{2\nu} = 0. \tag{8.27}$$

If $f=\gamma_\nu$, as defined by (8.1), then $\gamma_\nu \in D(F_\nu)$ and it may be seen that $K_\nu(f)=f(1)=1\neq 0$.

From the second limit result of (8.22), since $\nu\in(0,1)$, if follows that $f'\in L^2(0,1)$ and so $f\in AC_{\text{loc}}[0,\infty)$; this now defines the value of f at the endpoint 0 and gives $f(0)=0$, otherwise there is a contradiction to the first limit result of (8.22).

From the first limit result of (8.22) we obtain the existence of a positive number C such that, for all $x\in(0,1]$,

$$|f(x)|\leqslant Cx^{\nu+1/2} \quad \text{and so} \quad x^{-1}|f(x)|\leqslant Cx^{\nu-1/2}.$$

This implies, since $\nu\in(0,1)$ that $x^{-1}f\in L^2(0,1)$.

This completes the proof of the theorem. □

Remark 8.5. The content of Remark 7.5 applies, with appropriate changes, to each of the cases $\nu\in(0,1)$.

8.8. The Dirichlet formulae

The following results apply to all the cases when $\nu\in(0,1)$. For an earlier discussion of these results see [9, Page 710; 10].

The endpoint classification in the space $L^2(0,\infty)$ of the differential expression M_ν is given in Section 2 above; thus:

(1) M_ν is limit-circle non-oscillatory at 0^+; for any $\lambda\in[0,\infty)$ the principal solution is $x^{1/2}J_\nu(x\sqrt{\lambda})$.
(2) M_ν is strong limit-point and Dirichlet at $+\infty$, and this condition implies:
 (i) $\lim_{x\to\infty} f(x)\overline{g}'(x)=0$ for all $f,g\in D(T_{\nu,1})$,
 (ii) $f'\in L^2(\beta,\infty)$ for all $\beta\in(0,\infty)$ and for all $f\in D(T_{\nu,1})$,
 (iii) $x^{-1}f\in L^2(\beta,\infty)$ for all $\beta\in(0,\infty)$ and for all $f\in D(T_{\nu,1})$.

Integration by parts and using item 2 above yields, for all $f,g\in D(T_{\nu,1})$ and for all $\kappa\in(0,\infty)$,

$$\begin{aligned}\int_\kappa^\infty &\left\{f'(x)\overline{g}'(x)+\left(\nu^2-\frac{1}{4}\right)x^{-2}f(x)\overline{g}(x)\right\}\mathrm{d}x\\ &=[f(x)\overline{g}'(x)]_\kappa^\infty-\int_\kappa^\infty\left\{f(x)\overline{g}''(x)+\left(\nu^2-\frac{1}{4}\right)x^{-2}f(x)\overline{g}(x)\right\}\mathrm{d}x\\ &=-f(\kappa)\overline{g}'(\kappa)+\int_\kappa^\infty f(x)\overline{M_\nu[g]}(x)\,\mathrm{d}x.\end{aligned} \tag{8.28}$$

This result is the general Dirichlet formula for M_ν on the domain $D(T_{\nu,1})$ of the maximal operator in $L^2(0,\infty)$.

Let $\nu\in(0,1)$, take $f\in D(F_\nu)$ and to be real-valued on $(0,\infty)$; also let $g=f$; then from (8.28)

$$\lim_{\kappa\to 0+}\int_\kappa^\infty\left\{f'(x)^2+\left(\nu^2-\frac{1}{4}\right)x^{-2}f(x)^2\right\}\mathrm{d}x=-\lim_{\kappa\to 0+} f(\kappa)f'(\kappa)+\int_0^\infty f(x)M_\nu[f](x)\,\mathrm{d}x.$$

From Theorem 8.1, in particular from (8.22), letting $\kappa\to 0^+$ gives

$$\int_0^\infty\left\{f'(x)^2+\left(\nu^2-\frac{1}{4}\right)x^{-2}f(x)^2\right\}\mathrm{d}x=\int_0^\infty f(x)M_\nu[f](x)\,\mathrm{d}x. \tag{8.29}$$

Here the integral on the left-hand side is a Lebesgue integral for all $f\in D(F_\nu)$, on using the properties of f given in Theorem 8.1, see (8.23).

For $\nu\in(0,1)$ this result (8.29) is the Dirichlet formula for M_ν on the domain $D(F_\nu)$ of the Friedrichs operator in $L^2(0,\infty)$.

Remark 8.6. With appropriate changes for the situation when $\nu \in (0, 1)$, but with $\nu \neq \frac{1}{2}$, item 2 of Remark 7.6 applies also to these cases.

The Cauchy–Schwarz inequality can be applied to (8.29) to give, for all $f \in D(F_\nu)$,

$$\left(\int_0^\infty \left\{\left|f'(x)\right|^2 + \left(\nu^2 - \frac{1}{4}\right)x^{-2}|f(x)|^2\right\}\mathrm{d}x\right)^2 \leqslant \int_0^\infty |f(x)|^2\,\mathrm{d}x \int_0^\infty |M_\nu[f](x)|^2\,\mathrm{d}x.$$

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Further Reading

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