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Computation of the scattering amplitude for a scattering wave produced by a disc — Approach by a fundamental solution method[☆]

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ABSTRACT

Our fundamental solution method gives an analytic representation of the approximate solution for the reduced wave problem in the exterior region of a disc. The asymptotic behavior of this representation yields an approximate formula for the scattering amplitude. An error estimate for this formula is given. We add two numerical tests: the numerical estimate of errors; and profiles of scattering cross sections and the far-field coefficient. Both tests include cases of high wave numbers.

1. Introduction

We study a fundamental solution method (FSM) [1–5] in order to construct an FEM–FSM combined method [6]. The FEM–FSM combined method solves the reduced wave problem in a domain exterior to a simple closed curve. Observing the behavior of an FSM approximate solution in the far field, we obtain an approximate formula for the scattering amplitude.

Our fundamental solution method is applied to the reduced wave problem in the exterior region of a disc. The approximate solution obtained is analytically represented with the parameters Q_j that show the intensity of the sources at the source points established in the approximate problem. The asymptotic behavior of this analytic representation gives the approximate formula, that is mentioned in the title, for the scattering amplitude.

The following results are obtained. Under a fairly general condition, we show a proof for the derivation of the scattering amplitude from the reduced wave problem in the exterior region of the disc. We give an approximate formula for the scattering amplitude and the error estimate of the approximate formula.

Let a be the radius of a disc and let ρ be the radius of a circle which is concentric and interior to the disc, containing all the source points employed in the approximate problem. We adopt a way of arranging collocation points and source points.

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Let N be a number of the collocation points. Our estimate shows that the convergence rate has the form of $O(\gamma^{N/2}N^{-1})$ with $\gamma = \rho/a$.

Two numerical tests are added for the approximate formula: the numerical estimate of errors; and profiles of scattering cross sections and the far-field coefficient [7]. Both tests include the cases of high frequencies. Here the frequency is implied through a normalized wave number. The normalized wave number κ is defined through κ = wave number $\times$ radius of a disc. We employ normalized wave numbers $\kappa = 1, 10, 100$ and 1000. Multiple-precision arithmetic is effective for these tests.

The organization of the rest of the paper is as follows. In Section 2, the setting of the continuous problem ($\mathrm{E_f}$) is shown. In Section 3, our previous results on our fundamental method of solution for ($\mathrm{E_f}$) are summarized. In Section 4, the main results are described: convergence of the formal solution of ($\mathrm{E_f}$); convergence of the series of the scattering amplitude; the limiting formula for the scattering amplitude; an approximate formula for the scattering amplitude; and an error estimate of the approximate formula. In Section 5, we prepare for proofs of the main results. In Sections 6–10, the main results are proved. In Section 11, two numerical tests are shown. And acknowledgements are stated. In the Appendix, Proposition 23 from Section 5 is proved.

2. Reduced wave problem and scattering amplitude

2.1. A reduced wave problem in the exterior region of a disc

The domain Ω_e outside of a disc with the radius of a and its boundary Γ_a are defined through

$$\Omega_e = \{\boldsymbol{r} \in \mathbb{R}^2;\ |\boldsymbol{r}| > a\}, \qquad \Gamma_a = \{\boldsymbol{a} \in \mathbb{R}^2;\ |\boldsymbol{a}| = a\},$$

where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^2$.

Consider the Helmholtz equation with the wave number k in Ω_e. We give the appropriate Dirichlet data f on Γ_a. And the Sommerfeld outgoing radiation condition at infinity is employed. Hence the problem ($\mathrm{E_f}$) is represented with the polar coordinates (r, θ) through

$$(\mathrm{E_f}) \quad \begin{cases} -\Delta u - k^2 u = 0 & \text{in } \Omega_e, \\ u = f & \text{on } \Gamma_a, \\ \lim\limits_{r\to\infty} \sqrt{r}\left\{\dfrac{\partial u}{\partial r} - \mathrm{i}ku\right\} = 0. \end{cases}$$

Then the solution of ($\mathrm{E_f}$) is given by

$$u(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} f_n \frac{H_n^{(1)}(kr)}{H_n^{(1)}(ka)} \mathrm{e}^{\mathrm{i}n\theta} \quad \text{with } f_n = \frac{1}{2\pi}\int_0^{2\pi} f(\boldsymbol{a})\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta, \tag{1}$$

where the points $\boldsymbol{r}$ and $\boldsymbol{a}$ correspond to the polar coordinates (a, θ) and (r, θ), respectively, and $H_n^{(1)}(\cdot)$ is the nth-order Hankel function of the first kind.

2.2. Scattering amplitude for the solution of ($\mathrm{E_f}$)

Definition 1. The scattering amplitude $A(\theta)$ of (1) is defined through

$$A(\theta) = \sum_{n=-\infty}^{\infty} \sqrt{\frac{2}{\pi k}} \mathrm{e}^{-\mathrm{i}\pi\frac{2n+1}{4}} \frac{f_n}{H_n^{(1)}(ka)} \mathrm{e}^{\mathrm{i}n\theta} \quad \text{for } \theta \in [0, 2\pi]. \tag{2}$$

The scattering wave in the far field is expected to behave as

$$u(\boldsymbol{r}) \sim \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} A(\theta) \quad \text{as } r \to \infty. \tag{3}$$

3. Approximate approaches to the reduced wave problem through a fundamental solution method

3.1. A fundamental solution method

Let N be an arbitrary fixed positive integer greater than 1. Then we use the notation θ_j defined through

$$\theta_1 = \frac{2\pi}{N}, \qquad \theta_j = j\theta_1, \quad 0 \le j \le N-1.$$

Let ρ and a be positive numbers such that $0 < \rho < a$. Points $\boldsymbol{\rho}_j$ and $\boldsymbol{a}_j$ correspond to the polar coordinates (ρ, θ_j) and (a, θ_j) for $0 \le j \le N-1$, respectively. The points $\boldsymbol{\rho}_j$ and $\boldsymbol{a}_j$ are said to be the source and the collocation points, respectively.

Now we introduce an approximate version $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$ of (E_f) through a fundamental solution method (FSM) as the following problem:

$$(\mathrm{E}_\mathrm{f}^{(\mathrm{N})}) \quad \begin{cases} u^{(N)}(\boldsymbol{r}) = \displaystyle\sum_{j=0}^{N-1} Q_j G_j(\boldsymbol{r}), \\ u^{(N)}(\boldsymbol{a}_j) = f(\boldsymbol{a}_j), \quad 0 \le j \le N-1. \end{cases}$$

In the problem $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$, we employ the basis function $G_j(\boldsymbol{r})$ as follows:

$$G_j(\boldsymbol{r}) = H_0^{(1)}(k|r\mathrm{e}^{\mathrm{i}\theta} - \rho\mathrm{e}^{\mathrm{i}\theta_j}|), \quad 0 \le j \le N-1,$$

where $H_0^{(1)}(\cdot)$ is the zeroth-order Hankel function of the first kind, and the complex numbers $r\mathrm{e}^{\mathrm{i}\theta}$ and $\rho\mathrm{e}^{\mathrm{i}\theta_j}$ correspond to the points (r, θ) and (ρ, θ_j), respectively.

It is noted that $G_j(\boldsymbol{r})$ is a constant multiple of the fundamental solution of the Helmholtz equation with the singularity at $\boldsymbol{r} = \boldsymbol{\rho}_j$ satisfying the Sommerfeld outgoing radiation condition at infinity. The problem $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$ should be understood as being that the unknown N quantities Q_j, $0 \le j \le N-1$, should be determined the collocation condition described as the second equation of $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$.

Hereafter the following notation is employed:

$$\gamma = \frac{\rho}{a}, \qquad \delta = \frac{r}{a}, \qquad \kappa = ka. \tag{4}$$

These numbers are characteristic numbers of the relevant problem, normalized by the radius a. Using this notation we can rewrite the basis function $G_j(\boldsymbol{r})$ as follows:

$$G_j(\boldsymbol{r}) = H_0^{(1)}(\kappa|\delta - \gamma\mathrm{e}^{-\mathrm{i}(\theta-\theta_j)}|), \quad 0 \le j \le N-1. \tag{5}$$

3.2. Solvability of the fundamental solution method

We introduce the kernel function $g(\theta)$ through

$$g(\theta) = H_0^{(1)}(\kappa|1 - \gamma\mathrm{e}^{-\mathrm{i}\theta}|). \tag{6}$$

Graf's addition theorem given on p. 361 of Watson [8] yields the Fourier series expansion of $g(\theta)$:

$$g(\theta) = \sum_{n=-\infty}^{\infty} H_n^{(1)}(\kappa) J_n(\gamma\kappa)\mathrm{e}^{\mathrm{i}n\theta}. \tag{7}$$

Define the coefficient g_n by

$$g_n = H_n^{(1)}(\kappa) J_n(\gamma\kappa) \quad \text{for } n \in \mathbb{Z}. \tag{8}$$

The following assumption is introduced.

Assumption 2. Let κ be fixed as an arbitrary positive number. Choose $\gamma \in (0, 1)$ appropriately so that the kernel function $g(\theta)$ with parameters κ and γ may satisfy the following condition $(\mathbf{g})$:

$$(\mathbf{g}) \quad g_n \ne 0 \quad \text{for } n \in \mathbb{Z}.$$

Remark 3. Let us note the following fact: Fix $\kappa > 0$. Then the condition $(\mathbf{g})$ is satisfied for any $\gamma \in (0, 1)$, except for finitely many values of γ depending on κ.

We obtain the following theorem for the solvability of the problem $(\mathrm{E}_\mathrm{f}^{(\mathrm{N})})$ in [4].

Theorem 4. *Under the condition* $(\mathbf{g})$, *there exists a positive number* N_0 *such that the following condition* $(\mathbf{G}^{(\mathbf{N})})$ *holds for an arbitrary integer* $N \ge N_0$, *where the positive integer* N_0 *depends on* κ *and* γ*:*

$$(\mathbf{G}^{(\mathbf{N})}) \quad G_n^{(N)} \ne 0 \quad \textit{for } n \in \mathbb{Z},$$

where

$$G_n^{(N)} = \frac{1}{N}\sum_{j=0}^{N-1} g(\theta_j)\mathrm{e}^{-\mathrm{i}n\theta_j}. \tag{9}$$

Under the condition ($\mathbf{G}^{(\mathbf{N})}$), Q_j is given through

$$Q_j = \frac{1}{N}\sum_{k=0}^{N-1}\frac{F_k^{(N)}}{G_k^{(N)}}\mathrm{e}^{\mathrm{i}j\theta_k} \quad \text{for } j \in \mathbb{Z}, \tag{10}$$

where

$$F_n^{(N)} = \frac{1}{N}\sum_{j=0}^{N-1} f(\theta_j)\mathrm{e}^{-\mathrm{i}n\theta_j} \quad \text{for } n \in \mathbb{Z}. \tag{11}$$

(See Section 3 of [2].)

4. Main results

Assumption 5. The density norm $|\!|\!| q |\!|\!|$ of function f is finite:

$$|\!|\!| q |\!|\!| = \sup_{n\in\mathbb{Z}}\left|\frac{f_n}{g_n}\right| < \infty,$$

where f_n and g_n are Fourier coefficients of $f(\boldsymbol{a})$ and $g(\theta)$ defined through

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(\boldsymbol{a})\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta, \qquad g_n = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\mathrm{e}^{-\mathrm{i}n\theta}\mathrm{d}\theta,$$

respectively.

Theorem 6. *Under Assumption 5, the scattering wave $u(\boldsymbol{r})$ converges absolutely and uniformly with respect to $r \ge a$ and $\theta \in [0, 2\pi]$.*

Theorem 7. *Under Assumption 5, the series $A(\theta)$ defined in the formula (2) converges absolutely and uniformly with respect to $\theta \in [0, 2\pi]$.*

Theorem 8. *Under Assumption 5, the following formula holds:*

$$A(\theta) = \lim_{r\to\infty}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}}\right)^{-1} u(\boldsymbol{r}). \tag{12}$$

The above limiting process is uniform with respect to $\theta \in [0, 2\pi]$.

Definition 9. The approximate scattering amplitude is defined by

$$A^{(N)}(\theta) = \lim_{r\to\infty}\left(\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}}\right)^{-1} u^{(N)}(\boldsymbol{r}) \quad \text{for } \theta \in [0, 2\pi].$$

Theorem 10. *Under the solvability condition ($\mathbf{G}^{(\mathbf{N})}$), the approximate formula for the scattering amplitude is represented through*

$$A^{(N)}(\theta) = \sum_{j=0}^{N-1}\sqrt{\frac{2}{\pi k}}\mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}Q_j\mathrm{e}^{-\mathrm{i}\kappa\gamma\cos(\theta-\theta_j)} \quad \textit{for } \theta \in [0, 2\pi], \tag{13}$$

where Q_j is given through (9)–(11).

Theorem 11. *Under Assumption 5, there is a positive integer N_0 such that the following estimate is valid:*

$$\sup_{0\le\theta\le 2\pi}|A(\theta) - A^{(N)}(\theta)| < \sqrt{\frac{2}{\pi k}}M_\kappa\frac{900|\!|\!| q |\!|\!|}{\pi(1-\gamma)}\frac{\gamma^{N/2}}{N} \quad \textit{for } N \ge N_0, \tag{14}$$

where the positive constant M_κ is given by

$$M_\kappa = \sup_{n\in\mathbb{Z}}\frac{1}{|H_n^{(1)}(\kappa)|} < \infty.$$

The positive integer N_0 depends on κ and γ, but does not depend on Dirichlet data f.

5. Preparations for proof of the main results

5.1. Properties of cylindrical functions

Lemma 12.

$$J_{-n}(x) = (-1)^n J_n(x), \qquad H_{-n}^{(1)}(x) = (-1)^n H_n^{(1)}(x) \quad \text{for } n \in \mathbb{Z}. \tag{15}$$

*(*9.1.5 *and* 9.1.6 *on p.* 358 *of Abramowitz and Stegun* [9].*)*

Lemma 13. *Asymptotic formulae by order of Hankel and Bessel functions are given through*

$$H_n^{(1)}(x) \sim -\mathrm{i}\sqrt{\frac{2}{\pi n}}\left(\frac{\mathrm{e}x}{2n}\right)^{-n}, \qquad J_n(x) \sim \sqrt{\frac{1}{2\pi n}}\left(\frac{\mathrm{e}x}{2n}\right)^{n} \quad \text{as } n \to \infty, \tag{16}$$

*where x is a fixed positive number. (*9.3.1 *on p.* 365 *of Abramowitz and Stegun* [9].*)*

Lemma 14. *For every $n \in \mathbb{Z}$ there are positive constants C_n and x_n such that*

$$\left| \frac{H_n^{(1)}(x)}{\sqrt{\frac{2}{\pi x}}\mathrm{e}^{\mathrm{i}\left(x - \frac{2n+1}{4}\pi\right)}} - 1 \right| < \frac{C_n}{x} \quad \text{for } x > x_n. \tag{17}$$

Proof. The following asymptotic expansion holds for every $n \in \mathbb{Z}$ ((I.203) on p. 53 of Bowman, Senior and Uslenghi [7]):

$$H_n^{(1)}(x) \sim \sqrt{\frac{2}{\pi x}}\mathrm{e}^{\mathrm{i}\left(x - \frac{2n+1}{4}\pi\right)} \sum_{m=0}^{\infty} \frac{(-1)^m (n,m)}{(2\mathrm{i}x)^m} \quad \text{as } x \to \infty,$$

where

$$(n,m) = \frac{\Gamma\left(\frac{1}{2} + n + m\right)}{\Gamma(m+1)\Gamma\left(\frac{1}{2} + n - m\right)}.$$

Then there are positive constants C_n and x_n such that

$$\left| H_n^{(1)}(x) - \sqrt{\frac{2}{\pi x}}\mathrm{e}^{\mathrm{i}\left(x - \frac{2n+1}{4}\pi\right)} \right| < \frac{C_n}{x}\sqrt{\frac{2}{\pi x}} \quad \text{for } x > x_n.$$

Therefore the statement of the lemma is obtained. □

Lemma 15. *Due to Graf's addition formula given on p.* 361 *of Watson* [8]*, the following equality holds:*

$$H_0^{(1)}(|x - y\mathrm{e}^{-\mathrm{i}\theta}|) = \sum_{n=-\infty}^{\infty} H_n^{(1)}(x) J_n(y) \mathrm{e}^{\mathrm{i}n\theta} \quad \text{for } x > y \geq 0 \text{ and } \theta \in [0, 2\pi]. \tag{18}$$

Lemma 16. *By Nicholson's integral on p.* 444 *of Watson* [8]*, $|H_n^{(1)}(x)|^2$ is represented through*

$$\begin{aligned} |H_n^{(1)}(x)|^2 &= J_n^2(x) + Y_n^2(x) \\ &= \frac{8}{\pi^2}\int_0^\infty K_0(2x\sinh t)\cosh 2nt\,\mathrm{d}t \quad \text{for } x > 0 \text{ and } n \in \mathbb{Z}, \end{aligned} \tag{19}$$

with

$$K_0(\xi) = \int_0^\infty \mathrm{e}^{-\xi\cosh t}\mathrm{d}t \quad \text{for } \xi > 0,$$

where $Y_n(x)$ is the nth-order Neumann function and $K_0(\xi)$ is the zeroth-order modified Bessel function of the second kind (see the bottom of p. 446 *of Watson* [8]*).*

5.2. Propositions

Proposition 17. *The Hankel function $H_n^{(1)}(x)$ does not vanish for $x > 0$ and $n \in \mathbb{Z}$.*

Proof. Lemma 16 shows that $|H_n^{(1)}(x)|^2$ is a positive decreasing function of $x > 0$. Therefore we obtain the statement of the proposition. □

Proposition 18. *Let b be a positive number such that $b > a$. Then there is a positive integer L such that*

$$\frac{1}{|H_n^{(1)}(\kappa)J_n(kb)|} \le \frac{3\pi|n|}{2}\left(\frac{a}{b}\right)^{|n|} \quad \text{for } n \in \mathbb{Z} \text{ with } |n| > L. \tag{20}$$

Proof. Due to Lemma 13, the following asymptotic formula holds:

$$\frac{1}{H_n^{(1)}(\kappa)J_n(kb)} \sim \mathrm{i}n\pi\left(\frac{a}{b}\right)^n \quad \text{as } n \to \infty.$$

Then there is a positive integer L such that

$$\left|\frac{1}{\mathrm{i}n\pi\left(\frac{a}{b}\right)^n H_n^{(1)}(\kappa)J_n(kb)} - 1\right| \le \frac{1}{2} \quad \text{for } n > L.$$

We have

$$\frac{1}{|H_n^{(1)}(\kappa)J_n(kb)|} \le \frac{3\pi n}{2}\left(\frac{a}{b}\right)^n \quad \text{for } n > L. \tag{21}$$

Using Lemma 12, we obtain (20). □

Proposition 19. *Under the condition* (**g**), *there is a positive integer L such that*

$$|g_n| \le \frac{3\gamma^{|n|}}{2\pi|n|} \quad \text{for } n \in \mathbb{Z} \text{ with } |n| > L. \tag{22}$$

Proof. Due to Lemma 13, the following asymptotic formula holds:

$$g_n = H_n^{(1)}(\kappa)J_n(\gamma\kappa) \sim -\mathrm{i}\frac{1}{\pi n}\gamma^n \quad \text{as } n \to \infty.$$

Due to this formula, we obtain the inequality of the proposition in the same manner as in the proof of Proposition 18. □

Proposition 20. *There is a positive constant $M_{\gamma,\kappa}$ such that*

$$|g_n| \le M_{\gamma,\kappa} \quad \text{for } n \in \mathbb{Z}. \tag{23}$$

Proof. Due to Proposition 19, there is a positive integer L such that g_n is bounded for $n \in \mathbb{Z}$ with $|n| > L$. And g_n is also bounded for $n \in \mathbb{Z}$ with $|n| \le L$. Therefore there is a positive constant $M_{\gamma,\kappa}$ that satisfies the inequality (23). □

Proposition 21. *The following inequality holds:*

$$0 < \left|\frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)}\right| \le 1 \quad \text{for } r \ge a \text{ and } n \in \mathbb{Z}. \tag{24}$$

Proof. The same proposition is proved as Proposition 6 in Section 6 of [2]. □

Proposition 22. *The reciprocal of $H_n^{(1)}(\kappa)$, $n \in \mathbb{Z}$, is bounded for every fixed $\kappa > 0$. Namely there is a positive constant M_κ such that*

$$M_\kappa = \sup_{n\in\mathbb{Z}} \frac{1}{\left|H_n^{(1)}(\kappa)\right|} < \infty. \tag{25}$$

Proof. Due to Lemma 13, the following asymptotic formula holds:

$$\frac{1}{H_n^{(1)}(\kappa)} \sim \mathrm{i}\sqrt{\frac{\pi n}{2}}\left(\frac{\mathrm{e}\kappa}{2n}\right)^n \quad \text{as } n \to \infty.$$

Then there is a positive integer L such that $1/|H_n^{(1)}(\kappa)| \le 1$ for $n > L$. Lemma 12 yields the formula: $1/|H_n^{(1)}(\kappa)| \le 1$ for $n \in \mathbb{Z}$ with $|n| > L$. Due to Proposition 17, $H_n^{(1)}(\kappa)$ does not vanish for $n \in \mathbb{Z}$ with $|n| \le L$. Therefore there is a positive constant M_κ that satisfies the formula (25). □

Proposition 23. *Let k and b be fixed positive numbers. There are positive constants C_0' and r_0 such that*

$$\left| \frac{H_0^{(1)}(|kr - kb\mathrm{e}^{-\mathrm{i}\theta}|)}{\sqrt{\frac{2}{k\pi r}}\mathrm{e}^{\mathrm{i}\left(kr - kb\cos\theta - \frac{\pi}{4}\right)}} - 1 \right| < \frac{C_0'}{r} \quad \text{for } r > r_0. \tag{26}$$

This inequality holds uniformly with respect to $\theta \in [0, 2\pi]$.

Proof. We give a proof of this proposition in the Appendix. □

Proposition 24. *Let k and b be fixed positive numbers. Then the following formula holds:*

$$\lim_{r\to\infty} \left(\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} \right)^{-1} H_0^{(1)}(|kr - kb\mathrm{e}^{-\mathrm{i}\theta}|) = \sqrt{\frac{2}{k\pi}}\mathrm{e}^{-\mathrm{i}\left(kb\cos\theta + \frac{\pi}{4}\right)}. \tag{27}$$

This limiting process holds uniformly with respect to $\theta \in [0, 2\pi]$.

Proof. Due to Proposition 23, there are positive constants C_0' and r_0 such that

$$\left| \left(\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} \right)^{-1} H_0^{(1)}(|kr - kb\mathrm{e}^{-\mathrm{i}\theta}|) - \sqrt{\frac{2}{k\pi}}\mathrm{e}^{-\mathrm{i}\left(kb\cos\theta + \frac{\pi}{4}\right)} \right| < \sqrt{\frac{2}{k\pi}}\frac{C_0'}{r} \quad \text{for } r > r_0.$$

This inequality holds uniformly with respect to $\theta \in [0, 2\pi]$. Therefore the statement of the proposition is obtained. □

Proposition 25. *Let k and b be fixed positive numbers. The following equality holds for $r > b$:*

$$H_n^{(1)}(kr)J_n(kb) = \frac{1}{2\pi}\int_0^{2\pi} H_0^{(1)}(|kr - kb\mathrm{e}^{-\mathrm{i}\varphi}|)\mathrm{e}^{-\mathrm{i}n\varphi}\mathrm{d}\varphi \quad \text{for } n \in \mathbb{Z}. \tag{28}$$

Proof. Lemma 15 shows that $H_n^{(1)}(kr)J_n(kb)$ is the nth Fourier coefficient of $H_0^{(1)}(|kr - kb\,\mathrm{e}^{-\mathrm{i}\varphi}|)$. □

Proposition 26. *Let b be a positive number such that $b > a$. Then there are positive constants $\tilde{C}$ and $\tilde{R}$ and a positive integer $\tilde{N}$ such that*

$$\left| \frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)} \right| < \tilde{C} \left(\frac{a}{b} \right)^{|n|} \frac{|n|}{\sqrt{r}} \quad \text{for } r > \tilde{R} \text{ and } n \in \mathbb{Z} \text{ with } |n| > \tilde{N},$$

where $\tilde{R}$ does not depend on $\tilde{N}$.

Proof. Due to Proposition 18, there is a positive integer $\tilde{N}$ such that

$$\frac{1}{|H_n^{(1)}(\kappa)J_n(kb)|} \le \frac{3\pi|n|}{2}\left(\frac{a}{b}\right)^{|n|} \quad \text{for } n \in \mathbb{Z} \text{ with } |n| > \tilde{N}. \tag{29}$$

Since $J_n(kb) \ne 0$ for $n \in \mathbb{Z}$ with $|n| > \tilde{N}$, Proposition 25 yields the following equality:

$$\frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)} = \frac{1}{H_n^{(1)}(\kappa)J_n(kb)}\frac{1}{2\pi}\int_0^{2\pi} H_0^{(1)}(|kr - kb\,\mathrm{e}^{-\mathrm{i}\varphi}|)\mathrm{e}^{-\mathrm{i}n\varphi}\mathrm{d}\varphi.$$

Due to Proposition 23, there are positive constants C_0' and $\tilde{R}$ such that

$$\left|H_0^{(1)}(|kr - kb\,\mathrm{e}^{-\mathrm{i}\varphi}|)\right| < \sqrt{\frac{2}{k\pi r}}\left(1 + \frac{C_0'}{r}\right) \quad \text{for } r > \tilde{R}.$$

Define a positive constant C_0'' such that

$$\sqrt{\frac{2}{k\pi r}}\left(1 + \frac{C_0'}{r}\right) < \sqrt{\frac{2}{k\pi r}}\left(1 + \frac{C_0'}{\tilde{R}}\right) = \frac{C_0''}{\sqrt{r}} \quad \text{for } r > \tilde{R}.$$

Then we have

$$\left|H_0^{(1)}(|kr - kb\,\mathrm{e}^{-\mathrm{i}\varphi}|)\right| < \frac{C_0''}{\sqrt{r}} \quad \text{for } r > \tilde{R}.$$

Hence the following inequalities hold for $r > \tilde{R}$ and $n \in \mathbb{Z}$ with $|n| > \tilde{N}$:

$$\begin{aligned}\left|\frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)}\right| &\le \frac{1}{|H_n^{(1)}(\kappa)J_n(kb)|}\frac{1}{2\pi}\int_0^{2\pi}\left|H_0^{(1)}(|kr - kb\,\mathrm{e}^{-\mathrm{i}\varphi}|)\mathrm{e}^{-\mathrm{i}n\varphi}\right|\mathrm{d}\varphi \\ &< \frac{1}{|H_n^{(1)}(\kappa)J_n(kb)|}\frac{1}{2\pi}\int_0^{2\pi}\frac{C_0''}{\sqrt{r}}\mathrm{d}\varphi = \frac{1}{|H_n^{(1)}(\kappa)J_n(kb)|}\frac{C_0''}{\sqrt{r}}.\end{aligned} \tag{30}$$

Due to (29) and (30), the following inequality holds:

$$\left|\frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)}\right| \le \tilde{C}\left(\frac{a}{b}\right)^{|n|}\frac{|n|}{\sqrt{r}} \quad \text{for } r > \tilde{R} \text{ and } n \in \mathbb{Z} \text{ with } |n| > \tilde{N},$$

where $\tilde{C} = 3\pi C_0''/2$, and $\tilde{R}$ does not depend on $\tilde{N}$. □

6. Proof of Theorem 6

Proof. Due to Assumption 5, Propositions 19 and 21, there is a positive integer L such that

$$\left|f_n\frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)}\mathrm{e}^{\mathrm{i}n\theta}\right| \le |||q|||\,|g_n| \le \frac{3|||q|||\gamma^{|n|}}{2\pi|n|} < \frac{3|||q|||\gamma^{|n|}}{2\pi L} \quad \text{for } r \ge a,\ \theta \in [0, 2\pi] \text{ and } n \in \mathbb{Z} \text{ with } |n| > L.$$

Then we have

$$\begin{aligned}\left|\sum_{|n|>L} f_n\frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)}\mathrm{e}^{\mathrm{i}n\theta}\right| &\le \sum_{|n|>L}\left|f_n\frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)}\mathrm{e}^{\mathrm{i}n\theta}\right| \\ &\le \sum_{|n|>L}\frac{3|||q|||\gamma^{|n|}}{2\pi L} < \frac{3|||q|||\gamma^{L+1}}{\pi L(1-\gamma)} \quad \text{for } r \ge a \text{ and } \theta \in [0, 2\pi].\end{aligned}$$

Let ϵ be an arbitrary positive number. Then there is a positive integer $N > L$ such that

$$\left|\sum_{|n|>N} f_n\frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)}\mathrm{e}^{\mathrm{i}n\theta}\right| < \frac{3|||q|||\gamma^{N+1}}{\pi N(1-\gamma)} < \epsilon \quad \text{for } r \ge a \text{ and } \theta \in [0, 2\pi].$$

The inequalities show that the series of $u(\boldsymbol{r})$ converges absolutely and uniformly with respect to $r \ge a$ and $\theta \in [0, 2\pi]$. □

7. Proof of Theorem 7

Proof. Due to Assumption 5, Propositions 21 and 22, there is a positive integer L such that

$$\left|\sqrt{\frac{2}{k\pi}}\mathrm{e}^{-\mathrm{i}\pi\frac{2n+1}{4}}\frac{f_n}{H_n^{(1)}(\kappa)}\mathrm{e}^{\mathrm{i}n\theta}\right| \le \sqrt{\frac{2}{k\pi}}\frac{3|||q|||M_\kappa\gamma^{|n|}}{2\pi|n|} < \sqrt{\frac{2}{k\pi}}\frac{3|||q|||M_\kappa\gamma^{|n|}}{2\pi L} \quad \text{for } n \in \mathbb{Z} \text{ with } |n| > L \text{ and } \theta \in [0, 2\pi].$$

Then we have

$$\begin{aligned}\left|\sum_{|n|>L}\sqrt{\frac{2}{k\pi}}\mathrm{e}^{-\mathrm{i}\pi\frac{2n+1}{4}}\frac{f_n}{H_n^{(1)}(\kappa)}\mathrm{e}^{\mathrm{i}n\theta}\right| &\le \sum_{|n|>L}\left|\sqrt{\frac{2}{k\pi}}\mathrm{e}^{-\mathrm{i}\pi\frac{2n+1}{4}}\frac{f_n}{H_n^{(1)}(\kappa)}\mathrm{e}^{\mathrm{i}n\theta}\right| \\ &\le \sum_{|n|>L}\sqrt{\frac{2}{k\pi}}\frac{3|||q|||M_\kappa\gamma^{|n|}}{2\pi L} < \sqrt{\frac{2}{k\pi}}\frac{3|||q|||M_\kappa\gamma^{L+1}}{\pi L(1-\gamma)} \quad \text{for } \theta \in [0, 2\pi].\end{aligned}$$

Let ϵ be an arbitrary positive number. Then there is a positive integer $N > L$ such that

$$\left| \sum_{|n|>N} \sqrt{\frac{2}{k\pi}} e^{-i\pi \frac{2n+1}{4}} \frac{f_n}{H_n^{(1)}(\kappa)} e^{in\theta} \right| < \sqrt{\frac{2}{k\pi}} \frac{3|||q|||M_\kappa \gamma^{N+1}}{\pi N(1-\gamma)} < \epsilon \quad \text{for } \theta \in [0, 2\pi].$$

The inequalities show that the series of $A(\theta)$ converges absolutely and uniformly with respect to $\theta \in [0, 2\pi]$. □

8. Proof of Theorem 8

8.1. Preparations

Let N be a positive integer. The series of $u(\boldsymbol{r})$ converges absolutely and uniformly with respect to $r \geq a$ and $\theta \in [0, 2\pi]$. Then we have

$$\left(\frac{e^{ikr}}{\sqrt{r}} \right)^{-1} u(\boldsymbol{r}) = \mathcal{P}_N(\boldsymbol{r}) + \mathcal{R}_N(\boldsymbol{r}) \quad \text{for } r \geq a \text{ and } \theta \in [0, 2\pi], \tag{31}$$

where $\mathcal{P}_N(\boldsymbol{r})$ and $\mathcal{R}_N(\boldsymbol{r})$ are defined through

$$\mathcal{P}_N(\boldsymbol{r}) = \sum_{|n|\leq N} f_n \left(\frac{e^{ikr}}{\sqrt{r}} \right)^{-1} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)} e^{in\theta} \tag{32}$$

$$\mathcal{R}_N(\boldsymbol{r}) = \sum_{|n|>N} f_n \left(\frac{e^{ikr}}{\sqrt{r}} \right)^{-1} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)} e^{in\theta}. \tag{33}$$

A partial sum of $A(\theta)$ is defined by

$$A_N(\theta) = \sum_{|n|\leq N} \sqrt{\frac{2}{k\pi}} e^{-i\pi \frac{2n+1}{4}} \frac{f_n}{H_n^{(1)}(\kappa)} e^{in\theta} \quad \text{for } \theta \in [0, 2\pi]. \tag{34}$$

Consider the following telescoping:

$$A(\theta) - \left(\frac{e^{ikr}}{\sqrt{r}} \right)^{-1} u(\boldsymbol{r}) = A(\theta) - A_N(\theta) - \mathcal{R}_N(\boldsymbol{r}) + A_N(\theta) - \mathcal{P}_N(\boldsymbol{r}).$$

Then we have

$$\left| A(\theta) - \left(\frac{e^{ikr}}{\sqrt{r}} \right)^{-1} u(\boldsymbol{r}) \right| \leq |A(\theta) - A_N(\theta)| + |\mathcal{R}_N(\boldsymbol{r})| + |A_N(\theta) - \mathcal{P}_N(\boldsymbol{r})|. \tag{35}$$

Choosing an arbitrary positive number ϵ, we estimate each term on the right hand side of (35). □

8.2. Estimate of first term

In Section 7, we show that the series of $A(\theta)$ converges absolutely and uniformly with respect to $\theta \in [0, 2\pi]$. Then there is a positive integer $\tilde{N}_1$ such that

$$|A(\theta) - A_N(\theta)| < \frac{\epsilon}{3} \quad \text{for } N > \tilde{N}_1. \tag{36}$$

This inequality holds uniformly with respect to $\theta \in [0, 2\pi]$. □

8.3. Estimate of second term

Due to Propositions 19 and 26, there are positive constants $\tilde{C}$ and $\tilde{R}_0$ and a positive integer $\tilde{N}_2 > \tilde{N}_1$ such that

$$|f_n| \leq |||q|||\,|g_n| \leq \frac{3}{2\pi|n|} |||q||| \gamma^{|n|}, \qquad \left| \frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)} \right| < \tilde{C} \left(\frac{a}{b} \right)^{|n|} \frac{|n|}{\sqrt{r}} \quad \text{for } r > \tilde{R}_0 \text{ and } n \in \mathbb{Z} \text{ with } |n| > \tilde{N}_2,$$

where $\tilde{R}_0$ does not depend on $\tilde{N}_2$. Then we have

$$\left| f_n \left(\frac{e^{ikr}}{\sqrt{r}} \right)^{-1} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)} e^{in\theta} \right| < C\, \tilde{\gamma}^{|n|},$$

where

$$C = \tilde{C} \times |||q||| \times \frac{3}{2\pi}, \qquad \tilde{\gamma} = \gamma \times \frac{a}{b} < 1.$$

The following inequalities hold:

$$\begin{aligned} |\mathcal{R}_{\tilde{N}_2}(\boldsymbol{r})| &\le \sum_{|n|>\tilde{N}_2} \left| f_n \left(\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} \right)^{-1} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)} \mathrm{e}^{\mathrm{i}n\theta} \right| \\ &< \sum_{|n|>\tilde{N}_2} C\tilde{\gamma}^{|n|} = \frac{2C\,\tilde{\gamma}^{\tilde{N}_2+1}}{1-\tilde{\gamma}} \quad \text{for } r > \tilde{R}_0 \text{ and } \theta \in [0, 2\pi]. \end{aligned}$$

Therefore there is a positive integer $\tilde{N}_3 > \tilde{N}_2$ such that

$$|\mathcal{R}_{\tilde{N}_3}(\boldsymbol{r})| < \frac{2C\,\tilde{\gamma}^{\tilde{N}_3+1}}{1-\tilde{\gamma}} < \frac{\epsilon}{3} \quad \text{for } r > \tilde{R}_0 \text{ and } \theta \in [0, 2\pi], \tag{37}$$

where the positive constant $\tilde{R}_0$ does not depend on $\tilde{N}_3$. The inequalities hold uniformly with respect to $\theta \in [0, 2\pi]$. □

8.4. Estimate of third term

We employ $\tilde{R}_0$ and $\tilde{N}_3$ of Section 8.3. Due to Lemma 14, for every $n \in \mathbb{Z}$ with $|n| \le \tilde{N}_3$ there are positive constants C_n and R_n such that

$$\left| H_n^{(1)}(kr) - \sqrt{\frac{2}{\pi kr}} \mathrm{e}^{\mathrm{i}\left(kr - \pi\frac{2n+1}{4}\right)} \right| < \sqrt{\frac{2}{\pi kr}} \frac{C_n}{kr} \quad \text{for } r > R_n. \tag{38}$$

Let $\tilde{C}$ and $\tilde{R}_1$ be positive constants such that

$$\tilde{C} = \max_{|n|\le\tilde{N}_3} \{C_n\}, \qquad \tilde{R}_1 = \max\left[\max_{|n|\le\tilde{N}_3} \{R_n\}, \tilde{R}_0 \right]. \tag{39}$$

Due to Assumption 5, Propositions 20 and 22, there are constants $|||q|||$, $M_{\gamma,\kappa}$ and M_κ such that

$$\left| \frac{f_n}{H_n^{(1)}(\kappa)} \right| \le |||q||| \times |g_n| \times M_\kappa \le |||q||| \times M_{\gamma,\kappa} \times M_\kappa \quad \text{for } n \in \mathbb{Z}. \tag{40}$$

The formulae (38)–(40) yield

$$\left| \sqrt{\frac{2}{\pi k}} \mathrm{e}^{-\mathrm{i}\pi\frac{2n+1}{4}} \frac{f_n}{H_n^{(1)}(\kappa)} \mathrm{e}^{\mathrm{i}n\theta} - f_n \left(\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} \right)^{-1} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)} \mathrm{e}^{\mathrm{i}n\theta} \right| < |||q||| \times M_{\gamma,\kappa} \times M_\kappa \times \sqrt{\frac{2}{\pi k}} \frac{\tilde{C}}{kr}$$
$$\text{for } r > \tilde{R}_1 \text{ and } n \in \mathbb{Z} \text{ with } |n| \le \tilde{N}_3.$$

Then there is a positive constant $\tilde{R}_2 > \tilde{R}_1$ such that

$$\left| \sqrt{\frac{2}{\pi k}} \mathrm{e}^{-\mathrm{i}\pi\frac{2n+1}{4}} \frac{f_n}{H_n^{(1)}(\kappa)} \mathrm{e}^{\mathrm{i}n\theta} - f_n \left(\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} \right)^{-1} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)} \mathrm{e}^{\mathrm{i}n\theta} \right| < \frac{\epsilon}{3(2\tilde{N}_3+1)} \quad \text{for } r > \tilde{R}_2 \text{ and } n \in \mathbb{Z} \text{ with } |n| \le \tilde{N}_3.$$

Therefore we have

$$\begin{aligned} |A_{\tilde{N}_3}(\theta) - \mathcal{P}_{\tilde{N}_3}(\boldsymbol{r})| &\le \sum_{|n|\le\tilde{N}_3} \left| \sqrt{\frac{2}{\pi k}} \mathrm{e}^{-\mathrm{i}\pi\frac{2n+1}{4}} \frac{f_n}{H_n^{(1)}(\kappa)} \mathrm{e}^{\mathrm{i}n\theta} - f_n \left(\frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}} \right)^{-1} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)} \mathrm{e}^{\mathrm{i}n\theta} \right| \\ &< (2\tilde{N}_3+1) \times \frac{\epsilon}{3(2\tilde{N}_3+1)} = \frac{\epsilon}{3} \quad \text{for } r > \tilde{R}_2. \end{aligned} \tag{41}$$

The inequalities hold uniformly with respect to $\theta \in [0, 2\pi]$. □

8.5. Completion of proof

Due to (35)–(37) and (41), for an arbitrary positive number ϵ there are a positive constant $\tilde{R}_2$ and a positive integer $\tilde{N}_3$ such that

$$\left|A(\theta) - \left(\frac{\mathrm{e}^{ikr}}{\sqrt{r}}\right)^{-1} u(\boldsymbol{r})\right| \leq |A(\theta) - A_{\tilde{N}_3}(\theta)| + |\mathcal{R}_{\tilde{N}_3}(\boldsymbol{r})| + |A_{\tilde{N}_3}(\theta) - \mathcal{P}_{\tilde{N}_3}(\boldsymbol{r})| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon \quad \text{for } r > \tilde{R}_2.$$

These inequalities hold uniformly with respect to $\theta \in [0, 2\pi]$. Hence, we obtain the statement of the theorem. □

9. Proof of Theorem 10

Proof. Let $b = \rho$. Then Definition 9 and Proposition 24 yield

$$\begin{aligned} A^{(N)}(\theta) &= \lim_{r\to\infty} \left(\frac{\mathrm{e}^{ikr}}{\sqrt{r}}\right)^{-1} \sum_{j=0}^{N-1} Q_j G_j(\boldsymbol{r}) \\ &= \sum_{j=0}^{N-1} Q_j \lim_{r\to\infty} \left(\frac{\mathrm{e}^{ikr}}{\sqrt{r}}\right)^{-1} H_0^{(1)}(|kr - k\rho \mathrm{e}^{-\mathrm{i}(\theta-\theta_j)}|) \\ &= \sum_{j=0}^{N-1} \sqrt{\frac{2}{k\pi}} \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}} Q_j \mathrm{e}^{-\mathrm{i}\gamma\kappa\cos(\theta-\theta_j)}. \quad \square \end{aligned}$$

10. Proof of Theorem 11

10.1. Infinite series of the approximate solution

Theorem 27. *Assume Theorem 4. The solution $u^{(N)}(\boldsymbol{r})$ of $(\mathrm{E}_\mathrm{f}^{(N)})$ is represented through*

$$u^{(N)}(\boldsymbol{r}) = \sum_{n=-\infty}^{\infty} F_n^{(N)} \frac{g_n}{G_n^{(N)}} \frac{H_n^{(1)}(kr)}{H_n^{(1)}(\kappa)} \mathrm{e}^{\mathrm{i}n\theta} \quad \textit{for } r \geq a \textit{ and } \theta \in [0, 2\pi]. \tag{42}$$

Proof. A proof of Theorem 27 is given in Section 5 of [2]. □

Theorem 28. *The approximate scattering amplitude $A^{(N)}(\theta)$ is expanded as follows:*

$$A^{(N)}(\theta) = \sum_{n=-\infty}^{\infty} \sqrt{\frac{2}{\pi k}} \mathrm{e}^{-\mathrm{i}\pi\frac{2n+1}{4}} F_n^{(N)} \frac{g_n}{G_n^{(N)}} \frac{1}{H_n^{(1)}(\kappa)} \mathrm{e}^{\mathrm{i}n\theta}. \tag{43}$$

This series converges absolutely and uniformly with respect to $\theta \in [0, 2\pi]$.

Proof. Due to Proposition 13 of [2], there is a positive integer L_{13} such that

$$\left|\frac{F_n^{(N)}}{G_n^{(N)}}\right| \leq 49|||q||| \quad \text{for } N > L_{13} \text{ and } n \in \mathbb{Z}.$$

Then we have

$$\left|F_n^{(N)} \frac{g_n}{G_n^{(N)}}\right| \leq 49|||q|||\,|g_n| \quad \text{for } N > L_{13} \text{ and } n \in \mathbb{Z}. \tag{44}$$

By (44), $f_n = F_n^{(1)} g_n / G_n^{(N)}$ satisfies Assumption 5. Due to Theorem 6, the infinite series (42) converges absolutely and uniformly with respect to $r \geq a$ and $\theta \in [0, 2\pi]$. Due to Theorem 7, the infinite series (43) converges absolutely and uniformly with respect to $\theta \in [0, 2\pi]$. Theorem 8 and Definition 9 show that the series (43) is equal to $A^{(N)}(\theta)$. □

10.2. Proof of Theorem 11

Proof. The series (12) and the series (43) absolutely and uniformly converge with respect to $\theta \in [0, 2\pi]$. Then we have

$$A(\theta) - A^{(N)}(\theta) = \sqrt{\frac{2}{\pi k}} e^{-i\frac{\pi}{4}} \sum_{n=-\infty}^{\infty} \frac{1}{H_n^{(1)}(\kappa)} \left(\frac{f_n}{g_n} - \frac{F_n^{(N)}}{G_n^{(N)}} \right) g_n e^{in(\theta - \frac{\pi}{2})}.$$

Using Proposition 22, we have

$$\begin{aligned} |A(\theta) - A^{(N)}(\theta)| &\le \sqrt{\frac{2}{\pi k}} \sum_{n=-\infty}^{\infty} \left| \frac{1}{H_n^{(1)}(\kappa)} \left(\frac{f_n}{g_n} - \frac{F_n^{(N)}}{G_n^{(N)}} \right) g_n \right| \\ &\le \sqrt{\frac{2}{\pi k}} M_\kappa \sum_{n=-\infty}^{\infty} \left| \left(\frac{f_n}{g_n} - \frac{F_n^{(N)}}{G_n^{(N)}} \right) g_n \right|. \end{aligned}$$

As was shown in the proof of the main theorem of [2], there is a positive integer N_0 such that

$$\sum_{n=-\infty}^{\infty} \left| \left(\frac{f_n}{g_n} - \frac{F_n^{(N)}}{G_n^{(N)}} \right) g_n \right| < \frac{900 |||q|||}{\pi(1-\gamma)} \frac{\gamma^{N/2}}{N} \quad \text{for } N \ge N_0.$$

Hence the following inequality holds uniformly with respect to $\theta \in [0, 2\pi]$:

$$|A(\theta) - A^{(N)}(\theta)| < \sqrt{\frac{2}{\pi k}} M_\kappa \frac{900 |||q|||}{\pi(1-\gamma)} \frac{\gamma^{N/2}}{N} \quad \text{for } N \ge N_0.$$

The positive integer N_0 depends on κ and γ, but does not depend on Dirichlet data f. $\square$

11. Numerical tests

11.1. Numerical error estimate

11.1.1. Dirichlet data

We employ the following Dirichlet boundary data f on Γ_a:

$$f = e^{i\kappa \cos\theta} \quad \text{with } \theta \in [0, 2\pi].$$

Letting $t = ie^{i\theta}$ in the formula (1) on p. 14 of Watson [8], we have

$$f = \sum_{n=-\infty}^{\infty} i^n J_n(\kappa) e^{in\theta}.$$

Let $f_n = i^n J_n(\kappa)$. Due to the formula (8) and Lemma 13, we have the following asymptotic behavior of $|f_n|$ as $n \to \pm\infty$:

$$|f_n| = \left| \frac{f_n}{g_n} \right| |g_n| = \left| \frac{J_n(\kappa)}{H_n^{(1)}(\kappa) J_n(\gamma\kappa)} \right| |g_n| \sim \sqrt{\frac{\pi |n|}{2}} \left(\frac{e\kappa}{2\gamma |n|} \right)^{|n|} |g_n|,$$

where we use Lemma 12. Under Assumption 2 there is a positive constant C such that

$$|f_n| \le C |g_n| \quad \text{for } n \in \mathbb{Z}.$$

Thus Assumption 5 holds, and Theorem 11 can be applied to the problem.

11.1.2. Numerical estimator of the error

Due to (9)–(11), Q_j can be computed using a fast Fourier transform. And due to (13), a value of the approximate scattering amplitude $A^{(N)}(\theta)$ is computed with Q_j at every evaluation point.

Let N be a number of collocation points, and NN a number of evaluation points. If it is easy to compute correct values of the scattering amplitude $A(\theta)$, the following definition of the numerical estimator of the error is reasonable:

$$E^{(NN)}(N) = \max_{0 \le j \le NN-1} \left| A(\tilde{\theta}_j) - A^{(N)}(\tilde{\theta}_j) \right| \quad \text{with } \tilde{\theta}_j = \frac{2\pi j}{NN},$$

where $\tilde{\theta}_j$ is an evaluation point on $[0, 2\pi]$. But, in this case, computation of $A(\theta)$ is difficult.

The approximate amplitude $A^{(2N)}(\tilde{\theta}_j)$ is expected to be close to $A(\tilde{\theta}_j)$ for sufficiently large N. Then we employ the following estimator:

$$\widetilde{E^{(NN)}}(N) = \max_{0\le j\le NN-1} \left|A^{(2N)}(\tilde{\theta}_j) - A^{(N)}(\tilde{\theta}_j)\right| \quad \text{with } \tilde{\theta}_j = \frac{2\pi j}{NN}.$$

The following estimate holds:

$$\begin{aligned}\widetilde{E^{(NN)}}(N) &= \max_{0\le j\le NN-1} \left|A^{(2N)}(\tilde{\theta}_j) - A(\tilde{\theta}_j) + A(\tilde{\theta}_j) - A^{(N)}(\tilde{\theta}_j)\right| \\ &\le E^{(NN)}(2N) + E^{(NN)}(N).\end{aligned}$$

For sufficiently large N, we have

$$E^{(NN)}(2N) \le E^{(NN)}(N).$$

The following inequality holds:

$$\widetilde{E^{(NN)}}(N) \le 2E^{(NN)}(N).$$

Therefore $\widetilde{E^{(NN)}}(N)$ is expected to behave like $E^{(NN)}(N)$ for sufficiently large N.

11.1.3. Results of the computation

The results of the computation are given as two columns of graphs in Fig. 2. The left column corresponds to 30 decimal digits arithmetic, and the right one to 3200 decimal digits arithmetic, respectively. In each column, four graphs correspond to $\kappa = ka = 1, 10, 100$ and 1000, in descending order. In each graph, five polygonal lines correspond to $\gamma = 0.1, 0.3, 0.5, 0.7$ and 0.9. And, the abscissa axis shows the number of collocation points, N, and the ordinate axis shows the common logarithm of errors, $\log_{10} \widetilde{E^{(NN)}}(N)$. Other values of parameters are listed in Table 1.

11.1.4. Observation of results

For $\kappa = 1$ and 10 the estimator decays exponentially when 3200 decimal digits arithmetic is employed. But for $\kappa = 100$ and 1000 the behavior of the estimator with the same precision does not reflect exponential decay of errors completely.

In the case of $\kappa = 1000$, the estimator decreases hardly at all for $N \le 1024$ if the number of decimal digits of arithmetic is insufficient. This tendency is remarkable for small γ.

However, in the case of $\kappa = 1000$, the accuracy within 10 decimal digits is guaranteed when 30 decimal digits arithmetic is employed for $\gamma = 0.9$ with $N = 4096$ or more.

11.2. Applications of the approximate formula

11.2.1. Scattering cross section

For an incident wave $u_i(\boldsymbol{r})$, the scattering wave $u(\boldsymbol{r})$ is obtained as the solution of ($\mathrm{E_f}$) with $f = -u_i(\boldsymbol{a})$. Then the scattering cross section $\sigma(\theta)$ of $u(\boldsymbol{r})$ is defined through

$$\sigma(\theta) = \lim_{r\to\infty} 2\pi r \left|\frac{u(\boldsymbol{r})}{u_i(\boldsymbol{r})}\right|^2.$$

((I.34) on p. 7 of Bowman, Senior and Uslenghi [7].)

We introduce an assumption as follows.

Assumption 29. $\lim_{r\to\infty} |u_i(\boldsymbol{r})| = 1$.

Due to Theorem 8, the scattering wave behaves as (3) in the far field. Under Assumption 29, $\sigma(\theta)$ is represented by

$$\sigma(\theta) = 2\pi |A(\theta)|^2. \tag{45}$$

Then we introduce an approximate formula for the scattering cross section as follows.

Definition 30. Under Assumption 29, an approximate scattering cross section $\sigma^{(N)}(\theta)$ is defined through

$$\sigma^{(N)}(\theta) = 2\pi |A^{(N)}(\theta)|^2. \tag{46}$$

Table 1
Parameters for the error estimate.

	k	γ	$N = 2^n$	NN	Digit
Left column	1, 10, 100, 1000	$0.1 \le \gamma \le 0.9$	$1 \le n \le 10$	4 096	30
Left column	100, 1000	$0.1 \le \gamma \le 0.9$	$11 \le n \le 13$	32 768	30
Right column	1, 10, 100, 1000	$0.1 \le \gamma \le 0.9$	$1 \le n \le 10$	4 096	3200
Right column	100, 1000	$0.1 \le \gamma \le 0.9$	$11 \le n \le 13$	32 768	3200

Table 2
Parameters for computation.

	a	k	γ	N	NN	Θ	NN_1	NN_2	Digit
A: $\kappa = 10$	1	10	0.9	1 024	1024	–	–	–	78
B: $\kappa = 100$	1	100	0.9	4 096	–	$\pi/4$	4 096	1024	78
C: $\kappa = 1000$	1	1000	0.9	16 384	–	$\pi/12$	16 384	1024	78

Table 3
Parameters for computation.

	a	k	κ	γ	N	NN	Precision
$\lvert P(\theta)\rvert$	1	10	10	0.9	1024	1024	Double

11.2.2. Profiles of scattering cross sections

Consider the following incident wave: $u_i(\boldsymbol{r}) = \mathrm{e}^{\mathrm{i}kr\cos\theta}$. Then we have $f = -\mathrm{e}^{\mathrm{i}\kappa\cos\theta}$ for the Dirichlet data.

As was shown in Section 11.1.1, the density norm $|||q|||$ of f is finite, if the kernel function g satisfies Assumption 5.

We employ $\kappa = 10$, 100 and 1000. If $\kappa = 10$, computation points θ_j are given through: $\theta_j = 2\pi j/NN$ with $0 \le j \le NN - 1$.

In the cases of $\kappa = 100$ and 1000, we choose positive constants Θ such that $\Theta < \pi$. Let NN_1 and NN_2 be positive integers. Then computation points θ_l and θ_m are given as follows:

$$\theta_l = -\frac{\Theta}{2} + \frac{\Theta \times l}{NN_1} \qquad \text{for } 0 \le l \le NN_1 - 1,$$
$$\theta_m = \frac{\Theta}{2} + \frac{(2\pi - \Theta) \times m}{NN_2} \qquad \text{for } 0 \le m \le NN_2 - 1,$$

where $-\Theta/2 \le \theta_l < \Theta/2$, and $\Theta/2 \le \theta_m < 2\pi - \Theta/2$.

By the observation in Section 11.1.4, parameters are listed in Table 2. Computation is done in about 78 decimal digits precision (256 bits precision [10]).

Profiles of scattering cross sections are shown as graphs in Fig. 3. The areas A, B and C correspond to $\kappa = 10$, 100 and 1000, respectively. And the area D shows a composite of the left sides of the three areas.

When κ increases, the profile comes to a thing like a cardioid in the neighborhood of the origin, and the bundle of lines becomes narrow and dense at the dimple.

11.2.3. Profile of the far-field coefficient

The far-field coefficient $P(\theta)$ of a scattering wave $u(\boldsymbol{r})$ is defined through

$$u(\boldsymbol{r}) \sim P(\theta)\sqrt{\frac{2}{\pi kr}}\mathrm{e}^{\mathrm{i}\left(kr - \frac{\pi}{4}\right)} \quad \text{as } r \to \infty. \tag{47}$$

((I.29) on p. 6 of Bowman, Senior and Uslenghi [7].)

The formulae (3) and (47) yield

$$P(\theta) = \sqrt{\frac{\pi k}{2}}\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}A(\theta). \tag{48}$$

Let an incident wave $u_i(\boldsymbol{r}) = \mathrm{e}^{-\mathrm{i}kx}$. Then we have $f = -\mathrm{e}^{-\mathrm{i}\kappa\cos\theta}$ for the Dirichlet data.

We compute $P(\theta)$ using the approximate formula. Let NN be a number of computation points. The computation points are defined through: $\theta_j = 2\pi j/NN$ with $0 \le j \le NN - 1$. Parameters for computation are listed in Table 3. We employ double-precision arithmetic.

The profile of $|P(\theta)|$ with $\kappa = ka = 10$ is shown in Fig. 1. The profile corresponds to the profile (c) in Figure 2.4 on p. 96 of Bowman, Senior and Uslenghi [7]. Our computation reproduces the profile (c).

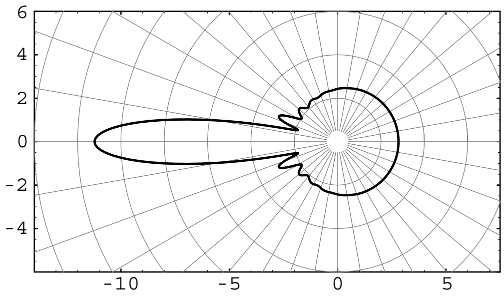

Fig. 1. Profile of $|P(\theta)|$ with $\kappa = ka = 10$.

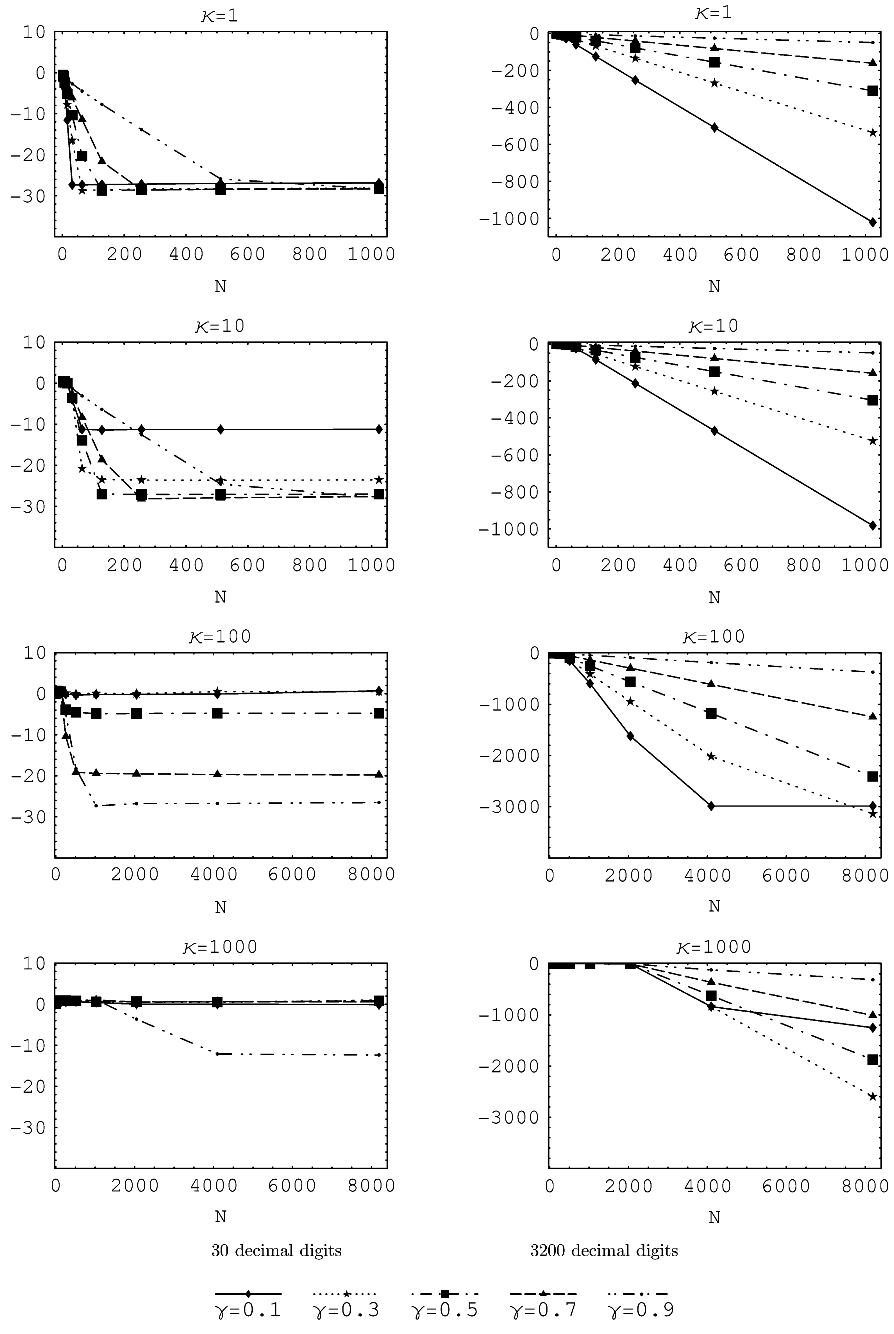

Fig. 2. Behavior of errors with a common logarithmic scaling ordinate.

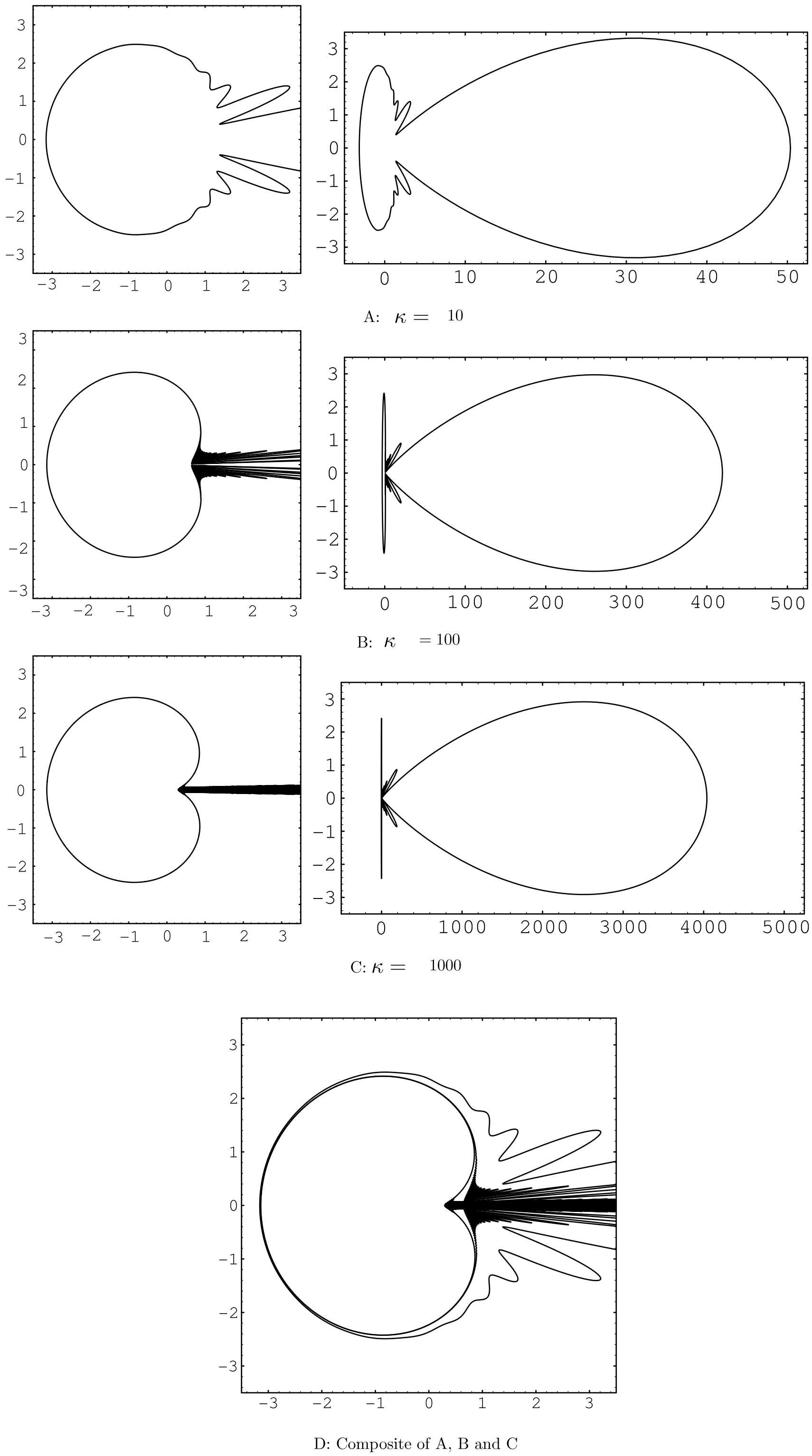

Fig. 3. Scattering cross sections.

11.3. Software for computation and visualization

Library programs MPFR [10] and GMP [11] are employed for multiple-precision computation. MPFR is a library for arbitrary precision floating point arithmetic and it is based on GMP. We owe our implementation of routines of multiple precision for cylindrical functions and for the fast Fourier transform to Ooura [12] and Brigham [13], respectively. Octave [14] is employed for the computation in double precision. For visualization of numerical data, we use Mathematica [15].

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Appendix. Proof of Proposition 23

A.1. Preparation

Lemma 31.

$$\frac{1}{4}|z| < |e^z - 1| < \frac{7}{4}|z| \quad \text{for } z \in \mathbb{C} \text{ with } 0 < |z| < 1. \tag{A.1}$$

*(*4.2.38 *on p.* 70 *of Abramowitz and Stegun* [9].*)*

Lemma 32.

$$(1+x)^{\frac{1}{2}} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128} + \cdots \quad \text{for } -1 < x < 1. \tag{A.2}$$

*(*3.6.11 *on p.* 15 *of Abramowitz and Stegun* [9].*)*

Proposition 33.

$$0 \le 1 + \frac{1}{2}x - (1+x)^{\frac{1}{2}} \le \frac{x^2}{8(1-|x|)} \quad \text{for } -1 < x < 1. \tag{A.3}$$

Proof. For $x \in \mathbb{R}$ with $|x| < 1$, we have

$$\left(1 + \frac{1}{2}x\right)^2 - (1+x) = 1 + x + \frac{1}{4}x^2 - 1 - x = \frac{1}{4}x^2 \ge 0. \tag{A.4}$$

Lemma 32 and (A.4) yield

$$0 \le 1 + \frac{1}{2}x - (1+x)^{\frac{1}{2}} \le \frac{x^2}{8}\sum_{n=0}^{\infty}|x|^n = \frac{x^2}{8(1-|x|)}. \quad \square$$

Let us introduce a symbol $D(r, \theta)$ through

$$D(r,\theta) = |kr - kb e^{-i\theta}|. \tag{A.5}$$

Proposition 34. *There is a positive number $\tilde{r}_1$ such that*

$$|D(r,\theta) - kr - kb\cos\theta| < \frac{11kb^2}{4r} \quad \text{for } r > \tilde{r}_1. \tag{A.6}$$

Proof. We have

$$D(r,\theta) = |kr - kbe^{-i\theta}| = kr\sqrt{1 - \frac{2b\cos\theta}{r} + \frac{b^2}{r^2}}. \tag{A.7}$$

A symbol ϵ is introduced by

$$\epsilon = -\frac{2b}{r}\cos\theta + \frac{b^2}{r^2}. \tag{A.8}$$

Due to (A.7) and (A.8), the following equalities hold:

$$\begin{aligned} D(r,\theta) - kr + kb\cos\theta &= kr(1+\epsilon)^{\frac{1}{2}} - kr - \frac{1}{2}kr\left(-\frac{2b}{r}\cos\theta + \frac{b^2}{r^2}\right) + \frac{kb^2}{2r} \\ &= \frac{kb^2}{2r} + kr\left((1+\epsilon)^{\frac{1}{2}} - 1 - \frac{1}{2}\epsilon\right). \end{aligned}$$

Choose a positive number $\tilde{r}_0$ such that $|\epsilon| < 1/2$ for $r > \tilde{r}_0$. Using Proposition 33, we have

$$\begin{aligned} |D(r,\theta) - kr + kb\cos\theta| &\le \frac{kb^2}{2r} + kr \times \frac{\epsilon^2}{8(1-|\epsilon|)} \\ &< \frac{kb^2}{2r} + kr \times \frac{\epsilon^2}{4} \quad \text{for } r > \tilde{r}_0, \end{aligned} \tag{A.9}$$

where $1/(1-|\epsilon|) < 2$.

Choose $\tilde{r}_1 > \tilde{r}_0$ such that $b^2/r^2 < b/r$ for $r > \tilde{r}_1$. Then we have

$$|\epsilon| = \left|-\frac{2b}{r}\cos\theta + \frac{b^2}{r^2}\right| \le \frac{2b}{r} + \frac{b^2}{r^2} < \frac{3b}{r} \quad \text{for } r > \tilde{r}_1. \tag{A.10}$$

Using (A.9) and (A.10), we have

$$|kr - kb\cos\theta - D(r,\theta)| < \frac{kb^2}{2r} + \frac{kr}{4} \times \left(\frac{3b}{r}\right)^2 = \frac{11kb^2}{4r} \quad \text{for } r > \tilde{r}_1. \quad \square$$

A.2. Proof of Proposition 23

A.2.1. Step 1

Consider the telescoping

$$\begin{aligned} \left|\frac{H_0^{(1)}(|kr - kbe^{-i\theta}|)}{\sqrt{\frac{2}{\pi kr}}e^{i\left(kr - kb\cos\theta - \frac{\pi}{4}\right)}} - 1\right| &\le \left|\frac{H_0^{(1)}(D(r,\theta))}{\sqrt{\frac{2}{\pi kr}}e^{i\left(kr - kb\cos\theta - \frac{\pi}{4}\right)}} - \frac{H_0^{(1)}(D(r,\theta))}{\sqrt{\frac{2}{\pi D(r,\theta)}}e^{i\left(D(r,\theta) - \frac{\pi}{4}\right)}}\right| + \left|\frac{H_0^{(1)}(D(r,\theta))}{\sqrt{\frac{2}{\pi D(r,\theta)}}e^{i\left(D(r,\theta) - \frac{\pi}{4}\right)}} - 1\right| \\ &= |H_0^{(1)}(D(r,\theta))|\left|\frac{1}{\sqrt{\frac{2}{\pi kr}}e^{i\left(kr - kb\cos\theta - \frac{\pi}{4}\right)}} - \frac{1}{\sqrt{\frac{2}{\pi D(r,\theta)}}e^{i\left(D(r,\theta) - \frac{\pi}{4}\right)}}\right| \\ &\quad + \left|\frac{H_0^{(1)}(D(r,\theta))}{\sqrt{\frac{2}{\pi D(r,\theta)}}e^{i\left(D(r,\theta) - \frac{\pi}{4}\right)}} - 1\right|. \end{aligned} \tag{A.11}$$

Employ Lemma 14 as $n = 0$. Then there are positive constants C_0 and x_0 such that

$$\left|\frac{H_0^{(1)}(x)}{\sqrt{\frac{2}{\pi x}}e^{i\left(x - \frac{\pi}{4}\right)}} - 1\right| \le \frac{C_0}{x} \quad \text{for } x > x_0.$$

Choose a positive number $r' > b$ such that

$$D(r,\theta) = |kr - kbe^{-i\theta}| \ge k(r-b) > x_0 \quad \text{for } r > r'. \tag{A.12}$$

Thus we have

$$\left|\frac{H_0^{(1)}(D(r,\theta))}{\sqrt{\frac{2}{\pi D(r,\theta)}}e^{i\left(D(r,\theta) - \frac{\pi}{4}\right)}} - 1\right| \le \frac{C_0}{D(r,\theta)} \le \frac{C_0}{k(r-b)} \quad \text{for } r > r'. \tag{A.13}$$

The inequalities (A.12) and (A.13) yield

$$|H_0^{(1)}(D(r,\theta))| \le \sqrt{\frac{2}{\pi D(r,\theta)}}\left(1 + \frac{C_0}{k(r-b)}\right) \le \sqrt{\frac{2}{\pi k(r-b)}}\left(1 + \frac{C_0}{k(r-b)}\right). \tag{A.14}$$

The formulae (A.11), (A.13) and (A.14) yield

$$\left|\frac{H_0^{(1)}(|kr-kbe^{-i\theta}|)}{\sqrt{\frac{2}{\pi kr}}e^{i(kr-kb\cos\theta-\frac{\pi}{4})}}-1\right| \le \sqrt{\frac{2}{\pi k(r-b)}}\times\left(1+\frac{C_0}{k(r-b)}\right) \times\left|\frac{1}{\sqrt{\frac{2}{\pi kr}}e^{i\left(kr-kb\cos\theta-\frac{\pi}{4}\right)}}-\frac{1}{\sqrt{\frac{2}{\pi D(r,\theta)}}e^{i\left(D(r,\theta)-\frac{\pi}{4}\right)}}\right|+\frac{C_0}{k(r-b)}. \tag{A.15}$$

Consider the first term on the right hand side of (A.15). We have

$$\left|\frac{1}{\sqrt{\frac{2}{\pi kr}}e^{i\left(kr-kb\cos\theta-\frac{\pi}{4}\right)}}-\frac{1}{\sqrt{\frac{2}{\pi D(r,\theta)}}e^{i\left(D(r,\theta)-\frac{\pi}{4}\right)}}\right| \le \left|\frac{1}{\sqrt{\frac{2}{\pi kr}}e^{i\left(kr-kb\cos\theta-\frac{\pi}{4}\right)}}-\frac{1}{\sqrt{\frac{2}{\pi kr}}e^{i\left(D(r,\theta)-\frac{\pi}{4}\right)}}\right| + \left|\frac{1}{\sqrt{\frac{2}{\pi kr}}e^{i\left(D(r,\theta)-\frac{\pi}{4}\right)}}-\frac{1}{\sqrt{\frac{2}{\pi D(r,\theta)}}e^{i\left(D(r,\theta)-\frac{\pi}{4}\right)}}\right|. \tag{A.16}$$

We introduce A and B through

$$A=\left|\frac{1}{\sqrt{\frac{2}{\pi kr}}e^{i\left(kr-kb\cos\theta-\frac{\pi}{4}\right)}}-\frac{1}{\sqrt{\frac{2}{\pi kr}}e^{i\left(D(r,\theta)-\frac{\pi}{4}\right)}}\right|, \tag{A.17}$$

$$B=\left|\frac{1}{\sqrt{\frac{2}{\pi kr}}e^{i\left(D(r,\theta)-\frac{\pi}{4}\right)}}-\frac{1}{\sqrt{\frac{2}{\pi D(r,\theta)}}e^{i\left(D(r,\theta)-\frac{\pi}{4}\right)}}\right|. \tag{A.18}$$

Due to (A.15)–(A.18), the following inequality holds:

$$\left|\frac{H_0^{(1)}(|kr-kbe^{-i\theta}|)}{\sqrt{\frac{2}{\pi kr}}e^{i\left(kr-kb\cos\theta-\frac{\pi}{4}\right)}}-1\right| \le \sqrt{\frac{2}{\pi k(r-b)}}\left(1+\frac{C_0}{k(r-b)}\right)(A+B)+\frac{C_0}{k(r-b)}. \tag{A.19}$$

We estimate A and B in Appendices A.2.2 and A.2.3, respectively. And the proof of Proposition 23 is completed in Appendix A.2.4. □

A.2.2. Step 2: Estimation of A

Due to Proposition 34, there is a positive number $r_0 > r'$ such that

$$|D(r,\theta)-kr-kb\cos\theta| < \frac{11kb^2}{4r} < 1 \quad \text{for } r > r_0.$$

Then Lemma 31 yields

$$\begin{aligned} A &= \sqrt{\frac{\pi kr}{2}}|e^{i(D(r,\theta)-kr+kb\cos\theta)}-1| \\ &< \frac{7}{4}\sqrt{\frac{\pi kr}{2}}|D(r,\theta)-kr+kb\cos\theta| \\ &< \frac{7}{4}\sqrt{\frac{\pi kr}{2}}\times\frac{11kb^2}{4r}=\frac{77}{16}\sqrt{\frac{\pi k}{2r}}kb^2 \quad \text{for } r > r_0. \quad \square \end{aligned} \tag{A.20}$$

A.2.3. Step 3: Estimation of B

The term B is deformed through

$$\begin{aligned} B &= \left|\frac{1}{\sqrt{\frac{2}{\pi kr}}e^{i\left(D(r,\theta)-\frac{\pi}{4}\right)}}-\frac{1}{\sqrt{\frac{2}{\pi D(r,\theta)}}e^{i\left(D(r,\theta)-\frac{\pi}{4}\right)}}\right| \\ &= \sqrt{\frac{\pi}{2}}\left|\sqrt{kr}-\sqrt{|kr-kbe^{-i\theta}|}\right| = \sqrt{\frac{\pi}{2}}\frac{|kr-|kr-kbe^{-i\theta}||}{\sqrt{kr}+\sqrt{|kr-kbe^{-i\theta}|}}, \end{aligned} \tag{A.21}$$

where $D(r,\theta) = |kr - kb\mathrm{e}^{-\mathrm{i}\theta}|$. The following estimate holds:

$$kr - kb \le |kr - kb\mathrm{e}^{-\mathrm{i}\theta}| \le kr + kb.$$

Then we have

$$\big|kr - |kr - kb\mathrm{e}^{-\mathrm{i}\theta}|\big| \le kb. \tag{A.22}$$

The inequalities (A.21) and (A.22) yield

$$B \le \sqrt{\frac{\pi}{2}} \frac{kb}{\sqrt{kr} + \sqrt{|kr - kb\mathrm{e}^{-\mathrm{i}\theta}|}} < \sqrt{\frac{\pi}{2}} \frac{kb}{\sqrt{kr}} = \sqrt{\frac{\pi k}{2r}} b \quad \text{for } r > r_0. \quad \square \tag{A.23}$$

A.2.4. Step 4: Completion of proof

Combining (A.19), (A.20) and (A.23), we have

$$\begin{aligned}
\left| \frac{H_0^{(1)}(|kr - kb\mathrm{e}^{-\mathrm{i}\theta}|)}{\sqrt{\frac{2}{\pi kr}}\mathrm{e}^{\mathrm{i}\left(kr - kb\cos\theta - \frac{\pi}{4}\right)}} - 1 \right| &< \sqrt{\frac{2}{\pi k(r-b)}} \left(1 + \frac{C_0}{k(r-b)}\right) \left\{ \sqrt{\frac{\pi k}{2r}} \left(\frac{77}{16}kb^2 + b\right) \right\} + \frac{C_0}{k(r-b)} \\
&< \sqrt{\frac{2}{\pi k(r-b)}} \left(1 + \frac{C_0}{k(r-b)}\right) \left\{ \sqrt{\frac{\pi k}{2(r-b)}} \left(\frac{77}{16}kb^2 + b\right) \right\} + \frac{C_0}{k(r-b)} \\
&= \left[\left(1 + \frac{C_0}{k(r-b)}\right) \left(\frac{77}{16}kb^2 + b\right) + \frac{C_0}{k} \right] \frac{1}{r-b} \quad \text{for } r > r_0.
\end{aligned} \tag{A.24}$$

Define a positive constant C_0' through

$$\begin{aligned}
&\left[\left(1 + \frac{C_0}{k(r-b)}\right) \left(\frac{77}{16}kb^2 + b\right) + \frac{C_0}{k} \right] \frac{1}{r-b} = \left[\left(1 + \frac{C_0}{k(r-b)}\right) \left(\frac{77}{16}kb^2 + b\right) + \frac{C_0}{k} \right] \frac{1}{1 - \frac{b}{r}} \times \frac{1}{r} \\
&\quad < \left[\left(1 + \frac{C_0}{k(r_0-b)}\right) \left(\frac{77}{16}kb^2 + b\right) + \frac{C_0}{k} \right] \frac{1}{1 - \frac{b}{r_0}} \times \frac{1}{r} = C_0' \times \frac{1}{r} \quad \text{for } r > r_0.
\end{aligned}$$

Thus we have

$$\left| \frac{H_0^{(1)}(|kr - kb\mathrm{e}^{-\mathrm{i}\theta}|)}{\sqrt{\frac{2}{\pi kr}}\mathrm{e}^{\mathrm{i}\left(kr - kb\cos\theta - \frac{\pi}{4}\right)}} - 1 \right| < \frac{C_0'}{r} \quad \text{for } r > r_0.$$

The discussion above shows that this inequality holds uniformly with respect to $\theta \in [0, 2\pi]$. $\square$

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