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Extremal solutions for the first order impulsive functional differential equations with upper and lower solutions in reversed order

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ABSTRACT

This paper studies the existence of solutions of first order impulsive functional differential equations with lower and upper solutions in the reversed order, obtains the sufficient conditions for the existence of solutions by establishing a new comparison principle and using the monotone iterative technique. A concrete example is presented and solved to illustrate the obtained results.

1. Introduction

Impulsive differential equations arise naturally from a wide variety of applications, such as control theory, physics, chemistry, population dynamics, biotechnology, industrial robotic, optimal control, etc. [1–3]. Therefore, it is very important to develop a general theory for differential equations with impulses including some basic aspects of this theory.

We consider the first order impulsive functional differential equations with deviating arguments:

$$\begin{cases} u'(t) = f(t, u(t), u(\alpha(t))), \quad t \in J', \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ g(u(0), u(T)) = 0, \end{cases} \tag{1.1}$$

where $t \in J = [0, T](T > 0), f \in C(J \times R \times R, R), g \in C(R \times R, R), I_k \in C(R, R), \alpha \in C(J, J), 0 = t_0 < t_1 < \cdots < t_k < \cdots < t_m < t_{m+1} = T, J' = J \setminus \{t_1, t_2, \ldots, t_m\}$, $\Delta u(t_k) = u(t_k^+) - u(t_k^-)$, where $u(t_k^+)$ and $u(t_k^-)$ denote the right and the left limit of $u(t)$ at $t = t_k (k = 1, 2, \ldots, m)$, respectively. Let $PC(J, R) = \{u : J \to R \mid u(t)$ be continuous at $t \neq t_k$, left continuous at $t = t_k$ and $u(t_k^+)$ exist, $k = 1, 2, \ldots, m\}$ and $PC^1(J, R) = \{u \in PC(J, E) | u(t)$ be continuously differentiable at $t \neq t_k$, $u'(t_k^+)$ and $u'(t_k^-)$ exist, $k = 1, 2, \ldots, m\}$. Evidently, $PC(J, R)$ and $PC^1(J, R)$ are the Banach spaces with respective norms

$$\|u\|_{PC} = \sup_{t \in J} |u(t)|, \qquad \|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\}.$$

Definition 1.1. We say that $u \in PC^1(J, R)$ is a solution of (1.1), if it satisfies (1.1).

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Definition 1.2. We say that $u \in PC^1(J, R)$ is called a lower solution of (1.1) if

$$\begin{cases} u'(t) \le f(t, u(t), u(\alpha(t))), \quad t \in J', \\ \Delta u(t_k) \le I_k(u(t_k)), \quad k = 1, 2, \ldots, m, \\ g(u(0), u(T)) \le 0, \end{cases} \tag{1.2}$$

and it is an upper solution of (1.1) if the above inequalities are reversed.

The method of upper and lower solutions coupled with the monotone iterative technique has been applied successfully to obtain the existence of solutions for nonlinear differential equations in recent years (see [4–16]). It is worthwhile mentioning that the main theorems in the above papers are formulated and proved in the presence of a lower solution u_0 and an upper solution v_0 with $u_0 \le v_0$. But in many cases, the lower and upper solutions occur in the reversed order, that is $u_0 \ge v_0$. This is a fundamentally different situation. However, only a few works discuss the existence results for the non-ordered case [17–22]. In this paper, we have considered boundary value problems for the first order impulsive functional differential equations with nonlinear boundary conditions and deviating arguments under the assumption of the existing upper and lower solutions in the reversed order.

Remark 1.1. Note that the nonlinear impulsive boundary value problems (1.1) reduce to periodic boundary value problems for $g(u(0), u(T)) = u(0) - u(T)$, anti-periodic boundary value problems for $g(u(0), u(T)) = u(0) + u(T)$ and nonlinear boundary value problems without impulse [21] for $g(u(0), u(T)) = h(u(0)) - u(T)$. Thus, problems (1.1) can be regarded as a generalization of the boundary value problems mentioned above.

Remark 1.2. It is important to indicate that, compared with other methods, the method of upper and lower solutions coupled with its associated monotone iteration scheme is an interesting and powerful mechanism that offers the theoretical as well as constructive existence results for nonlinear problems in a closed set, generated by the lower and upper solutions. Usually other methods need to satisfy a two-sided Lipschitz condition, but with the methods mentioned above, it is just needed to satisfy a one-sided Lipschitz condition; for instance, see [4–12].

Remark 1.3. In this paper, we apply the lower and upper solutions in reversed order, which is fundamentally different from the classical lower and upper solutions used in [4–16].

Remark 1.4. If (1.1) does not contain impulsive arguments $I_k(u(t_k))$, and let $g(u(0), u(T)) = h(u(0)) - u(T)$, then, as a particular case of problem (1.1), in 2009, by using the method of upper and lower solutions and a monotone iterative technique, Wang, Yang and Shen [21] have considered the existence of extreme solutions of the following functional differential equations without impulse

$$\begin{cases} u'(t) = f(t, u(t), u(\alpha(t))), \quad t \in J, \\ h(u(0)) = u(T), \end{cases}$$

where $t \in J = [0, T](T > 0), f \in C(J \times R \times R, R), \alpha \in C(J, J)$. They demand that the nonlinear term f satisfies a one-sided Lipschitz condition with corresponding constants and the nonlinear term h satisfies strongly restricted conditions $h \in C^1(R, R), h(0) \le 0$ and $0 < h'(t) < \left(1 + N \int_0^T \mathrm{e}^{M(\alpha(t)-t)}\mathrm{d}t\right) \mathrm{e}^{MT}$. In this paper, we will delete the strongly restricted condition for the nonlinear boundary value condition and extend their constant coefficients on a one-sided Lipschitz condition to functional coefficients.

The paper is organized as follows. In Section 2, we establish a new comparison principle and discuss the uniqueness of the solutions to linear impulsive differential equations. In Section 3, the main theorem is formulated and proved. In Section 4, we give an example about boundary value problems for impulsive functional differential equations (1.1).

2. Several lemmas

Lemma 2.1. *Suppose that* $u \in PC^1(J, R)$ *satisfies*

$$\begin{cases} u'(t) \ge M(t)u(t) + N(t)u(\alpha(t)), \quad t \in J', \\ \Delta u(t_k) \ge L_k u(t_k), \quad k = 1, 2, \ldots, m, \\ u(0) \ge ru(T), \end{cases} \tag{2.1}$$

where $M, N \in C(J, [0, +\infty)), L_k \ge 0, r > 0$ *satisfy*

(i) $r\left[1 + \int_0^T N(s)\mathrm{e}^{\int_s^{\alpha(s)} M(\tau)\mathrm{d}\tau}\mathrm{d}s + \sum_{k=1}^m L_k\right]\mathrm{e}^{\int_0^T M(\tau)\mathrm{d}\tau} > 1,$

(ii) $H \le \frac{1}{r+1}$, *here* $H \equiv \int_0^T [M(t) + N(t)]\mathrm{d}t + \sum_{k=1}^m L_k$.

Then $u(t) \le 0, t \in J$.

Proof. Suppose that $u(t) \le 0,\ t \in J$ is not true, then, we have the following two possible cases:

(1) $u(t) \ge 0, \forall t \in J$.
(2) There exist $t^*, t_* \in J$ such that $u(t^*) > 0$ and $u(t_*) < 0$.

Case (1). Let $v(t) = u(t)e^{-\int_0^t M(\tau)d\tau}$. Then we have

$$\begin{cases} v'(t) \ge N(t)v(\alpha(t))e^{\int_t^{\alpha(t)} M(\tau)d\tau}, & t \in J', \\ \Delta v(t_k) \ge L_k v(t_k), & k = 1, 2, \ldots, m, \\ v(0) \ge rv(T)e^{\int_0^T M(\tau)d\tau}. \end{cases} \tag{2.2}$$

By (2.2), we know that $v(t)$ is nondecreasing on J. So, we have

$$\begin{aligned} v(t) &= v(0) + \int_0^t v'(r)dr + \sum_{0<t_k<t} [v(t_k^+) - v(t_k)] \\ &\ge v(0) + \int_0^t N(s)v(\alpha(s))e^{\int_s^{\alpha(s)} M(\tau)d\tau} ds + \sum_{0<t_k<t} L_k v(t_k) \\ &\ge v(0) + v(0)\int_0^t N(s)e^{\int_s^{\alpha(s)} M(\tau)d\tau} ds + v(0)\sum_{0<t_k<t} L_k \\ &= \left[1 + \int_0^T N(s)e^{\int_s^{\alpha(s)} M(\tau)d\tau} ds + \sum_{0<t_k<t} L_k\right] v(0). \end{aligned}$$

Thus,

$$v(0) \ge rv(T)e^{\int_0^T M(\tau)d\tau} \ge r\left[1 + \int_0^T N(s)e^{\int_s^{\alpha(s)} M(\tau)d\tau} ds + \sum_{0<t_k<T} L_k\right] v(0)e^{\int_0^T M(\tau)d\tau}.$$

By condition (i), we have $v(0) = 0$. In addition, $rv(T)e^{\int_0^T M(\tau)d\tau} \le v(0) = 0$ implies $v(T) \le 0$. Since $v(t)$ is nondecreasing on J, then we have $v(t) \equiv 0, \forall t \in J$, i.e. $u(t) \equiv 0, \forall t \in J$.

Case (2). Let $\inf_{t\in J} u(t) = -\lambda$, then $\lambda > 0$, and for some $i \in \{1, 2, \ldots, m\}$, there exists a $t_* \in (t_i, t_{i+1}]$, such that $u(t_*) = -\lambda$ or $u(t_i^+) = -\lambda$. We only consider $u(t_*) = -\lambda$, for the case $u(t_i^+) = -\lambda$, the proof is similar.

By (2.1), we have

$$\begin{aligned} u(t) &= u(0) + \int_0^t u'(s)ds + \sum_{0<t_k<t} [u(t_k^+) - u(t_k)] \\ &\ge u(0) - \lambda\left\{\int_0^t [M(s) + N(s)]ds + \sum_{0<t_k<t} L_k\right\}. \end{aligned} \tag{2.3}$$

Let $t = t_*$ in (2.3), we have

$$\begin{aligned} -\lambda &\ge u(0) - \lambda\left\{\int_0^{t_*} [M(s) + N(s)]ds + \sum_{0<t_k<t_*} L_k\right\} \\ &\ge u(0) - \lambda\left\{\int_0^T [M(s) + N(s)]ds + \sum_{0<t_k<T} L_k\right\}. \end{aligned}$$

Thus,

$$u(0) \le -\lambda + \lambda H. \tag{2.4}$$

On the other hand,

$$u(t) = u(T) - \int_t^T u'(s)ds - \sum_{t\le t_k<T} [u(t_k^+) - u(t_k)]. \tag{2.5}$$

Let $t = t^*$ in (2.5), then

$$0 < u(t^*) = u(T) - \int_{t^*}^T u'(s)ds - \sum_{t^*\le t_k<T} [u(t_k^+) - u(t_k)].$$

So,

$$\begin{aligned} u(T) &> \int_{t^*}^{T} u'(s)\mathrm{d}s + \sum_{t^* \le t_k < T} [u(t_k^+) - u(t_k)] \\ &\ge -\lambda \int_0^T [M(s) + N(s)]\mathrm{d}s - \lambda \sum_{0<t_k<T} L_k \\ &= -\lambda H. \end{aligned} \tag{2.6}$$

By (2.1), (2.4) and (2.6), we have

$$-\lambda + \lambda H \ge u(0) \ge ru(T) > -r\lambda H.$$

So, $H > \frac{1}{r+1}$, which contradicts condition (ii). Hence, $u(t) \le 0$ on J. □

Corollary 2.1. *Assume that $M, N \in C(J, [0, +\infty))$, $\int_0^T M(t)\mathrm{d}t > 0$, $L_k \ge 0$, $r \ge 1$ and condition (ii) in Lemma 2.1 hold. Let $u \in PC^1(J, R)$ satisfy (2.1). Then $u(t) \le 0$, $t \in J$.*

The proof of Corollary 2.1 is easy, so we omit it. □

Remark 2.1. Corollary 2.1 holds for $r > 1$ if we delete $\int_0^T M(t)\mathrm{d}t > 0$.

Consider the problem:

$$\begin{cases} u'(t) = \sigma(t) + M(t)u(t) + N(t)u(\alpha(t)), & t \in J', \\ \Delta u(t_k) = \gamma_k + L_k u(t_k), & k = 1, 2, \ldots, m, \\ u(0) = ru(T) + a, \end{cases} \tag{2.7}$$

where $\sigma \in PC(J, R)$, $\gamma_k, a \in R$.

Definition 2.1. We say that $u \in PC^1(J, R)$ is a solution of (2.7), if it satisfies (2.7).

Definition 2.2. We say that $u \in PC^1(J, R)$ is called a lower solution of (2.7) if

$$\begin{cases} u'(t) \le \sigma(t) + M(t)u(t) + N(t)u(\alpha(t)), & t \in J', \\ \Delta u(t_k) \le \gamma_k + L_k u(t_k), & k = 1, 2, \ldots, m, \\ u(0) \le ru(T) + a, \end{cases}$$

and it is an upper solution of (2.7) if the above inequalities are reversed.

Lemma 2.2. *Let all assumptions of Lemma 2.1 hold. In addition assume that $u_0, v_0 \in PC^1(J, R)$ are lower and upper solutions of (2.7), respectively, and $v_0(t) \le u_0(t)$, $\forall t \in J$. Then problem (2.7) has a unique solution $w \in PC^1(J, R)$.*

Proof. Firstly, we show that (2.7) has a solution through three steps.

Step 1. Consider the equation

$$\begin{cases} u'(t) = \sigma(t) + M(t)u(t) + N(t)u(\alpha(t)), & t \in J', \\ \Delta u(t_k) = \gamma_k + L_k u(t_k), & k = 1, 2, \ldots, m, \\ u(T) = \lambda, \end{cases} \tag{2.8}$$

where $\lambda \in R$ we will show that (2.8) has a unique solution $u(t, \lambda)$ and $u(t, \lambda)$ is continuous in λ.

Eq. (2.8) is equivalent to the following equation:

$$u(t) = \lambda - \int_t^T [\sigma(s) + M(s)u(s) + N(s)u(\alpha(s))]\mathrm{d}s - \sum_{t \le t_k < T} [\gamma_k + L_k u(t_k)].$$

Let

$$Au(t) = \lambda - \int_t^T [\sigma(s) + M(s)u(s) + N(s)u(\alpha(s))]\mathrm{d}s - \sum_{t \le t_k < T} [\gamma_k + L_k u(t_k)].$$

Then $A : PC(J, R) \to PC(J, R)$, and for any $u, v \in PC(J, R)$, we have

$$\|Au - Av\|_{PC} \left[\int_0^T [M(t) + N(t)]\mathrm{d}t + \sum_{0<t_k<T} L_k \right] \|(u - v)\|_{PC} \equiv H\|(u - v)\|_{PC}.$$

By assumption (ii) in Lemma 2.1 and the Banach fixed point theorem, we get that (2.8) has a unique solution.

Let $u(t,\lambda_1)$, $u(t,\lambda_2)$ be the solution of

$$\begin{cases} u'(t)=\sigma(t)+M(t)u(t)+N(t)u(\alpha(t)), \quad t\in J', \\ \Delta u(t_k)=\gamma_k+L_k u(t_k), \quad k=1,2,\ldots,m, \\ u(T)=\lambda_i, \quad i=1,2, \end{cases}$$

then

$$u(t,\lambda_i)=\lambda_i-\int_t^T[\sigma(s)+M(s)u(s,\lambda_i)+N(s)u(\alpha(s),\lambda_i)]\mathrm{d}s-\sum_{t\le t_k<T}[\gamma_k+L_k u(t_k,\lambda_i)],$$

$$\max|u(t,\lambda_1)-u(t,\lambda_2)|\le\frac{1}{1-H}|\lambda_1-\lambda_2|.$$

Hence, $u(t,\lambda)$ is continuous in λ.

Step 2. We show that

$$v_0(0)\le u(0,\lambda)\le u_0(0), \tag{2.9}$$

where $\lambda\in\left[\frac{1}{r}(v_0(0)-a),\frac{1}{r}(u_0(0)-a)\right]$, $u(t,\lambda)$ is the unique solution of (2.8).

Let $p(t)=v_0(t)-u(t,\lambda)$. Assume that $u(0,\lambda)<v_0(0)$, then

$$\begin{cases} p(0)>0, P(T)=v_0(T)-u(T,\lambda)\le g(v_0(0))-u(T,\lambda)\le 0, \\ \Delta p(t_k)\ge L_k p(t_k), \quad k=1,2,\ldots,m, \\ p'(t)\ge M(t)p(t)+N(t)p(\alpha(t)), \quad t\in J'. \end{cases}$$

By Lemma 2.1, we can get $p(t)\le 0,\forall t\in J$ which contradicts $p(0)>0$. So, we have $v_0(0)\le u(0,\lambda)$. Let $q(t)=u(t,\lambda)-u_0(t)$. By a similar process as above, we can get $u(0,\lambda)\le u_0(0)$.

Step 3. Let $F(\lambda)=\frac{1}{r}[u(0,\lambda)-a]-\lambda$, where $u(t,\lambda)$ is the unique solution of (2.8). We have

$$F\left(\frac{1}{r}(v_0(0)-a)\right)\cdot F\left(\frac{1}{r}(u_0(0)-a)\right)\le 0.$$

Since function F is continuous in λ, then there exists a $\lambda_0\in\left[\frac{1}{r}(v_0(0)-a),\frac{1}{r}(u_0(0)-a)\right]$ such that $\frac{1}{r}[u(0,\lambda_0)-a]=\lambda_0$. Obviously, $u(t,\lambda_0)$ is a solution of (2.7).

Finally, we show that (2.7) has only a solution $w\in PC^1(J,R)$.

Let $u,v\in PC^1(J,R)$ be two different solutions of (2.7). Put $p=u-v$, then p satisfies the following problem:

$$\begin{cases} p'(t)=M(t)p(t)+N(t)p(\alpha(t)), \quad t\in J', \\ \Delta p(t_k)=L_k p(t_k), \quad k=1,2,\ldots,m, \\ p(0)=rp(T). \end{cases}$$

By Lemma 2.1, we have $p(t)\le 0,\forall t\in J$. Similarly, if put $p=v-u$, by Lemma 2.1, we have $p(t)\ge 0,\forall t\in J$. So, we have $p(t)=0,\forall t\in J$ i.e. $u(t)=v(t),\forall t\in J$. □

3. Main result

We list for convenience the following assumptions.

(H_1) $u_0,v_0\in PC^1(J,R)$ are lower and upper solutions of (1.1), respectively, and $v_0(t)\le u_0(t),\forall t\in J$.

(H_2) The functions f, $I_k (k=1,2,\ldots,m)$ satisfy

$$f(t,u,v)-f(t,\bar{u},\bar{v})\le M(t)(u-\bar{u})+N(t)(v-\bar{v}),$$
$$I_k(x)-I_k(y)\le L_k(x-y)$$

where $v_0(t)\le\bar{u}\le u\le u_0(t)$, $v_0(\alpha(t))\le\bar{v}\le v\le u_0(\alpha(t))$, $v_0(t_k)\le y\le x\le u_0(t_k)$, $\forall t\in J$.

(H_3) There exist positive constants K,L such that

$$g(u,v)-g(\bar{u},\bar{v})\ge K(u-\bar{u})-L(v-\bar{v}),$$

where $v_0(0)\le\bar{u}\le u\le u_0(0)$, $v_0(T)\le\bar{v}\le v\le u_0(T)$, $r=L/K$.

Theorem 3.1. *Let all assumptions of Lemma 2.1 and (H_1)–(H_3) hold. Then there exist monotone iterative sequences $\{u_n\}$, $\{v_n\}$, which converge uniformly on J to the extremal solutions of (1.1) in $[v_0,u_0]=\{u\in PC^1(J,R): v_0(t)\le u(t)\le u_0(t)\}$.*

Proof. For any $\eta \in [v_0, u_0]$, we consider the problem:

$$\begin{cases} u'(t) = \sigma_\eta(t) + M(t)u(t) + N(t)u(\alpha(t)), \quad t \in J', \\ \Delta u(t_k) = \gamma_k + L_k u(t_k), \quad k = 1, 2, \ldots, m, \\ u(0) = ru(T) + a, \end{cases} \tag{3.1}$$

where $\sigma_\eta(t) = f(t, \eta(t), \eta(\alpha(t))) - M(t)\eta(t) - N(t)\eta(\alpha(t))$, $\gamma_k = I_k(\eta(t_k)) - L_k\eta(t_k)$, $a = \eta(0) - r\eta(T) - \frac{1}{K}g(\eta(0), \eta(T))$.

By (H_1)–(H_3), we have

$$\begin{cases} u_0'(t) \le f(t, u_0(t), u_0(\alpha(t))) \\ \quad \le f(t, \eta(t), \eta(\alpha(t))) - M(t)\eta(t) - N(t)\eta(\alpha(t)) + M(t)u_0(t) + N(t)u_0(\alpha(t)) \\ \quad = \sigma_\eta(t) + M(t)u_0(t) + N(t)u_0(\alpha(t)), \quad t \in J', \\ \Delta u_0(t_k) \le I_k(u_0(t_k)) \le I_k(\eta(t_k)) - L_k\eta(t_k) + L_k u_0(t_k) = \gamma_k + L_k u_0(t_k), \quad k = 1, 2, \ldots, m, \\ u_0(0) \le \eta(0) + \dfrac{L}{K}(u_0(T) - \eta(T)) + \dfrac{1}{K}(g(u_0(0), u_0(T)) - g(\eta(0), \eta(T))) \\ \quad \le ru_0(T) + \eta(0) - r\eta(T) - \dfrac{1}{K}g(\eta(0), \eta(T)) \\ \quad = ru_0(T) + a, \end{cases}$$

and

$$\begin{cases} v_0'(t) \ge \sigma_\eta(t) + M(t)v_0(t) + N(t)v_0(\alpha(t)), \quad t \in J', \\ \Delta v_0(t_k) \ge \gamma_k + L_k v_0(t_k), \quad k = 1, 2, \ldots, m, \\ v_0(0) \ge rv_0(T) + a. \end{cases}$$

So, u_0, v_0 are lower and upper solutions of (3.1). By Lemma 2.2, we know that (3.1) has a unique solution $w \in PC^1(J, R)$. Now, we prove that $w \in [v_0, u_0]$. Since u_0, v_0 are lower and upper solutions of (3.1), let $p = w - u_0$, we can get

$$\begin{cases} p'(t) \ge M(t)p(t) + N(t)p(\alpha(t)), \quad t \in J', \\ \Delta p(t_k) \ge L_k p(t_k), \quad k = 1, 2, \ldots, m, \\ p(0) \ge rp(T). \end{cases}$$

By Lemma 2.1, we have that $p(t) \le 0$, $\forall t \in J$. That is, $w \le u_0$. Similarly, we can show that $v_0 \le w$. Therefore, we have $w \in [v_0, u_0]$.

Next, we denote an operator $A : [v_0, u_0] \to [v_0, u_0]$ by $u = A\eta$. Let $\eta_1, \eta_2 \in [v_0, u_0]$ such that $\eta_1 \le \eta_2$. Setting $p = u_1 - u_2$, $u_1 = A\eta_1$, $u_2 = A\eta_2$, by (H_2) and (H_3), we obtain

$$\begin{cases} p'(t) = f(t, \eta_1(t), \eta_1(\alpha(t))) - M(t)\eta_1(t) - N(t)\eta_1(\alpha(t)) + M(t)u_1(t) + N(t)u_1(\alpha(t)) \\ \qquad - f(t, \eta_2(t), \eta_2(\alpha(t))) + M(t)\eta_2(t) + N(t)\eta_2(\alpha(t)) - M(t)u_2(t) - N(t)u_2(\alpha(t)) \\ \quad \ge M(t)p(t) + N(t)p(\alpha(t)), \quad t \in J', \\ \Delta p(t_k) = I_k(\eta_1(t_k)) - L_k\eta_1(t_k) + L_k u_1(t_k) - I_k(\eta_2(t_k)) + L_k\eta_2(t_k) - L_k u_2(t_k) \\ \quad \ge L_k p(t_k), \quad k = 1, 2, \ldots, m, \\ p(0) = ru_1(T) + \eta_1(0) - r\eta_1(T) - \dfrac{1}{K}g(\eta_1(0), \eta_1(T)) - ru_2(T) \\ \qquad - \eta_2(0) + r\eta_2(T) + \dfrac{1}{K}g(\eta_2(0), \eta_2(T)) \\ \quad \ge rp(T) + \eta_1(0) - \eta_2(0) - r\eta_1(T) + r\eta_2(T) + (\eta_2(0) - \eta_1(0)) - r(\eta_2(T) - \eta_1(T)) \\ \quad = rp(T). \end{cases}$$

By Lemma 2.1, we know that $p(t) \le 0$ on J, i.e. A is nondecreasing.

Now, let $u_n = Au_{n-1}$, $v_n = Av_{n-1}$, $n = 1, 2, \ldots$, then we have

$$v_0 \le v_1 \le \cdots \le v_n \le \cdots \le u_n \le \cdots \le u_1 \le u_0, \quad n = 1, 2, \ldots. \tag{3.2}$$

Obviously, $u_n, v_n (n = 1, 2, \ldots)$ satisfy

$$\begin{cases} u_n'(t) = f(t, u_{n-1}(t), u_{n-1}(\alpha(t))) + M(t)(u_n - u_{n-1})(t) + N(t)(u_n - u_{n-1})(\alpha(t)), \quad t \in J', \\ \Delta u_n(t_k) = I_k(u_{n-1}(t_k)) + L_k(u_n - u_{n-1})(t_k), \quad k = 1, 2, \ldots, m, \\ (u_n - u_{n-1})(0) = r(u_n - u_{n-1})(T) - \dfrac{1}{K}g(u_{n-1}(0), u_{n-1}(T)), \end{cases}$$

and

$$\begin{cases} v_n'(t) = f(t, v_{n-1}(t), v_{n-1}(\alpha(t))) + M(t)(v_n - v_{n-1})(t) + N(t)(v_n - v_{n-1})(\alpha(t)), \quad t \in J', \\ \Delta v_n(t_k) = I_k(v_{n-1}(t_k)) + L_k(v_n - v_{n-1})(t_k), \quad k = 1, 2, \ldots, m, \\ (v_n - v_{n-1})(0) = r(v_n - v_{n-1})(T) - \dfrac{1}{K}g(v_{n-1}(0), v_{n-1}(T)). \end{cases}$$

Therefore, there exist u^*, v^* such that

$$\lim_{n\to\infty} u_n(t) = u^*(t), \qquad \lim_{n\to\infty} v_n(t) = v^*(t)$$

uniformly on J, and the limit functions u^*, v^* satisfy (1.1). Moreover, $u^*, v^* \in [v_0, u_0]$.

Finally, we prove that u^*, v^* are the extremal solutions of (1.1) in $[v_0, u_0]$. Let $w \in [v_0, u_0]$ be any solution of (1.1), then $Aw = w$. By $v_0 \le w \le u_0$ and the properties of A, we have

$$v_n \le w \le u_n, \quad n = 1, 2, \ldots. \tag{3.3}$$

Thus, taking limit in (3.3) as $n \to \infty$, we have $v^* \le w \le u^*$. That is, u^*, v^* are the extremal solutions of (1.1) in $[v_0, u_0]$. □

Theorem 3.2. *Let all assumptions of Corollary 2.1 and* (H_1)–(H_3) *hold. Then there exist monotone iterative sequences* $\{u_n\}, \{v_n\}$, *which converge uniformly on J to the extremal solutions of (1.1) in* $[v_0, u_0] = \{u \in PC^1(J, R) : v_0(t) \le u(t) \le u_0(t)\}$.

Proof. The proof is almost the same as that of Theorem 3.1, so we omit it. □

4. Example

Example 4.1. Consider the following boundary value problem:

$$\begin{cases} u'(t) = \dfrac{2}{3}t^3u^2(t) + t^2u(t) + \dfrac{1}{10}t^3e^{u(\sqrt{t})}, \quad t \in J = [0, 1],\ t \ne \dfrac{1}{2}, \\ \Delta u\left(\dfrac{1}{2}\right) = \dfrac{1}{10}u\left(\dfrac{1}{2}\right), \\ \dfrac{1}{2}u^3(0) + 8u(0) - 9u(1) - c = 0, \quad 0 \le c \le \dfrac{1}{2}, \end{cases} \tag{4.1}$$

where $m = 1,\ t_1 = \frac{1}{2},\ \alpha(t) = \sqrt{t},\ \forall t \in J$.

Obviously, $u_0 = 0$, $v_0 = -1$ are lower and upper solutions of (4.1), respectively, and $v_0 \le u_0$.

Let

$$f(t, u, v) = \frac{2}{3}t^3u^2 + t^2u + \frac{1}{10}t^3e^v,$$

$$g(u, v) = \frac{1}{2}u^3 + 8u - 9v - c,$$

we have

$$f(t, u, v) - f(t, \bar{u}, \bar{v}) \le t^2(u - \bar{u}) + \frac{1}{10}t^3(v - \bar{v}),$$

where $v_0(t) \le \bar{u} \le u \le u_0(t),\ v_0(\alpha(t)) \le \bar{v} \le v \le u_0(\alpha(t)),\ \forall t \in J$.

$$g(u, v) - g(\bar{u}, \bar{v}) \ge 8(u - \bar{u}) - 9(v - \bar{v}),$$

where $v_0(0) \le \bar{u} \le u \le u_0(0),\ v_0(1) \le \bar{v} \le v \le u_0(1)$.

For $M(t) = t^2,\ N(t) = \frac{1}{10}t^3,\ L_1 = \frac{1}{10},\ r = L/K = \frac{9}{8}$, it is easy to verify that conditions (i) and (ii) hold. Therefore, (4.1) satisfies all conditions of Theorem 3.1. By Theorem 3.1, there exist monotone iterative sequences $\{u_n\}, \{v_n\}$, which converge uniformly on J to the extremal solutions of (4.1) in $[v_0, u_0]$.

Example 4.2. Consider the following boundary value problem:

$$\begin{cases} u'(t) = \dfrac{t^3}{5\pi}\arctan\left[\dfrac{t}{t^2+1}e^{\frac{t}{2}} + \dfrac{t^3}{10}\sin^2(\ln(1+t))u(t)\right] + \dfrac{t^3e^{u(t)}\sin t}{5[2 + e^{(\frac{t}{2} + \cos\ln(1+t))}]} \\ \qquad + \dfrac{t^4u(\alpha(t))}{10(1+\sin t)} - \dfrac{t^3}{5\pi}\arctan\dfrac{1+e^{-t}}{2}u^2(\alpha(t)), \quad t \in J = [0, 1],\ t \ne t_1, \\ \Delta u(t_1) = a\sin u(t_1), \quad 0 \le a \le \dfrac{8}{19}, \\ \cos u(0) + 9u(0) - 10u(1) + e^{u(1)} - c = 0, \quad 2 \le c \le 2 + \dfrac{1}{e}, \end{cases} \tag{4.2}$$

where $\alpha \in C(J, J),\ 0 < t_1 < 1,\ m = 1$.

It is not difficult to verify that $u_0 = 0$, $v_0 = -1$ are lower and upper solutions of (4.2), respectively, and $v_0 \le u_0$.

Let

$$f(t,u,v) = \frac{t^3}{5\pi}\arctan\left[\frac{t}{t^2+1}e^{\frac{t}{2}} + \frac{t^3}{10}\sin^2(\ln(1+t))u\right] + \frac{t^3 e^u \sin t}{5\left[2+e^{\left(\frac{t}{2}+\cos\ln(1+t)\right)}\right]} + \frac{t^4 v}{10(1+\sin t)} - \frac{t^3}{5\pi}\arctan\frac{1+e^{-t}}{2}v^2,$$

$$I_1(u) = a\sin u,$$

$$g(u,v) = \cos u + 9u - 10v + e^v - c,$$

we have

$$f(t,u,v) - f(t,\overline{u},\overline{v}) \le \frac{t^3}{5}(u-\overline{u}) + \frac{t^3}{5}(v-\overline{v}),$$

where $v_0(t) \le \overline{u} \le u \le u_0(t),\ v_0(\alpha(t)) \le \overline{v} \le v \le u_0(\alpha(t)),\ \forall t \in J.$

$$I_1(x) - I_1(y) \le a(x-y),$$

where $v_0(t_1) \le y \le x \le u_0(t_1)$.

$$g(u,v) - g(\overline{u},\overline{v}) \ge 9(u-\overline{u}) - 10(v-\overline{v}),$$

where $v_0(0) \le \overline{u} \le u \le u_0(0),\ v_0(1) \le \overline{v} \le v \le u_0(1)$.

For $M(t) = N(t) = \frac{t^3}{5},\ L_1 = a,\ r = L/K = \frac{10}{9}$, we see that conditions (i) and (ii) hold. Therefore, (4.2) satisfies all conditions of Theorem 3.1. By Theorem 3.1, there exist monotone iterative sequences $\{u_n\}, \{v_n\}$, which converge uniformly on J to the extremal solutions of (4.2) in $[v_0, u_0]$.

Remark 4.1. For appropriate and suitable choices of a, c, t_1 and $\alpha(t)$, it is easy to see that problem (4.2) includes a number of differential equations, differential equations with deviating arguments.

Example 4.3. Consider the following Logistic model with a variable coefficient which is widely used in biology:

$$\begin{cases} u'(t) = t^3 u(t) - \dfrac{t^4\sin t}{10}u^2(t) + \dfrac{t^2\cos t}{5}u(\alpha(t)), \quad t \in J = [0,1],\ t \ne t_1, \\ \Delta u(t_1) = L_1 u(t_1), \\ bu^2(0) + u(0) - u(1) + c = 0, \end{cases} \tag{4.3}$$

where $\alpha \in C(J,J),\ 0 < t_1 < 1,\ m = 1,\ 0 \le L_1 \le \frac{11}{60},\ 0 \le b \le \frac{1}{4},\ 0 \le c \le \frac{1}{4} - b.$

Put $u_0(t) = 1 + \frac{t^4}{4}, v_0(t) = 0, \forall t \in J$. Then $u_0(t), v_0(t)$ are lower and upper solutions of (4.3), respectively, and $u_0(t) \ge v_0(t)$.

Let

$$f(t,u,v) = t^3 u - \frac{t^4\sin t}{10}u^2 + \frac{t^2\cos t}{5}v, \qquad g(u,v) = bu^2 + u - v + c,$$

then

$$f(t,u,v) - f(t,\overline{u},\overline{v}) \le t^3(u-\overline{u}) + \frac{t^2}{5}(v-\overline{v}),$$

where $v_0(t) \le \overline{u} \le u \le u_0(t),\ v_0(\alpha(t)) \le \overline{v} \le v \le u_0(\alpha(t)),\ \forall t \in J.$

$$g(u,v) - g(\overline{u},\overline{v}) \ge (u-\overline{u}) - (v-\overline{v}),$$

where $v_0(0) \le \overline{u} \le u \le u_0(0),\ v_0(1) \le \overline{v} \le v \le u_0(1)$.

For $M(t) = t^3,\ N(t) = \frac{t^2}{5},\ r = L/K = 1$, we see that conditions (i) and (ii) hold. Thus, (4.3) satisfies all conditions of Theorem 3.1. By Theorem 3.1, there exist monotone iterative sequences $\{u_n\}, \{v_n\}$, which converge uniformly on J to the extremal solutions of (4.3) in $[v_0, u_0]$.

Remark 4.2. For appropriate and suitable choices of b, c, t_1, L_1 and $\alpha(t)$, we see that problem (4.3) has a very general form. For example, we can take $b = \frac{1}{8}, c = \frac{1}{9}, t_1 = \frac{2}{3}, L_1 = \frac{1}{6}$ and $\alpha(t) = t^3, \forall t \in J$.

Remark 4.3. The three examples mentioned above satisfy all conditions of Theorem 3.2. Thus, by Theorem 3.2, there exist monotone iterative sequences $\{u_n\}, \{v_n\}$, respectively, which converge uniformly on J to the extremal solutions of the corresponding problem in $[v_0, u_0]$.

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