Journal of Computational and Applied Mathematics 239 (2013) 93–102

Contents lists available at SciVerse ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Estimating the lifetime performance index with Weibull distribution based on progressive first-failure censoring scheme

Mohammad Vali Ahmadi, Mahdi Doostparast *, Jafar Ahmadi

Department of Statistics, Ordered and Spatial Data Center of Excellence, Ferdowsi University of Mashhad, P. O. Box 91775-1159, Mashhad, Iran

ARTICLE INFO

Article history:
Received 12 September 2011
Received in revised form 4 September 2012

Keywords:
Confidence interval
Maximum likelihood estimator
Process capability indices
Pitman measure of closeness
Weibull distribution

ABSTRACT

An important topic in the manufacturing industries is the assessing of the lifetime performance. In this paper, it is supposed the lifetimes of products are independent and have a common Weibull distribution with a known shape parameter. The lifetime performance index (C_L) proposed by Montgomery (1985) [1], is used for evaluating the performance of a process with respect to a lower specification limit (L). The maximum likelihood estimate of C_L is obtained on the basis of the progressive first-failure censored data. This estimate is then used for developing a confidence interval for C_L. The behavior of the confidence interval for the parameter C_L given a significance level is investigated and also two illustrative examples and a sensitivity analysis are given. For the exponential distribution as a special case of Weibull distribution, a comparison study for various estimates of C_L based on mean squared error (MSE) and Pitman measure of closeness (PMC) criteria is done.

1. Introduction

For measuring the performance of a process in the manufacturing and services industry, process capability indices (PCIs) are used to examine the level of product quality. There are various PCIs in the literature. For more details see for example Montgomery [1] and references therein. Throughout this paper, the lifetime performance index (C_L), defined in [1], is used for evaluating the performance of the process given a lower specification limit (L). Statistical inference for C_L on the basis of various lifetime distributions has been considered. For example, Tong et al. [2] constructed the uniformly minimum variance unbiased estimator (UMVUE) for C_L and considered the problem of the hypothesis testing procedure for the one-parameter exponential distribution based on a complete sample.

Usually, in life testing experiments, there are time limits and/or other restrictions including money, material resources, mechanical or experimental difficulties on data collection. In these situations, the exact lifetime of products put on a test is not observable and we have censored data. There are several types of censoring schemes in survival analysis and the Type-II censoring scheme is one of the most common for consideration, see for example [3]. A generalization of Type-II censoring is the *progressive Type*-II *censoring* which allows for units to be removed from the test at points other than the final termination point. The description of the progressive Type-II censoring is as follows. A group of n products is placed on a test and the test is terminated at the time of the m-th failure. When the i-th item fails ($i = 1, 2, \dots, m-1$), R_i of the surviving items are removed randomly from the test. Finally, all of the remaining items $R_m = n - m - \sum_{j=1}^{m-1} R_j$ are removed from the test when the m-th failure occurs. Notice that m and $\mathbf{R} = (R_1, \dots, R_m)$ are pre-assigned. See [4] for theory and applications about progressive Type-II censoring. In recent years, there have been many works on the statistical inference for C_L based on the usual Type-II and progressive Type-II censoring schemes with various lifetime distributions. See, for example [5–10].

* Corresponding author. Tel.: +98 9155098659; fax: +98 5118828605.
E-mail addresses: ahmadi_m85@yahoo.com (M.V. Ahmadi), doostparast@math.um.ac.ir, doustparast@um.ac.ir (M. Doostparast), ahmadi-j@um.ac.ir (J. Ahmadi).

0377-0427/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2012.09.006

Also, Lee et al. [11] have constructed a credible interval for C_L using a Bayesian approach and proposed a Bayesian test for evaluating the lifetime performance of the products.

In some cases, the lifetime of products is quite long and so the experimental time of the Type-II censoring scheme can still be too long. In order to give an efficient experiment, the other test methods are proposed by statisticians where one of them is the first-failure censoring scheme, introduced in [12] and further illustrated in [13]. In this censoring scheme, $m \times n$ items are divided to m equal groups. Then, the life test is conducted by testing each of these groups simultaneously and terminated when the first failure is observed in each groups. As mentioned in [14], based on this censoring scheme, a considerable amount of time and money can be saved. Statistical inference on the parameters of various distributions under the first-failure censoring scheme have been studied by several authors. For example, Wu et al. [15] and Wu and Yu [16] obtained maximum likelihood (ML) estimates, exact confidence intervals and confidence regions for the parameters of Gompertz and Burr type XII distributions on the basis of the first-failure censored data, respectively.

Since both of the above mentioned schemes improve the efficiency of the test, Wu and Kus [14] combined these schemes in order to propose a new life test plan called the *progressive first-failure censoring scheme* which is more efficient in lifetime studies. Also, assuming the two-parameter Weibull distribution for the lifetime data, Wu and Kus [14] proved that the progressive first-failure censoring scheme had shorter expected test times than the progressive Type-II censoring scheme. In the progressive first-failure censoring scheme, n disjoint groups with k items within each group ($N = n \times k$) are placed on a test at time zero. The life test is terminated at the time of the m-th failure. When the i-th item fails ($i = 1, 2, \ldots, m-1$), randomly selected R_i groups and the group including the i-th failure are removed from the test. When the m-th failure occur, all of the remaining groups are removed from the test. Note that: (i) for $k = 1$, the progressive first-failure censoring scheme is reduced to the case of progressive Type-II censoring, (ii) if $R_i = 0$ for $i = 1, 2, \ldots, m$, we have the first-failure censoring, (iii) if $k = 1$, $R_i = 0$ for $i = 1, 2, \ldots, m-1$ and hence $R_m = n - m$, this scheme is reduced to the Type-II right censoring and (iv) if $k = 1$ and $R_i = 0$ for $i = 1, 2, \ldots, m$, this scheme is simplified to the complete sample.

The Weibull model with known shape parameter has been considered in literature and applied in practice. See, for example [17–19]. Notice that the exponential and Rayleigh distributions are two special cases of this model which are used widely in reliability analysis [20]. Based on the Type-II right censoring scheme, Lee [21] used the *max p-value method* to select the optimum value of the shape parameter β of the Weibull distribution and hence supposed that β is known. Then, he constructed the ML estimator for C_L and developed a testing procedure for the lifetime performance index of the products with Weibull distribution on the basis of the Type-II right censored sample. The main aim of this paper is to develop a confidence interval and the ML estimator for C_L based on the progressive first-failure censored sample under Weibull distribution with a known shape parameter. Therefore, the organization of this paper is as follows. In Section 2, we introduce some properties of the lifetime performance index C_L when the lifetime of the products is coming from the Weibull distribution. The relationship between the lifetime performance index C_L and the *conforming rate* (the ratio of conforming products) is discussed in Section 2. The ML estimate of the lifetime performance index C_L and some of the corresponding statistical properties are investigated in Section 3. Section 4 develops a lower bound for the lifetime performance index C_L. Two illustrative examples and a sensitivity study via a Monte Carlo method are conducted in Section 5. In Section 6, a comparison study for various estimates of C_L based on MSE and PMC criteria is done under the exponential model. Section 7 concludes.

2. The lifetime performance index

Let X be the lifetime of products, then following [1], C_{L_X} is defined as

$$C_{L_X} = \frac{\mu_X - L_X}{\sigma_X}, \tag{1}$$

where μ_X and σ_X are, respectively, mean and standard deviation of X while L_X is a lower specification limit based on X's random variable. To assess the lifetime performance of products, C_{L_X} can be considered as the lifetime performance index. Throughout this paper, we suppose that the random variable X of products follows a two-parameter Weibull distribution with the shape and scale parameters β and λ, respectively. Hence, the probability density function (p.d.f.) and the cumulative distribution function (c.d.f.) of the lifetime X of products are

$$f_X(x; \lambda, \beta) = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1} \exp\left\{-\left(\frac{x}{\lambda}\right)^{\beta}\right\}, \quad x > 0,\ \lambda > 0,\ \beta > 0, \tag{2}$$

and

$$F_X(x; \lambda, \beta) = 1 - \exp\left\{-\left(\frac{x}{\lambda}\right)^{\beta}\right\}, \quad x > 0,\ \lambda > 0,\ \beta > 0, \tag{3}$$

respectively. Since the mean and the standard deviation of the process are, respectively, given by $\mu_X = E(X) = \lambda\,\Gamma(1 + 1/\beta)$ and $\sigma_X = \sqrt{\mathrm{Var}(X)} = \lambda\sqrt{\Gamma(1 + 2/\beta) - (\Gamma(1 + 1/\beta))^2}$, then from (1) the lifetime performance index is given by

$$C_{L_X} = \frac{1}{A}\left[\Gamma\left(1 + \frac{1}{\beta}\right) - \frac{L_X}{\lambda}\right], \quad -\infty < C_{L_X} < \frac{1}{A}\,\Gamma\left(1 + \frac{1}{\beta}\right), \tag{4}$$

Table 1
The lifetime performance index C_{L_X} versus the conforming rate P_r for $\beta = 1.89$.

C_{L_X}	P_r	C_{L_X}	P_r	C_{L_X}	P_r	C_{L_X}	P_r
$-\infty$	0.00000	−0.6	0.25456	0.2	0.52713	1.0	0.83828
−1.3	0.10944	−0.5	0.28273	0.3	0.56690	1.1	0.87121
−1.2	0.12491	−0.4	0.31279	0.4	0.60715	1.2	0.90137
−1.1	0.14202	−0.3	0.34467	0.5	0.64755	1.3	0.92832
−1.0	0.16084	−0.2	0.37828	0.6	0.68773	1.4	0.95162
−0.9	0.18146	−0.1	0.41350	0.7	0.72732	1.5	0.97086
−0.8	0.20393	0.0	0.45019	0.8	0.76589	1.6	0.98562
−0.7	0.22829	0.1	0.48814	0.9	0.80302	1.7	0.99547

where

$$A = \left[\Gamma\left(1+\frac{2}{\beta}\right) - \left(\Gamma\left(1+\frac{1}{\beta}\right)\right)^2\right]^{1/2}, \tag{5}$$

and $\Gamma(\cdot)$ is the complete gamma function. Obviously, for $\lambda\Gamma(1+1/\beta) > L_X$, we have $C_{L_X} > 0$. Also, the failure rate function $r(x;\lambda,\beta)$ is readily obtained as

$$r(x;\lambda,\beta) = \frac{f_X(x;\lambda,\beta)}{1-F_X(x;\lambda,\beta)} = \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1}, \quad x>0,\ \lambda>0,\ \beta>0. \tag{6}$$

From (4) and (6), we see that the failure rate function is decreasing in λ while the lifetime performance index C_{L_X} is increasing in λ. Therefore, C_{L_X} is appropriate for representing the lifetime performance of products.

Throughout this paper, if $X > (<) L_X$, then the product is called the conforming (nonconforming) product. Therefore, the ratio of conforming products is known as the *conforming rate* which is defined as

$$P_r = P(X \ge L_X) = \exp\left\{-\left(\Gamma\left(1+\frac{1}{\beta}\right) - A\,C_{L_X}\right)^{\beta}\right\}, \quad -\infty < C_{L_X} < \frac{1}{A}\Gamma\left(1+\frac{1}{\beta}\right). \tag{7}$$

Obviously, a strictly increasing relationship exists between the conforming rate P_r and the lifetime performance index C_{L_X}, for given $\beta > 0$. In Table 1, some numerical values of C_{L_X} and the corresponding conforming rates P_r for $\beta = 1.89$ are presented. One can easily obtain P_r from (7) for given C_{L_X} and β.

3. ML estimate of C_L

Let $X_{1:m:n:k} < X_{2:m:n:k} < \cdots < X_{m:m:n:k}$ be the progressive first-failure censored sample from a continuous population with p.d.f. and c.d.f $f_X(\cdot;\theta)$ and $F_X(\cdot;\theta)$, respectively, where θ is a vector of parameters. Following [14], the associated likelihood function of the observed data $\mathbf{x} = (x_1, x_2, \ldots, x_m)$ reads

$$L(\theta;\mathbf{x}) = C\,k^m \prod_{i=1}^{m} f_X(x_i;\theta)[1-F_X(x_i;\theta)]^{k(R_i+1)-1}, \tag{8}$$

where $0 < x_1 < \cdots < x_m < \infty$ and

$$C = n(n-R_1-1)(n-R_1-R_2-2)\cdots\left(n-\sum_{i=1}^{m-1}R_i - m + 1\right).$$

Upon substituting (2) and (3) into (8), the likelihood function becomes as

$$L(\lambda,\beta;\mathbf{x}) = C\left(\frac{k\beta}{\lambda^\beta}\right)^m \prod_{i=1}^{m} x_i^{\beta-1}\exp\left(-k\sum_{i=1}^{m}(R_i+1)\left(\frac{x_i}{\lambda}\right)^{\beta}\right). \tag{9}$$

For known β, the ML estimate of λ is readily derived from (9) as

$$\hat{\lambda} = \left(\frac{k}{m}\sum_{i=1}^{m}(R_i+1)\,X_{i:m:n:k}^{\beta}\right)^{\frac{1}{\beta}}. \tag{10}$$

By using the invariance property of ML estimators (see [22]), the ML estimate of C_{L_X} is given by

$$\hat{C}_{L_X} = \frac{1}{A}\left[\Gamma\left(1+\frac{1}{\beta}\right) - \frac{L_X}{\hat{\lambda}}\right] = \frac{1}{A}\left[\Gamma\left(1+\frac{1}{\beta}\right) - L_X\left(\frac{m}{W}\right)^{\frac{1}{\beta}}\right], \tag{11}$$

where $W = k\sum_{i=1}^{m}(R_i+1)\,X_{i:m:n:k}^{\beta}$.

Remark 3.1. Wu and Kus [14] proved that $(2W/\lambda^{\beta})$ has a chi-squared distribution with $2m$ degrees of freedom. Hence, it can be shown that the ML estimator $\hat{C}_{L_X}$ is an asymptotically unbiased estimator, when $m \longrightarrow \infty$. In addition, it is proved that this estimator is consistent for C_{L_X}.

Remark 3.2. Since the parameter β is known, from (9) and the fact that W has a gamma distribution with shape and scale parameters m and λ^{β}, respectively, one can conclude that the statistic W is sufficient and complete for λ.

4. Confidence interval for C_L

In this section, a $100 \times (1-\alpha)\%$ lower bound for C_{L_X} is obtained by using the ML estimate given by (11) and then, based on this lower bound, a hypothesis testing procedure is developed in order to determine whether the lifetime performance index of products meets the predetermined level. To this end, let c_X denote the lower bound of C_{L_X}. Notice that C_{L_X} of products must be larger than c_X. Our aim is to test the null hypothesis $H_0 : C_{L_X} \le c_X$ (the product is unreliable) against the alternative $H_1 : C_{L_X} > c_X$ (the product is reliable). Since β is known and by Remark 3.1 and using the fact that $(2W/\lambda^{\beta}) \sim \chi^2_{2m}$, we have

$$\begin{aligned}
1-\alpha &= P\left(\frac{2W}{\lambda^{\beta}} \le \chi^2_{1-\alpha,2m}\right) \\
&= P\left(\frac{1}{A}\left[\Gamma\left(1+\frac{1}{\beta}\right) - \frac{L_X}{\lambda}\right] \ge \frac{1}{A}\left[\Gamma\left(1+\frac{1}{\beta}\right) - L_X\left(\frac{\chi^2_{1-\alpha,2m}}{2W}\right)^{\frac{1}{\beta}}\right]\right) \\
&= P\left(C_{L_X} \ge \frac{1}{A}\left[\Gamma\left(1+\frac{1}{\beta}\right) - L_X\left(\frac{\chi^2_{1-\alpha,2m}}{2W}\right)^{\frac{1}{\beta}}\right]\right) \\
&= P\left(C_{L_X} \ge \frac{1}{A}\left[\Gamma\left(1+\frac{1}{\beta}\right) - \left(\frac{\chi^2_{1-\alpha,2m}}{2m}\right)^{\frac{1}{\beta}}\left(\Gamma\left(1+\frac{1}{\beta}\right) - A\hat{C}_{L_X}\right)\right]\right),
\end{aligned} \tag{12}$$

or a $100 \times (1-\alpha)\%$ lower bound for C_{L_X} is

$$\underline{LB}_X = \frac{1}{A}\left[\Gamma\left(1+\frac{1}{\beta}\right) - \left(\frac{\chi^2_{1-\alpha,2m}}{2m}\right)^{\frac{1}{\beta}}\left(\Gamma\left(1+\frac{1}{\beta}\right) - A\hat{C}_{L_X}\right)\right], \tag{13}$$

where $\chi^2_{\gamma,\nu}$ is the γ-th percentile of the chi-square distribution with ν degrees of freedom. Since $W \sim \Gamma(m, \lambda^{\beta})$ and C_{L_X} in (4) is a strictly increasing function of λ, by Corollary 3.4.1 of [23], a uniformly most powerful (UMP) test for testing the null hypothesis $H_0 : C_{L_X} \le c_X$ against the alternative $H_1 : C_{L_X} > c_X$ has the rejection region of the form

$$W > \omega \tag{14}$$

where ω is a constant and determined by the level of the test. From Remark 3.1, the rejection region (14) is reduced to $(-\infty, \underline{LB}_X)$. Thus, by Theorem 3.5.1 of [23], the lower bound $\underline{LB}_X$ given by (13) is a uniformly most accurate lower (UMAL) bound for C_{L_X}.

One may use the following computationally more simple approach. Using the transformation $Y = X^{\beta}$, $Y_{1:m:n:k} < Y_{2:m:n:k} < \cdots < Y_{m:m:n:k}$ will be the corresponding progressive first-failure censored sample from the one-parameter exponential distribution with p.d.f. and c.d.f.

$$f_Y(y;\theta) = \theta \exp(-\theta x), \quad x > 0,\ \theta > 0, \tag{15}$$

and

$$F_Y(y;\theta) = 1 - \exp(-\theta x), \quad x > 0,\ \theta > 0, \tag{16}$$

respectively, where $\theta = \lambda^{-\beta}$ and β is known. Thus, the lifetime performance index reduces to

$$C_{L_Y} = \frac{\mu_Y - L_Y}{\sigma_Y} = 1 - \theta L_Y, \quad -\infty < C_{L_Y} < 1, \tag{17}$$

where $\mu_Y = \sigma_Y = 1/\theta$ and $L_Y = L_X^{\beta}$. Substituting (15) and (16) into (8), the associated likelihood function of the observed data $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ becomes as

$$L(\theta;\mathbf{y}) = C\,(k\theta)^m \exp\left(-k\theta \sum_{i=1}^{m}(R_i+1)y_i\right), \quad 0 < y_1 < \cdots < y_m < \infty. \tag{18}$$

Table 2
Failures of 23 ball bearings data for Example 1.

17.88	28.92	33.00	41.52	42.12	45.60	48.48	51.84	51.96	54.12	55.56	67.80
68.64	68.64	68.88	84.12	93.12	98.64	105.12	105.84	127.92	128.04	173.40	

The ML estimates of θ and C_{L_Y} are

$$\hat{\theta} = \frac{m}{k\sum_{i=1}^{m}(R_i+1)Y_{i:m:n:k}}, \tag{19}$$

and

$$\hat{C}_{L_Y} = 1 - \frac{m}{W'} L_Y, \tag{20}$$

respectively, where

$$W' = k\sum_{i=1}^{m}(R_i+1)Y_{i:m:n:k}. \tag{21}$$

Notice that $2\theta W' \sim \chi^2_{2m}$. Thus

$$\begin{aligned} 1-\alpha &= P\left(2\theta W' \le \chi^2_{1-\alpha,2m}\right) \\ &= P\left(C_{L_Y} \ge 1 - \frac{\chi^2_{1-\alpha,2m}}{2m}(1-\hat{C}_{L_Y})\right). \end{aligned} \tag{22}$$

From (22), the $100 \times (1-\alpha)\%$ lower bound for C_{L_Y} can be derived as:

$$\underline{LB}_Y = 1 - \frac{\chi^2_{1-\alpha,2m}}{2m}(1-\hat{C}_{L_Y}). \tag{23}$$

Similarly, the rejection region for a UMP test of the null hypothesis $H'_0 : C_{L_Y} \le c_Y$ against the alternative $H'_1 : C_{L_Y} > c_Y$ is $(-\infty, \underline{LB}_Y)$. This yields that $\underline{LB}_Y$ in (23) is a UMAL bound for C_{L_Y}.

5. Illustrative examples

For illustration of the proposed procedures, we consider a real data set, due to Lawless [24], a simulated first-failure progressive censored sample and a sensitivity study via a Monte Carlo method.

Example 1 (*Ball Bearing Data Set*). In Table 2, the number of millions of revolutions before failing for 23 ball bearings in a life test were presented, due to Lawless [24, p. 99]. Lawless [24] observed that the Weibull distribution is appropriate for these failure times. In order to estimate the shape parameter β in the Weibull distribution, the Gini statistic is suggested (see [25]). Similar to Lee [21], the Gini statistic is defined as

$$G_n = \frac{\sum_{i=1}^{n-1} iD_{i+1}}{(n-1)\sum_{i=1}^{n} D_i},$$

where $D_i = (n-i+1)(T_{i:n} - T_{i-1:n})$ for $i = 2, 3, \ldots, n$ and $D_1 = nT_{1:n}$ while $T_{i:n} = X^{\beta}_{i:n}$. For $n > 20$, $(12(n-1))^{1/2}(G_n - 0.5)$ tends to the standard normal distribution $N(0, 1)$. Hence, the p-value $= P\left\{|Z| > |(12(n-1))^{1/2}(g_n - 0.5)|\right\}$, where g_n is the observed value of G_n and Z has an approximation of $N(0, 1)$. So, by using the maximum p-value method, the optimum value of β is selected and then we suppose β is known. For the data set in Table 2, the values of β and the corresponding p-values are shown in Table 3.

Table 3 indicates that $\beta = 1.89$ is very close to the optimum value and the maximum p-value $= 0.99797$. So, we assume that the failure times of ball bearings follow a Weibull distribution with the shape parameter $\beta = 1.89$.

A progressive first-failure censoring scheme was conducted with $k = 1, m = 10$ and $(R_1, \ldots, R_m) = (0, 0, 0, 1, 2, 2, 2, 2, 2, 2)$ and the observed values were presented in Table 4. Let $L_X = 28.58$ and to deal with the product purchasers' concerns about lifetime performance, the conforming rate P_r of products was assumed to exceed 80%. From (7) and Table 1, the C_{L_X} will exceed 0.9. Therefore, the performance index value is supposed to be $c_X = 0.9$. Hence, the testing hypothesis

Table 3
Numerical values of p-values for ball bearing data set.

β	p-value	β	p-value	β	p-value	β	p-value
1.70	0.50983	1.82	0.81304	1.94	0.86507	2.06	0.57537
1.71	0.53310	1.83	0.83990	1.95	0.83909	2.07	0.55388
1.72	0.55683	1.84	0.86686	1.96	0.81336	2.08	0.53287
1.73	0.58098	1.85	0.89389	1.97	0.78793	2.09	0.51234
1.74	0.60554	1.86	0.92095	1.98	0.76280	2.10	0.49231
1.75	0.63047	1.87	0.94802	1.99	0.73801	2.11	0.47277
1.76	0.65574	1.88	0.97505	2.00	0.71357	2.12	0.45373
1.77	0.68134	**1.89**	**0.99797**	2.01	0.68950	2.13	0.43519
1.78	0.70722	1.90	0.97109	2.02	0.66583	2.14	0.41717
1.79	0.73337	1.91	0.94433	2.03	0.64256	2.15	0.39966
1.80	0.75974	1.92	0.91772	2.04	0.61972	2.16	0.38265
1.81	0.78630	1.93	0.89129	2.05	0.59732	2.17	0.36616

Table 4
Progressive first-failure censored sample for ball bearing data set.

i	1	2	3	4	5	6	7	8	9	10
$X_{i:m:n:k}$	17.88	28.92	33.00	41.52	42.12	45.60	51.84	51.96	55.56	105.12
R_i	0	0	0	1	2	2	2	2	2	2

Table 5
Transformed progressive first-failure censored sample with transformation $Y_{i:m:n:k} = (X_{i:m:n:k})^{1.89}$ based on the data in Table 4.

i	1	2	3	4	5	6	7	8	9	10
$Y_{i:m:n:k}$	232.8	577.7	741.3	1144.2	1175.7	1366.0	1740.7	1748.3	1984.3	6622.0
R_i	0	0	0	1	2	2	2	2	2	2

Table 6
Simulated progressive first-failure censored sample.

i	1	2	3	4	5	6	7	8	9	10	11
$X_{i:m:n:k}$	0.1401	0.1684	0.2691	0.2860	0.2964	0.2967	0.3287	0.4513	0.6113	0.6241	0.6657
R_i	0	1	0	1	0	1	0	1	0	1	0
i	12	13	14	15	16	17	18	19	20	21	22
$X_{i:m:n:k}$	0.7395	0.7806	0.7916	0.8252	0.8753	0.9379	0.9584	1.0068	1.1725	1.2230	1.2341
R_i	1	0	1	0	1	0	1	0	1	0	1
i	23	24	25	26	27	28	29	30	31	32	33
$X_{i:m:n:k}$	1.2447	1.2731	1.3228	1.3648	1.3773	1.5918	1.5989	1.7877	1.9148	1.9501	1.9656
R_i	0	1	0	1	0	1	0	1	0	1	6

$H_0 : C_{L_X} \leq 0.9$ against the alternative $H_1 : C_{L_X} > 0.9$ is concerned. From (13), the 95% lower bound for C_{L_X} is obtained as $\underline{LB}_X = 0.9772$. Since $c_X = 0.9 \notin [0.9772, \infty)$, so we reject the null hypothesis $H_0 : C_{L_X} \leq 0.9$ in favor of $H_1 : C_{L_X} > 0.9$.

As discussed in Section 4, the proposed testing procedure for C_{L_Y} can be stated. To this end, the values of $Y_{i:m:n:k} = (X_{i:m:n:k})^{1.89}$ were presented in Table 5. By assumption $L_X = 28.58$, then L_Y was obtained as 564.88. Thus, using the expression $P_r = P(Y \geq L_Y) = e^{C_{L_Y}-1}$, the performance index value is supposed to be $c_Y = 0.78$ and the testing hypothesis $H_0' : C_{L_Y} \leq 0.78$ versus $H_1' : C_{L_Y} > 0.78$ is constructed. From (23), the 95% lower bound for C_{L_Y} is derived as $\underline{LB}_Y = 0.9906$. Since $c_Y = 0.78 \notin [0.9906, \infty)$, we reject the null hypothesis H_0.

Based on the two methods, the lifetime performance index of products meets the required level.

Example 2 (*Simulated Data Set*)**.** A progressive first-failure censored sample with $n = 220$, $k = 4$, $m = 33$ and $(R_1, \ldots, R_m) = (0, 1, 0, 1, \ldots, 0, 1, 6)$ was generated from a Weibull distribution with p.d.f. (2) and $(\beta, \lambda) = (1.89, 3.35)$. The observed data were reported in Table 6.

Based on the censored data in Table 6 and assuming $L_X = 1.5$, $C_{L_X} = 0.9$ and $P_r = 0.9$, we obtained $c_X = 1.2$ and $\underline{LB}_X = 0.7538$. Since $c_X = 1.2 \in [0.7538, \infty)$, the null hypothesis $H_0 : C_{L_X} \leq 1.2$ is not rejected. Similarly, the proposed testing procedure for C_{L_Y} can be conducted. To this end, let the transformation $Y_{i:m:n:k} = (X_{i:m:n:k})^{1.89}$, $i = 1, 2, \ldots, 33$. Thus, $L_Y = 2.152$ and $c_Y = 0.89$. From (23), the 95% lower bound for C_{L_Y} is obtained as $\underline{LB}_Y = 0.7099$. Since $c_Y = 0.89 \in [0.7099, \infty)$, the null hypothesis $H_0' : C_{L_Y} \leq 0.89$ is not rejected.

Example 3 (*Sensitivity Analysis*)**.** To carry out the effect of the shape parameter β and the censoring scheme $(R_1, R_2, \ldots, R_m)$, a simulation study was conducted for the different values of $\beta = 0.5, 1, 1.89, 2,$ and 3 and the various censoring schemes with $n = 69$, $k = 3$ and $m = 7$, 13, and 18. We also consider $\lambda = 1$ and $L = 1.5$. The simulation algorithm of $(1-\alpha)\%$ lower bound is given in the following steps:

1. A random sample of size n represented by $X_1, X_2, \ldots, X_n$ is generated from the Weibull distribution with parameters λ and β.
2. The progressive first-failure censored data $(X_{1:m:n:k}, X_{2:m:n:k}, \ldots, X_{m:m:n:k})$ is simulated from a random sample obtained in Step 1, and then by transformation $Y_{i:m:n:k} = (X_{i:m:n:k})^{\beta}$, the values $(Y_{1:m:n:k}, Y_{2:m:n:k}, \ldots, Y_{m:m:n:k})$ are obtained.
3. The 95% lower bounds $\underline{LB}_{X,i}$ and $\underline{LB}_{Y,i}$ are calculated from (13) and (23).
4. If $C_{L_X} > \underline{LB}_{X,i}$ then $\text{Count}_{X,i} = 1$ else $\text{Count}_{X,i} = 0$. Similarly, if $C_{L_Y} > \underline{LB}_{Y,i}$, it's supposed that $\text{Count}_{Y,i} = 1$ else $\text{Count}_{Y,i} = 0$.
5. Steps 1–4 are repeated $t = 5 \times 10^4$ times and the mean of lower bounds based on censored data (X) and transformed censored data (Y) as well as the coverage probabilities of the confidence intervals of C_{L_X} and C_{L_Y} are computed from the following equations.

$$ME(\underline{LB}_X) = \frac{1}{t}\sum_{i=1}^{t} \underline{LB}_{X,i},$$

$$ME(\underline{LB}_Y) = \frac{1}{t}\sum_{i=1}^{t} \underline{LB}_{Y,i},$$

$$ME(CO_X) = \frac{1}{t}\sum_{i=1}^{t} \text{Count}_{X,i},$$

$$ME(CO_Y) = \frac{1}{t}\sum_{i=1}^{t} \text{Count}_{Y,i}.$$

The results of the simulation study were reported in Table 7. From empirical evidence in Table 7, we have:

- The lower bounds for C_{L_X} and C_{L_Y} are quite sensitive to the value of β. Also, when β is underestimated (<1.89), the actual coverage probabilities of the confidence lower bounds for C_{L_X} and C_{L_Y} are less than the nominal level, while the overestimated value of β (>1.89) leads to the larger values for these coverage probabilities relative to the nominal level. Thus, the exact determination of β seems very important.
- If β is properly estimated, it seems that the results obtained are stable with respect to the different progressive first-failure censoring employed. But the incorrect estimation of β causes the results obtained to be sensitive with respect to the choice of censoring scheme, especially when m is small.
- As we expected, for all of the progressive first-failure censoring schemes, the results based on the censored data and the transformed censored data are identical.

6. More results for exponential model

Due to the Weibull distribution with known shape parameter is considered as the transformation of the exponential distribution, in this section, using the data generated by the exponential distribution and first-failure progressive censoring, we will compare the various estimates of C_{L_Y} on the basis of two popular criteria, i.e. MSE and PMC. In order to obtain the best estimate based on the MSE criterion, we confine ourselves to the class $\mathcal{C}$ of the form

$$\mathcal{C} = \left\{\delta_a \middle| \delta_a = 1 - \frac{a}{W'}L\right\}, \tag{24}$$

where W' is defined by (21).

In what follows, we show that the estimator $C^*_{L_Y} \equiv 1 - \frac{m-2}{W'}L$ has the minimum MSE in the class of estimators $\mathcal{C}$ in (24). To see this, by definition, for every $\delta_a \in \mathcal{C}$, we have

$$\text{MSE}_\theta(\delta_a) = E_\theta(\delta_a - C_{L_Y})^2 = \text{Var}_\theta(\delta_a) + \left[E_\theta(\delta_a) - C_{L_Y}\right]^2. \tag{25}$$

Since $2\theta W' \sim \chi^2_{2m}$, after some algebraic manipulations, we have

$$E_\theta(\delta_a) = 1 - aL\frac{\theta}{m-1}, \quad \text{and} \quad \text{Var}_\theta(\delta_a) = \frac{(aL\theta)^2}{(m-1)^2(m-2)}. \tag{26}$$

Substituting (26) in (25), the MSE of δ_a is obtained as

$$\text{MSE}_\theta(\delta_a) = \left(\frac{L\theta}{m-1}\right)^2\left[\frac{a^2}{m-2} + (m-1-a)^2\right]. \tag{27}$$

By differentiating from $\text{MSE}_\theta(\delta_a)$ with respect to a, the desired result follows.

Table 7
The mean of lower bounds and the coverage probabilities of the confidence intervals of C_{L_X} and C_{L_Y} for the different values of β and the various progressive first-failure censoring schemes with $n = 69$, $k = 3$, $L = 1.5$ and $\lambda = 1$.

m	$(R_1, \ldots, R_m)$	β	$ME(\underline{LB}_X)$	$ME(CO_X)$	$ME(\underline{LB}_Y)$	$ME(CO_Y)$
7	$(16, 0, \ldots, 0)$	0.5	0.3782	0.0001	0.4544	0.0001
		1	−0.3456	0.2820	−0.3456	0.2820
		1.89	−2.4919	0.9504	−3.2407	0.9504
		2	−2.6991	0.9574	−3.7856	0.9574
		3	−4.1080	0.9664	−11.9302	0.9664
	$(2, 2, 2, 4, 2, 2, 2)$	0.5	0.4051	0.0000	0.5691	0.0000
		1	−0.0181	0.0372	−0.0181	0.0372
		1.89	−2.4883	0.9484	−3.2356	0.9484
		2	−2.8271	0.9706	−4.0554	0.9706
		3	−5.5840	0.9990	−22.0118	0.9990
	$(0, \ldots, 0, 16)$	0.5	0.4104	0.0000	0.5966	0.0000
		1	0.0591	0.0159	0.0591	0.0159
		1.89	−2.4971	0.9501	−3.2526	0.9501
		2	−2.8515	0.9727	−4.1023	0.9727
		3	−6.2442	0.9999	−27.8175	0.9999
13	$(2, 2, 2, 2, 2, 0, \ldots, 0)$	0.5	0.3553	0.0000	0.3619	0.0000
		1	−0.2799	0.1438	−0.2799	0.1438
		1.89	−2.1083	0.9505	−2.4881	0.9505
		2	−2.2927	0.9618	−2.8853	0.9618
		3	−3.5925	0.9816	−8.4353	0.9816
	$(0, \ldots, 0, 2, 2, 2, 2, 2, 0, \ldots, 0)$	0.5	0.3677	0.0000	0.4057	0.0000
		1	−0.1792	0.0599	−0.1792	0.0599
		1.89	−2.1031	0.9500	−2.4796	0.9500
		2	−2.3235	0.9646	−2.9435	0.9646
		3	−3.9386	0.9909	−10.0825	0.9909
	$(0, \ldots, 0, 2, 2, 2, 2, 2)$	0.5	0.3732	0.0000	0.4266	0.0000
		1	−0.1232	0.0308	−0.1232	0.0308
		1.89	−2.1053	0.9498	−2.4826	0.9498
		2	−2.3499	0.9708	−2.9903	0.9708
		3	−4.4131	0.9990	−12.5240	0.9990
18	$(1, 1, 1, 1, 1, 0, \ldots, 0)$	0.5	0.3219	0.0000	0.2534	0.0000
		1	−0.3546	0.2013	−0.3546	0.2013
		1.89	−1.9671	0.9509	−2.2359	0.9509
		2	−2.1220	0.9610	−2.5525	0.9610
		3	−3.2451	0.9791	−6.7743	0.9791
	$(0, \ldots, 0, 1, 1, 1, 1, 1, 0, \ldots, 0)$	0.5	0.3327	0.0000	0.2861	0.0000
		1	−0.2882	0.1122	−0.2882	0.1122
		1.89	−1.9616	0.9514	−2.2265	0.9514
		2	−2.1399	0.9636	−2.5837	0.9636
		3	−3.4265	0.9872	−7.5042	0.9872
	$(0, \ldots, 0, 1, 1, 1, 1, 1)$	0.5	0.3370	0.0000	0.2995	0.0000
		1	−0.2538	0.0756	−0.2538	0.0756
		1.89	−1.9604	0.9482	−2.2252	0.9482
		2	−2.1526	0.9664	−2.6064	0.9664
		3	−3.6692	0.9966	−8.5336	0.9966

Remark 6.1. Note that the pdf of $(Y_1, \ldots, Y_m)$ in (18) can be written in the form of the exponential family. Hence, based on Theorem 1.11 and Corollary 1.12 of [26], the UMVUE for C_{L_Y} is derived as

$$\tilde{C}_{L_Y} = 1 - \frac{m-1}{W'}L. \tag{28}$$

Remark 6.2. Notice that δ_m is reduced to $\hat{C}_{L_Y}$ in (20). Since $\tilde{C}_{L_Y} \in \mathcal{C}$, we have

$$\mathrm{MSE}_\theta(C^*_{L_Y}) < \mathrm{MSE}_\theta(\tilde{C}_{L_Y}) < \mathrm{MSE}_\theta(\hat{C}_{L_Y}). \tag{29}$$

The last inequality follows from (27), since $\frac{d}{d\,a}\mathrm{MSE}_\theta(\delta_a) > 0$ for $a > m - 2$. It is worth mentioning that (29) holds for all of θ and L.

In the following, we will utilize the PMC criterion for comparing the estimates $C^*_{L_Y}$, $\tilde{C}_{L_Y}$ and $\hat{C}_{L_Y}$ of C_{L_Y}. For more details, see [27].

Table 8
The values of $a_m \equiv P(Z > 2m-1)$ where $Z \sim \chi^2_{2m}$.

m	a_m	m	a_m	m	a_m	m	a_m
1	0.6065	15	0.5176	29	0.5125	43	0.5102
2	0.5578	16	0.5170	30	0.5123	44	0.5101
3	0.5438	17	0.5165	31	0.5121	45	0.5100
4	0.5366	18	0.5160	32	0.5119	46	0.5099
5	0.5321	19	0.5156	33	0.5117	47	0.5098
6	0.5289	20	0.5151	34	0.5115	48	0.5097
7	0.5265	21	0.5148	35	0.5114	49	0.5096
8	0.5246	22	0.5144	36	0.5112	50	0.5095
9	0.5231	23	0.5141	37	0.5110	60	0.5086
10	0.5218	24	0.5138	38	0.5109	70	0.5080
11	0.5207	25	0.5135	39	0.5107	80	0.5075
12	0.5198	26	0.5132	40	0.5106	90	0.5070
13	0.5190	27	0.5130	41	0.5105	100	0.5067
14	0.5182	28	0.5127	42	0.5103	200	0.5047

Definition 6.3. Under the PMC criterion, the estimator $\hat{\delta}_1$ is better than the estimator $\hat{\delta}_2$ if

$$\mathrm{PMC}(\hat{\delta}_1, \hat{\delta}_2) \equiv P(|\hat{\delta}_1 - \theta| < |\hat{\delta}_2 - \theta|) > \frac{1}{2},$$

where θ is the parameter of interest.

One can prove that, based on the PMC criterion, $\hat{C}_{L_Y}$ is better than $\tilde{C}_{L_Y}$ for C_{L_Y}. To this end, it is sufficient to show

$$\mathrm{PMC}(\hat{C}_{L_Y}, \tilde{C}_{L_Y}) > \frac{1}{2}. \tag{30}$$

From (17), (20), (28) and (30), and after some algebraic computations, we have

$$\begin{aligned}
\mathrm{PMC}(\hat{C}_{L_Y}, \tilde{C}_{L_Y}) &= P\left(\left|\theta - \frac{m-1}{W'}\right| > \left|\theta - \frac{m}{W'}\right|\right)\\
&= P\left(\left|\theta - \frac{2\theta(m-1)}{Z}\right| > \left|\theta - \frac{2\theta m}{Z}\right|\right)\\
&= P\left(\left|1 - \frac{2(m-1)}{Z}\right| > \left|1 - \frac{2m}{Z}\right|\right)\\
&= P(Z > 2m-1),
\end{aligned}$$

where $Z = 2\theta W' \sim \chi^2_{2m}$. By assumption $a_m \equiv P(Z > 2m-1)$, the values of a_m for different values of m reported in Table 8. It seems that a_m is a decreasing function of m and tends to $\frac{1}{2}$. Also, based on the central limit theorem, Z have a normal distribution with mean $2m$ and variance $4m$ as m goes to infinity. Hence, as $m \to \infty$, we have

$$P(Z > 2m-1) = P\left(\frac{Z-2m}{2\sqrt{m}} > \frac{-1}{2\sqrt{m}}\right) \gtrsim \frac{1}{2}.$$

This conclusion along with the results obtained in Table 8 imply that $P(Z > 2m-1) > \frac{1}{2}$, i.e. for every θ, $\hat{C}_{L_Y}$ dominates $\tilde{C}_{L_Y}$ based on the PMC criterion.

Similarly, it can easily be shown that

$$\mathrm{PMC}(\hat{C}_{L_Y}, C^*_{L_Y}) = P(Z > 2m-2),$$

and

$$\mathrm{PMC}(\tilde{C}_{L_Y}, C^*_{L_Y}) = P(Z > 2m-3).$$

Since $P(Z > 2m-1) > \frac{1}{2}$ for any m, then we have

$$P(Z > 2m-3) > P(Z > 2m-2) > \frac{1}{2}.$$

Therefore, we have

$$\hat{C}_{L_Y} >_{\mathrm{PMC}} \tilde{C}_{L_Y} >_{\mathrm{PMC}} C^*_{L_Y},$$

where $\hat{\delta}_1 >_{\mathrm{PMC}} \hat{\delta}_2$ means that $P(|\hat{\delta}_1 - \theta| < |\hat{\delta}_1 - \theta|) > \frac{1}{2}$.

7. Conclusions

The lifetime performance index C_L was used for assessing the performance of a process, in which L denotes the lower specification limit. In this paper, we considered the problem of estimating C_L on the basis of a progressive first-failure censored sample. Assuming the Weibull distribution for the lifetime of the products with a known shape parameter, explicit expressions for the ML estimates of the unknown parameters were obtained. The ML estimate of C_L was utilized to develop a $100 \times (1-\alpha)\%$ one-sided confidence interval for it. This lower bound may be used for assessing whether the product performance meets customer expectations. For the exponential distribution as a special case of the Weibull distribution, a comparison study for various estimates of C_L based on MSE and PMC criteria was done. Moreover, using Lehmann and Casella [26], it can be easily shown that the UMVUE for C_{L_X} is of the form

$$\tilde{C}_{L_X} = \frac{1}{A}\left[\Gamma\left(1+\frac{1}{\beta}\right) - \frac{\Gamma(m)L_X}{\Gamma(m-\frac{1}{\beta})W^{\frac{1}{\beta}}}\right].$$

Using the confidence interval for C_{L_X} based on $\tilde{C}_{L_X}$, one can evaluate whether the product quality meets the required level.

Acknowledgments

The authors are grateful to the Editor-in-Chief, the associate editor and two anonymous referees for their useful comments and suggestions on an earlier version of this paper.

References

[1] D.C. Montgomery, Introduction to Statistical Quality Control, John Wiley and Sons Inc., New York, 1985.
[2] L.I. Tong, K.S. Chen, H.T. Chen, Statistical testing for assessing the performance of lifetime index of electronic components with exponential distribution, International Journal of Quality and Reliability Management 19 (2002) 812–824.
[3] H.A. David, H.N. Nagaraja, Order Statistics, third ed., John Wiley and Sons Inc., New York, 2003.
[4] N. Balakrishnan, R. Aggarwala, Progressive Censoring: Theory, Methods and Applications, Birkhäuser, Boston, 2000.
[5] C.W. Hong, J.W. Wu, C.H. Cheng, Computational procedure of performance assessment of lifetime index of Pareto lifetime businesses based on confidence interval, Applied Soft Computing 8 (2008) 698–705.
[6] C.W. Hong, J.W. Wu, C.H. Cheng, Implementing lifetime performance index for the pareto lifetime businesses of the service industries, Quality and Quantity 43 (2009) 291–304.
[7] W.C. Lee, J.W. Wu, C.W. Hong, Assessing the lifetime performance index of products from progressively type-II right censored data using Burr XII model, Mathematics and Computers in Simulation 79 (2009) 2167–2179.
[8] W.C. Lee, J.W. Wu, C.W. Hong, Assessing the lifetime performance index of products with the exponential distribution under progressively type-II right censored samples, Journal of Computational and Applied Mathematics 231 (2009) 648–656.
[9] W.C. Lee, J.W. Wu, C.L. Lei, Evaluating the lifetime performance index for the exponential lifetime products, Applied Mathematical Modelling 34 (2010) 1217–1224.
[10] H.M. Lee, J.W. Wu, C.L. Lei, W.L. Hung, Implementing lifetime performance index of products with two-parameter exponential distribution, International Journal of Systems Science 42 (2010) 1305–1321.
[11] W.C. Lee, J.W. Wu, M.L. Hong, L.S. Lin, R.L. Chan, Assessing the lifetime performance index of Rayleigh products based on the Bayesian estimation under progressive type-II right censored samples, Journal of Computational and Applied Mathematics 235 (2011) 1676–1688.
[12] L.G. Johnson, Theory and Technique of Variation Research, Elsevier, Amsterdam, 1964.
[13] U. Balasooriya, Failure-censored reliability sampling plans for the exponential distribution, Journal of Statistical Computation and Simulation 52 (1995) 337–349.
[14] S.J. Wu, C. Kus, On estimation based on progressive first-failure censored sampling, Computational Statistics and Data Analysis 53 (2009) 3659–3670.
[15] J.W. Wu, W.L. Hong, C.H. Tsai, Estimation of the parameters of the Gompertz distribution under the first-failure censored sampling plan, Statistics 37 (2003) 517–525.
[16] J.W. Wu, H.Y. Yu, Statistical inference about the shape parameter of the Burr type XII distribution under the failure-censored sampling plan, Applied Mathematics and Computation 163 (2005) 443–482.
[17] Y.M. Kantar, B. Senoglu, A comparative study for the location and scale parameters of the Weibull distribution with given shape parameter, Computers and Geosciences 34 (2008) 1900–1909.
[18] D. Nordman, W.Q. Meeker, Weibull prediction intervals for a future number of failures, Technometrics 44 (2002) 15–23.
[19] Y. Zhang, W.Q. Meeker, Bayesian life test planning for the Weibull distribution with given shape parameter, Annals of Mathematical Statistics 61 (2005) 237–249.
[20] N. Balakrishnan, A.P. Basu (Eds.), Exponential Distribution: Theory, Methods and Applications, Taylor & Francis, Newark, New Jersey, 1995.
[21] W.C. Lee, Inferences on the lifetime performance index for Weibull distribution based on censored observations using the max p-value method, International Journal of Systems Science 42 (2011) 931–937.
[22] P.W. Zehna, Invariance of maximum likelihood estimation, Annals of Mathematical Statistics 37 (1966) 744–744.
[23] E.L. Lehmann, J. Romano, Testing Statistical Hypotheses, third ed., Springer, New York, 2005.
[24] J.F. Lawless, Statistical Models and Methods for Lifetime Data, second ed., John Wiley and Sons Inc., New York, 2003.
[25] M.H. Gail, J.L. Gastwirth, A scale-free goodness-of-fit test for the exponential distribution based on the Gini statistic, Journal of the Royal Statistical Society, Series B 40 (1978) 350–357.
[26] E.L. Lehmann, G. Casella, Theory of Point Estimation, second ed., Springer, New York, 1998.
[27] J.P. Keating, R.L. Mason, P.K. Sen, Pitman's Measure of Closeness—A Comparison of Statistical Estimators, Cambridge University Press, 1993.