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A system of variational inclusions with P–η-accretive operators

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Abstract

In this paper, we introduce and study a system of variational inclusions with P–η-accretive operators in real q-uniformly smooth Banach spaces. By using the resolvent operator technique associated with P–η-accretive operators, we prove the existence and uniqueness of solutions for this system of variational inclusions and construct a Mann iterative algorithm to approximate the unique solution. The results in this paper extend and improve some known results in the literature.

MSC: 49J40; 47J20

Keywords: P–η-accretive operators; System of variational inclusions; Iterative algorithm; Resolvent operator technique; q-Uniformly smooth Banach spaces

1. Introduction

Variational inclusion problems are among the most interesting and intensively studied classes of mathematical problems and have wide applications in the fields of optimization and control, economics and transportation equilibrium, engineering science. For the past years, many existence results and iterative algorithms for various variational inequality and variational inclusion problems have been studied. For details, please see [1–6,8–35] and the references therein.

Recently, some new and interesting problems, which are called systems of variational inequality problems were introduced and studied. Pang [27], Cohen and Chaplais [11], Bianchi [6] and Ansari and Yao [5] considered a system of scalar variational inequalities and Pang showed that the traffic equilibrium problem, the spatial equilibrium problem, the Nash equilibrium, and the general equilibrium programming problem can be modeled as a variational inequality. He decomposed the original variational inequality into a system of variational inequalities which are easy to solve and studied the convergence of such methods. Ansari et al. [4] introduced and studied a system of vector equilibrium problems and a system of vector variational inequalities by a fixed point theorem. Allevi et al. [3] considered a system of generalized vector variational inequalities and established some existence results under relative pseudomonotonicity.

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Kassay and Kolumbán [20] introduced a system of variational inequalities and proved an existence theorem by the Ky Fan lemma. Kassay et al. [21] studied Minty and Stampacchia variational inequality systems with the help of the Kakutani–Fan–Glicksberg fixed point theorem. Peng [28,30] introduced a system of quasi-variational inequality problems and proved its existence theorem by maximal element theorems. Verma [31–35] introduced and studied some systems of variational inequalities and developed some iterative algorithms for approximating the solutions of system of variational inequalities in Hilbert spaces. Kim and Kim [24] introduced a new system of generalized nonlinear quasi-variational inequalities and obtained some existence and uniqueness results of solution for this system of generalized nonlinear quasi-variational inequalities in Hilbert spaces. Cho et al. [10] introduced a new system of nonlinear variational inequalities and proved some existence and uniqueness theorems of solutions for this system of nonlinear variational inequalities in Hilbert spaces. As generalizations of system of variational inequalities, Agarwal et al. [2] introduced a system of generalized nonlinear mixed quasi-variational inclusions and investigated the sensitivity of solutions for this system of generalized nonlinear mixed quasi-variational inclusions in Hilbert spaces. Kazmi and Bhat [22] introduced a system of nonlinear variational-like inclusions and gave an iterative algorithm for finding its approximate solution. Fang and Huang [16,14], and Fang et al. [17] introduced and studied some new systems of variational inclusions involving P-accretive operators, P-monotone operators, and P–η-monotone operators, respectively.

On the other hand, Kazmi and Khan [23] introduced and studied the class of P–η-accretive operators which generalizes and unifies the classes of η–m-accretive operators in [9], P–η-monotone operators in [17] and P-accretive operators in [13] as special cases.

Inspired and motivated by the above results, the purpose of this paper is to study some properties of the class of P–η-accretive operators in q-uniformly smooth Banach spaces. We also introduce and study a system of variational inclusions with P–η-accretive operators. By using the resolvent technique for the P–η-accretive operators, we prove the existence and uniqueness of solutions for this system of variational inclusions. We also prove the convergence of a Mann iterative algorithm which approximate the solution to this system of variational inclusions. The results in this paper extend and improve the corresponding results about the existence and uniqueness of solutions for systems of variational inclusions or systems of variational inequalities and the convergence of some iterative algorithms which approximate the solution to these systems of variational inclusions or systems of variational inequalities in [17,16,14,31–35,24,10].

2. Preliminaries

Following Xu [36], Xu and Roach [37] and Chang [7], let E be a real Banach space with dual space and norm denoted by E^* and $\|\cdot\|$. For x in E and f^* in E^*, let $\langle x, f^*\rangle$ be the value of f^* at x ($\langle\cdot,\cdot\rangle$ is the generalized dual pair between E and E^*). Let 2^E denote the family of all the nonempty subsets of E, $\mathrm{CB}(E)$ denote the families of all nonempty closed bounded subsets of E, and the generalized duality mapping $J_q : E \to 2^{E^*}$ be defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|f^*\| \cdot \|x\|, \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2} J_2(x)$, for all $x \neq 0$, and J_q is single-valued if E^* is strictly convex. The modulus of smoothness of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\{\tfrac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leqslant 1, \|y\| \leqslant t\}.$$

A Banach space E is said to be uniformly smooth if

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0.$$

E is said to be q-uniformly smooth if there exists a constant $c > 0$, such that

$$\rho_E(t) \leqslant ct^q, \quad q > 1.$$

Note that J_q is single-valued if E is uniformly smooth.

Xu [36] and Xu and Roach [37] proved the following result.

Lemma 2.1. *Let E be a real uniformly smooth Banach space. Then, E is q-uniformly smooth if and only if there exists a constants $c_q > 0$, such that for all $x, y \in E$,*

$$\|x+y\|^q \leqslant \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

We recall some definitions needed later.

Definition 2.1 (*see Fang and Huang [13]*). Let E be a real q-uniformly smooth Banach space, and $P, g : E \longrightarrow E$ be two single-valued operators. P is said to be:

(i) accretive if

$$\langle P(x) - P(y), J_q(x-y)\rangle \geqslant 0, \quad \forall x, y \in E;$$

(ii) strictly accretive if P is accretive and

$$\langle P(x) - P(y), J_q(x-y)\rangle = 0 \text{ if and only if } x = y;$$

(iii) strongly accretive if there exists a constant $r > 0$ such that

$$\langle P(x) - P(y), J_q(x-y)\rangle \geqslant r\|x-y\|^q, \quad \forall x, y \in E;$$

(iv) Lipschitz continuous if there exists a constant $s > 0$ such that

$$\|P(x) - P(y)\| \leqslant s\|x-y\|, \quad \forall x, y \in E;$$

(v) g-strongly accretive if there exists a constant $\gamma > 0$ such that

$$\langle P(x) - P(y), J_q(g(x) - g(y))\rangle \geqslant \gamma\|x-y\|^q, \quad \forall x, y \in E.$$

Definition 2.2 (*see Kazmi and Khan [23], Chidume et al. [9]*). Let E be a real q-uniformly smooth Banach space, $P : E \longrightarrow E$ and $\eta : E \times E \longrightarrow E$ be two single-valued operators. P is said to be η-accretive if

$$\langle P(x) - P(y), J_q(\eta(x,y))\rangle \geqslant 0, \quad \forall x, y \in E,$$

or equivalently,

$$\langle P(x) - P(y), J_2(\eta(x,y))\rangle \geqslant 0, \quad \forall x, y \in E.$$

Definition 2.3 (*see Fang and Huang [13]*). Let E be a real q-uniformly smooth Banach space, $P : E \longrightarrow E$ be a single-valued operator and $M : E \longrightarrow 2^E$ be a multi-valued operator. M is said to be:

(i) accretive if

$$\langle u - v, J_q(x-y)\rangle \geqslant 0, \quad \forall x, y \in E, u \in M(x), v \in M(y);$$

(ii) m-accretive if M is accretive and $(I + \rho M)(E) = E$ holds for all $\rho > 0$, where I is the identity map on E;

(iii) P-accretive if M is accretive and $(P + \rho M)(E) = E$ holds for all $\rho > 0$.

Definition 2.4 (*see Kazmi and Khan [23]*). Let E be a real q-uniformly smooth Banach space, $P : E \longrightarrow E$ and $\eta : E \times E \longrightarrow E$ be two single-valued operators and $M : E \longrightarrow 2^E$ be a multi-valued operator. M is said to be:

(i) η-accretive if

$$\langle u - v, J_q(\eta(x,y))\rangle \geqslant 0, \quad \forall x, y \in E, u \in M(x), v \in M(y),$$

or equivalently,

$$\langle u - v, J_2(\eta(x,y))\rangle \geqslant 0, \quad \forall x, y \in E, u \in M(x), v \in M(y);$$

(ii) η–m-accretive if M is η-accretive and $(I + \rho M)(E) = E$ holds for all $\rho > 0$;

(iii) P–η-accretive if M is η-accretive and $(P + \rho M)(E) = E$ holds for all $\rho > 0$.

Remark 2.1. (i) If $\eta(x, y)=x-y, \forall x, y \in E$, then the definition of the P–η-accretive operator is that of the P-accretive operator introduced by Fang and Huang [13]. If $P = I$, then the definition of the (I, η)-accretive operator is that of the η–m-accretive operator introduced by Chidume et al. [9].

(ii) If $E = \mathscr{H}$ is a real Hilbert space, then the definition of the P–η-accretive operator becomes that of the P–η-monotone operator in [17] and the definition of the P-accretive operator becomes that of the P-monotone operator in [15].

(iii) The following examples illustrate that a P–η-accretive operator maybe neither a P-accretive operators nor an η–m-accretive operator. And so the definition of the P–η-accretive operator is a real generalization of those of the P-accretive operator and the η–m-accretive operator and their special cases.

Example 2.1. The following new example shows that a P–η-accretive operator may not be a P-accretive operator.

Let $E = R$ and $P : E \longrightarrow E, \eta : E \times E \longrightarrow E, M : E \longrightarrow 2^E$ be defined as follows: $P(x) = x^5, \eta(x, y) = x^4 - y^4$ and $M(x) = \{x^2, x^4, x^8\}, \forall x, y \in E$. It is easy to verify that M is P–η-accretive. However, M is not an accretive operator, and so M is not a P-accretive operator.

Example 2.2. The following new example shows that a P–η-accretive operator may not be an η–m-accretive operator.

Let $E = R$ and $P : E \longrightarrow E, \eta : E \times E \longrightarrow E, N : E \longrightarrow 2^E$ be defined as follows: $P(x) = x^5, \eta(x, y) = x^4 - y^4$ and $N(x)=\{x^2, x^2+1/4, 2x^2+3\}, \forall x, y \in E$. It is easy to verify that N is P–η-accretive. However, $(I+\rho M)(E) \neq E$, and so N is not an η–m-accretive operator.

Kazmi and Khan [23] present some properties for P–η-accretive operators as follows.

Lemma 2.2. *Let E be a real q-uniformly smooth Banach space, $\eta : E \times E \longrightarrow E$ be a single-valued operator, $P : E \longrightarrow E$ be a strictly η-accretive single-valued operator, and $M : E \longrightarrow 2^E$ be a P–η-accretive operator, and $x, u \in E$ be two given points. If $\langle u - v, J_q(\eta(x, y))\rangle \geqslant 0$ holds, for all $(y, v) \in$ Graph M, then $u \in M(x)$, where* Graph $M = \{(x, u) \in E \times E : u \in M(x)\}$.

Lemma 2.3. *Let E be a real q-uniformly smooth Banach space, $\eta : E \times E \longrightarrow E$ be a single-valued operator, $P : E \longrightarrow E$ be a strictly η-accretive single-valued operator, and $M : E \longrightarrow 2^E$ be a P–η-accretive operator. Then, the operator $(P + \lambda M)^{-1}$ is single-valued, where $\lambda > 0$ is a constant.*

Based on Lemma 2.3, we can define the resolvent operator $R^{P,\eta}_{M,\lambda}$ associated with P, η, M, λ as follows.

Definition 2.5. Let E be a real q-uniformly smooth Banach space, $\eta : E \times E \longrightarrow E$ be a single-valued operator, $P : E \longrightarrow E$ be a strictly η-accretive single-valued operator, and $M : E \longrightarrow 2^E$ be a P–η-accretive operator, $\lambda > 0$ be a constant. The resolvent operator $R^{P,\eta}_{M,\lambda} : E \longrightarrow E$ associated with P, η, M, λ is defined by

$$R^{P,\eta}_{M,\lambda}(u) = (P + \lambda M)^{-1}(u), \quad \forall u \in E.$$

Definition 2.6. Let E be a real q-uniformly smooth Banach space, $P : E \longrightarrow E$ and $\eta : E \times E \longrightarrow E$ be two single-valued operators. P is said to strongly η-accretive if there exists a constant $r > 0$ such that

$$\langle P(x) - P(y), J_q(\eta(x, y))\rangle \geqslant r\|x - y\|^q, \quad \forall x, y \in E. \tag{2.1}$$

Remark 2.2. It follows from the definition of generalized duality mapping J_q that $\langle P(x) - P(y), J_q(\eta(x, y))\rangle = \|\eta(x, y)\|^{q-2}\langle P(x) - P(y), J_2(\eta(x, y))\rangle$, Hence, (2.1) is not equivalent to the following formula (see [23, Definition 2.1(iii)]):

$$\langle P(x) - P(y), J_2(\eta(x, y))\rangle \geqslant r\|x - y\|^2, \quad \forall x, y \in E, \tag{2.2}$$

and this is the reason why the following Lemma 2.4 is different from Theorem 2.2 in [23].

Lemma 2.4. *Let E be a real q-uniformly smooth Banach space, $\eta : E \times E \longrightarrow E$ be Lipschitz continuous with a constant $\tau > 0$ (i.e., $\|\eta(x, y)\| \leqslant \tau\|x - y\|, \forall x, y \in E$), $P : E \longrightarrow E$ be strongly η-accretive with a constant $\gamma > 0$,* *and $M : E \longrightarrow 2^E$ be a P–η-accretive operator, $\lambda > 0$ be a constant. Then the resolvent operator $R^{P,\eta}_{M,\lambda}$ is Lipschitz continuous with a constant τ^{q-1}/γ, i.e.,*

$$\|R^{P,\eta}_{M,\lambda}(u) - R^{P,\eta}_{M,\lambda}(v)\| \leqslant (\tau^{q-1}/\gamma)\|u - v\|, \quad \forall u, v \in E.$$

Proof. Let u, v be any given points in E, it follows from Definition 2.5 that

$$R^{P,\eta}_{M,\lambda}(u) = (P + \lambda M)^{-1}(u) \quad \text{and} \quad R^{P,\eta}_{M,\lambda}(v) = (P + \lambda M)^{-1}(v).$$

This implies that

$$\frac{1}{\lambda}(u - P(R^{P,\eta}_{M,\lambda}(u))) \in M(R^{P,\eta}_{M,\lambda}(u))$$

and

$$\frac{1}{\lambda}(v - P(R^{P,\eta}_{M,\lambda}(v))) \in M(R^{P,\eta}_{M,\lambda}(v)).$$

Since M is P–η-accretive, we have

$$\begin{aligned}&\frac{1}{\lambda}\langle u - P(R^{P,\eta}_{M,\lambda}(u)) - (v - P(R^{P,\eta}_{M,\lambda}(v))), J_q(\eta(R^{P,\eta}_{M,\lambda}(u), R^{P,\eta}_{M,\lambda}(v)))\rangle\\ &\quad = \frac{1}{\lambda}\langle u - v - (P(R^{P,\eta}_{M,\lambda}(u)) - P(R^{P,\eta}_{M,\lambda}(v))), J_q(\eta(R^{P,\eta}_{M,\lambda}(u), R^{P,\eta}_{M,\lambda}(v)))\rangle \geqslant 0.\end{aligned}$$

The inequality above implies that

$$\begin{aligned}\|u - v\| \cdot \|\eta(R^{P,\eta}_{M,\lambda}(u), R^{P,\eta}_{M,\lambda}(v))\|^{q-1} &= \|u - v\| \cdot \|J_q(\eta(R^{P,\eta}_{M,\lambda}(u), R^{P,\eta}_{M,\lambda}(v)))\|\\ &\geqslant \langle u - v, J_q(\eta(R^{P,\eta}_{M,\lambda}(u), R^{P,\eta}_{M,\lambda}(v)))\rangle\\ &\geqslant \langle P(R^{P,\eta}_{M,\lambda}(u)) - P(R^{P,\eta}_{M,\lambda}(v)), J_q(\eta(R^{P,\eta}_{M,\lambda}(u), R^{P,\eta}_{M,\lambda}(v)))\rangle\\ &\geqslant \gamma\|R^{P,\eta}_{M,\lambda}(u) - R^{P,\eta}_{M,\lambda}(v)\|^q. \end{aligned} \tag{2.3}$$

Since η is Lipschitz continuous with a constant τ, we have

$$\|\eta(R^{P,\eta}_{M,\lambda}(u), R^{P,\eta}_{M,\lambda}(v))\| \leqslant \tau\|R^{P,\eta}_{M,\lambda}(u) - R^{P,\eta}_{M,\lambda}(v)\|. \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\|u - v\| \cdot \tau^{q-1}\|R^{P,\eta}_{M,\lambda}(u) - R^{P,\eta}_{M,\lambda}(v)\|^{q-1} \geqslant \gamma\|R^{P,\eta}_{M,\lambda}(u) - R^{P,\eta}_{M,\lambda}(v)\|^q.$$

Hence, we get

$$\|R^{P,\eta}_{M,\lambda}(u) - R^{P,\eta}_{M,\lambda}(v)\| \leqslant (\tau^{q-1}/\gamma)\|u - v\|.$$

The proof is complete. □

Remark 2.3. If E is 2-uniformly smooth Banach space, then Lemma 2.4 becomes Theorem 2.2 in [23]. Hence, Lemma 2.4 generalizes and unifies Theorem 2.2 in [23], Lemma 2.2 in [17], Theorem 2.3 in [13], Theorem 2.2 in [15].

3. A system of variational inclusions

In this section, we will introduce a new system of variational inclusions with P–η-accretive operators. In what follows, unless specified otherwise, we always suppose that E_1 and E_2 are two real q-uniformly smooth Banach spaces, $P_1 : E_1 \longrightarrow E_1$, $P_2 : E_2 \longrightarrow E_2$, $\eta_1 : E_1 \times E_1 \longrightarrow E_1$, $\eta_2 : E_2 \times E_2 \longrightarrow E_2$, $F : E_1 \times E_2 \longrightarrow E_1$, $G : E_1 \times E_2 \longrightarrow E_2$ are all single-valued mappings. Let $M : E_1 \longrightarrow 2^{E_1}$ be a P_1–η_1-accretive operator and $N : E_2 \longrightarrow 2^{E_2}$ be a P_2–η_2-accretive operator. We consider the following problem of finding $(x, y) \in E_1 \times E_2$ such that

$$\begin{cases} 0 \in F(x, y) + M(x), \\ 0 \in G(x, y) + N(y). \end{cases} \tag{3.1}$$

Problem (3.1) is called a system of variational inclusions with P–η-accretive operators.

Below are some special cases of problem (3.1).

(i) If $\eta_1(x_1, y_1) = x_1 - y_1$ for all $x_1, y_1 \in E_1$, $\eta_2(x_2, y_2) = x_2 - y_2$ for all $x_2, y_2 \in E_2$, then M and N, respectively, become a P_1-accretive operator and a P_2-accretive operator, and problem (3.1) becomes the system of variational inclusions with P-accretive operators in Banach spaces introduced and studied by Fang and Huang [16]. Moreover, if $E_1 = \mathscr{H}_1$ and $E_2 = \mathscr{H}_2$ are Hilbert spaces, then M and N, respectively, become a P_1-monotone operator and a P_2-monotone operator, and problem (3.1) becomes the system of variational inclusions with P-monotone operators in Hilbert spaces introduced and studied by Fang and Huang [14].

(ii) If $E_1 = \mathscr{H}_1$ and $E_2 = \mathscr{H}_2$ are Hilbert spaces, then M and N, respectively, become a P_1–η_1-monotone operator and a P_2–η_2-monotone operator, and problem (3.1) becomes the system of variational inclusions with P–η-monotone operators in Hilbert spaces introduced and studied by Fang et al. [4].

(iii) Let $E_1 = \mathscr{H}_1$ and $E_2 = \mathscr{H}_2$ be two Hilbert spaces, $M(x) = \partial_{\eta_1}\varphi(x)$ and $N(y) = \partial_{\eta_2}\phi(y)$ for all $x \in E_1$ and $y \in E_2$, where $\varphi : E_1 \longrightarrow R \cup \{\infty\}$ is a proper lower semi-continuous and η_1-subdifferentiable function and $\phi : E_2 \longrightarrow R \cup \{\infty\}$ is a proper lower semi-continuous and η_2-subdifferentiable function, $\partial_{\eta_1}\varphi(x)$ is the η_1-subdifferential of φ at x and $\partial_{\eta_2}\phi(y)$ is the η_2-subdifferential of ϕ at y, then problem (3.1) reduces to the following system of variational-like inequalities, which is to find $(x, y) \in E_1 \times E_2$ such that

$$\begin{cases} \langle F(x, y), \eta_1(a, x)\rangle + \varphi(a) - \varphi(x) \geqslant 0, & \forall\, a \in E_1, \\ \langle G(x, y), \eta_2(b, y)\rangle + \phi(b) - \phi(y) \geqslant 0, & \forall\, b \in E_2. \end{cases} \tag{3.2}$$

If $\eta_1(a, x) = a - x$ for all $a, x \in E_1$, $\eta_2(b, y) = b - y$ for all $b, y \in E_2$, $M(x) = \partial\varphi(x)$ is the subdifferential of φ at x and $N(x) = \partial\phi(y)$ is the subdifferential of ϕ at y, then problem (3.2) becomes the following system of variational inequalities, which is to find $(x, y) \in E_1 \times E_2$ such that

$$\begin{cases} \langle F(x, y), a - x\rangle + \varphi(a) - \varphi(x) \geqslant 0, & \forall\, a \in E_1, \\ \langle G(x, y), b - y\rangle + \phi(b) - \phi(y) \geqslant 0, & \forall\, b \in E_2. \end{cases} \tag{3.3}$$

Problem (3.3) was introduced and studied by Cho et al. [10].

If $M(x) = \partial\delta_{K_1}(x)$ and $N(y) = \partial\delta_{K_2}(y)$ for all $x \in K_1$ and $y \in K_2$, where $K_1 \subset E_1$ and $K_2 \subset E_2$ are two nonempty, closed and convex subsets, δ_{K_1} and δ_{K_2} denote the indicator functions of K_1 and K_2, respectively, then problem (3.3) reduces to the following system of variational inequalities, which is to find $(x, y) \in E_1 \times E_2$ such that

$$\begin{cases} \langle P(x, y), a - x\rangle \geqslant 0, & \forall\, a \in E_1, \\ \langle Q(x, y), b - y\rangle \geqslant 0, & \forall\, b \in E_2. \end{cases} \tag{3.4}$$

Problem (3.4) is just the problem in [20] with P and Q being single-valued.

If $E_1 = E_2 = \mathscr{H}$ is a Hilbert space, $K_1 = K_2 = K$ is a nonempty, closed and convex subset, $P(x, y) = \rho T(y, x) + x - y$ and $Q(x, y) = \lambda T(x, y) + y - x$ for all $x, y \in K$, where $T : K \times K \longrightarrow \mathscr{H}$ is a mapping on $K \times K$, $\rho, \lambda > 0$ are two numbers, then problem (3.4) reduces to the following problem: find $x, y \in K$ such that

$$\begin{cases} \langle \rho T(y, x) + x - y, a - x\rangle \geqslant 0, & \forall\, a \in K, \\ \langle \lambda T(x, y) + y - x, a - y\rangle \geqslant 0, & \forall\, a \in K. \end{cases} \tag{3.5}$$

Problem (3.5) was introduced and studied by Verma [34].

If $E_1 = E_2 = \mathscr{H}$ is a Hilbert space, $K_1 = K_2 = K$ is a nonempty, closed and convex subset, $P(x, y) = \rho T(y) + x - y$ and $Q(x, y) = \lambda T(x) + y - x$ for all $x, y \in K$, where $T : K \longrightarrow \mathscr{H}$ is a mapping on K, $\rho, \lambda > 0$ are two numbers, then problem (3.4) reduces to the following problem: find $x, y \in K$ such that

$$\begin{cases} \langle \rho T(y) + x - y, a - x \rangle \geqslant 0, & \forall a \in K, \\ \langle \lambda T(x) + y - x, a - y \rangle \geqslant 0, & \forall a \in K. \end{cases} \tag{3.6}$$

Problem (3.10) was introduced and studied by Verma [31–33,35].

4. Existence and uniqueness

In this section, we will prove existence and uniqueness for solutions of problem (3.1). For our main results, we give a characterization of the solution of problem (3.1) as follows.

Lemma 4.1. *Let E_1 and E_2 be real q-uniformly smooth Banach spaces, $\eta_1 : E_1 \times E_1 \longrightarrow E_1, \eta_2 : E_2 \times E_2 \longrightarrow E_2$ be two single-valued operators, $P_1 : E_1 \longrightarrow E_1$ be a strictly η_1-accretive operator and $P_2 : E_2 \longrightarrow E_2$ be a strictly η_2-accretive operator and $M : E_1 \longrightarrow 2^{E_1}$ be a P_1–η_1-accretive operator, $N : E_2 \longrightarrow 2^{E_2}$ be a P_2–η_2-accretive operator. Then $(x, y) \in E_1 \times E_2$ is a solution of problem* (3.1) *if and only if*

$$x = R_{M,\lambda}^{P_1,\eta_1}(P_1(x) - \lambda F(x, y)),$$

$$y = R_{N,\rho}^{P_2,\eta_2}(P_2(y) - \rho G(x, y)),$$

where $R_{M,\lambda}^{P_1,\eta_1} = (P_1 + \lambda M)^{-1}$, $R_{N,\rho}^{P_2,\eta_2} = (P_2 + \rho N)^{-1}$, $\lambda > 0$ and $\rho > 0$ are constants.

Proof. The fact follows directly from Definition 2.5. □

Theorem 4.1. *Let E_1 and E_2 be real q-uniformly smooth Banach spaces. For $i = 1, 2$, let $\eta_i : E_i \times E_i \longrightarrow E_i$ be Lipschitz continuous with constant τ_i, $P_i : E_i \longrightarrow E_i$ be strongly η_i-accretive and Lipschitz continuous with constants γ_i and δ_i, respectively. Let $F : E_1 \times E_2 \longrightarrow E_1$ be a nonlinear operator such that for any given $(a, b) \in E_1 \times E_2$, $F(., b)$ is P_1-strongly accretive and Lipschitz continuous with constants r_1 and $s_1 > 0$, respectively, and $F(a, .)$ is Lipschitz continuous with constant $\xi_1 > 0$. Let $G : E_1 \times E_2 \longrightarrow E_2$ be a nonlinear operator such that for any given $(x, y) \in E_1 \times E_2$, $G(x, .)$ is P_2-strongly accretive and Lipschitz continuous with constants r_2 and $s_2 > 0$, respectively, and $G(., y)$ is Lipschitz continuous with constant $\xi_2 > 0$. Assume that $M : E_1 \longrightarrow 2^{E_1}$ is a P_1–η_1-accretive operator and $N : E_2 \longrightarrow 2^{E_2}$ is a P_2–η_2-accretive operator.*

If there exist constants $\lambda > 0$ and $\rho > 0$ such that

$$\begin{cases} \dfrac{\tau_1^{q-1}}{\gamma_1} \sqrt[q]{\delta_1^q - q\lambda r_1 + c_q \lambda^q s_1^q} + \dfrac{\xi_2 \rho \tau_2^{q-1}}{\gamma_2} < 1, \\ \dfrac{\tau_2^{q-1}}{\gamma_2} \sqrt[q]{\delta_2^q - q\rho r_2 + c_q \rho^q s_2^q} + \dfrac{\xi_1 \lambda \tau_1^{q-1}}{\gamma_1} < 1. \end{cases} \tag{4.1}$$

Then problem (3.1) *admits a unique solution.*

Proof. For any given $\lambda > 0$ and $\rho > 0$, define $T_\lambda : E_1 \times E_2 \longrightarrow E_1$ and $S_\rho : E_1 \times E_2 \longrightarrow E_2$ by

$$T_\lambda(u, v) = R_{M,\lambda}^{P_1,\eta_1}[P_1(u) - \lambda F(u, v)] \quad \text{and} \quad S_\rho(u, v) = R_{N,\rho}^{P_2,\eta_2}[P_2(v) - \rho G(u, v)], \tag{4.2}$$

for all $(u, v) \in E_1 \times E_2$.

For any $(u_1, v_1), (u_2, v_2) \in E_1 \times E_2$, it follows from (4.2) and Lemma 2.4 that

$$\begin{aligned}\|T_\lambda(u_1, v_1) - T_\lambda(u_2, v_2)\| &\leqslant \frac{\tau_1^{q-1}}{\gamma_1}\|P_1(u_1) - P_1(u_2) - \lambda(F(u_1, v_1) - F(u_2, v_2))\| \\ &\leqslant \frac{\tau_1^{q-1}}{\gamma_1}\|P_1(u_1) - P_1(u_2) - \lambda(F(u_1, v_1) - F(u_2, v_1))\| \\ &\quad + \frac{\lambda\tau_1^{q-1}}{\gamma_1}\|F(u_2, v_1) - F(u_2, v_2)\| \end{aligned} \tag{4.3}$$

and

$$\begin{aligned}\|S_\rho(u_1, v_1) - S_\rho(u_2, v_2)\| &\leqslant \frac{\tau_2^{q-1}}{\gamma_2}\|P_2(v_1) - P_2(v_2) - \rho(G(u_1, v_1) - G(u_2, v_2))\| \\ &\leqslant \frac{\tau_2^{q-1}}{\gamma_2}\|P_2(v_1) - P_2(v_2) - \rho(G(u_1, v_1) - G(u_1, v_2))\| \\ &\quad + \frac{\rho\tau_2^{q-1}}{\gamma_2}\|G(u_1, v_2) - G(u_2, v_2)\|. \end{aligned} \tag{4.4}$$

By assumption, we have

$$\begin{aligned}&\|P_1(u_1) - P_1(u_2) - \lambda(F(u_1, v_1) - F(u_2, v_1))\|^q \\ &\quad \leqslant \|P_1(u_1) - P_1(u_2)\|^q - q\lambda\langle F(u_1, v_1) - F(u_2, v_1), J_q(P_1(u_1) - P_1(u_2))\rangle \\ &\qquad + \lambda^q c_q\|F(u_1, v_1) - F(u_2, v_1)\|^q \\ &\quad \leqslant (\delta_1^q - q\lambda r_1 + c_q\lambda^q s_1^q)\|u_1 - u_2\|^q \end{aligned} \tag{4.5}$$

and

$$\begin{aligned}&\|P_2(v_1) - P_2(v_2) - \rho(G(u_1, v_1) - G(u_1, v_2))\|^q \\ &\quad \leqslant \|P_2(v_1) - P_2(v_2)\|^q - q\rho\langle G(u_1, v_1) - G(u_1, v_2), J_q(P_2(v_1) - P_2(v_2))\rangle \\ &\qquad + c_q\rho^q\|G(u_1, v_1) - G(u_1, v_2)\|^q \\ &\quad \leqslant (\delta_2^q - q\rho r_2 + c_q\rho^q s_2^q)\|v_1 - v_2\|^q. \end{aligned} \tag{4.6}$$

Furthermore,

$$\|F(u_2, v_1) - F(u_2, v_2)\| \leqslant \xi_1\|v_1 - v_2\| \tag{4.7}$$

and

$$\|G(u_1, v_2) - G(u_2, v_2)\| \leqslant \xi_2\|u_1 - u_2\|. \tag{4.8}$$

It follows from (4.3) and (4.8) that

$$\|T_\lambda(u_1, v_1) - T_\lambda(u_2, v_2)\| \leqslant \frac{\tau_1^{q-1}}{\gamma_1}\sqrt[q]{\delta_1^q - q\lambda r_1 + c_q\lambda^q s_1^q}\|u_1 - u_2\| + \frac{\xi_1\lambda\tau_1^{q-1}}{\gamma_1}\|v_1 - v_2\| \tag{4.9}$$

and

$$\|S_\rho(u_1, v_1) - S_\rho(u_2, v_2)\| \leqslant \frac{\tau_2^{q-1}}{\gamma_2}\sqrt[q]{\delta_2^q - q\rho r_2 + c_q\rho^q s_2^q}\|v_1 - v_2\| + \frac{\xi_2\rho\tau_2^{q-1}}{\gamma_2}\|u_1 - u_2\|. \tag{4.10}$$

Now (4.9) and (4.10) jointly imply that

$$\begin{aligned}
&\|T_\lambda(u_1, v_1) - T_\lambda(u_2, v_2)\| + \|S_\rho(u_1, v_1) - S_\rho(u_2, v_2)\| \\
&\quad \leqslant \left[\frac{\tau_1^{q-1}}{\gamma_1}\sqrt[q]{\delta_1^q - q\lambda r_1 + c_q\lambda^q s_1^q} + \frac{\xi_2\rho\tau_2^{q-1}}{\gamma_2}\right]\|u_1 - u_2\| \\
&\qquad + \left[\frac{\tau_2^{q-1}}{\gamma_2}\sqrt[q]{\delta_2^q - q\rho r_2 + c_q\rho^q s_2^q} + \frac{\xi_1\lambda\tau_1^{q-1}}{\gamma_1}\right]\|v_1 - v_2\| \\
&\quad \leqslant \theta(\|u_1 - u_2\| + \|v_1 - v_2\|),
\end{aligned} \tag{4.11}$$

where

$$\theta = \max\left\{\frac{\tau_1^{q-1}}{\gamma_1}\sqrt[q]{\delta_1^q - q\lambda r_1 + c_q\lambda^q s_1^q} + \frac{\xi_2\rho\tau_2^{q-1}}{\gamma_2}, \frac{\tau_2^{q-1}}{\gamma_2}\sqrt[q]{\delta_2^q - q\rho r_2 + c_q\rho^q s_2^q} + \frac{\xi_1\lambda\tau_1^{q-1}}{\gamma_1}\right\}.$$

Define $\|\cdot\|_1$ on $E_1 \times E_2$ by

$$\|(u, v)\|_1 = \|u\| + \|v\|, \quad \forall (u, v) \in E_1 \times E_2.$$

It is easy to see that $(E_1 \times E_2, \|\cdot\|_1)$ is a Banach space. For any given $\lambda > 0$ and $\rho > 0$, define $Q_{\lambda,\rho} : E_1 \times E_2 \longrightarrow E_1 \times E_2$ by

$$Q_{\lambda,\rho}(u, v) = (T_\lambda(u, v), S_\rho(u, v)), \quad \forall (u, v) \in E_1 \times E_2.$$

By (4.1), we know that $0 < \theta < 1$. It follows from (4.11) that

$$\|Q_{\lambda,\rho}(u_1, v_1) - Q_{\lambda,\rho}(u_2, v_2)\|_1 \leqslant \theta\|(u_1, v_1) - (u_2, v_2)\|_1.$$

This proves that $Q_{\lambda,\rho} : E_1 \times E_2 \longrightarrow E_1 \times E_2$ is a contraction operator. Hence, there exists a unique $(x, y) \in E_1 \times E_2$, such that

$$Q_{\lambda,\rho}(x, y) = (x, y),$$

that is,

$$x = R_{M,\lambda}^{P_1,\eta_1}(P_1(x) - \lambda F(x, y)),$$

$$y = R_{N,\rho}^{P_2,\eta_2}(P_2(y) - \rho G(x, y)).$$

By Lemma 4.1, (x, y) is the unique solution of problem (3.1). □

Remark 4.1. If E_1 and E_2 are 2-uniformly smooth Banach spaces, then (4.1) becomes the following formula:

$$\begin{cases}
\dfrac{\tau_1}{\gamma_1}\sqrt{\delta_1^2 - 2\lambda r_1 + c_2\lambda^2 s_1^2} + \dfrac{\xi_2\rho\tau_2}{\gamma_2} < 1, \\
\dfrac{\tau_2}{\gamma_2}\sqrt{\delta_2^2 - 2\rho r_2 + c_2\rho^2 s_2^2} + \dfrac{\xi_1\lambda\tau_1}{\gamma_1} < 1.
\end{cases}$$

Moreover, if both E_1 and E_2 are Hilbert spaces, the above formula becomes (4.1) in [17]. Hence, Theorem 4.1 generalizes and unifies Theorem 4.1 in [17] and Theorem 4.1 in [16].

5. An iterative algorithm and convergence

In this section, we construct the Mann iterative algorithm for approximating the unique solution of problem (3.1) and discuss the convergence of the algorithm.

Theorem 5.1. *Let E_1 and E_2 be real q-uniformly smooth Banach spaces. For $i = 1, 2$, let $\eta_i : E_i \times E_i \longrightarrow E_i$ be Lipschitz continuous with constant τ_i, $P_i : E_i \longrightarrow E_i$ be strongly η_i-accretive and Lipschitz continuous with constants γ_i and δ_i, respectively. Let $F : E_1 \times E_2 \longrightarrow E_1$ be a nonlinear operator such that for any given $(a, b) \in E_1 \times E_2$, $F(., b)$ is P_1-strongly accretive and Lipschitz continuous with constants r_1 and $s_1 > 0$, respectively, and $F(a, .)$ is Lipschitz continuous with constant $\xi_1 > 0$. Let $G : E_1 \times E_2 \longrightarrow E_2$ be a nonlinear operator such that for any given $(x, y) \in E_1 \times E_2$, $G(x, .)$ is P_2-strongly accretive and Lipschitz continuous with constants r_2 and $s_2 > 0$, respectively, and $G(., y)$ is Lipschitz continuous with constant $\xi_2 > 0$. Assume that $M : E_1 \longrightarrow 2^{E_1}$ is a P_1–η_1-accretive operator and $N : E_2 \longrightarrow 2^{E_2}$ is a P_2–η_2-accretive operator. If there exist constants $\lambda > 0$ and $\rho > 0$ such that* (4.1) *holds. For any given $(x_0, y_0) \in E_1 \times E_2$, define the Mann iterative sequence $\{(x_n, y_n)\}$ by*

$$\begin{cases} x_{n+1} = \alpha_n x_n + (1 - \alpha_n) R^{P_1,\eta_1}_{M,\lambda}[P_1(x_n) - \lambda F(x_n, y_n)], \\ y_{n+1} = \alpha_n y_n + (1 - \alpha_n) R^{P_2,\eta_2}_{N,\rho}[P_2(y_n) - \rho G(x_n, y_n)], \end{cases} \tag{5.1}$$

where

$$0 \leqslant \alpha_n < 1 \quad \text{and} \quad \limsup_n \alpha_n < 1. \tag{5.2}$$

Then $\{(x_n, y_n)\}$ converges strongly to the unique solution (x, y) of problem (3.1).

Proof. By Theorem 4.1, problem (3.1) admits a unique solution (x, y). It follows from Lemma 4.1 that

$$\begin{cases} x = \alpha_n x + (1 - \alpha_n) R^{P_1,\eta_1}_{M,\lambda}[P_1(x) - \lambda F(x, y)], \\ y = \alpha_n y + (1 - \alpha_n) R^{P_2,\eta_2}_{N,\rho}[P_2(y) - \rho G(x, y)]. \end{cases} \tag{5.3}$$

By (5.1) and (5.3), we have

$$\begin{aligned} \|x_{n+1} - x\| &\leqslant \alpha_n \|x_n - x\| + (1 - \alpha_n) \|R^{P_1,\eta_1}_{M,\lambda}[P_1(x_n) - \lambda F(x_n, y_n)] - R^{P_1,\eta_1}_{M,\lambda}[P_1(x) - \lambda F(x, y)]\| \\ &\leqslant \alpha_n \|x_n - x\| + (1 - \alpha_n) \frac{\tau_1^{q-1}}{\gamma_1} \|P_1(x_n) - P_1(x) - \lambda[F(x_n, y_n) - F(x, y)]\| \\ &\leqslant \alpha_n \|x_n - x\| + (1 - \alpha_n) \frac{\tau_1^{q-1}}{\gamma_1} \|P_1(x_n) - P_1(x) - \lambda[F(x_n, y_n) - F(x, y_n)]\| \\ &\quad + (1 - \alpha_n) \frac{\lambda \tau_1^{q-1}}{\gamma_1} \|F(x, y_n) - F(x, y)\| \\ &\leqslant \alpha_n \|x_n - x\| + (1 - \alpha_n) \frac{\tau_1^{q-1}}{\gamma_1} \sqrt[q]{\delta_1^q - q\lambda r_1 + c_q \lambda^q s_1^q} \|x_n - x\| + (1 - \alpha_n) \frac{\lambda \xi_1 \tau_1^{q-1}}{\gamma_1} \|y_n - y\| \end{aligned} \tag{5.4}$$

and

$$\begin{aligned} \|y_{n+1} - y\| &\leqslant \alpha_n \|y_n - y\| + (1 - \alpha_n) \|R^{P_2,\eta_2}_{N,\rho}[P_2(y_n) - \rho G(x_n, y_n)] - R^{P_2,\eta_2}_{N,\rho}[P_2(y) - \rho G(x, y)]\| \\ &\leqslant \alpha_n \|y_n - x\| + (1 - \alpha_n) \frac{\tau_2^{q-1}}{\gamma_2} \|P_2(y_n) - P_2(y) - \rho[G(x_n, y_n) - G(x, y)]\| \\ &\leqslant \alpha_n \|y_n - y\| + (1 - \alpha_n) \frac{\tau_2^{q-1}}{\gamma_2} \|P_2(y_n) - P_2(y) - \rho[G(x_n, y_n) - G(x_n, y)]\| \\ &\quad + (1 - \alpha_n) \frac{\rho \tau_2^{q-1}}{\gamma_2} \|G(x_n, y) - G(x, y)\| \\ &\leqslant \alpha_n \|y_n - y\| + (1 - \alpha_n) \frac{\tau_2^{q-1}}{\gamma_2} \sqrt[q]{\delta_2^q - q\rho r_2 + c_q \rho^q s_2^q} \|y_n - y\| + (1 - \alpha_n) \frac{\rho \xi_2 \tau_2^{q-1}}{\gamma_2} \|x_n - x\|. \end{aligned} \tag{5.5}$$

It follows from (5.4) and (5.5) that

$$\begin{aligned}\|x_{n+1}-x\|+\|y_{n+1}-y\| &\leqslant \alpha_n(\|x_n-x\|+\|y_n-y\|)+(1-\alpha_n)\theta(\|x_n-x\|+\|y_n-y\|)\\ &=(\theta+(1-\theta)\alpha_n)(\|x_n-x\|+\|y_n-y\|),\end{aligned} \tag{5.6}$$

where $0\leqslant\theta<1$ is defined by (4.11). Let

$$c_n=\|x_n-x\|+\|y_n-y\| \quad \text{and} \quad k_n=\theta+(1-\theta)\alpha_n.$$

Then (5.6) can be rewritten as

$$c_{n+1}\leqslant k_n c_n, \quad n=0,1,2,\ldots .$$

By (5.2), we know that $\lim\ \sup_n k_n<1$. It follows from Lemma 5.1 in [16] that

$$\|x_n-x\|+\|y_n-y\|\longrightarrow 0 \quad \text{as } n\longrightarrow\infty.$$

Therefore, (x_n, y_n) converges strongly to the unique solution (x, y) of problem (3.1). □

Remark 5.1. Theorem 5.1 extends and unifies Theorem 5.1 in [17] and Theorem 5.1 in [16].

Remark 5.2. By Theorems 4.1 and 5.1, it is easy to obtain the existence and convergence results for the special cases of problem (3.1). Hence, Theorems 4.1 and 5.1 generalize the main results in [31–35,24,10,14].

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