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Local convergence of an adaptive scalar method and its application in a nonoverlapping domain decomposition scheme

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ABSTRACT

In this paper, we demonstrate a local convergence of an adaptive scalar solver which is practical for strongly diagonal dominant Jacobian problems such as in some systems of nonlinear equations arising from the application of a nonoverlapping domain decomposition method. The method is tested to a nonlinear interface problem of a multichip heat conduction problem. The numerical results show that the method performs slightly better than a Newton–Krylov method.

1. Introduction

This paper is concerned with the solving of a class of systems of nonlinear equations. Interest is given to solving nonlinear equations arising from the application of a nonoverlapping domain decomposition problem.

Consider the problem:

$$\begin{aligned} \mathbf{L}u &= f && \text{in } \Omega \\ u &= \varphi_D && \text{on } \Gamma_D \\ \frac{\partial u}{\partial \mathbf{n}^*} &= \varphi_N && \text{on } \Gamma_N \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\Re^d$, $d = 2, 3$, with a Lipschitz boundary $\partial\Omega$; the subsets Γ_D and Γ_N provide a partition of $\partial\Omega$, and $\mathbf{n}^*$ is the unit outward normal vector on $\partial\Omega$. Here, $\mathbf{L}$ is a nonlinear elliptical partial differential operator, $f \in L^2(\Omega)$, $\varphi_D \in L^2(\Gamma_D)$, and $\varphi_N \in L^2(\Gamma_N)$ are given functions. Assume that Ω is partitioned into two nonoverlapping subdomains Ω_1 and Ω_2 with boundaries $\partial\Omega_1$ and $\partial\Omega_2$, and denote the interface by $\Gamma := \partial\Omega_1 \cap \partial\Omega_2$ and boundary partitions by $\Gamma_{i,D} := \partial\Omega_i \cap \Gamma_D$, $\Gamma_{i,N} := \partial\Omega_i \cap \Gamma_N$, $i = 1, 2$, with the corresponding functions $\varphi_{i,D} := \varphi_{D|\Gamma_{i,D}}$ and $\varphi_{i,N} := \varphi_{N|\Gamma_{i,N}}$. Now let u_i be the restriction of u to Ω_i, the construction of a nonoverlapping domain decomposition method [1] for the above problem using a defect correction scheme [2,3] is expressed in the following iterative fashion:

$k := 0$; Initial guess: $\lambda^{(0)}$;

Do

$$\text{Solve} \begin{cases} \mathbf{L}u_1^{(k+1)} = f & \text{in } \Omega_1 \\ u_1^{(k+1)} = \varphi_{1,D} & \text{on } \Gamma_{1,D} \\ \dfrac{\partial u_1^{(k+1)}}{\partial \mathbf{n}^*} = \varphi_{1,N} & \text{on } \Gamma_{1,N} \\ u_1^{(k+1)} = \lambda^{(k)} & \text{on } \Gamma \end{cases} \quad (\text{Step} - 1a)$$

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$$\text{Solve}\begin{cases}\mathbf{L}u_2^{(k+1)} = f & \text{in } \Omega_2\\ u_2^{(k+1)} = \varphi_{2,D} & \text{on } \Gamma_{2,D}\\ \dfrac{\partial u_2^{(k+1)}}{\partial \mathbf{n}^*} = \varphi_{2,N} & \text{on } \Gamma_{2,N}\\ u_2^{(k+1)} = \lambda^{(k)} & \text{on } \Gamma\end{cases} \quad (\text{Step} - 1b)$$

Compute D; (Step − 2)
Update $\lambda^{(k+1)}$ based on D; (Step − 3)
$k := k + 1$;
Until $D = 0$.

For each $i = 1, 2$, u_i is an extension of λ to Ω_i, and can be denoted by $u_i(\lambda)$. The defect value D couples the solution of $u_1(\lambda)$ of Ω_1 and $u_2(\lambda)$ of Ω_2 along the interface and this scheme is aimed at solving the constraint condition $D = 0$. If the operator $\mathbf{L}$ represents the general elliptic operator

$$\mathbf{L}w := \sum_{l,j}^{d} D_l(a_{lj}D_j w) + a_0 w,$$

where D_j denotes the partial derivative with respect to x_j, $j = 1, \ldots, d$, the constraint condition is equivalent to the continuity of conormal derivative [4] at the interface Γ:

$$D(\lambda) = \frac{\partial u_1(\lambda)}{\partial \mathbf{n_L}} - \frac{\partial u_2(\lambda)}{\partial \mathbf{n}_L} = 0 \quad \text{on } \Gamma$$

where the conormal derivative $\frac{\partial w}{\partial \mathbf{n}_L}$ is defined as

$$\frac{\partial w}{\partial \mathbf{n}_L} := \sum_{l,j}^{d} a_{lj}\, D_j w\, n_l \tag{2}$$

with $\mathbf{n}$ being the unit outward normal vector on either $\partial\Omega_1 \cap \Gamma$ or $\partial\Omega_2 \cap \Gamma$. It is obvious that, when $\mathbf{L} = \Delta$, the conormal derivative coincides with the normal derivative $\frac{\partial u}{\partial \mathbf{n}}$, while for heat conduction problems it coincides with the heat flux. Given the nonlinear operator (1) and interface boundary value λ, the subdomain solutions $u_1(\lambda)$ and $u_2(\lambda)$ will be implicitly nonlinear in λ. Thus, the coupling of $u_1(\lambda)$ and $u_2(\lambda)$ at the interface through the transmission condition generates a nonlinear defect equation

$$D(\lambda) = 0.$$

In the remaining part of this paper, the problem is presented in a finite dimensional framework. Therefore, $\lambda \in \Re^N$ and $\mathbf{D} : \Re^N \to \Re^N$ represent the finite dimensional approximation of the interface variable and the defect function on N interface grid points. The next section briefly describes the adaptive α-method which will be used to solve the above nonlinear problem, while Section 3 demonstrates a local convergence of this method. In Section 4, a blackbox nonlinear problem arising from the domain decomposition implementation in a sandwich-like multichip module is solved with the method starting from some different initial conditions.

2. Adaptive-α method

When the system of nonlinear equations $D(\lambda) = 0$ is solved with the Newton method, the step-3 in the defect correction scheme is solely:

$$x_{n+1} = x_n - J(x_n)^{-1}\, F(x_n)$$

with x and F now replacing λ and D, whereas J and n represent the Jacobian matrix and the iteration number respectively. In parallel computation, step-1a and step-1b are carried out simultaneously while step-2 and step-3 must be performed sequentially. When using the Newton method or other quasi-Newton methods which need computation of the Jacobian matrix and its inverse, step-3 will be time and memory-consuming thus making the benefit of parallelisation of step-1a and step-1b less significant. In the parallel implementation of the defect correction scheme, it is then desired to have a quick interface computation. For this, a quasi-Newton is proposed here where it only uses a scalar as the approximation of Jacobian matrix:

$$x_{n+1} = x_n - \alpha_n^{-1}\, F(x_n) \tag{3}$$

where α_n^{-1} is an adaptive rate approximating $J(x_n)^{-1}$.

The following update for the value of α_n is used:

$$\alpha_{n+1} = \alpha_n \frac{\|F(x_{n+1}) - F(x_n)\|}{\|F(x_n)\|}. \tag{4}$$

It is called the adaptive-α method and it is first proposed in [5], yet without any convergence analysis. The following section shows the local convergence of this adaptive-α method in solving a system of nonlinear equations.

3. Local convergence analysis

For a system of nonlinear equations with a diagonal dominant Jacobian matrix and positive diagonal entries, a scalar may be used as a Jacobian approximation which leads a sufficiently good initial condition into convergence. We show in the following some propositions which support that. They are built from the standard assumptions and a theorem in [6].

Consider the system of nonlinear equations

$$F(x^*) = 0 \tag{5}$$

where $F : R^N \to R^N$ and $x^* \in R^N$ is the solution to the equations. The ith component of F is denoted by f_i. If the components of F are differentiable at $x \in R^N$, the Jacobian matrix $F'(x)$ is defined by

$$F'(x)_{ij} = \frac{\partial f_i}{\partial x_j}(x).$$

The fundamental theorem of calculus may be expressed as follows [6]:

Lemma 1. *Let F be differentiable in an open set $\Omega \subset R^N$ and let $x^* \in \Omega$ is.*
Then for all $x^* \in \Omega$ sufficiently near x^*,

$$F(x) - F(x^*) = \int_0^1 F'(x^* + t(x - x^*))(x - x^*)\mathrm{d}t. \tag{6}$$

In this section, $\|.\|$ will denote a norm on R^N as well as the induced matrix norm.

Definition 1. Let $\|.\|$ be a norm on R^N. The induced matrix norm of an $N \times N$ matrix A is defined by

$$\|A\| = \max_{\|x\|=1} \|Ax\|.$$

If $\|.\|_p$ is the l^p norm, the norm of a vector x is defined by

$$\|x\|_p = \left(\left| \sum_{i=1}^{N} (x)_i \right|^p \right)^{1/p}.$$

There are some assumptions are made on F. One of them is the smoothness assumption on $F'(x)$ in order to estimate the error approximation. For this, the notion of Lipschitz continuity is needed.

Definition 2. Let $\Omega \subset R^N$ and let $G : \Omega \to R^M$. G is Lipschitz continuous on Ω with Lipschitz constant γ if

$$\|G(x) - G(y)\| \le \gamma \|x - y\|$$

for all $x, y \in \Omega$.

Standard assumptions.

1. Eq. (5) has a solution x^*.
2. $F' : \Omega \to R^{N\times N}$ is Lipschitz continuous with Lipschitz constant γ.
3. $F'(x^*)$ is nonsingular.

Iterative methods can be classified by the rate of convergence. In many quasi-Newton methods, the rate is given by the linear convergence.

Definition 3. Let $x_n \subset R^N$ and $x^* \in R^N$. Then $x_n \to x^*$ q-linearly with q-factor $\sigma \in (0, 1)$ if $\|x_{n+1} - x^*\| \le \sigma \|x_n - x^*\|$

A theorem in [6] is used as the starting point of the analysis of the adaptive-α method. In what follows, $\mathcal{B}(r)$ denotes the ball of radius r about the solution x^*, and if $x_n \in R^N$ then $e_n = x_n - x^*$ denotes the error.

Theorem 1. *Let the standard assumptions hold. Then there are $K_B > 0$, $\delta > 0$, and $\delta_1 > 0$ such that if $x_0 \in \mathcal{B}(\delta)$ and the matrix-valued function $B(x)$ satisfies*

$$\|I - B(x)F'(x^*)\| = \rho(x) \le \delta_1 \tag{7}$$

for all $x \in \mathcal{B}(\delta)$ then the iteration

$$x_{n+1} = x_n - B(x_n)F(x_n) \tag{8}$$

converges q-linearly to x^* and

$$\|e_{n+1}\| \le K_B(\rho(x_n) + \|e_n\|)\|e_n\|.$$

Proof. This proof is rewritten from [6] with slight modification, because it will be used in subsequent analysis. First, by (7) we have

$$\begin{aligned}\|B(x)\| &= \|B(x)F'(x^*)F'(x^*)^{-1}\| \le \|B(x)F'(x^*)\|\,\|F'(x^*)^{-1}\| \\ &\le M_B = (1+\delta_1)\|F'(x^*)^{-1}\|.\end{aligned} \tag{9}$$

Using (6), the error of the iteration (8) can be written in the following equation:

$$\begin{aligned}e_{n+1} &= e_n - B(x_n)F(x_n) = \int_0^1 (I - B(x_n)F'(x^*+te_n))e_n \mathrm{d}t \\ &= (I - B(x_n)F'(x^*))e_n + B(x_n)\int_0^1 (F'(x^*) - F'(x^*+te_n))e_n \mathrm{d}t.\end{aligned}$$

From the above equation and (9), it follows that

$$\|e_{n+1}\| \le \rho(x_n)\|e_n\| + M_B\gamma\|e_n\|^2/2. \tag{10}$$

A local linear convergence is obtained in the neighbourhood $\mathcal{B}(\delta)$ with

$$\delta_1 < 1$$

and

$$\delta < \frac{2(1-\delta_1)}{\gamma M_B} \tag{11}$$

since

$$\rho(x_n) + M_B\gamma\delta/2 < \delta_1 + M_B\gamma\delta/2 < 1.$$

Continuing (10),

$$\begin{aligned}\|e_{n+1}\| &\le \rho(x_n)\|e_n\| + M_B\gamma\|e_n\|^2/2 \\ &\le \rho(x_n)\|e_n\| + M_B\gamma\|e_n\|^2/2 + M_B\gamma\rho(x_n)\|e_n\|/2 + \|e_n\|^2 \\ &\le K_B(\rho(x_n) + \|e_n\|)\|e_n\|.\end{aligned}$$

The proof completes with $K_B = 1 + M_B\gamma/2$. □

The closer the Jacobian approximation $B(x)$ is to $F'(x^*)$, the better is the local convergence of the quasi-Newton method. However, the rate of convergence notifies that $\|I - B(x)F'(x^*)\| \ge 1$ will not satisfy the sufficiency of the convergence. The notion of approximate inverse becomes relevant here.

Definition 4. Let A and B be $N \times N$ matrices. Then B is an approximate inverse of A if $\|I - BA\| < 1$

Corollary 1. *Let the assumptions of Theorem 1 hold. If $B(x_n)$ is an approximate inverse of $F'(x^*)$ for each n, then the conclusions of Theorem 1 also hold.*

Proof. The proof is immediate. Since $B(x_n)$ is an approximate inverse for each n,

$$\delta_1 = \max_n \|I - B(x_n)F'(x^*)\| < 1.$$

It follows from (11) that $\delta > 0$ can be preserved. □

There is a situation where the diagonal elements represent a good approximation of the Jacobian matrix. This happens in a strictly diagonal dominant $F'(x^*)$.

Corollary 2. *Let the standard assumptions hold and let $F'(x^*)$ be strictly diagonal dominant. Then there are $K > 0$ and $\delta > 0$ such that if $x_0 \in \mathcal{B}(\delta)$ and D is the diagonal of $F'(x^*)$, then the iteration*

$$x_{n+1} = x_n - D^{-1}F(x_n)$$

converges q-linearly to x^* in ∞-norm with

$$\|e_{n+1}\|_\infty \le K\|e_n\|_\infty.$$

Proof. Let δ be small enough that the conclusions of Theorem 1 hold. Let $F'(x^*) = D + E$, where $D = (d_{ii})$ is the diagonal matrix and $E = (e_{ij})$ is the matrix with off-diagonal elements of $F'(x^*)$. It follows that

$$\|I - D^{-1}F'(x^*)\| = \|I - D^{-1}(D+E)\| = \|D^{-1}E\|.$$

In the ∞-norm, the property of strictly diagonal dominance gives:

$$\|D^{-1}E\|_\infty = \max_i \frac{\sum_{j=1,j\neq i}^{n} |e_{ij}|}{d_{ii}} < 1.$$

So D is an approximate inverse of $F'(x^*)$. The local convergence, K and δ then follow from Corollary 1 in ∞-norm. $\square$

The use of α-method can be viewed as approximating the Jacobian as a diagonal matrix with a uniform value. Definitely, this approximation will not be as good as the diagonal of the real Jacobian matrix. Nevertheless, for a more strictly diagonal dominant Jacobian, a uniform diagonal matrix can still obtain a local convergence.

Theorem 2. *Let the standard assumptions hold and let $F'(x^*) = (f_{ij})$ be strictly diagonal dominant with positive diagonal entries. Then there are $\delta > 0$, $\delta_2 > 0$ and $\alpha > 0$ such that if $x_0 \in \mathcal{B}(\delta)$ and $\max_i \frac{\sum_{j=1,j\neq i}^{n} |f_{ij}|}{f_{ii}} = \rho < \delta_2$, then the iteration*

$$x_{n+1} = x_n - \alpha^{-1}F(x_n)$$

converges q-linearly to x^* in ∞-norm.

Proof. Let $\Delta = I - \alpha^{-1}F'(x^*)$ and $F'(x^*) = D + E$ where D and E are the diagonal and the off-diagonal of $F'(x^*)$. Then

$$\begin{aligned}\|\Delta\| &= \|I - \alpha^{-1}F'(x^*)\| \leq \|I - \alpha^{-1}D\| + \|\alpha^{-1}D\|\,\|D^{-1}E\| \\ &\leq \max_i \left|1 - \frac{f_{ii}}{\alpha}\right| + \frac{\max_i |f_{ii}|}{\alpha}\|D^{-1}E\|\end{aligned}$$

m Let $m_1 = \min_i(|f_{ii}|)$ and $m_2 = \max_i(|f_{ii}|)$ and let $\rho = \|D^{-1}E\|$. The first term on the right-hand side of the above inequality is subject to the position of α with respect to m_1 and m_2. There are two possible cases:

- If $\alpha \in [\frac{m_1+m_2}{2}, \infty)$, the norm of Δ can be expressed as

$$\|\Delta\| \leq \frac{1}{\alpha}(\alpha - m_1) + \frac{\rho m_2}{\alpha} \leq \frac{1}{\alpha}(\alpha - m_1 + \rho m_2) = 1 - \frac{m_1 - \rho m_2}{\alpha}. \tag{12}$$

  If $\rho < m_1/m_2$, then $\|\Delta\| < 1$ will be obtained in this range of α.
- If $\alpha \in (0, \frac{m_1+m_2}{2}]$, the inequality can be written in

$$\|\Delta\| \leq \frac{1}{\alpha}(m_2 - \alpha) + \frac{\rho m_2}{\alpha} \leq \frac{1}{\alpha}(m_2 - \alpha + \rho m_2) = \frac{(1+\rho)m_2}{\alpha} - 1. \tag{13}$$

  It is easy to verify that $\rho < m_1/m_2$ must be preserved in the inequality. Furthermore, given $\rho \in [0, m_1/m_2)$, setting $(\rho m_2 + m_2)/2 < \alpha \leq (m_1 + m_2)/2$ satisfies $\|\Delta\| < 1$.

Combining both, it is contained that $\alpha > (m_2 + \rho m_2)/2$, with $\rho = \|D^{-1}E\| < m_1/m_2$ is the sufficient condition for $\|I - \alpha^{-1}F'(x^*)\| < 1$ under the corresponding assumptions. In ∞-norm, there holds

$$\|D^{-1}E\|_\infty = \max_i \frac{\sum_{j=1,j\neq i}^{n} |f_{ij}|}{f_{ii}} < m_1/m_2.$$

The proof completes where the local linear convergence and δ follows from Corollary 1 in the ∞-norm. $\square$

Although there is no upper bound for α in the above theorem, a larger α will lead to a bigger $\|\Delta\|$ and $x_{n+1} = x_n + \alpha^{-1}F(x_n)$ shows that the iteration will be very slow. It is clear from (12) and (13) that $\alpha = (m_1 + m_2)/2$ gives the smallest $\|\Delta\|$. On the other hand it is not an easy task to estimate the minimum and the maximum diagonal element of $F'(x^*)$. The following lemma is the combination of two lemmas in [7] which will be of very much help later. The proof is given here since the proof of the second lemma is not given in [7].

Lemma 2. *Let $F : \Re^n \to \Re^m$ be continuously differentiable in the open convex set $D \subset \Re^n$,, let $x \in D$, and let F' be Lipschitz continuous at in the neighbourhood D, using a vector norm and the induced matrix operator norm and the constant γ, and assume that $F'(x)^{-1}$ exists. Then there exists $\epsilon > 0$, $0 < \mu < \beta$, such that*

$$\mu\|v - u\| \leq \|F(v) - F(u)\| \leq \beta\|v - u\|$$

for all $v, u \in D$ for which max $\|v - x\|, \|u - x\| \leq \epsilon$.

Proof. With the help of the equation of fundamental calculus (6), for u sufficiently close to v,

$$F(v) - F(u) = \int_0^1 F'(u + t(v-u))(v-u)\mathrm{d}t.$$

Therefore the following can be expressed:

$$F(v) - F(u) - F'(x)(v-u) = \int_0^1 F'(u^* + t(v-u))(v-u)\mathrm{d}t$$
$$- \int_0^1 F'(u^* + t(x-u))(v-u)\mathrm{d}t + \int_0^1 F'(u^* + t(x-u))(v-u)\mathrm{d}t - \int_0^1 F'(x)(v-u)\mathrm{d}t.$$

An inequality in norm can be presented in

$$\begin{aligned}
\|F(v) - F(u) - F'(x)(v-u)\| &\le \int_0^1 \left\|F'(u + t(v-u)) - F'(u + t(x-u))\right\| \, \|v-u\|\mathrm{d}t \\
&\quad + \int_0^1 \|F'(u + t(x-u)) - F'(x)\| \|v-u\|\mathrm{d}t \\
&\le \int_0^1 \gamma \|t((v-u) - (x-u))\| \|v-u\|\mathrm{d}t + \int_0^1 \gamma \|u + t(x-u) - x\| \|v-u\|\mathrm{d}t \\
&\le \gamma \int_0^1 \|t(v-x)\| \|v-u\|\mathrm{d}t + \gamma \int_0^1 \|(1-t)(u-x)\| \|v-u\|\mathrm{d}t \\
&\le \gamma \|v-x\| \|v-u\| \int_0^1 t\mathrm{d}t + \gamma \|u-x\| \|v-u\| \int_0^1 (1-t)\mathrm{d}t \\
&\le \gamma \frac{\|v-x\| + \|u-x\|}{2} \|v-u\|.
\end{aligned} \tag{14}$$

Using the triangle inequality and (14),

$$\begin{aligned}
\|F(v) - F(u)\| &\le \|F'(x)(v-u)\| + \|F(v) - F(u) - F'(x)(v-u)\| \\
&\le \left[\|F'(x)\| + \frac{\gamma}{2}(\|v-x\| + \|u-x\|)\right] \|v-u\| \\
&\le [\|F'(x)\| + \gamma\epsilon] \|v-u\|
\end{aligned}$$

which proves the upper bound with $\beta = \|F'(x)\| + \gamma\epsilon$. Similarly,

$$\begin{aligned}
\|F(v) - F(u)\| &\ge \|F'(x)(v-u)\| - \|F(v) - F(u) - F'(x)(v-u)\| \\
&\ge \left(\frac{1}{\|F'(x)^{-1}\|} - \frac{\gamma}{2}(\|v-x\| + \|u-x\|)\right) \|v-u\| \\
&\ge \left(\frac{1}{\|F'(x)^{-1}\|} - \gamma\epsilon\right) \|v-u\|.
\end{aligned}$$

Thus if $\epsilon < 1/(\|F'(x)^{-1}\|\gamma)$, the lower bound holds with

$$\mu = \left(\frac{1}{\|F'(x)^{-1}\|}\right) - \gamma\epsilon > 0. \quad \square$$

Theorem 3. *Let the standard assumptions hold, let $F'(x^*) = (f_{ij})$ be strictly diagonal dominant with positive diagonal entries, and let $\|.\|$ be the ∞-norm. Then there are $\delta > 0$, $\delta_2 > 0$, and δ_3 such that if $\max_i \frac{\sum_{j=1, j\neq i}^{n} |f_{ij}|}{f_{ii}} = \rho < \delta_2$, $\|F'(x^*)^{-1}\| < \delta_3$, and $x_0, x_{-1} \in B(\delta)$ with $x_0 \neq x_{-1}$, then the iteration*

$$x_{n+1} = x_n - \alpha_n^{-1} F(x_n)$$

with

$$\alpha_n = \alpha_{n-1} \frac{\|F(x_n) - F(x_{n-1})\|}{\|F(x_{n-1})\|} \tag{15}$$

converges q-linearly to x^* in ∞-norm.

Proof. This theorem could be regarded as a corollary to Theorem 2 and Lemma 2. From

$$x_n - x_{n-1} = \alpha_{n-1}^{-1} F(x_{n-1})$$

it follows that

$$\|x_n - x_{n-1}\| = \alpha_{n-1}^{-1} \|F(x_{n-1})\|.$$

Therefore

$$\alpha_n = \alpha_{n-1} \frac{\|F(x_n) - F(x_{n-1})\|}{\|F(x_{n-1})\|} = \frac{\|F(x_n) - F(x_{n-1})\|}{\|x_n - x_{n-1}\|}.$$

Let δ be small enough that $\mathcal{B}(\delta) \subset D$ where D is the open convex set in Lemma 1. Assume that $x_{n-1}, x_n \in \mathcal{B}(\delta)$ and $x_{n-1} \neq x_n$. The lower bound of Lemma 2 then gives

$$\alpha_n \geq \mu \geq \frac{1}{\|F(x^*)^{-1}\|} - \gamma\delta.$$

Now choose $\alpha \in \Re$ such that $\alpha > \frac{1+\rho}{2} m_2$. If $\|F'(x^*)^{-1}\| < 1/\alpha$, then $\theta = \frac{1}{\gamma}(\frac{1}{\|F(x^*)^{-1}\|} - \alpha) > 0$. Reduce δ if necessary so that $\delta \leq \theta$. Then

$$\alpha_n \geq \frac{1}{\|F'(x^*)^{-1}\|} - \gamma\delta \geq \alpha > \frac{1+\rho}{2} m_2.$$

Notice that a general norm is still valid until this point. From the proof of Theorem 2, it can be obtained that

$$\|I - \alpha_n^{-1} F'(x^*)\|_\infty \leq \frac{(1+\rho)m_2}{\alpha_n} - 1 \leq \frac{(1+\rho)m_2}{\alpha} - 1 < 1$$

because in ∞-norm,

$$\rho = \|D^{-1}E\|_\infty = \max_i \frac{\sum_{j=1, j\neq i}^{n} |f_{ij}|}{f_{ii}} < m_1/m_2.$$

The local linear convergence of (15) and the final δ then follows from Theorem 2 in ∞-norm. $\square$

Notice that the norm in the computation of α_n can be different from the norm of the local convergence analysis. In the local convergence, ∞-norm is used whereas any other norm can be employed in obtaining α_n. From this can arise two different Lipschitz constants, each for the corresponding norm. Suppose α_n is computed in a p-norm, $< 1 < p < \infty$, with the associated Lipschitz constant γ_p and the Lipschitz constant associated with the ∞-norm is denoted by γ, then the local neighbourhood $\mathcal{B}(\delta)$ which satisfies Theorem 3 is given by

$$\delta = min \left\{ \frac{2(1-\delta_1)}{\gamma_p (1+\delta_1) \|F'(x^*)^{-1}\|_p}, \frac{1}{\gamma}\left(\frac{1}{\|F(x^*)^{-1}\|_\infty} - \alpha\right) \right\}$$

where α and δ_1 can be concluded from the proof of Theorem 3 as $\delta_1 = \frac{(1+\rho)m_2}{\alpha_n} - 1$ and $\alpha > \frac{1+\rho}{2} m_2$. Although the application of Theorem 3 is restricted to a relatively smaller neighbourhood than many other Newton-related methods, it can be quicker in a certain problem such as the one shown in the next section.

4. A numerical test

The adaptive-α method is applied to a blackbox system of nonlinear equations generated by the defect correction scheme in a multi-subdomain test problem where the following steady state nonlinear heat conduction process occurs:

$$\nabla.(k(u)\nabla u) = \frac{\partial}{\partial x}\left(k(u)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(k(u)\frac{\partial u}{\partial y}\right) = 0. \tag{16}$$

In this equation, u denotes the temperature and k is the heat conductivity which is a function of temperature, thus making the equation nonlinear. This process occurs in a multichip module of electronic device. The physical domain has 5 layers where 2 solder joint layers (Solder-1 and Solder-2) are used to connect 3 board layers (Board-1, Board-2, and Board-3). The construction of this domain is given in Fig. 1 and Table 1. The complete description of the domain is referenced in [8].

Boundary conditions are applied with 100 °C along the top surface and 10 °C along the top surface. The left side of the model is symmetry and for all other boundaries, a convective heat boundary condition of ambient temperature of 25 °C with a heat transfer coefficient of 10 W/m^{2}C. The nonlinearity of the process is reflected by the heat conductivity, which takes

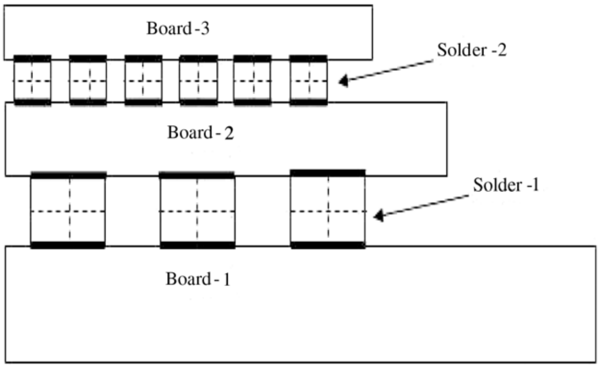

Fig. 1. Multichip geometry.

Table 1
Geometry dimension of components.

	Length (mm)	Height (mm)	Gap interval (mm)
Board-1	19.5	3.0	
Solder-1	2.0	2.0	3.0
Board-2	15.5	2.0	
Solder-2	1.0	1.0	1.0
Board-3	10.5	1.5	

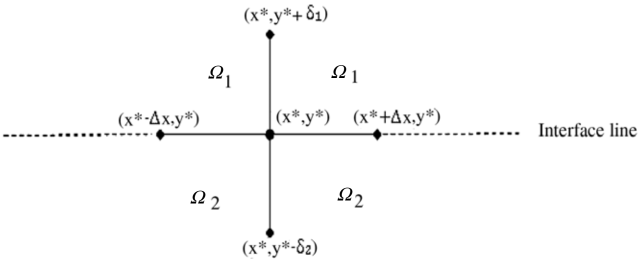

Fig. 2. Five point stencil at the interface.

$k_{board}(u) = 0.005 + 0.0013u$, in board layers
$k_{solder}(u) = 280(0.005 + 0.0013u)$, in solder layers.

The discretisation in each subdomain is carried out using the standard central difference scheme and iterated with Picard linearisation method [9] by using a preconditioned conjugate gradient method [10] as the linear solver. It can be ensured that, at every domain decomposition iteration, the subproblem (16) is solved accurately to a small tolerance in each subdomain. Since the emphasis is given at the application of the adaptive-α method in nonlinear equations generated at the interface segments between subdomains (the segments are shown in dark solid lines in Fig. 1, the setting of defect equation $D(\lambda)$ is the most significant here. In view of (2), the defect equation associated with (16) is obviously represented by the continuity of flux at the interface Γ between two subdomains Ω_1 and Ω_2:

$$D(\lambda) = k_1 \frac{\partial u_1}{\partial \mathbf{n}} - k_2 \frac{\partial u_2}{\partial \mathbf{n}} = 0 \quad \text{on } \Gamma.$$

The finite dimensional representation of the defect equation is obtained by discretising each flux term, $\sigma_i = k_i \frac{\partial u_i}{\partial \mathbf{n}}$. Suppose the mesh size is uniform in the x-direction, whereas that in the y-direction is given by δ_1 and δ_2 as indicated in Fig. 2, a second order accurate defect equation at an interface point (x^*, y^*) can be derived following the difference scheme in [11] by using the following flux approximations:

$$\begin{aligned}\sigma_1(x^*, y^*) = a_1 \Bigg[& k_1(x^*, y^*) \left(\frac{u_1(x^*, y^*) - u_1(x^*, y^* + \delta_1)}{\delta_1} \right) \\ & + \frac{\delta_1}{2} \Bigg(k_1(x^*, y^*) \frac{-u_1(x^* + \Delta x, y^*) + 2u_1(x^*, y^*) - u_1(x^* - \Delta x, y^*)}{\Delta y^2} \\ & - \frac{k_1(x^* + \Delta x, y^*) - k^+(x^* - \Delta x, y^*)}{2\Delta x} . \frac{u_1(x^* + \Delta x, y^*) - u_1(x^* - \Delta x, y^*)}{2\Delta x} \Bigg) \Bigg] \end{aligned} \tag{17}$$

Table 2
The number of domain decomposition iterations.

h (mm)	α-method	Newton-GMRES
0.50	1 622	6 245
0.25	4 316	9 382
0.125	10 164	11 659
0.0625	24 871	26 652

where $a_1 = \frac{2k_1(x^*,y^*)}{3k_1(x^*,y^*)-k_1(x^*,y^*+\delta_1)}$

$$\begin{aligned}\sigma_2(x^*, y^*) = a_2 \Bigg[& k_2(x^*, y^*) \left(\frac{u_2(x^*, y^*) - u_2(x^*, y^* - \delta_2)}{-\delta_2} \right) \\ & - \frac{\delta_2}{2} \Bigg(k_2(x^*, y^*) \frac{-u_2(x^* + \Delta x, y^*) + 2u_2(x^*, y^*) - u_2(x^* - \Delta x, y^*)}{\Delta y^2} \\ & - \frac{k_2(x^* + \Delta x, y^*) - k_2(x^* - \Delta x, y^*)}{2\Delta x} \cdot \frac{u_2(x^* + \Delta x, y^*) - u_2(x^* - \Delta x, y^*)}{2\Delta x} \Bigg) \Bigg] \end{aligned} \tag{18}$$

where $a_2 = \frac{2k_2(x^*,y^*)}{3k_2(x^*,y^*)-k_2(x^*,y^*+\delta_2)}$.

In the defect correction iterative procedure, the continuity of variable u is enforced at the interface Γ. Therefore the following hold:

$$\begin{aligned} & u_1(x^*, y^*) = u_2(x^*, y^*) = \lambda(x^*, y^*) \\ & u_1(x^* + \Delta x, y^*) = u_2(x^* + \Delta x, y^*) = \lambda(x^* + \Delta x, y^*) \\ & u_1(x^* - \Delta x, y^*) = u_2(x^* - \Delta x, y^*) = \lambda(x^* - \Delta x, y^*). \end{aligned}$$

If the discretisation is performed using equidistant square mesh in the entire domain ($\delta_1 = \delta_2 = \Delta x = h$), a rough inspection of the coefficients of $\lambda(x^* - \Delta x, y^*)$, $\lambda(x^*, y^*)$, and $\lambda(x^* + \Delta x, y^*)$ indicates a tendency of diagonal dominance of the interface problem. The iterative procedure of the defect correction scheme is carried out by starting with an initial guess of 20 °C throughout the entire domain. Thecomputations are run under different mesh specifications ($h = 0.5$, $h = 0.25$, $h = 0.125$, $h = 0.06125$ mm). The stopping criterion is chosen whena relative residual of 5E-3 is achieved. The relative residual is defined by the ratio of the current L^2-norm of $D(\lambda)$ and that of the initial defect. The value of 5E-3 is chosen since, for some mesh specifications, the minimum relative residual achieved is bigger than 1E-3.

The performance of the adaptive-α method is compared with the Newton-GMRES method [12] [13] as both methods circumvent the computation of Jacobian. Armijo's rule [14] is used as the line search method in the latter method. A set of Newton-GMRES' parameters is chosen from the most optimal results from various numerical experiments, hence it is better than the estimate suggested in [13]. Yet Table 2 indicates that, under the four mesh specifications, the adaptive-α method performs slightly better than the Newton-GMRES in terms of the number of domain decomposition iterations needed to achieve the relative residual of 5E-3.

Note also that the adaptive-α method only needs to perform simple calculations as opposed to the Newton-GMRES which requires a great deal of matrix computation for the GMRES steps in each nonlinear iteration as well as some line search steps.

5. Conclusion

The local convergence analysis of the adaptive-α method has been demonstrated. The method is handy for parallel computations of very strongly diagonal dominant Jacobian problems. It is tested on a system of blackbox nonlinear equations arising from the application of a nonoverlapping domain decomposition method in a multichip nonlinear heat conduction problem. The results show that the method performs slightly better than the Newton-GMRES method.

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