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Methods of critical value reduction for type-2 fuzzy variables and their applications

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ABSTRACT

A type-2 fuzzy variable is a map from a fuzzy possibility space to the real number space; it is an appropriate tool for describing type-2 fuzziness. This paper first presents three kinds of critical values (CVs) for a regular fuzzy variable (RFV), and proposes three novel methods of reduction for a type-2 fuzzy variable. Secondly, this paper applies the reduction methods to data envelopment analysis (DEA) models with type-2 fuzzy inputs and outputs, and develops a new class of generalized credibility DEA models. According to the properties of generalized credibility, when the inputs and outputs are mutually independent type-2 triangular fuzzy variables, we can turn the proposed fuzzy DEA model into its equivalent parametric programming problem, in which the parameters can be used to characterize the degree of uncertainty about type-2 fuzziness. For any given parameters, the parametric programming model becomes a linear programming one that can be solved using standard optimization solvers. Finally, one numerical example is provided to illustrate the modeling idea and the efficiency of the proposed DEA model.

1. Introduction

The concept of a type-2 fuzzy set was first proposed in [1] as an extension of an ordinary fuzzy set. Since then, many researchers have employed the theory in their studies. For example, Mitchell [2] used the concept of an embedded type-1 fuzzy number to give a method for ranking type-2 fuzzy numbers; Liang and Mendel [3] proposed the concept of an interval type-2 fuzzy set for dealing with the operations via interval arithmetics; Zeng and Liu [4] described the important advances concerning type-2 fuzzy sets for pattern recognition, and in [5] explored the calculation of the union and intersection of concave type-2 fuzzy sets using the minimum t-norm and the maximum t-conorm. From the computational viewpoint, type-2 fuzziness is more difficult to deal with than type-1 fuzziness because the possibility of a type-2 fuzzy variable taking on a crisp value is a fuzzy number in [0, 1]. To avoid this difficulty, some type reduction approaches have been proposed in the literature for dealing with type-2 fuzziness, for example: [6] proposed a defuzzification method with the concept of a centroid of a type-2 fuzzy set; Liu [7] employed a centroid type reduction strategy for a general type-2 fuzzy logic system, and Qiu et al. [8] developed a statistical method for deciding on interval-valued fuzzy membership functions and a probability type reduction reasoning method for use with the interval-valued fuzzy logic system. In this paper, we attempt to present some novel reduction methods based on CVs of RFVs. According to the fuzzy integral [9], we first define three kinds of CVs for an RFV, which are referred to as the optimistic CV, the pessimistic CV and the CV. Some numerical examples are provided to illustrate the concepts, and the properties of CVs of trapezoidal, triangular, normal and gamma RFVs are also

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discussed. Furthermore, we develop three methods of reduction for type-2 fuzzy variables, which are called the optimistic, the pessimistic and the CV reduction methods, respectively.

In the literature, DEA technology was first proposed in [10]. One of the advantages of the DEA method is that it does not require either a priori weights for the inputs and outputs or the explicit specification of functional relations between the multiple inputs and outputs; therefore DEA has been widely used in many areas (see, e.g., [11–15]). A number of researchers have developed, in addition to the CCR model, some other meaningful DEA models, including the BCC model [16], the FDH (free disposal hull) model [17], the SBM (slack-based measure of efficiency) model [18], the RAM model [19] and so on. More advanced treatments may be found in [20,21]. On the basis of these models, some researchers extended the crisp inputs and outputs of traditional DEA models to stochastic data and developed some stochastic DEA models. For example, Sengupta [22] incorporated stochastic input and output variations into the DEA model; Banker [23] incorporated stochastic variables into DEA and developed a nonparametric approach; Cooper et al. [24] and Land et al. [25] developed a chance-constrained programming for DEA problems in order to accommodate the stochastic variations in the data. On other hand, fuzzy DEA with the inputs and outputs as fuzzy data has also been an area of active investigation. For instance, Sengupta [26] explored the use of fuzzy set-theoretic measures in the context of data envelopment analysis, and utilized a nonparametric approach for measuring efficiency; Triantis and Girod [27] suggested a mathematical programming approach to transforming fuzzy input and output data into crisp data by using membership function values; Wang and Yang [28], and Wang et al. [29] developed some methods for measuring the performance of DMUs, in which the efficiency is measured within the range of an interval; Wen and Li [30] established a DEA model in fuzzy environments and provided a ranking method for comparing the efficiencies of DMUs. On the basis of fuzzy possibility theory [31], this paper also considers data variations, and models fuzzy DEA from a new viewpoint, in which the inputs and outputs are characterized by type-2 fuzzy variables with known secondary possibility distributions. From the computational viewpoint, type-2 fuzziness is very complex compared with type-1 fuzziness. To overcome this difficulty, we first employ the proposed reduction methods in order to reduce the type-2 fuzzy inputs and outputs, then formulate a generalized credibility DEA model. When the inputs and outputs are mutually independent type-2 triangular fuzzy variables, we can turn the established DEA model into its equivalent parametric programming form, where the parameters can be used to characterize the degree of uncertainty as regards type-2 fuzziness. For any given parameters, the equivalent parametric programming model becomes a linear programming one that can be solved using standard optimization solvers. At the end of this paper, we provide one numerical example to illustrate the modeling idea and the efficiency in the proposed model by adjusting parameters with different values.

The rest of this paper is organized as follows. Section 2 introduces some concepts of type-2 fuzzy theory. In Section 3, we define three kinds of CVs for a fuzzy variable via the fuzzy integral and discuss the properties of CVs. In Section 4, we first develop the CV-based reduction methods for type-2 fuzzy variables, then discuss the fundamental properties for generalized credibility. In Section 5, we apply our reduction methods to the DEA model with type-2 fuzzy coefficients. Section 6 provides one numerical example to illustrate the modeling idea and the efficiency in the proposed DEA model. Section 7 gives our conclusions.

2. Fundamental concepts

Let Γ be the universe of discourse. An ample field [32] $\mathcal{A}$ on Γ is a class of subsets of Γ that is closed under arbitrary unions, intersections, and complements in Γ.

Let $\text{Pos} : \mathcal{A} \mapsto [0, 1]$ be a set function on the ample field $\mathcal{A}$. Pos is said to be a possibility measure [32] if it satisfies the following conditions:

(P1) $\text{Pos}(\emptyset) = 0$ and $\text{Pos}(\Gamma) = 1$.

(P2) For any subclass $\{A_i \mid i \in I\}$ of $\mathcal{A}$ (finite, countable or uncountable),

$$\text{Pos}\left(\bigcup_{i\in I} A_i\right) = \sup_{i\in I} \text{Pos}(A_i).$$

The triplet $(\Gamma, \mathcal{A}, \text{Pos})$ is referred to as a possibility space, in which a credibility measure [33] is defined as

$$\text{Cr}(A) = \frac{1}{2}(1 + \text{Pos}(A) - \text{Pos}(A^c)), \quad A \in \mathcal{A}.$$

If $(\Gamma, \mathcal{A}, \text{Pos})$ is a possibility space, then an m-ary regular fuzzy vector $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$ is defined as a measurable map from Γ to the space $[0, 1]^m$ in the sense that for every $t = (t_1, t_2, \ldots, t_m) \in [0, 1]^m$, one has

$$\{\gamma \in \Gamma \mid \xi(\gamma) \le t\} = \{\gamma \in \Gamma \mid \xi_1(\gamma) \le t_1, \xi_2(\gamma) \le t_2, \ldots, \xi_m(\gamma) \le t_m\} \in \mathcal{A}.$$

When m=1, ξ is called a regular fuzzy variable (RFV).

In this paper, we denote by $\mathcal{R}([0, 1])$ the collection of all RFVs on [0, 1]. In the following, we provide several common RFVs.

Example 1. If ξ has the following possibility distribution:

$$\xi \sim \begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ \mu_1 & \mu_2 & \cdots & \mu_n \end{pmatrix},$$

where for each $i = 1, 2, \ldots, n$, $r_i \in [0, 1]$, $\mu_i > 0$, and $\max_{i=1}^n \mu_i = 1$, then ξ is a discrete RFV.

If $\xi = (r_1, r_2, r_3, r_4)$ with $0 \le r_1 < r_2 < r_3 < r_4 \le 1$, then ξ is a trapezoidal RFV.

If $\xi = (r_1, r_2, r_3)$ with $0 \le r_1 < r_2 < r_3 \le 1$, then ξ is a triangular RFV.

If the possibility distribution of ξ is as follows:

$$\mu_\xi(x) = \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in [0, 1],$$

where $0 \le \mu \le 1$ and $\sigma > 0$, then ξ is a normal RFV.

If the possibility distribution of ξ is as follows:

$$\mu_\xi(t) = \left(\frac{t}{\lambda r}\right)^r \exp\left(r - \frac{t}{\lambda}\right), \quad t \in [0, 1],$$

where the parameter $0 < r \le 1$, and $0 < \lambda \le 1/r$, then ξ is a gamma RFV.

Definition 1 (*Liu and Liu [31]*)**.** Let $\tilde{\mathrm{P}}os : \mathcal{A} \mapsto \mathcal{R}([0, 1])$ be a set function defined on $\mathcal{A}$ such that $\{\tilde{\mathrm{P}}os(A) \mid \mathcal{A} \ni A \text{ atom}\}$ is a family of mutually independent RFVs. We call $\tilde{\mathrm{P}}os$ a fuzzy possibility measure if it satisfies the following conditions:

($\tilde{\mathrm{P}}$1) $\tilde{\mathrm{P}}os(\emptyset) = \tilde{0}$.

($\tilde{\mathrm{P}}$2) For any subclass $\{A_i \mid i \in I\}$ of $\mathcal{A}$ (finite, countable or uncountable),

$$\tilde{\mathrm{P}}os\left(\bigcup_{i \in I} A_i\right) = \sup_{i \in I} \tilde{\mathrm{P}}os(A_i).$$

Moreover, if $\mu_{\tilde{\mathrm{P}}os(\Gamma)}(1) = 1$, then we call $\tilde{\mathrm{P}}os$ a regular fuzzy possibility measure.

The triplet $(\Gamma, \mathcal{A}, \tilde{\mathrm{P}}os)$ is referred to as a fuzzy possibility space (FPS), in which a map $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, \cdots, \tilde{\xi}_m) : \Gamma \mapsto \Re^m$ is called an m-ary type-2 fuzzy vector if for any $r = (r_1, r_2, \ldots, r_m) \in \Re^m$, the set $\{\gamma \in \Gamma \mid \tilde{\xi}(\gamma) \le r\}$ is an element of $\mathcal{A}$, i.e.,

$$\{\gamma \in \Gamma \mid \tilde{\xi}(\gamma) \le r\} = \{\gamma \in \Gamma \mid \tilde{\xi}_1(\gamma) \le r_1, \tilde{\xi}_2(\gamma) \le r_2, \ldots, \tilde{\xi}_m(\gamma) \le r_m\} \in \mathcal{A}.$$

As $m = 1$, the map $\tilde{\xi} : \Gamma \mapsto \Re$ is called a type-2 fuzzy variable.

For example, if $\tilde{\xi}$ is defined as

$$\tilde{\xi} = \begin{cases} 1, & \text{with possibility } (0.1, 0.2, 0.4) \\ 4, & \text{with possibility } \tilde{1} \\ 8, & \text{with possibility } (0.1, 0.3, 0.5, 0.7), \end{cases}$$

then $\tilde{\xi}$ is a type-2 fuzzy variable that takes on the values 1, 4 and 8 with possibilities (0.1, 0.2, 0.4), $\tilde{1}$ and (0.1, 0.3, 0.5, 0.7), respectively.

Definition 2 (*Liu and Liu [31]*)**.** Let $\tilde{\xi} = (\tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_m)$ be a type-2 fuzzy vector defined on an FPS $(\Gamma, \mathcal{A}, \tilde{\mathrm{P}}os)$. The secondary possibility distribution function $\tilde{\mu}_{\tilde{\xi}}(x)$ of $\tilde{\xi}$ is a map $\Re^m \mapsto \mathcal{R}[0, 1]$ such that

$$\tilde{\mu}_{\tilde{\xi}}(x) = \tilde{\mathrm{P}}os\{\gamma \in \Gamma \mid \tilde{\xi}(\gamma) = x\}, \quad x \in \Re^m,$$

while the type-2 possibility distribution function $\tilde{\mu}_{\tilde{\xi}}(x, u)$ of $\tilde{\xi}$ is a map $\Re^m \times J_x \mapsto [0, 1]$ such that

$$\mu_{\tilde{\xi}}(x, u) = \mathrm{Pos}\{\tilde{\mu}_{\tilde{\xi}}(x) = u\}, \quad (x, u) \in \Re^m \times J_x,$$

where Pos is the possibility measure induced by the distribution of $\tilde{\mu}_{\tilde{\xi}}(x)$, and $J_x \subset [0, 1]$ is the support of $\tilde{\mu}_{\tilde{\xi}}(x)$, i.e., $J_x = \{u \in [0, 1] \mid \mu_{\tilde{\xi}}(x, u) > 0\}$.

The support of a type-2 fuzzy vector $\tilde{\xi}$ is denoted as

$$\mathrm{supp}\,\tilde{\xi} = \left\{(x, u) \in \Re^m \times [0, 1] \mid \mu_{\tilde{\xi}}(x, u) > 0\right\},$$

where $\mu_{\tilde{\xi}}(x, u)$ is the type-2 possibility distribution function of $\tilde{\xi}$.

Definition 3 (*Liu and Liu [31]*). Let $\tilde{\xi}_i$, $i = 1, 2, \ldots, m$, be type-2 fuzzy variables defined on an FPS $(\Gamma, \mathcal{A}, \widetilde{\text{P}}os)$. They are said to be mutually independent if

$$\widetilde{\text{P}}os(\{\gamma \in \Gamma \mid \tilde{\xi}_i(\gamma) \in B_i, 1 \le i \le m\}) = \min_{1\le i\le m} \widetilde{\text{P}}os(\{\gamma \in \Gamma \mid \tilde{\xi}_i(\gamma) \in B_i\})$$

for any $B_i \subset \Re$, $i = 1, 2, \ldots, m$, where the $\widetilde{\text{P}}os(\{\gamma \in \Gamma \mid \tilde{\xi}_i(\gamma) \in B_i\})$ are supposed to be mutually independent RFVs.

In the following example, we give three kinds of common type-2 fuzzy variables.

Example 2. A type-2 fuzzy variable $\tilde{\xi}$ is called triangular if its secondary possibility distribution $\tilde{\mu}_{\tilde{\xi}}(x)$ is

$$\left(\frac{x-r_1}{r_2-r_1} - \theta_l \min\left\{\frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1}\right\}, \frac{x-r_1}{r_2-r_1}, \frac{x-r_1}{r_2-r_1} + \theta_r \min\left\{\frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1}\right\}\right)$$

for any $x \in [r_1, r_2]$, and

$$\left(\frac{r_3-x}{r_3-r_2} - \theta_l \min\left\{\frac{r_3-x}{r_3-r_2}, \frac{x-r_2}{r_3-r_2}\right\}, \frac{r_3-x}{r_3-r_2}, \frac{r_3-x}{r_3-r_2} + \theta_r \min\left\{\frac{r_3-x}{r_3-r_2}, \frac{x-r_2}{r_3-r_2}\right\}\right)$$

for any $x \in (r_2, r_3]$, where $\theta_l, \theta_r \in [0, 1]$ are two parameters characterizing the degree of uncertainty of $\tilde{\xi}$ taking the value x. For simplicity, we denote the type-2 triangular fuzzy variable $\tilde{\xi}$ with the above distribution by $(\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r)$.

A type-2 fuzzy variable $\tilde{\xi}$ is called normal if its secondary possibility distribution $\tilde{\mu}_{\tilde{\xi}}(x)$ is

$$\left(\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) - \theta_l \min\left\{1-\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\}, \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\right.$$
$$\left.\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) + \theta_r \min\left\{1-\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\}\right)$$

for any $x \in \Re$, where $\mu \in \Re$, $\sigma > 0$, and $\theta_l, \theta_r \in [0, 1]$ are two parameters characterizing the degree of uncertainty of $\tilde{\xi}$ taking the value x. For simplicity, the type-2 normal fuzzy variable $\tilde{\xi}$ with the above distribution is denoted by $\tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$.

A type-2 fuzzy variable $\tilde{\xi}$ is called gamma if its secondary possibility distribution $\tilde{\mu}_{\tilde{\xi}}(x)$ is

$$\left(\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right) - \theta_l \min\left\{1-\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right\}, \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right),\right.$$
$$\left.\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right) + \theta_r \min\left\{1-\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right\}\right)$$

for any $x \in \Re$, where $\lambda > 0$, r is a fixed constant, and $\theta_l, \theta_r \in [0, 1]$ are two parameters characterizing the degree of uncertainty of $\tilde{\xi}$ taking the value x. For simplicity, the type-2 gamma fuzzy variable $\tilde{\xi}$ with the above distribution is denoted by $\tilde{\gamma}(\lambda, r; \theta_l, \theta_r)$.

3. Critical values for RFVs

In this section, we will define three kinds of CVs for an RFV by using a fuzzy integral [9].

Definition 4. Let ξ be an RFV. Then the optimistic CV of ξ, denoted by $\text{CV}^*[\xi]$, is defined as

$$\text{CV}^*[\xi] = \sup_{\alpha\in[0,1]} [\alpha \wedge \text{Pos}\{\xi \ge \alpha\}], \tag{1}$$

while the pessimistic CV of ξ, denoted by $\text{CV}_*[\xi]$, is defined as

$$\text{CV}_*[\xi] = \sup_{\alpha\in[0,1]} [\alpha \wedge \text{Nec}\{\xi \ge \alpha\}]. \tag{2}$$

The CV of ξ, denoted by $\text{CV}[\xi]$, is defined as

$$\text{CV}[\xi] = \sup_{\alpha\in[0,1]} [\alpha \wedge \text{Cr}\{\xi \ge \alpha\}]. \tag{3}$$

Example 3. Let ξ be a discrete RFV with the following possibility distribution:

$$\xi \sim \begin{pmatrix} 0.1 & 0.3 & 0.6 & 0.8 \\ 0.2 & 1 & 0.5 & 0.7 \end{pmatrix}.$$

Then it is easy to compute that

$$\mathrm{Pos}\{\xi \geq \alpha\} = \begin{cases} 1, & \text{if } \alpha \leq 0.3 \\ 0.7, & \text{if } 0.3 < \alpha \leq 0.8 \\ 0, & \text{if } 0.8 < \alpha \leq 1, \end{cases}$$

$$\mathrm{Nec}\{\xi \geq \alpha\} = \begin{cases} 1, & \text{if } \alpha \leq 0.1 \\ 0.8, & \text{if } 0.1 < \alpha \leq 0.3 \\ 0, & \text{if } 0.3 < \alpha \leq 1, \end{cases}$$

and

$$\mathrm{Cr}\{\xi \geq \alpha\} = \begin{cases} 1, & \text{if } \alpha \leq 0.1 \\ 0.9, & \text{if } 0.1 < \alpha \leq 0.3 \\ 0.35, & \text{if } 0.3 < \alpha \leq 0.8 \\ 0, & \text{if } 0.8 < \alpha \leq 1. \end{cases}$$

Therefore, by the definitions of CVs, we have

$$\begin{aligned} \mathrm{CV}^*[\xi] &= \sup_{\alpha\in[0,1]} [\alpha \wedge \mathrm{Pos}\{\xi \geq \alpha\}] \\ &= \sup_{\alpha\in[0,0.3]} [\alpha \wedge 1] \vee \sup_{\alpha\in(0.3,0.8]} [\alpha \wedge 0.7] \vee \sup_{\alpha\in(0.8,1]} [\alpha \wedge 0] \\ &= 0.3 \vee 0.7 \vee 0 = 0.7, \\ \mathrm{CV}_*[\xi] &= \sup_{\alpha\in[0,1]} [\alpha \wedge \mathrm{Nec}\{\xi \geq \alpha\}] \\ &= \sup_{\alpha\in[0,0.1]} [\alpha \wedge 1] \vee \sup_{\alpha\in(0.1,0.3]} [\alpha \wedge 0.8] \vee \sup_{\alpha\in(0.3,1]} [\alpha \wedge 0] \\ &= 0.1 \vee 0.3 \vee 0 = 0.3, \end{aligned}$$

and

$$\begin{aligned} \mathrm{CV}[\xi] &= \sup_{\alpha\in[0,1]} [\alpha \wedge \mathrm{Cr}\{\xi \geq \alpha\}] \\ &= \sup_{\alpha\in[0,0.1]} [\alpha \wedge 1] \vee \sup_{\alpha\in(0.1,0.3]} [\alpha \wedge 0.9] \vee \sup_{\alpha\in(0.3,0.8]} [\alpha \wedge 0.35] \vee \sup_{\alpha\in(0.8,1]} [\alpha \wedge 0] \\ &= 0.1 \vee 0.3 \vee 0.35 \vee 0 = 0.35. \end{aligned}$$

The following theorem presents the formulas for CVs of a trapezoidal RFV.

Theorem 1. *Let $\xi = (r_1, r_2, r_3, r_4)$ be a trapezoidal RFV. Then we have:*

(i) *The optimistic CVof ξ is* $\mathrm{CV}^*[\xi] = r_4/(1 + r_4 - r_3)$.
(ii) *The pessimistic CVof ξ is* $\mathrm{CV}_*[\xi] = r_2/(1 + r_2 - r_1)$.
(iii) *The CVof ξ is*

$$\mathrm{CV}[\xi] = \begin{cases} \dfrac{2r_2 - r_1}{1 + 2(r_2 - r_1)}, & \text{if } r_2 > \dfrac{1}{2} \\ \dfrac{1}{2}, & \text{if } r_2 \leq \dfrac{1}{2} < r_3 \\ \dfrac{r_4}{1 + 2(r_4 - r_3)}, & \text{if } r_3 \leq \dfrac{1}{2}. \end{cases} \quad (4)$$

Proof. (i) According to the distribution of ξ, for any $\alpha \in [0, 1]$, we have

$$\mathrm{Pos}\{\xi \geq \alpha\} = \begin{cases} 1, & \text{if } \alpha \leq r_3 \\ \dfrac{r_4 - \alpha}{r_4 - r_3}, & \text{if } r_3 < \alpha \leq r_4 \\ 0, & \text{if } \alpha > r_4. \end{cases}$$

Thus, it follows from the definition of the optimistic CV that

$$\begin{aligned}
\mathrm{CV}^*[\xi] &= \sup_{\alpha\in[0,1]} [\alpha \wedge \mathrm{Pos}\{\xi \geq \alpha\}] \\
&= \sup_{\alpha\in[0,r_3]} [\alpha \wedge 1] \vee \sup_{\alpha\in(r_3,r_4]} \left[\alpha \wedge \frac{r_4-\alpha}{r_4-r_3}\right] \\
&= r_3 \vee \frac{r_4}{1+r_4-r_3} \\
&= \frac{r_4}{1+r_4-r_3}.
\end{aligned}$$

(ii) By the distribution of ξ, for any $\alpha \in [0, 1]$, we have

$$\mathrm{Nec}\{\xi \geq \alpha\} = \begin{cases} 1, & \text{if } \alpha \leq r_1 \\ \dfrac{r_2-\alpha}{r_2-r_1}, & \text{if } r_1 < \alpha \leq r_2 \\ 0, & \text{if } r > r_2. \end{cases}$$

Thus, according to the definition of the pessimistic CV, we have

$$\begin{aligned}
\mathrm{CV}_*[\xi] &= \sup_{\alpha\in[0,1]} [\alpha \wedge \mathrm{Nec}\{\xi \geq \alpha\}] \\
&= \sup_{\alpha\in[0,r_1]} [\alpha \wedge 1] \vee \sup_{\alpha\in(r_1,r_2]} \left[\alpha \wedge \frac{r_2-\alpha}{r_2-r_1}\right] \\
&= r_1 \vee \frac{r_2}{1+r_2-r_1} \\
&= \frac{r_2}{1+r_2-r_1}.
\end{aligned}$$

(iii) Using the distribution of ξ, for any $\alpha \in [0, 1]$, we have

$$\mathrm{Cr}\{\xi \geq \alpha\} = \begin{cases} 1, & \text{if } \alpha \leq r_1 \\ \dfrac{2r_2-r_1-\alpha}{2(r_2-r_1)}, & \text{if } r_1 < \alpha \leq r_2 \\ \dfrac{1}{2}, & \text{if } r_2 < \alpha \leq r_3 \\ \dfrac{r_4-\alpha}{2(r_4-r_3)}, & \text{if } r_3 < \alpha \leq r_4 \\ 0, & \text{if } \alpha > r_4. \end{cases} \tag{5}$$

As a consequence,

$$\begin{aligned}
\mathrm{CV}[\xi] &= \sup_{\alpha\in[0,1]} [\alpha \wedge \mathrm{Cr}\{\xi \geq \alpha\}] \\
&= \sup_{\alpha\in[0,r_1]} [\alpha \wedge 1] \vee \sup_{\alpha\in(r_1,r_2]} \left[\alpha \wedge \frac{2r_2-r_1-\alpha}{2(r_2-r_1)}\right] \vee \sup_{\alpha\in(r_2,r_3]} \left[\alpha \wedge \frac{1}{2}\right] \vee \sup_{\alpha\in(r_3,r_4]} \left[\alpha \wedge \frac{r_4-\alpha}{2(r_4-r_3)}\right] \\
&= r_1 \vee \sup_{\alpha\in(r_1,r_2]} \left[\alpha \wedge \frac{2r_2-r_1-\alpha}{2(r_2-r_1)}\right] \vee \left[r_3 \wedge \frac{1}{2}\right] \vee \sup_{\alpha\in(r_3,r_4]} \left[\alpha \wedge \frac{r_4-\alpha}{2(r_4-r_3)}\right] \\
&= \begin{cases} \dfrac{2r_2-r_1}{1+2(r_2-r_1)}, & \text{if } r_2 > \dfrac{1}{2} \\ \dfrac{1}{2}, & \text{if } r_2 \leq \dfrac{1}{2} < r_3 \\ \dfrac{r_4}{1+2(r_4-r_3)}, & \text{if } r_3 \leq \dfrac{1}{2}. \end{cases}
\end{aligned}$$

The proof of the theorem is complete. □

Example 4. Let ξ be the trapezoidal RFV $(0.1, 0.3, 0.4, 0.6)$. Then according to Theorem 1, we have

$$\mathrm{CV}^*[\xi] = \frac{1}{2}, \mathrm{CV}_*[\xi] = \frac{1}{4}, \mathrm{CV}[\xi] = \frac{3}{7}.$$

As a corollary of Theorem 1, we have:

Corollary 1. *Let $\xi = (r_1, r_2, r_3)$ be a triangular RFV. Then we have:*

(i) *The optimistic CVof ξ is $\mathrm{CV}^*[\xi] = r_3/(1 + r_3 - r_2)$.*
(ii) *The pessimistic CVof ξ is $\mathrm{CV}_*[\xi] = r_2/(1 + r_2 - r_1)$.*
(iii) *The CVof ξ is*

$$\mathrm{CV}[\xi] = \begin{cases} \dfrac{2r_2 - r_1}{1 + 2(r_2 - r_1)}, & \text{if } r_2 > \dfrac{1}{2} \\ \dfrac{r_3}{1 + 2(r_3 - r_2)}, & \text{if } r_2 \le \dfrac{1}{2}. \end{cases} \tag{6}$$

Example 5. Let ξ be the triangular RFV (0.1, 0.6, 0.9). It follows from Corollary 1 that

$$\mathrm{CV}^*[\xi] = \frac{9}{13}, \qquad \mathrm{CV}_*[\xi] = \frac{2}{5}, \qquad \mathrm{CV}[\xi] = \frac{11}{20}.$$

The following theorem gives the equations satisfied by CVs of a normal RFV.

Theorem 2. *Let ξ be a normal RFV with the following possibility distribution:*

$$\mu_\xi(x) = \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right), \quad x \in [0, 1].$$

(i) *If $\mu = 1$, then $\mathrm{CV}^*[\xi] = 1$, and if $0 \le \mu < 1$, then $\mathrm{CV}^*[\xi]$ is the solution of the following equation:*
$$(\alpha - \mu)^2 + 2\sigma^2 \ln \alpha = 0.$$
(ii) *If $\mu = 0$, then $\mathrm{CV}_*[\xi] = 0$, and if $0 < \mu \le 1$, then $\mathrm{CV}_*[\xi]$ is the solution of the following equation:*
$$(\alpha - \mu)^2 + 2\sigma^2 \ln(1 - \alpha) = 0.$$
(iii) *If $0 \le \mu < 0.5$, then $\mathrm{CV}[\xi]$ is the solution of the following equation:*
$$(\alpha - \mu)^2 + 2\sigma^2 \ln 2\alpha = 0;$$
if $\mu = 0.5$, then $\mathrm{CV}[\xi] = 0.5$, and if $0.5 < \mu \le 1$, then $\mathrm{CV}[\xi]$ is the solution of the following equation:
$$(\alpha - \mu)^2 - 2\sigma^2 \ln 2(1 - \alpha) = 0.$$

Proof. We only prove (i). The rest can be proved similarly. According to the distribution of ξ, for any $\alpha \in [0, 1]$, we have

$$\mathrm{Pos}\{\xi \ge \alpha\} = \begin{cases} 1, & \text{if } 0 \le \alpha \le \mu \\ \exp\left(-\dfrac{(\alpha-\mu)^2}{2\sigma^2}\right), & \text{if } \mu < \alpha \le 1. \end{cases}$$

It follows from the definition of the optimistic CV that

$$\begin{aligned} \mathrm{CV}^*[\xi] &= \sup_{\alpha \in [0,1]} [\alpha \wedge \mathrm{Pos}\{\xi \ge \alpha\}] \\ &= \sup_{\alpha \in [0,\mu]} [\alpha \wedge 1] \vee \sup_{\alpha \in (\mu,1]} \left[\alpha \wedge \exp\left(-\frac{(\alpha-\mu)^2}{2\sigma^2}\right)\right] \\ &= \mu \vee \sup_{\alpha \in (\mu,1]} \left[\alpha \wedge \exp\left(-\frac{(\alpha-\mu)^2}{2\sigma^2}\right)\right] \\ &= \sup_{\alpha \in (\mu,1]} \left[\alpha \wedge \exp\left(-\frac{(\alpha-\mu)^2}{2\sigma^2}\right)\right]. \end{aligned}$$

Therefore, if $\mu = 1$, then $\mathrm{CV}^*[\xi] = 1$; if $0 \le \mu < 1$, then $\mathrm{CV}^*[\xi]$ is the solution of the following equation:

$$\exp\left(-\frac{(\alpha-\mu)^2}{2\sigma^2}\right) - \alpha = 0,$$

i.e.,

$$(\alpha - \mu)^2 + 2\sigma^2 \ln \alpha = 0.$$

The proof of the theorem is complete. $\square$

Remark 1. The CVs of a normal RFV can be evaluated by the bisection method (see [34]). Consider the following possibility distribution:

$$\mu_\xi(x) = \exp\left(-\frac{(x-0.6)^2}{0.02}\right).$$

Then according to Theorem 2, $\mathrm{CV}^*[\xi]$ is the solution of the following equation:

$$(\alpha-\mu)^2 + 2\sigma^2 \ln\alpha = 0.$$

With the bisection method, we obtain that $\mathrm{CV}^*[\xi] = 0.686699$.

In a similar way, we have $\mathrm{CV}_*[\xi] = 0.484825$, and $\mathrm{CV}[\xi] = 0.552772$.

For the case of a gamma RFV, we have:

Theorem 3. *Let ξ be a gamma RFV with the following possibility distribution:*

$$\mu_\xi(t) = \left(\frac{t}{\lambda r}\right)^r \exp\left(r - \frac{t}{\lambda}\right), \quad t \in (0, 1],$$

where the parameters $0 < r \le 1$, and $0 < \lambda \le 1/r$. Then we have:

(i) $\mathrm{CV}^*[\xi]$ *is the solution of the following equation:*

$$\left(\frac{\alpha}{\lambda r}\right)^r \exp\left(r - \frac{\alpha}{\lambda}\right) - \alpha = 0.$$

(ii) $\mathrm{CV}_*[\xi]$ *is the solution of the following equation:*

$$1 - \left(\frac{\alpha}{\lambda r}\right)^r \exp\left(r - \frac{\alpha}{\lambda}\right) - \alpha = 0.$$

(iii) *If $0 < \lambda r < 0.5$, then $\mathrm{CV}[\xi]$ is the solution of the following equation:*

$$\frac{1}{2}\left(\frac{\alpha}{\lambda r}\right)^r \exp\left(r - \frac{\alpha}{\lambda}\right) - \alpha = 0;$$

if $\lambda r = 0.5$, then $\mathrm{CV}[\xi] = 0.5$, and if $0.5 < \lambda r \le 1$, then $\mathrm{CV}[\xi]$ is the solution of the following equation:

$$1 - \frac{1}{2}\left(\frac{\alpha}{\lambda r}\right)^r \exp\left(r - \frac{\alpha}{\lambda}\right) - \alpha = 0.$$

Proof. We only prove (iii). The rest can be proved similarly. By the distribution of ξ, for any $\alpha \in [0, 1]$, we have

$$\mathrm{Cr}\{\xi \ge \alpha\} = \begin{cases} 1 - \dfrac{1}{2}\left(\dfrac{\alpha}{\lambda r}\right)^r \exp\left(r - \dfrac{\alpha}{\lambda}\right), & \text{if } 0 \le \alpha \le \lambda r \\ \dfrac{1}{2}\left(\dfrac{\alpha}{\lambda r}\right)^r \exp\left(r - \dfrac{\alpha}{\lambda}\right), & \text{if } \lambda r < \alpha \le 1. \end{cases} \tag{7}$$

Thus, the CV of ξ is

$$\begin{aligned} \mathrm{CV}[\xi] &= \sup_{\alpha\in[0,1]} [\alpha \wedge \mathrm{Cr}\{\xi \ge \alpha\}] \\ &= \sup_{\alpha\in[0,\lambda r]} \left[\alpha \wedge \left(1 - \frac{1}{2}\left(\frac{\alpha}{\lambda r}\right)^r \exp\left(r - \frac{\alpha}{\lambda}\right)\right)\right] \vee \sup_{\alpha\in(\lambda r,1]} \left[\alpha \wedge \frac{1}{2}\left(\frac{\alpha}{\lambda r}\right)^r \exp\left(r - \frac{\alpha}{\lambda}\right)\right] \\ &= \begin{cases} \sup\limits_{\alpha\in(\lambda r,1]} \left[\alpha \wedge \dfrac{1}{2}\left(\dfrac{\alpha}{\lambda r}\right)^r \exp\left(r - \dfrac{\alpha}{\lambda}\right)\right], & \text{if } 0 < \lambda r \le 0.5 \\ \sup\limits_{\alpha\in[0,\lambda r]} \left[\alpha \wedge \left(1 - \dfrac{1}{2}\left(\dfrac{\alpha}{\lambda r}\right)^r \exp\left(r - \dfrac{\alpha}{\lambda}\right)\right)\right], & \text{if } 0.5 < \lambda r \le 1. \end{cases} \end{aligned}$$

Consequently, if $0 < \lambda r \le 0.5$, then $\mathrm{CV}[\xi]$ is the solution of the following equation:

$$\frac{1}{2}\left(\frac{\alpha}{\lambda r}\right)^r \exp\left(r - \frac{\alpha}{\lambda}\right) - \alpha = 0,$$

from which we conclude that if $\lambda r = 0.5$, then $\mathrm{CV}[\xi] = 0.5$. If $0.5 < \lambda r \le 1$, then $\mathrm{CV}[\xi]$ is the solution of the following equation:

$$1 - \frac{1}{2}\left(\frac{\alpha}{\lambda r}\right)^r \exp\left(r - \frac{\alpha}{\lambda}\right) - \alpha = 0.$$

The proof of the theorem is complete. □

Remark 2. The CVs of a gamma RFV can be evaluated by the bisection method. Consider the following possibility distribution:

$$\mu_\xi(t) = \left(\frac{t}{0.8}\right)^{0.2} \exp\left(0.2 - \frac{t}{4}\right).$$

Then according to Theorem 3, $\mathrm{CV}[\xi]$ is the solution of the following equation:

$$1 - \frac{1}{2}\left(\frac{t}{0.8}\right)^{0.2} \exp\left(0.2 - \frac{t}{4}\right) - \alpha = 0.$$

Solving the above equation with bisection method, we obtain that $\mathrm{CV}[\xi] = 0.508770$.

In a similar way, we have $\mathrm{CV}^*[\xi] = 0.994903$, and $\mathrm{CV}_*[\xi] = 0.154361$.

4. Methods of reduction for type-2 fuzzy variables

4.1. CV-based reduction methods

Due to the fuzzy membership function of a type-2 fuzzy number, the computation complexity is very high in practical applications. To avoid this difficulty, some defuzzification methods have been proposed in the literature (see [6–8]). In this section, we propose some new methods of reduction for a type-2 fuzzy variable. Compared with the existing methods, the new methods are very much easier to implement when we employ them to build a mathematical model with type-2 fuzzy coefficients.

Let $(\Gamma, \mathcal{A}, \widetilde{\mathrm{P}}os)$ be a fuzzy possibility space and $\tilde{\xi}$ a type-2 fuzzy variable with a known secondary possibility distribution function $\tilde{\mu}_{\tilde{\xi}}(x)$. To reduce the type-2 fuzziness, one approach is to give a representing value for RFV $\tilde{\mu}_{\tilde{\xi}}(x)$. For this purpose, we suggest employing the CVs of $\widetilde{\mathrm{P}}os\{\gamma \in \Gamma \mid \tilde{\xi}(\gamma) = x\}$ as the representing values.

We call the above methods the CV-based methods of reduction for the type-2 fuzzy variable $\tilde{\xi}$.

To specify the proposed reduction methods, we next provide an example as an illustration.

Example 6. Let $\tilde{\xi}$ be the type-2 fuzzy variable defined as

$$\tilde{\xi}(\gamma) = \begin{cases} 3, & \text{with possibility } (0.1, 0.4, 0.7) \\ 4, & \text{with possibility } (0.9, 1, 1) \\ 5, & \text{with possibility } (0.1, 0.3, 0.4, 0.6). \end{cases}$$

That is, $\tilde{\xi}$ takes on the values 3, 4 and 5 with possibilities (0.1, 0.4, 0.7), (0.9, 1, 1) and (0.1, 0.3, 0.4, 0.6), respectively. Since $\tilde{\mu}_{\tilde{\xi}}(3) = (0.1, 0.4, 0.7)$, $\tilde{\mu}_{\tilde{\xi}}(4) = (0.9, 1, 1)$ and $\tilde{\mu}_{\tilde{\xi}}(5) = (0.1, 0.3, 0.4, 0.6)$, it follows from Theorem 1 and its corollary that

$$\mathrm{CV}^*[\tilde{\mu}_{\tilde{\xi}}(3)] = \frac{7}{13}, \qquad \mathrm{CV}^*[\tilde{\mu}_{\tilde{\xi}}(4)] = 1, \qquad \mathrm{CV}^*[\tilde{\mu}_{\tilde{\xi}}(5)] = \frac{1}{2}.$$
$$\mathrm{CV}_*[\tilde{\mu}_{\tilde{\xi}}(3)] = \frac{4}{13}, \qquad \mathrm{CV}_*[\tilde{\mu}_{\tilde{\xi}}(4)] = \frac{10}{11}, \qquad \mathrm{CV}_*[\tilde{\mu}_{\tilde{\xi}}(5)] = \frac{1}{4}.$$
$$\mathrm{CV}[\tilde{\mu}_{\tilde{\xi}}(3)] = \frac{7}{16}, \qquad \mathrm{CV}[\tilde{\mu}_{\tilde{\xi}}(4)] = \frac{11}{12}, \qquad \mathrm{CV}[\tilde{\mu}_{\tilde{\xi}}(5)] = \frac{3}{7}.$$

That is, by the optimistic CV reduction method, the type-2 fuzzy variable $\tilde{\xi}$ is reduced to the following fuzzy variable:

$$\begin{pmatrix} 3 & 4 & 5 \\ 7/13 & 1 & 1/2 \end{pmatrix}.$$

By the pessimistic CV reduction method, $\tilde{\xi}$ is reduced to the following fuzzy variable:

$$\begin{pmatrix} 3 & 4 & 5 \\ 4/13 & 10/11 & 1/4 \end{pmatrix}.$$

By the CV reduction method, $\tilde{\xi}$ is reduced to the following fuzzy variable:

$$\begin{pmatrix} 3 & 4 & 5 \\ 7/16 & 11/12 & 3/7 \end{pmatrix}.$$

In the following, we discuss the reductions for three common type-2 fuzzy variables.

Theorem 4. *Let $\tilde{\xi}$ be a type-2 triangular fuzzy variable defined as $\tilde{\xi} = (\tilde{r}_1, \tilde{r}_2, \tilde{r}_3; \theta_l, \theta_r)$. Then we have:*

(i) *Using the optimistic CV reduction method, the reduction ξ_1 of $\tilde{\xi}$ has the following possibility distribution:*

$$\mu_{\xi_1}(x) = \begin{cases} \dfrac{(1+\theta_r)(x-r_1)}{r_2-r_1+\theta_r(x-r_1)}, & \text{if } x \in \left[r_1, \dfrac{r_1+r_2}{2}\right] \\ \dfrac{(1-\theta_r)x+\theta_r r_2 - r_1}{r_2-r_1+\theta_r(r_2-x)}, & \text{if } x \in \left(\dfrac{r_1+r_2}{2}, r_2\right] \\ \dfrac{(-1+\theta_r)x-\theta_r r_2+r_3}{r_3-r_2+\theta_r(x-r_2)}, & \text{if } x \in \left(r_2, \dfrac{r_2+r_3}{2}\right] \\ \dfrac{(1+\theta_r)(r_3-x)}{r_3-r_2+\theta_r(r_3-x)}, & \text{if } x \in \left(\dfrac{r_2+r_3}{2}, r_3\right]. \end{cases} \tag{8}$$

(ii) *Using the pessimistic CV reduction method, the reduction ξ_2 of $\tilde{\xi}$ has the following possibility distribution:*

$$\mu_{\xi_2}(x) = \begin{cases} \dfrac{x-r_1}{r_2-r_1+\theta_l(x-r_1)}, & \text{if } x \in \left[r_1, \dfrac{r_1+r_2}{2}\right] \\ \dfrac{x-r_1}{r_2-r_1+\theta_l(r_2-x)}, & \text{if } x \in \left(\dfrac{r_1+r_2}{2}, r_2\right] \\ \dfrac{r_3-x}{r_3-r_2+\theta_l(x-r_2)}, & \text{if } x \in \left(r_2, \dfrac{r_2+r_3}{2}\right] \\ \dfrac{r_3-x}{r_3-r_2+\theta_l(r_3-x)}, & \text{if } x \in \left(\dfrac{r_2+r_3}{2}, r_3\right]. \end{cases} \tag{9}$$

(iii) *Using the CV reduction method, the reduction ξ_3 of $\tilde{\xi}$ has the following possibility distribution:*

$$\mu_{\xi_3}(x) = \begin{cases} \dfrac{(1+\theta_r)(x-r_1)}{r_2-r_1+2\theta_r(x-r_1)}, & \text{if } x \in \left[r_1, \dfrac{r_1+r_2}{2}\right] \\ \dfrac{(1-\theta_l)x+\theta_l r_2 - r_1}{r_2-r_1+2\theta_l(r_2-x)}, & \text{if } x \in \left(\dfrac{r_1+r_2}{2}, r_2\right] \\ \dfrac{(-1+\theta_l)x-\theta_l r_2+r_3}{r_3-r_2+2\theta_l(x-r_2)}, & \text{if } x \in \left(r_2, \dfrac{r_2+r_3}{2}\right] \\ \dfrac{(1+\theta_r)(r_3-x)}{r_3-r_2+2\theta_r(r_3-x)}, & \text{if } x \in \left(\dfrac{r_2+r_3}{2}, r_3\right]. \end{cases} \tag{10}$$

Proof. We only prove (i). The rest can be proved similarly. Note that the secondary possibility distribution $\tilde{\mu}_{\tilde{\xi}}(x)$ of $\tilde{\xi}$ is the triangular RFV

$$\left(\frac{x-r_1}{r_2-r_1} - \theta_l \min\left\{\frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1}\right\}, \frac{x-r_1}{r_2-r_1}, \frac{x-r_1}{r_2-r_1} + \theta_r \min\left\{\frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1}\right\}\right)$$

for $x \in [r_1, r_2]$, and

$$\left(\frac{r_3-x}{r_3-r_2} - \theta_l \min\left\{\frac{r_3-x}{r_3-r_2}, \frac{x-r_2}{r_3-r_2}\right\}, \frac{r_3-x}{r_3-r_2}, \frac{r_3-x}{r_3-r_2} + \theta_r \min\left\{\frac{r_3-x}{r_3-r_2}, \frac{x-r_2}{r_3-r_2}\right\}\right)$$

for $x \in (r_2, r_3]$. Let ξ_1 be the reduction of $\tilde{\xi}$ obtained by the optimistic CV reduction method. Then according to Corollary 1, we have

$$\begin{aligned} \mu_{\xi_1}(x) &= \mathrm{Pos}\{\xi_1 = x\} \\ &= \begin{cases} \dfrac{\frac{x-r_1}{r_2-r_1} + \theta_r \min\left\{\frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1}\right\}}{1+\theta_r \min\left\{\frac{x-r_1}{r_2-r_1}, \frac{r_2-x}{r_2-r_1}\right\}}, & \text{if } x \in [r_1, r_2] \\ \dfrac{\frac{r_3-x}{r_3-r_2} + \theta_r \min\left\{\frac{r_3-x}{r_3-r_2}, \frac{x-r_2}{r_3-r_2}\right\}}{1+\theta_r \min\left\{\frac{r_3-x}{r_3-r_2}, \frac{x-r_2}{r_3-r_2}\right\}}, & \text{if } x \in (r_2, r_3] \end{cases} \end{aligned}$$

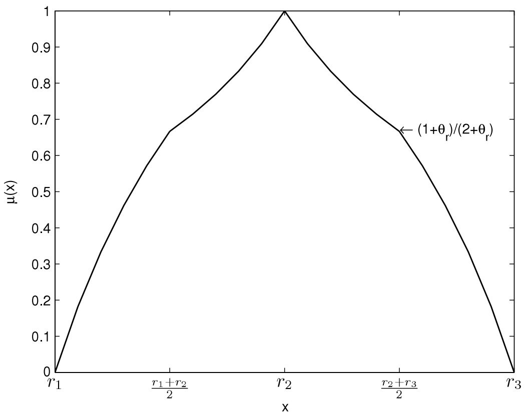

Fig. 1. The possibility distribution $\mu_{\xi_1}(x)$ of ξ_1.

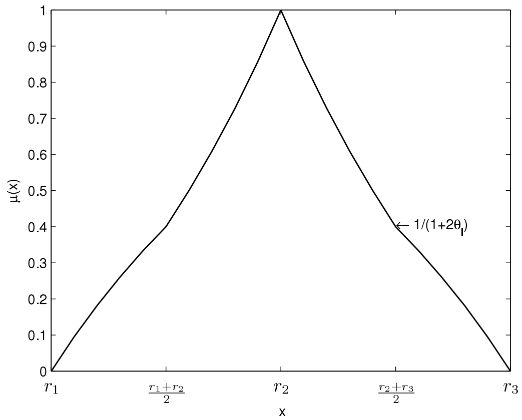

Fig. 2. The possibility distribution $\mu_{\xi_2}(x)$ of ξ_2.

$$= \begin{cases} \dfrac{(1+\theta_r)(x-r_1)}{r_2-r_1+\theta_r(x-r_1)}, & \text{if } x \in \left[r_1, \dfrac{r_1+r_2}{2}\right] \\ \dfrac{(1-\theta_r)x+\theta_r r_2-r_1}{r_2-r_1+\theta_r(r_2-x)}, & \text{if } x \in \left(\dfrac{r_1+r_2}{2}, r_2\right] \\ \dfrac{(-1+\theta_r)x-\theta_r r_2+r_3}{r_3-r_2+\theta_r(x-r_2)}, & \text{if } x \in \left(r_2, \dfrac{r_2+r_3}{2}\right] \\ \dfrac{(1+\theta_r)(r_3-x)}{r_3-r_2+\theta_r(r_3-x)}, & \text{if } x \in \left(\dfrac{r_2+r_3}{2}, r_3\right], \end{cases}$$

which completes the proof of assertion (i). □

The possibility distributions of ξ_1, ξ_2, and ξ_3 are shown in Figs. 1, 2 and 3, respectively.

Example 7. Let $\tilde{\xi} = (\tilde{2}, \tilde{3}, \tilde{4}; 0.5, 1)$, and its support be as shown in Fig. 4. Suppose ξ_1, ξ_2 and ξ_3 are the reductions of $\tilde{\xi}$ obtained by the optimistic, pessimistic and CV reduction methods, respectively. Then according to Theorem 4, we have

$$\mu_{\xi_1}(x) = \begin{cases} 2 - \dfrac{2}{x-1}, & \text{if } x \in \left[2, \dfrac{5}{2}\right] \\ \dfrac{1}{4-x}, & \text{if } x \in \left(\dfrac{5}{2}, 3\right] \\ \dfrac{1}{x-2}, & \text{if } x \in \left(3, \dfrac{7}{2}\right] \\ 2 - \dfrac{2}{5-x}, & \text{if } x \in \left(\dfrac{7}{2}, 4\right], \end{cases}$$

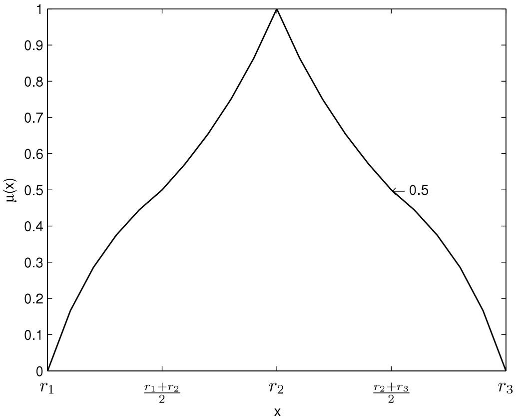

Fig. 3. The possibility distribution $\mu_{\xi_3}(x)$ of ξ_3.

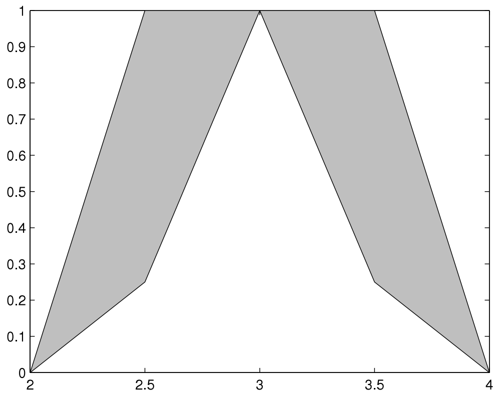

Fig. 4. The support of $\tilde{\xi}$ in Example 7.

$$\mu_{\xi_2}(x) = \begin{cases} 2 - \dfrac{4}{x}, & \text{if } x \in \left[2, \dfrac{5}{2}\right] \\ \dfrac{6}{5-x} - 2, & \text{if } x \in \left(\dfrac{5}{2}, 3\right] \\ \dfrac{6}{x-1} - 2, & \text{if } x \in \left(3, \dfrac{7}{2}\right] \\ 2 - \dfrac{4}{6-x}, & \text{if } x \in \left(\dfrac{7}{2}, 4\right], \end{cases}$$

and

$$\mu_{\xi_3}(x) = \begin{cases} \dfrac{2(x-2)}{2x-3}, & \text{if } x \in \left[2, \dfrac{5}{2}\right] \\ \dfrac{x-1}{8-2x}, & \text{if } x \in \left(\dfrac{5}{2}, 3\right] \\ \dfrac{5-x}{2(x-2)}, & \text{if } x \in \left(3, \dfrac{7}{2}\right] \\ \dfrac{2(4-x)}{9-2x}, & \text{if } x \in \left(\dfrac{7}{2}, 4\right]. \end{cases}$$

Theorem 5. *Let $\tilde{\eta}$ be a type-2 normal fuzzy variable $\tilde{n}(\mu, \sigma^2; \theta_l, \theta_r)$. Then we have:*

(i) *Using the optimistic CV reduction method, the reduction η_1 of $\tilde{\eta}$ has the following possibility distribution:*

$$\mu_{\eta_1}(x) = \begin{cases} \dfrac{(1+\theta_r)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{1+\theta_r\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}, & \text{if } x \le \mu-\sigma\sqrt{2\ln 2} \text{ or } x \ge \mu+\sigma\sqrt{2\ln 2} \\ \dfrac{\theta_r+(1-\theta_r)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{1+\theta_r-\theta_r\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}, & \text{if } \mu-\sigma\sqrt{2\ln 2} < x < \mu+\sigma\sqrt{2\ln 2}. \end{cases}$$

(ii) *Using the pessimistic CV reduction method, the reduction η_2 of $\tilde{\eta}$ has the following possibility distribution:*

$$\mu_{\eta_2}(x) = \begin{cases} \dfrac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{1+\theta_l\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}, & \text{if } x \le \mu-\sigma\sqrt{2\ln 2} \text{ or } x \ge \mu+\sigma\sqrt{2\ln 2} \\ \dfrac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{1+\theta_l-\theta_l\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}, & \text{if } \mu-\sigma\sqrt{2\ln 2} < x < \mu+\sigma\sqrt{2\ln 2}. \end{cases}$$

(iii) *Using the CV reduction method, the reduction η_3 of $\tilde{\eta}$ has the following possibility distribution:*

$$\mu_{\eta_3}(x) = \begin{cases} \dfrac{(1+\theta_r)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{1+2\theta_r\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}, & \text{if } x \le \mu-\sigma\sqrt{2\ln 2} \text{ or } x \ge \mu+\sigma\sqrt{2\ln 2} \\ \dfrac{\theta_l+(1-\theta_l)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{1+2\theta_l-2\theta_l\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}, & \text{if } \mu-\sigma\sqrt{2\ln 2} < x < \mu+\sigma\sqrt{2\ln 2}. \end{cases}$$

Proof. We only prove (iii). The rest can be proved similarly. Note that the secondary possibility distribution $\tilde{\mu}_{\tilde{\eta}}(x)$ of $\tilde{\eta}$ is the triangular RFV

$$\left(\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)-\theta_l\min\left\{1-\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\},\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\right.$$
$$\left.\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)+\theta_r\min\left\{1-\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\}\right)$$

for any $x \in \Re$. If we denote η_3 as the reduction of $\tilde{\eta}$ obtained by the CV reduction method, then by Corollary 1, we have

$$\begin{aligned} \mu_{\eta_3}(x) &= \text{Pos}\{\eta_3 = x\} \\ &= \begin{cases} \dfrac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)+\theta_r\min\left\{1-\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\}}{1+2\theta_r\min\left\{1-\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\}}, & \text{if } \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right) \le \dfrac{1}{2} \\ \dfrac{\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)+\theta_l\min\left\{1-\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\}}{1+2\theta_l\min\left\{1-\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)\right\}}, & \text{if } \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right) > \dfrac{1}{2} \end{cases} \\ &= \begin{cases} \dfrac{(1+\theta_r)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{1+2\theta_r\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}, & \text{if } x \le \mu-\sigma\sqrt{2\ln 2} \text{ or } x \ge \mu+\sigma\sqrt{2\ln 2} \\ \dfrac{\theta_l+(1-\theta_l)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}{1+2\theta_l-2\theta_l\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)}, & \text{if } \mu-\sigma\sqrt{2\ln 2} < x < \mu+\sigma\sqrt{2\ln 2}. \end{cases} \end{aligned}$$

The proof of the theorem is complete. □

The possibility distributions of RFVs η_1, η_2 and η_3 are plotted in Figs. 5, 6 and 7, respectively.

Example 8. Let $\tilde{\eta}$ be the type-2 normal fuzzy variable $\tilde{n}(3, 1; 0.5, 1)$, and its support be as shown in Fig. 8. Suppose η_1, η_2 and η_3 are the reductions of $\tilde{\eta}$ obtained by the optimistic, pessimistic and CV reduction methods, respectively. Then according

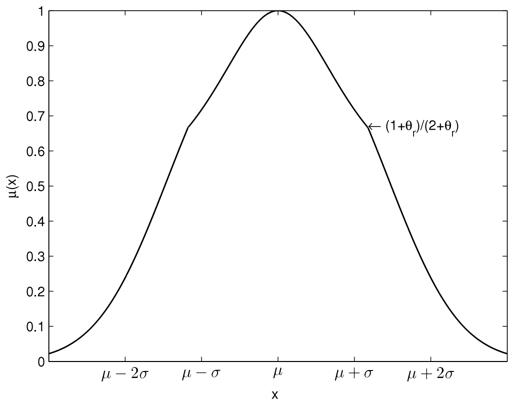

Fig. 5. The possibility distribution $\mu_{\eta_1}(x)$ of η_1.

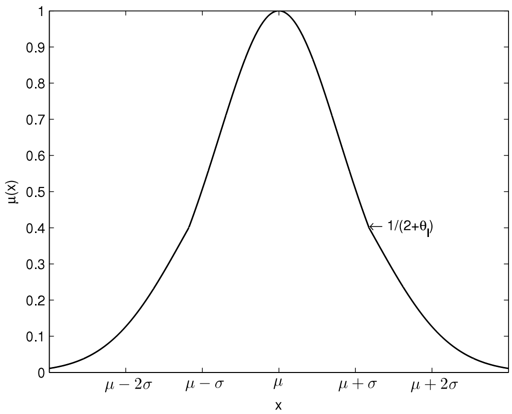

Fig. 6. The possibility distribution $\mu_{\eta_2}(x)$ of η_2.

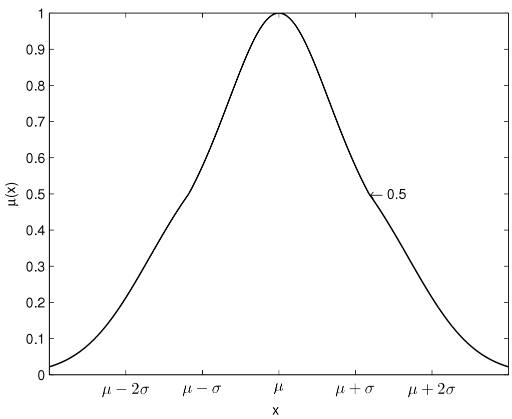

Fig. 7. The possibility distribution $\mu_{\eta_3}(x)$ of η_3.

to Theorem 5, we have

$$\mu_{\eta_1}(x) = \begin{cases} \dfrac{2\exp\left(-\frac{(x-3)^2}{2}\right)}{1+\exp\left(-\frac{(x-3)^2}{2}\right)}, & \text{if } x \le 3-\sqrt{2\ln 2} \text{ or } x \ge 3+\sqrt{2\ln 2} \\ \dfrac{1}{2-\exp\left(-\frac{(x-3)^2}{2}\right)}, & \text{if } 3-\sqrt{2\ln 2} < x < 3+\sqrt{2\ln 2}, \end{cases}$$

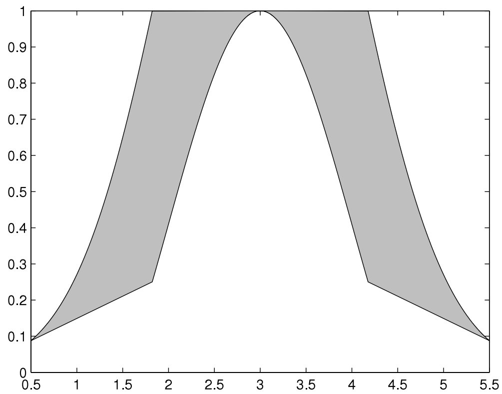

Fig. 8. The support of $\tilde{\eta}$ in Example 8.

$$\mu_{\eta_2}(x) = \begin{cases} \dfrac{2\exp\left(-\frac{(x-3)^2}{2}\right)}{2+\exp\left(-\frac{(x-3)^2}{2}\right)}, & \text{if } x \le 3-\sqrt{2\ln 2} \text{ or } x \ge 3+\sqrt{2\ln 2} \\ \dfrac{2\exp\left(-\frac{(x-3)^2}{2}\right)}{3-\exp\left(-\frac{(x-3)^2}{2}\right)}, & \text{if } 3-\sqrt{2\ln 2} < x < 3+\sqrt{2\ln 2}, \end{cases}$$

and

$$\mu_{\eta_3}(x) = \begin{cases} \dfrac{2\exp\left(-\frac{(x-3)^2}{2}\right)}{1+2\exp\left(-\frac{(x-3)^2}{2}\right)}, & \text{if } x \le 3-\sqrt{2\ln 2} \text{ or } x \ge 3+\sqrt{2\ln 2} \\ \dfrac{1+\exp\left(-\frac{(x-3)^2}{2}\right)}{4-2\exp\left(-\frac{(x-3)^2}{2}\right)}, & \text{if } 3-\sqrt{2\ln 2} < x < 3+\sqrt{2\ln 2}. \end{cases}$$

Theorem 6. *Let $\tilde{\xi}$ be a type-2 gamma fuzzy variable defined as $\tilde{\gamma}(\lambda, r; \theta_l, \theta_r)$. Then we have:*

(i) *Using the optimistic CV reduction method, the reduction ξ_1 of $\tilde{\xi}$ has the following possibility distribution:*

$$\mu_{\xi_1}(x) = \begin{cases} \dfrac{(1+\theta_r)\left(\frac{x}{\lambda r}\right)^r \exp(r-\frac{x}{\lambda})}{1+\theta_r\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}, & \text{if } \left(\dfrac{x}{\lambda r}\right)^r \exp\left(r-\dfrac{x}{\lambda}\right) \le \dfrac{1}{2} \\ \dfrac{\theta_r+(1-\theta_r)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}{1+\theta_r-\theta_r\left(\frac{x}{\lambda r}\right)^r \exp(r-\frac{x}{\lambda})}, & \text{if } \left(\dfrac{x}{\lambda r}\right)^r \exp\left(r-\dfrac{x}{\lambda}\right) > \dfrac{1}{2}. \end{cases}$$

(ii) *Using the pessimistic CV reduction method, the reduction ξ_2 of $\tilde{\xi}$ has the following possibility distribution:*

$$\mu_{\xi_2}(x) = \begin{cases} \dfrac{\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}{1+\theta_l\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}, & \text{if } \left(\dfrac{x}{\lambda r}\right)^r \exp\left(r-\dfrac{x}{\lambda}\right) \le \dfrac{1}{2} \\ \dfrac{\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}{1+\theta_l-\theta_l\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}, & \text{if } \left(\dfrac{x}{\lambda r}\right)^r \exp\left(r-\dfrac{x}{\lambda}\right) > \dfrac{1}{2}. \end{cases}$$

(iii) *Using the CV reduction method, the reduction ξ_3 of $\tilde{\xi}$ has the following possibility distribution:*

$$\mu_{\xi_3}(x) = \begin{cases} \dfrac{(1+\theta_r)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}{1+2\theta_r\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}, & \text{if } \left(\dfrac{x}{\lambda r}\right)^r \exp\left(r-\dfrac{x}{\lambda}\right) \le \dfrac{1}{2} \\ \dfrac{\theta_l+(1-\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(1-\frac{x}{\lambda}\right)}{1+2\theta_l-2\theta_l\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}, & \text{if } \left(\dfrac{x}{\lambda r}\right)^r \exp\left(r-\dfrac{x}{\lambda}\right) > \dfrac{1}{2}. \end{cases}$$

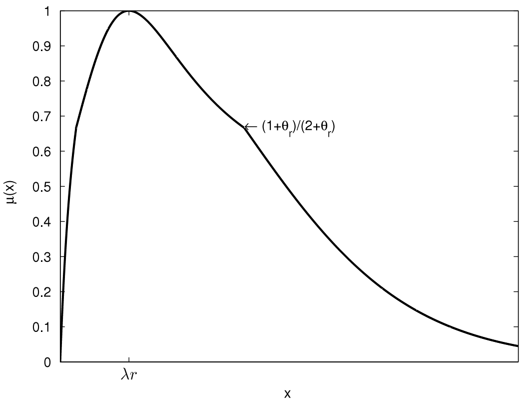

Fig. 9. The possibility distribution $\mu_{\zeta_1}(x)$ of ζ_1.

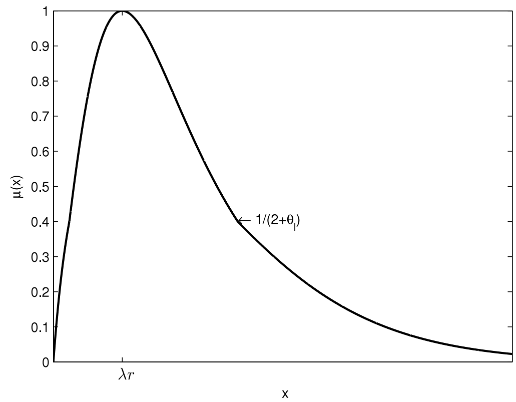

Fig. 10. The possibility distribution $\mu_{\zeta_2}(x)$ of ζ_2.

Proof. We only prove (iii). The rest can be proved similarly. Note that the secondary possibility distribution $\tilde{\mu}_{\tilde{\zeta}}(x)$ of $\tilde{\zeta}$ is the triangular RFV

$$\left(\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right) - \theta_l \min\left\{1-\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right\}, \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right),\right.$$
$$\left.\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right) + \theta_r \min\left\{1-\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right\}\right)$$

for any $x \in \Re$. Then according to Corollary 1, the distribution of ζ_3 is as follows:

$$\begin{aligned}
\mu_{\zeta_3}(x) &= \mathrm{Pos}\{\zeta_3 = x\} \\
&= \begin{cases}
\dfrac{\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right) + \theta_r \min\left\{1-\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right\}}{1+2\theta_r \min\left\{1-\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right\}}, & \text{if } \left(\dfrac{x}{\lambda r}\right)^r \exp\left(-\dfrac{x}{\lambda}\right) \le \dfrac{1}{2} \\
\dfrac{\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right) + \theta_l \min\left\{1-\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right\}}{1+2\theta_l \min\left\{1-\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right), \left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)\right\}}, & \text{if } \left(\dfrac{x}{\lambda r}\right)^r \exp\left(r-\dfrac{x}{\lambda}\right) > \dfrac{1}{2}
\end{cases} \\
&= \begin{cases}
\dfrac{(1+\theta_r)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}{1+2\theta_r\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}, & \text{if } \left(\dfrac{x}{\lambda r}\right)^r \exp\left(r-\dfrac{x}{\lambda}\right) \le \dfrac{1}{2} \\
\dfrac{\theta_l + (1-\theta_l)\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}{1+2\theta_l-2\theta_l\left(\frac{x}{\lambda r}\right)^r \exp\left(r-\frac{x}{\lambda}\right)}, & \text{if } \left(\dfrac{x}{\lambda r}\right)^r \exp\left(r-\dfrac{x}{\lambda}\right) > \dfrac{1}{2},
\end{cases}
\end{aligned}$$

which completes the proof of assertion (iii). □

The possibility distributions of ζ_1, ζ_2, and ζ_3 are as described in Figs. 9, 10 and 11, respectively.

Example 9. Let $\tilde{\zeta}$ be the type-2 gamma fuzzy variable $\tilde{\gamma}(5, 1; 0.5, 0.8)$, and its support be as shown in Fig. 12. Suppose ζ_1, ζ_2 and ζ_3 are the reductions of $\tilde{\zeta}$ obtained by the optimistic, pessimistic and CV reduction methods, respectively. Then

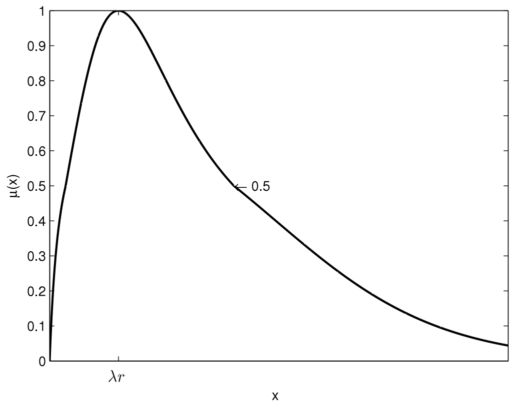

Fig. 11. The possibility distribution $\mu_{\zeta_3}(x)$ of ζ_3.

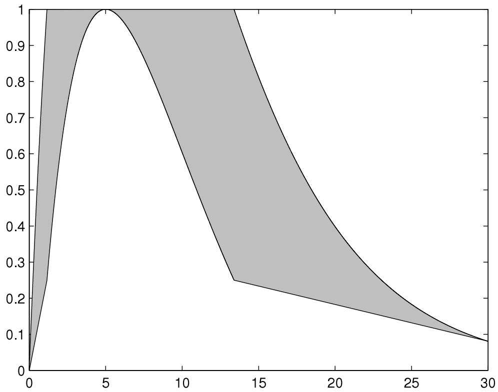

Fig. 12. The support of $\tilde{\zeta}$ in Example 9.

according to Theorem 6, we have

$$\mu_{\zeta_1}(x)=\begin{cases}\dfrac{9x\exp\left(1-\frac{x}{5}\right)}{25+4x\exp\left(1-\frac{x}{5}\right)}, & \text{if } \dfrac{x}{5}\exp\left(1-\dfrac{x}{5}\right)\le\dfrac{1}{2}\\ \dfrac{20+x\exp\left(1-\frac{x}{5}\right)}{45-4x\exp\left(1-\frac{x}{5}\right)}, & \text{if } \dfrac{x}{5}\exp\left(1-\dfrac{x}{5}\right)>\dfrac{1}{2},\end{cases}$$

$$\mu_{\zeta_2}(x)=\begin{cases}\dfrac{2x\exp\left(1-\frac{x}{5}\right)}{10+x\exp\left(1-\frac{x}{5}\right)}, & \text{if } \dfrac{x}{5}\exp\left(1-\dfrac{x}{5}\right)\le\dfrac{1}{2}\\ \dfrac{2x\exp\left(1-\frac{x}{5}\right)}{15-x\exp\left(1-\frac{x}{5}\right)}, & \text{if } \dfrac{x}{5}\exp\left(1-\dfrac{x}{5}\right)>\dfrac{1}{2},\end{cases}$$

and

$$\mu_{\zeta_3}(x)=\begin{cases}\dfrac{9x\exp\left(1-\frac{x}{5}\right)}{25+8x\exp\left(1-\frac{x}{5}\right)}, & \text{if } \dfrac{x}{5}\exp\left(1-\dfrac{x}{5}\right)\le\dfrac{1}{2}\\ \dfrac{5+x\exp\left(1-\frac{x}{5}\right)}{20-2x\exp\left(1-\frac{x}{5}\right)}, & \text{if } \dfrac{x}{5}\exp\left(1-\dfrac{x}{5}\right)>\dfrac{1}{2}.\end{cases}$$

4.2. Generalized credibility and its properties

As shown in Example 6, the fuzzy variables obtained via CV-based reduction methods aren't always normalized. In such cases, the credibility measure defined in [33] couldn't be used in the current development; it is necessary to extend the concept to general fuzzy variables.

Suppose ξ is a general fuzzy variable with the distribution μ. The generalized credibility measure $\tilde{\mathrm{C}}r$ of the event $\{\xi \geq r\}$ is defined by

$$\tilde{\mathrm{C}}r\{\xi \geq r\} = \frac{1}{2}\left(\sup_{x\in\Re}\mu(x) + \sup_{x\geq r}\mu(x) - \sup_{x<r}\mu(x)\right), \quad r \in \Re.$$

Therefore, if ξ is normalized, it is easy to check that $\mathrm{Cr}\{\xi \geq r\} + \mathrm{Cr}\{\xi < r\} = \sup_{x\in\Re}\mu_\xi(x) = 1$; then $\tilde{\mathrm{C}}r$ coincides with the usual credibility measure.

The concept of independence for normalized fuzzy variables and its properties were discussed in [35]. In the following, we also need to extend independence to general fuzzy variables.

The general fuzzy variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be mutually independent if and only if

$$\tilde{\mathrm{C}}r\{\xi_i \in B_i, i = 1, 2, \ldots, n\} = \min_{1\leq i\leq n}\tilde{\mathrm{C}}r\{\xi_i \in B_i\}$$

for any subsets B_i, $i = 1, 2, \ldots, n$ of $\Re$.

Like the α-optimistic value of the normalized fuzzy variable [36], the α-optimistic value of general fuzzy variables can be defined through the generalized credibility measure. Let ξ be a fuzzy variable (not necessary normalized). Then

$$\xi_{\sup}(\alpha) = \sup\left\{r \mid \tilde{\mathrm{C}}r\,\{\xi \geq r\} \geq \alpha\right\}, \quad \alpha \in (0, 1]$$

is called the α-optimistic value of ξ, while

$$\xi_{\inf}(\alpha) = \inf\left\{r \mid \tilde{\mathrm{C}}r\,\{\xi \leq r\} \geq \alpha\right\}, \quad \alpha \in (0, 1]$$

is called the α-pessimistic value of ξ.

In the following, we discuss the fundamental properties of generalized credibility.

Theorem 7. *Let ξ_i be the reduction of the type-2 fuzzy variable $\tilde{\xi}_i = (\tilde{r}_1^i, \tilde{r}_2^i, \tilde{r}_3^i; \theta_{l,i}, \theta_{r,i})$ obtained by the CV reduction method for $i = 1, 2, \ldots, n$. Suppose $\xi_1, \xi_2, \ldots, \xi_n$ are mutually independent, and $k_i \geq 0$ for $i = 1, 2, \ldots, n$.*

(i) *Given the generalized credibility level $\alpha \in (0, 0.5]$, if $\alpha \in (0, 0.25]$, then $\tilde{\mathrm{C}}r\{\sum_{i=1}^n k_i\xi_i \leq t\} \geq \alpha$ is equivalent to*

$$\sum_{i=1}^{n}\frac{(1-2\alpha+(1-4\alpha)\theta_{r,i})k_ir_1^i + 2\alpha k_ir_2^i}{1+(1-4\alpha)\theta_{r,i}} \leq t,$$

and if $\alpha \in (0.25, 0.5]$, then $\tilde{\mathrm{C}}r\{\sum_{i=1}^n k_i\xi_i \leq t\} \geq \alpha$ is equivalent to

$$\sum_{i=1}^{n}\frac{(1-2\alpha)k_ir_1^i + (2\alpha+(4\alpha-1)\theta_{l,i})k_ir_2^i}{1+(4\alpha-1)\theta_{l,i}} \leq t.$$

(ii) *Given the generalized credibility level $\alpha \in (0.5, 1]$, if $\alpha \in (0.5, 0.75]$, then $\tilde{\mathrm{C}}r\{\sum_{i=1}^n k_i\xi_i \leq t\} \geq \alpha$ is equivalent to*

$$\sum_{i=1}^{n}\frac{(2\alpha-1)k_ir_3^i + (2(1-\alpha)+(3-4\alpha)\theta_{l,i})k_ir_2^i}{1+(3-4\alpha)\theta_{l,i}} \leq t,$$

and if $\alpha \in (0.75, 1]$, then $\tilde{\mathrm{C}}r\{\sum_{i=1}^n k_i\xi_i \leq t\} \geq \alpha$ is equivalent to

$$\sum_{i=1}^{n}\frac{(2\alpha-1+(4\alpha-3)\theta_{r,i})k_ir_3^i + 2(1-\alpha)k_ir_2^i}{1+(4\alpha-3)\theta_{r,i}} \leq t.$$

Proof. We only prove (ii), as (i) can be proved similarly. For each $i = 1, 2, \ldots, n$, since ξ_i is the reduction of the type-2 triangular fuzzy variable $\tilde{\xi}_i$ obtained by the CV reduction method, we know that the fuzzy variable ξ_i has the following possibility distribution:

$$\mu_{\xi_i}(x) = \begin{cases} \dfrac{(1+\theta_{r,i})(x-r_1^i)}{r_2^i - r_1^i + 2\theta_{r,i}(x-r_1^i)}, & \text{if } x \in \left[r_1^i, \dfrac{r_1^i+r_2^i}{2}\right] \\ \dfrac{(1-\theta_{l,i})x + \theta_{l,i}r_2^i - r_1^i}{r_2^i - r_1^i + 2\theta_{l,i}(r_2^i - x)}, & \text{if } x \in \left(\dfrac{r_1^i+r_2^i}{2}, r_2^i\right] \\ \dfrac{(-1+\theta_{l,i})x - \theta_{l,i}r_2^i + r_3^i}{r_3^i - r_2^i + 2\theta_{l,i}(x-r_2^i)}, & \text{if } x \in \left(r_2^i, \dfrac{r_2^i+r_3^i}{2}\right] \\ \dfrac{(1+\theta_{r,i})(r_3^i - x)}{r_3^i - r_2^i + 2\theta_{r,i}(r_3^i - x)}, & \text{if } x \in \left(\dfrac{r_2^i+r_3^i}{2}, r_3^i\right] \end{cases}$$

for $i = 1, 2, \ldots, n$.

Write $\xi = \sum_{i=1}^{n} k_i\xi_i$. If $\alpha > 0.5$, then we have

$$\tilde{\mathrm{C}}\mathrm{r}\{\xi \le t\} = \frac{1}{2}\left(1 + \sup_{x\le t}\mu_\xi(x) - \sup_{x>t}\mu_\xi(x)\right) = \frac{1}{2}\left(1 + 1 - \sup_{x>t}\mu_\xi(x)\right).$$

Therefore $\tilde{\mathrm{C}}\mathrm{r}\{\xi \le t\} \ge \alpha$ is equivalent to

$$\sup_{x>t}\mu_\xi(x) \le 2 - 2\alpha.$$

If we define $\xi_{\sup}(\alpha) = \sup\{r \mid \sup_{x>r}\mu_\xi(x) \ge \alpha\}$ for $\alpha \in (0, 1]$, then we have

$$\xi_{\sup}(2-2\alpha) \le t.$$

Since $\xi_1, \xi_2, \ldots, \xi_n$ are mutually independent, we have

$$\xi_{\sup}(2-2\alpha) = \left(\sum_{i=1}^{n} k_i\xi_i\right)_{\sup}(2-2\alpha) = \sum_{i=1}^{n} k_i\xi_{i,\sup}(2-2\alpha) \le t.$$

Note that $\mu_{\xi_i}((r_2^i + r_3^i)/2) = 0.5$. If $2 - 2\alpha \ge 0.5$, i.e., $\alpha \in (0.5, 0.75]$, then for each i, $\xi_{i,\sup}(2-2\alpha)$ is the solution of the following equation:

$$\frac{(-1+\theta_{l,i})x - \theta_{l,i}r_2^i + r_3^i}{r_3^i - r_2^i + 2\theta_{l,i}(x - r_2^i)} = 2 - 2\alpha.$$

Solving the above equation, we have

$$\xi_{i,\sup}(2-2\alpha) = \frac{(2\alpha-1)r_3^i + (2(1-\alpha) + (3-4\alpha)\theta_{l,i})r_2^i}{1 + (3-4\alpha)\theta_{l,i}}.$$

Therefore, when $\alpha \in (0.5, 0.75]$, $\tilde{\mathrm{C}}\mathrm{r}\{\sum_{i=1}^{n} k_i\xi_i \le t\} \ge \alpha$ is equivalent to

$$\sum_{i=1}^{n} \frac{(2\alpha-1)k_ir_3^i + (2(1-\alpha) + (3-4\alpha)\theta_{l,i})k_ir_2^i}{1 + (3-4\alpha)\theta_{l,i}} \le t.$$

On other hand, if $2 - 2\alpha < 0.5$, i.e., $\alpha \in (0.75, 1]$, then for each i, $\xi_{i,\sup}(2-2\alpha)$ is the solution of the following equation:

$$\frac{(1+\theta_{r,i})(r_3^i - x)}{r_3^i - r_2^i + 2\theta_{r,i}(r_3^i - x)} = 2 - 2\alpha.$$

Solving the above equation gives

$$\xi_{i,\sup}(2-2\alpha) = \frac{(2\alpha - 1 + (4\alpha-3)\theta_{r,i})r_3^i + 2(1-\alpha)r_2^i}{1 + (4\alpha-3)\theta_{r,i}}.$$

Therefore, when $\alpha \in (0.75, 1]$, $\tilde{\mathrm{C}}\mathrm{r}\{\sum_{i=1}^{n} k_i\xi_i \le t\} \ge \alpha$ is equivalent to

$$\sum_{i=1}^{n} \frac{(2\alpha - 1 + (4\alpha-3)\theta_{r,i})k_ir_3^i + 2(1-\alpha)k_ir_2^i}{1 + (4\alpha-3)\theta_{r,i}} \le t.$$

The proof of the theorem is complete. □

Theorem 8. *Let ξ_i be the reduction of the type-2 fuzzy variable $\tilde{\xi}_i = \tilde{n}(\mu_i, \sigma_i^2; \theta_{l,i}, \theta_{r,i})$ obtained by the CV reduction method for $i = 1, 2, \ldots, n$. Suppose $\xi_1, \xi_2, \ldots, \xi_n$ are mutually independent, and $k_i \ge 0$ for $i = 1, 2, \ldots, n$.*

(i) *Given the generalized credibility level $\alpha \in (0, 0.5]$, if $\alpha \in (0, 0.25]$, then $\tilde{\mathrm{C}}\mathrm{r}\{\sum_{i=1}^{n} k_i\xi_i \le t\} \ge \alpha$ is equivalent to*

$$\sum_{i=1}^{n} k_i\left(\mu_i - \sigma_i\sqrt{2\ln(1 + (1-4\alpha)\theta_{r,i}) - 2\ln 2\alpha}\right) \le t,$$

and if $\alpha \in (0.25, 0.5]$, then $\tilde{\mathrm{C}}\mathrm{r}\{\sum_{i=1}^{n} k_i\xi_i \le t\} \ge \alpha$ is equivalent to

$$\sum_{i=1}^{n} k_i\left(\mu_i - \sigma_i\sqrt{2\ln(1 + (4\alpha-1)\theta_{l,i}) - 2\ln(2\alpha + (4\alpha-1)\theta_{l,i})}\right) \le t.$$

(ii) *Given the generalized credibility level* $\alpha \in (0.5, 1]$, *if* $\alpha \in (0.5, 0.75]$, *then* $\tilde{C}r\{\sum_{i=1}^{n} k_i\xi_i \le t\} \ge \alpha$ *is equivalent to*

$$\sum_{i=1}^{n} k_i \left(\mu_i + \sigma_i\sqrt{2\ln(1+(3-4\alpha)\theta_{l,i}) - 2\ln 2(1-\alpha) + (3-4\alpha)\theta_{l,i}}\right) \le t,$$

and if $\alpha \in (0.75, 1]$, *then* $\tilde{C}r\{\sum_{i=1}^{n} k_i\xi_i \le t\} \ge \alpha$ *is equivalent to*

$$\sum_{i=1}^{n} k_i \left(\mu_i + \sigma_i\sqrt{2\ln(1+(4\alpha-3)\theta_{r,i}) - 2\ln 2(1-\alpha)}\right) \le t.$$

Proof. We only prove (i), as (ii) can be proved similarly. For each $i = 1, 2, \dots, n$, since ξ_i is the reduction of the type-2 normal fuzzy variable $\tilde{\xi}_i$ obtained by the CV reduction method, we know that the fuzzy variable ξ_i has the following possibility distribution:

$$\mu_{\xi_i}(x) = \begin{cases} \dfrac{(1+\theta_{r,i})\exp\left(-\frac{(x-\mu_i)^2}{2\sigma_i^2}\right)}{1+2\theta_{r,i}\exp\left(-\frac{(x-\mu_i)^2}{2\sigma_i^2}\right)}, & \text{if } x \le \mu_i - \sigma_i\sqrt{2\ln 2} \text{ or } x \ge \mu_i + \sigma_i\sqrt{2\ln 2} \\ \dfrac{\theta_{l,i} + (1-\theta_{l,i})\exp\left(-\frac{(x-\mu_i)^2}{2\sigma_i^2}\right)}{1+2\theta_{l,i} - 2\theta_{l,i}\exp\left(-\frac{(x-\mu_i)^2}{2\sigma_i^2}\right)}, & \text{if } \mu_i - \sigma_i\sqrt{2\ln 2} < x < \mu_i + \sigma_i\sqrt{2\ln 2} \end{cases}$$

for $i = 1, 2, \dots, n$.

Write $\xi = \sum_{i=1}^{n} k_i\xi_i$. Then for $\alpha \le 0.5$, one has

$$\tilde{C}r\{\xi \le t\} = \frac{1}{2}\left(1 + \sup_{x\le t}\mu_\xi(x) - \sup_{x>t}\mu_\xi(x)\right) = \frac{1}{2}\left(1 + \sup_{x\le t}\mu_\xi(x) - 1\right) = \frac{1}{2}\sup_{x\le t}\mu_\xi(x).$$

Therefore, $\tilde{C}r\{\xi \le t\} \ge \alpha$ is equivalent to

$$\sup_{x\le t}\mu_\xi(x) \ge 2\alpha.$$

If we define $\xi_{\inf}(\alpha) = \inf\{r \mid \sup_{x\le r}\mu_\xi(x) \ge \alpha\}$ for $\alpha \in (0, 1]$, then we have

$$\xi_{\inf}(2\alpha) \le t.$$

Since $\xi_1, \xi_2, \dots, \xi_n$ are mutually independent, we have

$$\xi_{\inf}(2\alpha) = \left(\sum_{i=1}^{n} k_i\xi_i\right)_{\inf}(2\alpha) = \sum_{i=1}^{n} k_i\xi_{i,\inf}(2\alpha) \le t.$$

Note that $\mu_{\xi_i}(\mu_i - \sigma_i\sqrt{2\ln 2}) = 0.5$. If $2\alpha \le 0.5$, i.e., $\alpha \in (0, 0.25]$, then for each i, $\xi_{i,\inf}(2\alpha)$ is the solution of the following equation:

$$\frac{(1+\theta_{r,i})\exp\left(-\frac{(x-\mu_i)^2}{2\sigma_i^2}\right)}{1+2\theta_{r,i}\exp\left(-\frac{(x-\mu_i)^2}{2\sigma_i^2}\right)} = 2\alpha.$$

Solving the above equation, we have

$$\xi_{i,\inf}(2\alpha) = \mu_i - \sigma_i\sqrt{2\ln(1+(1-4\alpha)\theta_{r,i}) - 2\ln 2\alpha}.$$

Therefore, if $\alpha \in (0, 0.25]$, then $\tilde{C}r\{\sum_{i=1}^{n} k_i\xi_i \le t\} \ge \alpha$ is equivalent to

$$\sum_{i=1}^{n} k_i(\mu_i - \sigma_i\sqrt{2\ln(1+(1-4\alpha)\theta_{r,i}) - 2\ln 2\alpha}) \le t.$$

On other hand, if $2\alpha > 0.5$, i.e., $\alpha \in (0.25, 0.5]$, then for each i, $\xi_{i,\inf}(2\alpha)$ is the solution of the following equation:

$$\frac{\theta_{l,i} + (1-\theta_{l,i})\exp\left(-\frac{(x-\mu_i)^2}{2\sigma_i^2}\right)}{1+2\theta_{l,i} - 2\theta_{l,i}\exp\left(-\frac{(x-\mu_i)^2}{2\sigma_i^2}\right)} = 2\alpha.$$

Solving the above equation, we have

$$\xi_{i,\inf}(2\alpha) = \mu_i - \sigma_i\sqrt{2\ln(1+(4\alpha-1)\theta_{l,i}) - 2\ln(2\alpha+(4\alpha-1)\theta_{l,i})}.$$

Therefore, if $\alpha \in (0.25, 0.5]$, then $\tilde{\mathrm{C}}r\{\sum_{i=1}^n k_i\xi_i \le t\} \ge \alpha$ is equivalent to

$$\sum_{i=1}^n k_i\left(\mu_i - \sigma_i\sqrt{2\ln(1+(4\alpha-1)\theta_{l,i}) - 2\ln(2\alpha+(4\alpha-1)\theta_{l,i})}\right) \le t.$$

The proof of the theorem is complete. □

5. Formulation of generalized credibility DEA models

DEA is a method for assessing the productive efficiency of DMUs which use the same kinds of resources (inputs) to produce the same kinds of goods or services (outputs). The traditional CCR model [10] was built as

$$\begin{cases} \max\limits_{u,v} & \dfrac{v^T y_0}{u^T x_0} \\ \text{subject to} & \dfrac{v^T y_i}{u^T x_i} \le 1, i = 1, 2, \ldots, n \\ & u \ge 0, u \ne 0 \\ & v \ge 0, v \ne 0. \end{cases} \tag{11}$$

The CCR model (11) can be used to evaluate the efficiency for each DMU when the inputs and outputs are known precisely. However, in many cases, the data cannot be known with certainty, and very often one can only obtain incomplete information about the data such as the distributions. In this section, we assume that we can only obtain the type-2 distributions of the data, i.e., the inputs and outputs are characterized by type-2 fuzzy variables. In such a case, the CCR model (11) becomes

$$\begin{cases} \max\limits_{u,v} & \dfrac{v^T \tilde{\eta}_0}{u^T \tilde{\xi}_0} \\ \text{subject to} & \dfrac{v^T \tilde{\eta}_i}{u^T \tilde{\xi}_i} \le 1, i = 1, 2, \ldots, n \\ & u \ge 0, u \ne 0 \\ & v \ge 0, v \ne 0, \end{cases} \tag{12}$$

where the $\tilde{\xi}_i$ $(i = 1, 2, \ldots, n)$ represent the type-2 fuzzy input column vector of DMU$_i$, $\tilde{\xi}_0$ represents the type-2 fuzzy input column vector of DMU$_0$; the $\tilde{\eta}_i$ $(i = 1, 2, \ldots, n)$ represent the type-2 fuzzy output column vector of DMU$_i$, and $\tilde{\eta}_0$ represents the type-2 fuzzy output column vector of DMU$_0$.

However, problem (12) is not well-defined since the meanings of "max" as well as of the constraints are not clear at all, if we think of taking a decision before knowing the values of uncertain parameters involved in the problem. Therefore a revision of the modeling process is necessary, by applying the proposed CV-based reduction method, leading to the following generalized credibility DEA model:

$$\begin{cases} \max\limits_{u,\,v} & \bar{f} \\ \text{subject to} & \tilde{\mathrm{C}}r\left\{\dfrac{v^T \eta_0}{u^T \xi_0} \ge \bar{f}\right\} \ge \alpha_0 \\ & \tilde{\mathrm{C}}r\left\{-u^T\xi_i + v^T\eta_i \le 0\right\} \ge \alpha_i, i = 1, 2, \ldots, n \\ & u \ge 0, u \ne 0 \\ & v \ge 0, v \ne 0, \end{cases} \tag{13}$$

where the notation in the model (13) are collected in Table 1.

In traditional CCR model (11), we use the value of $v^T y_0/u^T x_0$ to illustrate the efficiency of DMU$_0$. DMU$_0$ is efficient if and only if the optimal value is equal to 1 and there exists at least one optimal solution (u^*, v^*) with $u^* > 0$, $v^* > 0$. But in model (13), we cannot define such efficiency due to the existence of uncertainty. So in model (13), we use the optimal value $\bar{f}$ as the α_0-efficient value of DMU$_0$ to illustrate the efficiency of DMU$_0$, and the bigger the value is, the more efficient it is.

In the following, we focus our attention on the equivalent forms of model (13) in some special cases.

Table 1
List of notation for model (13).

Notation	Definition
ξ_i	the reduction of $\tilde{\xi}_i$ according to the CV-based reduction methods, $i = 1, 2, \ldots, n$
ξ_0	the reduction of $\tilde{\xi}_0$ according to the CV-based reduction methods
η_i	the reduction of $\tilde{\eta}_i$ according to the CV-based reduction methods, $i = 1, 2, \ldots, n$
η_0	the reduction of $\tilde{\eta}_0$ according to the CV-based reduction methods
$u \in \Re^m$	the weights of the type-2 fuzzy input column vector
v$\in \Re^s$	the weights of the type-2 fuzzy output column vector
$\alpha_i \in (0, 1]$	the predetermined generalized credibility levels, $i = 0, 1, \ldots, n$

Suppose the $\tilde{\xi}_i$, $\tilde{\eta}_i$ ($i = 1, 2, \ldots, n$) in model (13) are mutually independent type-2 triangular fuzzy vectors with their elements defined as

$$\tilde{\xi}_{j,i} = (\tilde{\xi}_{j,i}^{r_1}, \tilde{\xi}_{j,i}^{r_2}, \tilde{\xi}_{j,i}^{r_3}; \theta_{l,j,i}, \theta_{r,j,i}), \quad j = 1, 2, \ldots, m,$$

and

$$\tilde{\eta}_{k,i} = (\tilde{\eta}_{k,i}^{r_1}, \tilde{\eta}_{k,i}^{r_2}, \tilde{\eta}_{k,i}^{r_3}; \bar{\theta}_{l,k,i}, \bar{\theta}_{r,k,i}), \quad k = 1, 2, \ldots, s$$

for $i = 1, 2, \ldots, n$. Then we have

$$-\tilde{\xi}_{j,i} = (-\tilde{\xi}_{j,i}^{r_3}, -\tilde{\xi}_{j,i}^{r_2}, -\tilde{\xi}_{j,i}^{r_1}; \theta_{r,j,i}, \theta_{l,j,i}), \quad j = 1, 2, \ldots, m,$$

and

$$-\tilde{\eta}_{k,i} = (-\tilde{\eta}_{k,i}^{r_3}, -\tilde{\eta}_{k,i}^{r_2}, -\tilde{\eta}_{k,i}^{r_1}; \bar{\theta}_{r,k,i}, \bar{\theta}_{l,k,i}), \quad k = 1, 2, \ldots, s$$

for $i = 1, 2, \ldots, n$.

Suppose ξ and η are the reductions of $\tilde{\xi}$ and $\tilde{\eta}$ obtained by the CV reduction method. Then $\xi_{j,i}$ and $-\eta_{k,i}$, $-\xi_{j,i}$ and $\eta_{k,i}$ are mutually independent fuzzy variables as shown in Fig. 3.

According to Theorem 7, when $\alpha_i > 0.5$, $i = 0, 1, 2, \ldots, n$, we can turn model (13) into its crisp equivalent parametric form.

Define $I = \{i \mid 0.5 < \alpha_i \le 0.75, i = 1, 2, \ldots, n\}$, and $J = \{i \mid 0.75 < \alpha_i \le 1, i = 1, 2, \ldots, n\}$. According to the above discussion, when $0.5 < \alpha_0 \le 0.75$, model (13) can be turned into the following parametric programming problem:

$$\begin{cases}
\max\limits_{u,\,v} & \dfrac{\sum\limits_{k=1}^{s}((2\alpha_0 - 1)v_k\eta_{k,0}^{r_1} + (2(1-\alpha_0) + (3-4\alpha_0)\bar{\theta}_{r,k,0})v_k\eta_{k,0}^{r_2})/(1 + (3-4\alpha_0)\bar{\theta}_{r,k,0})}{\sum\limits_{j=1}^{m}((2\alpha_0 - 1)u_j\xi_{j,0}^{r_3} + (2(1-\alpha_0) + (3-4\alpha_0)\theta_{l,j,0})u_j\xi_{j,0}^{r_2})/(1 + (3-4\alpha_0)\theta_{l,j,0})} \\
\text{subject to} & -\sum\limits_{j=1}^{m}\dfrac{(2\alpha_i - 1)u_j\xi_{j,i}^{r_1} + (2(1-\alpha_i) + (3-4\alpha_i)\theta_{r,j,i})u_j\xi_{j,i}^{r_2}}{1 + (3-4\alpha_i)\theta_{r,j,i}} \\
& \quad + \sum\limits_{k=1}^{s}\dfrac{(2\alpha_i - 1)v_k\eta_{k,i}^{r_3} + (2(1-\alpha_i) + (3-4\alpha_i)\bar{\theta}_{l,k,i})v_k\eta_{k,i}^{r_2}}{1 + (3-4\alpha_i)\bar{\theta}_{l,j,i}} \le 0, i \in I \\
& -\sum\limits_{j=1}^{m}\dfrac{(2\alpha_i - 1 + (4\alpha_i - 3)\theta_{l,j,i})u_j\xi_{j,i}^{r_1} + 2(1-\alpha_i)u_j\xi_{j,i}^{r_2}}{1 + (4\alpha_i - 3)\theta_{l,j,i}} \\
& \quad + \sum\limits_{k=1}^{s}\dfrac{(2\alpha_i - 1 + (4\alpha_i - 3)\bar{\theta}_{r,k,i})v_k\eta_{k,i}^{r_3} + 2(1-\alpha_i)v_k\eta_{k,i}^{r_2}}{1 + (4\alpha_i - 3)\bar{\theta}_{r,k,i}} \le 0, i \in J \\
& u_j \ge 0, u_j \ne 0, j = 1, 2, \ldots, m \\
& v_k \ge 0, v_k \ne 0, k = 1, 2, \ldots, s,
\end{cases} \tag{14}$$

which is equivalent to

$$\begin{cases} \max\limits_{u,v} & \sum_{k=1}^{s} \dfrac{(2\alpha_0-1)v_k\eta_{k,0}^{r_1}+(2(1-\alpha_0)+(3-4\alpha_0)\bar{\theta}_{r,k,0})v_k\eta_{k,0}^{r_2}}{1+(3-4\alpha_0)\bar{\theta}_{r,k,0}} \\ \text{subject to} & \sum_{j=1}^{m} \dfrac{(2\alpha_0-1)u_j\xi_{j,0}^{r_3}+(2(1-\alpha_0)+(3-4\alpha_0)\theta_{l,j,0})u_j\xi_{j,0}^{r_2}}{1+(3-4\alpha_0)\theta_{l,j,0}}=1 \\ & -\sum_{j=1}^{m} \dfrac{(2\alpha_i-1)u_j\xi_{j,i}^{r_1}+(2(1-\alpha_i)+(3-4\alpha_i)\theta_{r,j,i})u_j\xi_{j,i}^{r_2}}{1+(3-4\alpha_i)\theta_{r,j,i}} \\ & \quad +\sum_{k=1}^{s} \dfrac{(2\alpha_i-1)v_k\eta_{k,i}^{r_3}+(2(1-\alpha_i)+(3-4\alpha_i)\bar{\theta}_{l,k,i})v_k\eta_{k,i}^{r_2}}{1+(3-4\alpha_i)\bar{\theta}_{l,j,i}} \le 0,\ i\in I \\ & -\sum_{j=1}^{m} \dfrac{(2\alpha_i-1+(4\alpha_i-3)\theta_{l,j,i})u_j\xi_{j,i}^{r_1}+2(1-\alpha_i)u_j\xi_{j,i}^{r_2}}{1+(4\alpha_i-3)\theta_{l,j,i}} \\ & \quad +\sum_{k=1}^{s} \dfrac{(2\alpha_i-1+(4\alpha_i-3)\bar{\theta}_{r,k,i})v_k\eta_{k,i}^{r_3}+2(1-\alpha_i)v_k\eta_{k,i}^{r_2}}{1+(4\alpha_i-3)\bar{\theta}_{r,k,i}} \le 0,\ i\in J \\ & u_j\ge 0,\ u_j\ne 0,\ j=1,2,\ldots,m \\ & v_k\ge 0,\ v_k\ne 0,\ k=1,2,\ldots,s. \end{cases} \tag{15}$$

On other hand, when $0.75 < \alpha_0 \le 1$, model (13) can be turned into the following equivalent parametric programming problem:

$$\begin{cases} \max\limits_{u,v} & \dfrac{\sum_{k=1}^{s}((2\alpha_0-1+(4\alpha_0-3)\bar{\theta}_{l,k,0})v_k\eta_{k,0}^{r_1}+2(1-\alpha_0)v_k\eta_{k,0}^{r_2})/(1+(4\alpha_0-3)\bar{\theta}_{l,k,0})}{\sum_{j=1}^{m}((2\alpha_0-1+(4\alpha_0-3)\theta_{r,j,0})u_j\xi_{j,0}^{r_3}+2(1-\alpha_0)u_j\xi_{j,0}^{r_2})/(1+(4\alpha_0-3)\theta_{r,j,0})} \\ \text{subject to} & -\sum_{j=1}^{m} \dfrac{(2\alpha_i-1)u_j\xi_{j,i}^{r_1}+(2(1-\alpha_i)+(3-4\alpha_i)\theta_{r,j,i})u_j\xi_{j,i}^{r_2}}{1+(3-4\alpha_i)\theta_{r,j,i}} \\ & \quad +\sum_{k=1}^{s} \dfrac{(2\alpha_i-1)v_k\eta_{k,i}^{r_3}+(2(1-\alpha_i)+(3-4\alpha_i)\bar{\theta}_{l,k,i})v_k\eta_{k,i}^{r_2}}{1+(3-4\alpha_i)\bar{\theta}_{l,j,i}} \le 0,\ i\in I \\ & -\sum_{j=1}^{m} \dfrac{(2\alpha_i-1+(4\alpha_i-3)\theta_{l,j,i})u_j\xi_{j,i}^{r_1}+2(1-\alpha_i)u_j\xi_{j,i}^{r_2}}{1+(4\alpha_i-3)\theta_{l,j,i}} \\ & \quad +\sum_{k=1}^{s} \dfrac{(2\alpha_i-1+(4\alpha_i-3)\bar{\theta}_{r,k,i})v_k\eta_{k,i}^{r_3}+2(1-\alpha_i)v_k\eta_{k,i}^{r_2}}{1+(4\alpha_i-3)\bar{\theta}_{r,k,i}} \le 0,\ i\in J \\ & u_j\ge 0,\ u_j\ne 0,\ j=1,2,\ldots,m \\ & v_k\ge 0,\ v_k\ne 0,\ k=1,2,\ldots,s, \end{cases} \tag{16}$$

which is equivalent to

$$\begin{cases} \max\limits_{u,v} & \sum_{k=1}^{s} \dfrac{(2\alpha_0-1+(4\alpha_0-3)\bar{\theta}_{l,k,0})v_k\eta_{k,0}^{r_1}+2(1-\alpha_0)v_k\eta_{k,0}^{r_2}}{1+(4\alpha_0-3)\bar{\theta}_{l,k,0}} \\ \text{subject to} & \sum_{j=1}^{m} \dfrac{(2\alpha_0-1+(4\alpha_0-3)\theta_{r,j,0})u_j\xi_{j,0}^{r_3}+2(1-\alpha_0)u_j\xi_{j,0}^{r_2}}{1+(4\alpha_0-3)\theta_{r,j,0}}=1 \\ & -\sum_{j=1}^{m} \dfrac{(2\alpha_i-1)u_j\xi_{j,i}^{r_1}+(2(1-\alpha_i)+(3-4\alpha_i)\theta_{r,j,i})u_j\xi_{j,i}^{r_2}}{1+(3-4\alpha_i)\theta_{r,j,i}} \\ & \quad +\sum_{k=1}^{s} \dfrac{(2\alpha_i-1)v_k\eta_{k,i}^{r_3}+(2(1-\alpha_i)+(3-4\alpha_i)\bar{\theta}_{l,k,i})v_k\eta_{k,i}^{r_2}}{1+(3-4\alpha_i)\bar{\theta}_{l,j,i}} \le 0,\ i\in I \\ & -\sum_{j=1}^{m} \dfrac{(2\alpha_i-1+(4\alpha_i-3)\theta_{l,j,i})u_j\xi_{j,i}^{r_1}+2(1-\alpha_i)u_j\xi_{j,i}^{r_2}}{1+(4\alpha_i-3)\theta_{l,j,i}} \\ & \quad +\sum_{k=1}^{s} \dfrac{(2\alpha_i-1+(4\alpha_i-3)\bar{\theta}_{r,k,i})v_k\eta_{k,i}^{r_3}+2(1-\alpha_i)v_k\eta_{k,i}^{r_2}}{1+(4\alpha_i-3)\bar{\theta}_{r,k,i}} \le 0,\ i\in J \\ & u_j\ge 0,\ u_j\ne 0,\ j=1,2,\ldots,m \\ & v_k\ge 0,\ v_k\ne 0,\ k=1,2,\ldots,s. \end{cases} \tag{17}$$

Table 2
Input data for five DMUs.

DMU_i	Input 1	Input 2	Input 3	Input 4
$i=1$	$(\widetilde{2.8}, \widetilde{3.0}, \widetilde{3.1}; \theta_{l,1,1}, \theta_{r,1,1})$	$(\widetilde{3.7}, \widetilde{3.8}, \widetilde{4.0}; \theta_{l,2,1}, \theta_{r,2,1})$	$(\widetilde{3.8}, \widetilde{4.1}, \widetilde{4.2}; \theta_{l,3,1}, \theta_{r,3,1})$	$(\widetilde{3.4}, \widetilde{3.7}, \widetilde{4.2}; \theta_{l,4,1}, \theta_{r,4,1})$
$i=2$	$(\widetilde{1.2}, \widetilde{1.4}, \widetilde{1.7}; \theta_{l,1,2}, \theta_{r,1,2})$	$(\widetilde{1.9}, \widetilde{2.0}, \widetilde{2.3}; \theta_{l,2,2}, \theta_{r,2,2})$	$(\widetilde{2.0}, \widetilde{2.3}, \widetilde{2.4}; \theta_{l,3,2}, \theta_{r,3,2})$	$(\widetilde{1.7}, \widetilde{2.0}, \widetilde{2.1}; \theta_{l,4,2}, \theta_{r,4,2})$
$i=3$	$(\widetilde{2.1}, \widetilde{2.4}, \widetilde{2.5}; \theta_{l,1,3}, \theta_{r,1,3})$	$(\widetilde{3.1}, \widetilde{3.2}, \widetilde{3.4}; \theta_{l,2,3}, \theta_{r,2,3})$	$(\widetilde{2.4}, \widetilde{2.6}, \widetilde{2.9}; \theta_{l,3,3}, \theta_{r,3,3})$	$(\widetilde{2.4}, \widetilde{2.8}, \widetilde{3.0}; \theta_{l,4,3}, \theta_{r,4,3})$
$i=4$	$(\widetilde{1.6}, \widetilde{1.8}, \widetilde{2.0}; \theta_{l,1,4}, \theta_{r,1,4})$	$(\widetilde{2.2}, \widetilde{2.4}, \widetilde{2.5}; \theta_{l,2,4}, \theta_{r,2,4})$	$(\widetilde{2.5}, \widetilde{2.8}, \widetilde{3.0}; \theta_{l,3,4}, \theta_{r,3,4})$	$(\widetilde{3.3}, \widetilde{3.4}, \widetilde{3.7}; \theta_{l,4,4}, \theta_{r,4,4})$
$i=5$	$(\widetilde{3.1}, \widetilde{3.5}, \widetilde{3.7}; \theta_{l,1,5}, \theta_{r,1,5})$	$(\widetilde{2.0}, \widetilde{2.5}, \widetilde{2.8}; \theta_{l,2,5}, \theta_{r,2,5})$	$(\widetilde{3.9}, \widetilde{4.0}, \widetilde{4.2}; \theta_{l,3,5}, \theta_{r,3,5})$	$(\widetilde{3.6}, \widetilde{4.1}, \widetilde{4.3}; \theta_{l,4,5}, \theta_{r,4,5})$

Table 3
Output data for five DMUs.

DMU_i	Output 1	Output 2	Output 3	Output 4
$i=1$	$(\widetilde{5.1}, \widetilde{5.5}, \widetilde{5.7}; \bar{\theta}_{l,1,1}, \bar{\theta}_{r,1,1})$	$(\widetilde{5.2}, \widetilde{5.3}, \widetilde{5.6}; \bar{\theta}_{l,2,1}, \bar{\theta}_{r,2,1})$	$(\widetilde{6.1}, \widetilde{6.2}, \widetilde{6.4}; \bar{\theta}_{l,3,1}, \bar{\theta}_{r,3,1})$	$(\widetilde{4.8}, \widetilde{5.2}, \widetilde{5.5}; \bar{\theta}_{l,4,1}, \bar{\theta}_{r,4,1})$
$i=2$	$(\widetilde{4.0}, \widetilde{4.1}, \widetilde{4.3}; \bar{\theta}_{l,1,2}, \bar{\theta}_{r,1,2})$	$(\widetilde{3.9}, \widetilde{4.4}, \widetilde{4.7}; \bar{\theta}_{l,2,2}, \bar{\theta}_{r,2,2})$	$(\widetilde{5.0}, \widetilde{5.1}, \widetilde{5.2}; \bar{\theta}_{l,3,2}, \bar{\theta}_{r,3,2})$	$(\widetilde{3.0}, \widetilde{3.3}, \widetilde{3.4}; \bar{\theta}_{l,4,2}, \bar{\theta}_{r,4,2})$
$i=3$	$(\widetilde{4.4}, \widetilde{4.5}, \widetilde{4.7}; \bar{\theta}_{l,1,3}, \bar{\theta}_{r,1,3})$	$(\widetilde{4.0}, \widetilde{4.1}, \widetilde{4.4}; \bar{\theta}_{l,2,3}, \bar{\theta}_{r,2,3})$	$(\widetilde{5.2}, \widetilde{5.5}, \widetilde{5.6}; \bar{\theta}_{l,3,3}, \bar{\theta}_{r,3,3})$	$(\widetilde{3.2}, \widetilde{3.3}, \widetilde{3.5}; \bar{\theta}_{l,4,3}, \bar{\theta}_{r,4,3})$
$i=4$	$(\widetilde{4.2}, \widetilde{4.3}, \widetilde{4.4}; \bar{\theta}_{l,1,4}, \bar{\theta}_{r,1,4})$	$(\widetilde{3.9}, \widetilde{4.0}, \widetilde{4.2}; \bar{\theta}_{l,2,4}, \bar{\theta}_{r,2,4})$	$(\widetilde{5.1}, \widetilde{5.3}, \widetilde{5.5}; \bar{\theta}_{l,3,4}, \bar{\theta}_{r,3,4})$	$(\widetilde{3.6}, \widetilde{3.7}, \widetilde{3.9}; \bar{\theta}_{l,4,4}, \bar{\theta}_{r,4,4})$
$i=5$	$(\widetilde{5.5}, \widetilde{5.7}, \widetilde{6.0}; \bar{\theta}_{l,1,5}, \bar{\theta}_{r,1,5})$	$(\widetilde{5.3}, \widetilde{5.5}, \widetilde{5.8}; \bar{\theta}_{l,2,5}, \bar{\theta}_{r,2,5})$	$(\widetilde{6.6}, \widetilde{6.7}, \widetilde{7.0}; \bar{\theta}_{l,3,5}, \bar{\theta}_{r,3,5})$	$(\widetilde{5.0}, \widetilde{5.3}, \widetilde{5.7}; \bar{\theta}_{l,4,5}, \bar{\theta}_{r,4,5})$

Problems (15) and (17) are parametric programming problems. Given parameters $\theta_{l,j,i}$, $\bar{\theta}_{l,k,i}$, $\theta_{r,j,i}$ and $\bar{\theta}_{r,k,i}$, both problems become linear programming ones that can be solved using standard optimization solvers.

6. Numerical experiments

To demonstrate the modeling idea and the efficiency of the proposed DEA model, we now consider a system composed of five DMUs, and each DMU with four inputs and four outputs that are characterized by type-2 triangular fuzzy vectors as shown in Tables 2 and 3. For convenience, we adopt the following notation in this system: $\alpha_0 = \alpha_1 = \alpha_3 = \alpha_5 = \alpha$ and $\alpha_2 = \alpha_4 = \beta$.

When $\alpha \in (0.75, 1]$ and $\beta \in (0.5, 0.75]$, the system can be modeled as the following parametric programming problem:

$$\begin{cases} \max\limits_{u,\,v} & f_0(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4) \\ \text{subject to} & g_i(u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4) \le 0 \\ & u_j \ge 0,\ u_j \ne 0,\ j = 1, 2, 3, 4 \\ & v_k \ge 0,\ v_k \ne 0,\ k = 1, 2, 3, 4, \end{cases} \tag{18}$$

where the objective is as follows when the ith DMU, $i = 1, 2, \ldots, 5$, is modeled as the target UMU_0:

$$\begin{aligned} f_1 = {} & \frac{5.1(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,1,1}) + 2\times 5.5(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,1,1}} v_1 + \frac{5.2(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,2,1}) + 2\times 5.3(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,2,1}} v_2 \\ & + \frac{6.1(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,3,1}) + 2\times 6.2(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,3,1}} v_3 + \frac{4.8(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,4,1}) + 2\times 5.2(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,4,1}} v_4 \end{aligned}$$

such that

$$\begin{aligned} & \frac{3.1(2\alpha-1+(4\alpha-3)\theta_{r,1,1}) + 2\times 3.0(1-\alpha)}{1+(4\alpha-3)\theta_{r,1,1}} u_1 + \frac{4.0(2\alpha-1+(4\alpha-3)\theta_{r,2,1}) + 2\times 3.8(1-\alpha)}{1+(4\alpha-3)\theta_{r,2,1}} u_2 \\ & \quad + \frac{4.2(2\alpha-1+(4\alpha-3)\theta_{r,3,1}) + 2\times 4.1(1-\alpha)}{1+(4\alpha-3)\theta_{r,3,1}} u_3 + \frac{4.2(2\alpha-1+(4\alpha-3)\theta_{r,4,1}) + 2\times 3.7(1-\alpha)}{1+(4\alpha-3)\theta_{r,4,1}} u_4 = 1. \end{aligned}$$

$$\begin{aligned} f_2 = {} & \frac{4.0(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,1,2}) + 2\times 4.1(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,1,2}} v_1 + \frac{3.9(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,2,2}) + 2\times 4.4(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,2,2}} v_2 \\ & + \frac{5.0(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,3,2}) + 2\times 5.1(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,3,2}} v_3 + \frac{3.0(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,4,2}) + 2\times 3.3(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,4,2}} v_4 \end{aligned}$$

such that

$$\begin{aligned} & \frac{1.7(2\alpha-1+(4\alpha-3)\theta_{r,1,2}) + 2\times 1.4(1-\alpha)}{1+(4\alpha-3)\theta_{r,1,2}} u_1 + \frac{2.3(2\alpha-1+(4\alpha-3)\theta_{r,2,2}) + 2\times 2.0(1-\alpha)}{1+(4\alpha-3)\theta_{r,2,2}} u_2 \\ & \quad + \frac{2.4(2\alpha-1+(4\alpha-3)\theta_{r,3,2}) + 2\times 2.3(1-\alpha)}{1+(4\alpha-3)\theta_{r,3,2}} u_3 + \frac{2.1(2\alpha-1+(4\alpha-3)\theta_{r,4,2}) + 2\times 2.0(1-\alpha)}{1+(4\alpha-3)\theta_{r,4,2}} u_4 = 1. \end{aligned}$$

$$f_3 = \frac{4.4(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,1,3})+2\times 4.5(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,1,3}}v_1+\frac{4.0(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,2,3})+2\times 4.1(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,2,3}}v_2 + \frac{5.2(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,3,3})+2\times 5.5(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,3,3}}v_3+\frac{3.2(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,4,3})+2\times 3.3(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,4,3}}v_4$$

such that

$$\frac{2.5(2\alpha-1+(4\alpha-3)\theta_{r,1,3})+2\times 2.4(1-\alpha)}{1+(4\alpha-3)\theta_{r,1,3}}u_1+\frac{3.4(2\alpha-1+(4\alpha-3)\theta_{r,2,3})+2\times 3.2(1-\alpha)}{1+(4\alpha-3)\theta_{r,2,3}}u_2 + \frac{2.9(2\alpha-1+(4\alpha-3)\theta_{r,3,3})+2\times 2.6(1-\alpha)}{1+(4\alpha-3)\theta_{r,3,3}}u_3+\frac{3.0(2\alpha-1+(4\alpha-3)\theta_{r,4,3})+2\times 2.8(1-\alpha)}{1+(4\alpha-3)\theta_{r,4,3}}u_4=1.$$

$$f_4 = \frac{4.2(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,1,4})+2\times 4.3(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,1,4}}v_1+\frac{3.9(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,2,4})+2\times 4.0(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,2,4}}v_2 + \frac{5.1(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,3,4})+2\times 5.3(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,3,4}}v_3+\frac{3.6(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,4,4})+2\times 3.7(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,4,4}}v_4$$

such that

$$\frac{2.0(2\alpha-1+(4\alpha-3)\theta_{r,1,4})+2\times 1.8(1-\alpha)}{1+(4\alpha-3)\theta_{r,1,4}}u_1+\frac{2.5(2\alpha-1+(4\alpha-3)\theta_{r,2,4})+2\times 2.4(1-\alpha)}{1+(4\alpha-3)\theta_{r,2,4}}u_2 + \frac{3.0(2\alpha-1+(4\alpha-3)\theta_{r,3,4})+2\times 2.8(1-\alpha)}{1+(4\alpha-3)\theta_{r,3,4}}u_3+\frac{3.7(2\alpha-1+(4\alpha-3)\theta_{r,4,4})+2\times 3.4(1-\alpha)}{1+(4\alpha-3)\theta_{r,4,4}}u_4=1,$$

and

$$f_5 = \frac{5.5(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,1,5})+2\times 5.7(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,1,5}}v_1+\frac{5.3(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,2,5})+2\times 5.5(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,2,5}}v_2 + \frac{6.6(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,3,5})+2\times 6.7(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,3,5}}v_3+\frac{5.0(2\alpha-1+(4\alpha-3)\bar{\theta}_{l,4,5})+2\times 5.3(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{l,4,5}}v_4$$

such that

$$\frac{3.7(2\alpha-1+(4\alpha-3)\theta_{r,1,5})+2\times 3.5(1-\alpha)}{1+(4\alpha-3)\theta_{r,1,5}}u_1+\frac{2.8(2\alpha-1+(4\alpha-3)\theta_{r,2,5})+2\times 2.5(1-\alpha)}{1+(4\alpha-3)\theta_{r,2,5}}u_2 + \frac{4.2(2\alpha-1+(4\alpha-3)\theta_{r,3,5})+2\times 4.0(1-\alpha)}{1+(4\alpha-3)\theta_{r,3,5}}u_3+\frac{4.3(2\alpha-1+(4\alpha-3)\theta_{r,4,5})+2\times 4.1(1-\alpha)}{1+(4\alpha-3)\theta_{r,4,5}}u_4=1.$$

In addition, the analytical expressions for the constraint functions g_i in model (18) are as follows:

$$g_1 = -\frac{2.8(2\alpha-1+(4\alpha-3)\theta_{l,1,1})+2\times 3.0(1-\alpha)}{1+(4\alpha-3)\theta_{l,1,1}}u_1-\frac{3.7(2\alpha-1+(4\alpha-3)\theta_{l,2,1})+2\times 3.8(1-\alpha)}{1+(4\alpha-3)\theta_{l,2,1}}u_2 - \frac{3.8(2\alpha-1+(4\alpha-3)\theta_{l,3,1})+2\times 4.1(1-\alpha)}{1+(4\alpha-3)\theta_{l,3,1}}u_3-\frac{3.4(2\alpha-1+(4\alpha-3)\theta_{l,4,1})+2\times 3.7(1-\alpha)}{1+(4\alpha-3)\theta_{l,4,1}}u_4 + \frac{5.7(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,1,1})+2\times 5.5(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,1,1}}v_1+\frac{5.6(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,2,1})+2\times 5.3(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,2,1}}v_2 + \frac{6.4(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,3,1})+2\times 6.2(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,3,1}}v_3+\frac{5.5(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,4,1})+2\times 5.2(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,4,1}}v_4,$$

$$g_2 = -\frac{1.2(2\beta-1)+1.4(2(1-\beta)+(3-4\beta)\theta_{r,1,2})}{1+(3-4\beta)\theta_{r,1,2}}u_1-\frac{1.9(2\beta-1)+2.0(2(1-\beta)+(3-4\beta)\theta_{r,2,2})}{1+(3-4\beta)\theta_{r,2,2}}u_2 - \frac{2.0(2\beta-1)+2.3(2(1-\beta)+(3-4\beta)\theta_{r,3,2})}{1+(3-4\beta)\theta_{r,3,2}}u_3-\frac{1.7(2\beta-1)+2.0(2(1-\beta)+(3-4\beta)\theta_{r,4,2})}{1+(3-4\beta)\theta_{r,4,2}}u_4 + \frac{4.3(2\beta-1)+4.1(2(1-\beta)+(3-4\beta)\bar{\theta}_{l,1,2})}{1+(3-4\beta)\bar{\theta}_{l,1,2}}v_1+\frac{4.7(2\beta-1)+4.4(2(1-\beta)+(3-4\beta)\bar{\theta}_{l,2,2})}{1+(3-4\beta)\bar{\theta}_{l,2,2}}v_2 + \frac{5.2(2\beta-1)+5.1(2(1-\beta)+(3-4\beta)\bar{\theta}_{l,3,2})}{1+(3-4\beta)\bar{\theta}_{l,3,2}}v_3+\frac{3.4(2\beta-1)+3.3(2(1-\beta)+(3-4\beta)\bar{\theta}_{l,4,2})}{1+(3-4\beta)\bar{\theta}_{l,4,2}}v_4,$$

Table 4
Evaluation result for each DMU with $(\theta_l, \theta_r) = (0.5, 0.5)$ under $\alpha = 0.9$ and $\beta = 0.7$.

DMUs	Optimal solution (u, v)	α-efficient value
DMU_1	(0.0000, 0.0000, 0.2390, 0.0000, 0.0000, 0.0000, 0.0000, 0.1569)	0.7629027
DMU_2	(0.0000, 0.0000, 0.4194, 0.0000, 0.0000, 0.0000, 0.1789, 0.0000)	0.8971282
DMU_3	(0.0000, 0.0077, 0.3413, 0.0000, 0.1828, 0.0000, 0.0000, 0.0000)	0.8072445
DMU_4	(0.0000, 0.0505, 0.2946, 0.0000, 0.0000, 0.0000, 0.0000, 0.2231)	0.8067061
DMU_5	(0.0000, 0.0369, 0.2155, 0.0000, 0.0000, 0.0000, 0.0000, 0.1632)	0.8236366

Table 5
Evaluation results for each DMU with different parameters (θ_l, θ_r) under $\alpha = 0.9$ and $\beta = 0.7$.

(θ_l,θ_r)	(0, 0)	(0.1, 0.2)	(0.3, 0.4)	(0.5, 0.5)	(0.7, 0.6)	(0.9, 0.8)	(1, 1)
DMU_1	0.7619975	0.7626904	0.7630024	0.7629027	0.7629323	0.7635817	0.7644179
DMU_2	0.8945820	0.8956042	0.8966210	0.8971282	0.8976606	0.8987352	0.8997756
DMU_3	0.8116787	0.8114339	0.8087777	0.8072445	0.8059549	0.8044095	0.8033815
DMU_4	0.8069665	0.8068084	0.8067367	0.8067061	0.8067528	0.8070330	0.8073965
DMU_5	0.8291510	0.8271533	0.8249915	0.8236366	0.8225233	0.8213728	0.8207225

$$
\begin{aligned}
g_3 = & -\frac{2.1(2\alpha-1+(4\alpha-3)\theta_{l,1,3})+2\times 2.4(1-\alpha)}{1+(4\alpha-3)\theta_{l,1,3}}u_1 - \frac{3.1(2\alpha-1+(4\alpha-3)\theta_{l,2,3})+2\times 3.2(1-\alpha)}{1+(4\alpha-3)\theta_{l,2,3}}u_2 \\
& - \frac{2.4(2\alpha-1+(4\alpha-3)\theta_{l,3,3})+2\times 2.6(1-\alpha)}{1+(4\alpha-3)\theta_{l,3,3}}u_3 - \frac{2.4(2\alpha-1+(4\alpha-3)\theta_{l,4,3})+2\times 2.8(1-\alpha)}{1+(4\alpha-3)\theta_{l,4,3}}u_4 \\
& + \frac{4.7(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,1,3})+2\times 4.5(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,1,3}}v_1 + \frac{4.4(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,2,3})+2\times 4.1(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,2,3}}v_2 \\
& + \frac{5.6(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,3,3})+2\times 5.5(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,3,3}}v_3 + \frac{3.5(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,4,3})+2\times 3.3(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,4,3}}v_4, \\
g_4 = & -\frac{1.6(2\beta-1)+1.8(2(1-\beta)+(3-4\beta)\theta_{r,1,4})}{1+(3-4\beta)\theta_{r,1,4}}u_1 - \frac{2.2(2\beta-1)+2.4(2(1-\beta)+(3-4\beta)\theta_{r,2,4})}{1+(3-4\beta)\theta_{r,2,4}}u_2 \\
& - \frac{2.5(2\beta-1)+2.8(2(1-\beta)+(3-4\beta)\theta_{r,3,4})}{1+(3-4\beta)\theta_{r,3,4}}u_3 - \frac{3.3(2\beta-1)+3.4(2(1-\beta)+(3-4\beta)\theta_{r,4,4})}{1+(3-4\beta)\theta_{r,4,4}}u_4 \\
& + \frac{4.4(2\beta-1)+4.3(2(1-\beta)+(3-4\beta)\bar{\theta}_{l,1,4})}{1+(3-4\beta)\bar{\theta}_{l,1,4}}v_1 + \frac{4.2(2\beta-1)+4.0(2(1-\beta)+(3-4\beta)\bar{\theta}_{l,2,4})}{1+(3-4\beta)\bar{\theta}_{l,2,4}}v_2 \\
& + \frac{5.5(2\beta-1)+5.3(2(1-\beta)+(3-4\beta)\bar{\theta}_{l,3,4})}{1+(3-4\beta)\bar{\theta}_{l,3,4}}v_3 + \frac{3.9(2\beta-1)+3.7(2(1-\beta)+(3-4\beta)\bar{\theta}_{l,4,4})}{1+(3-4\beta)\bar{\theta}_{l,4,4}}v_4,
\end{aligned}
$$

and

$$
\begin{aligned}
g_5 = & -\frac{3.1(2\alpha-1+(4\alpha-3)\theta_{l,1,5})+2\times 3.5(1-\alpha)}{1+(4\alpha-3)\theta_{l,1,5}}u_1 - \frac{2.0(2\alpha-1+(4\alpha-3)\theta_{l,2,5})+2\times 2.5(1-\alpha)}{1+(4\alpha-3)\theta_{l,2,5}}u_2 \\
& - \frac{3.9(2\alpha-1+(4\alpha-3)\theta_{l,3,5})+2\times 4.0(1-\alpha)}{1+(4\alpha-3)\theta_{l,3,5}}u_3 - \frac{3.6(2\alpha-1+(4\alpha-3)\theta_{l,4,5})+2\times 4.1(1-\alpha)}{1+(4\alpha-3)\theta_{l,4,5}}u_4 \\
& + \frac{6.0(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,1,5})+2\times 5.7(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,1,5}}v_1 + \frac{5.8(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,2,5})+2\times 5.5(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,2,5}}v_2 \\
& + \frac{7.0(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,3,5})+2\times 6.7(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,3,5}}v_3 + \frac{5.7(2\alpha-1+(4\alpha-3)\bar{\theta}_{r,4,5})+2\times 5.3(1-\alpha)}{1+(4\alpha-3)\bar{\theta}_{r,4,5}}v_4.
\end{aligned}
$$

To solve model (18), we set parameters $\theta_{l,j,i} = \bar{\theta}_{l,k,i} = \theta_l$, and $\theta_{r,j,i} = \bar{\theta}_{r,k,i} = \theta_r$ for each i, j, k. When $(\theta_l, \theta_r) = (0.5, 0.5)$, and the credibility levels are $\alpha = 0.9$ and $\beta = 0.7$, we can get the evaluation results shown in Table 4 for each DMU with Lingo software. From the solution results, we can learn information about each DMU. DMU_2 has the biggest α-efficient value 0.8971282, followed by DMU_5 and DMU_3, which implies that DMU_2 is the most efficient DMU.

Furthermore, the evaluation results as regards the efficiency for each DMU with different values of parameters θ_l and θ_r are reported in Table 5, from which we can see that the efficiency for each DMU varies when we adjust the values of parameters θ_l and θ_r in the unit interval $[0, 1]$. The most efficient values or the least efficient values for all the DMUs aren't reached at the same parameters (θ_l, θ_r). Therefore, with the method proposed in this paper, the decision makers can obtain more precise information so that they can take a better decision.

7. Conclusions and future work

In fuzzy possibility theory, the type-2 fuzzy variable is an appropriate tool for describing type-2 fuzziness. This paper attempted to propose some methods of reduction for type-2 fuzzy variables, and applied the methods to the DEA model with type-2 fuzzy inputs and outputs. The major new results of the paper include the following four aspects.

(i) The optimistic CV, pessimistic CV and CV for RFVs were presented, and the properties of CVs for trapezoidal, triangular, normal and gamma RFVs were discussed (Theorems 1–3, and Corollary 1).
(ii) On the basis of the properties of the CVs, the optimistic, pessimistic and CV reduction methods were proposed. The reductions for type-2 triangular, normal and gamma fuzzy variables were discussed in Theorems 4–6.
(iii) For general fuzzy variables, we defined a generalized credibility measure, and discussed the properties of the reduced fuzzy variables of type-2 triangular and normal fuzzy variables (see Theorems 7 and 8).
(iv) Using the proposed reduction methods, a new class of generalized credibility DEA models was established. According to the properties of the generalized credibility measure, when the inputs and outputs are mutually independent type-2 triangular fuzzy variables, we can turn the proposed DEA model into its equivalent parametric programming form. One numerical example was also provided to demonstrate the modeling idea and the efficiency of DMUs via different parameter values involved in the proposed DEA model.

Type-2 fuzzy theory is a fertile field for research. This paper focuses on theoretical and computational issues as regards type-2 fuzzy variables. While some issues have been resolved, some new ones have been exposed, such as that we may consider how to reduce a general type-2 fuzzy variable, and apply the proposed reduction methods in various practical management problems.

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