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# An efficient algorithm based on eigenfunction expansions for some optimal timing problems in finance

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ABSTRACT

This paper considers the optimal switching problem and the optimal multiple stopping problem for one-dimensional Markov processes in a finite horizon discrete time framework. We develop a dynamic programming procedure to solve these problems and provide easy-to-verify conditions to characterize connectedness of switching and exercise regions. When the transition or Feynman–Kac semigroup of the Markov process has discrete spectrum, we develop an efficient algorithm based on eigenfunction expansions that explicitly solves the dynamic programming problem. We also prove that the algorithm converges exponentially in the series truncation level. Our method is applicable to a rich family of Markov processes which are widely used in financial applications, including many diffusions as well as jump–diffusions and pure jump processes that are constructed from diffusion through time change. In particular, many of these processes are often used to model mean-reversion. We illustrate the versatility of our method by considering three applications: valuation of combination shipping carriers, interest-rate chooser flexible caps and commodity swing options. Numerical examples show that our method is highly efficient and has significant computational advantages over standard numerical PDE methods that are typically used to solve such problems.



## 1. Introduction

Problems in finance often involve timing decisions, and many of them can be formulated as the optimal switching problem or the optimal multiple stopping problem. In the former problem, there are several regimes and the decision maker decides when and where to switch to maximize expected payoffs from each regime, minus any costs incurred for switching. In the latter problem, the decision maker holds multiple exercise rights and her goal is to maximize the expected payoffs from all exercises. This problem is an extension of the classical optimal stopping problem with only one exercise right (see e.g., [1] for the classical optimal stopping problem with applications). Applications of the optimal switching problem include but are not limited to, Brennan and Schwartz [2] and Dixit and Pindyck [3] for the valuation of natural resource mines, Dixit [4] for the production decision of a company, and Sødal et al. [5] for the valuation of combination shipping carriers which can carry different types of cargo. For the multiple stopping problem, two well-known examples are interest-rate chooser flexible caps and floors, and commodity swing options. These derivatives are important tools for managing interest rate risk (e.g., [6,7]) and commodity volume risk (e.g., [8,9]), respectively.

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In this paper, we assume the underlying uncertainty is modeled by a one-dimensional Markov process, and consider finite horizon optimal switching and optimal multiple stopping problems with decisions made in discrete time. This setting is appropriate in many real-world applications. For example, in reality, mining rights expire in finite time and shipping carries have finite useful life. Exercise in discrete time is often a contractual requirement, as in the case of interest-rate chooser flexible caps/floors and some commodity swing options. In the case of combination shipping carriers, it is impossible to realize decisions in continuous time since the ship cannot be switched to carry another type of cargo until it finishes its current trip.

We treat both problems in a unified way as the optimal multiple stopping problem can be formulated as an optimal switching problem with constraints. Under some minimal integrability conditions, we derive a dynamic programming procedure to solve these problems and characterize the optimal strategy.

In general, the dynamic programming problem must be solved numerically. When the underlying uncertainty is modeled by a one-dimensional Markov process, a popular choice in practice is the lattice method due to its intuitiveness and flexibility in incorporating dynamic programming. For example, binomial or trinomial trees are used by Pedersen and Sidenius [6] and Ito et al. [10] for pricing chooser flexible caps and by Thompson [11], Lari-Lavassani et al. [12] and Jaillet et al. [8] for valuation of swing options. More generally, implicit schemes for PDE/PIDE can be used, which are more efficient than the lattice method that corresponds to explicit finite difference schemes.

While numerical PDE/PIDE schemes are general-purpose algorithms, many stochastic models in finance have special features, based on which more efficient computational methods can be developed. An important case is when the characteristic function of the underlying Markov process is known, which is true for Lévy processes in particular. In this case, the method of Fourier-cosine series expansions and fast Hilbert transform are both highly efficient (see [13] for another efficient method). For the development and applications of the Fourier-cosine expansion method, see [14] for European options, Fang and Oosterlee [15] for Bermudan and discretely monitored barrier options, and Zhang and Oosterlee [16] for swing options. The fast Hilbert transform method has been developed and applied by Feng and Linetsky [17,18] for discretely monitored barrier and lookback options, and by Feng and Lin [19] for Bermudan options.

This paper considers another important case in finance where the transition semigroup or Feynman–Kac semigroup of the underlying Markov process defined on the Hilbert space of square-integrable payoffs can be represented by an eigenfunction expansion (see Assumption 1; we consider Feynman–Kac semigroup to accommodate interest rate applications where the short rate is stochastic). Many diffusion processes that are commonly used in financial modeling possess discrete spectrum with explicit eigenvalues and eigenfunctions. Well-known examples include the CEV process [20], the Ornstein–Uhlenbeck (OU) process [21], the CIR process [22], the 3/2 process [23] and the Jacobi process [24]. The last four processes are frequently used to model mean-reversion, which is a key feature in the dynamics of many quantities of interests, such as the short rate, commodity spot prices, exchange rates in a target zone and the price difference between two assets. Moreover, this setting includes a rich class of jump–diffusions and pure jump processes that are constructed from diffusions with discrete spectrum through Bochner's subordination and additive subordination (i.e., time changing diffusions with independent Lévy or additive subordinators). We refer readers to Li and Linetsky [25] and Li et al. [26] for detailed discussions of these processes, which feature state-dependent jumps in general and if additive subordination is used, jumps are also time-dependent. For example, applying Bochner's/additive subordination to a mean-reverting diffusion results in a process with state-dependent jumps which also contribute to mean-reversion. It is shown that these processes improve the realism of diffusion processes while retaining analytical tractability. In particular, models based on Bochner's subordination are able to calibrate volatility smiles of a single maturity while those based on additive subordination can calibrate the entire implied volatility surface. For applications in financial modeling, see [27] for equity, Li and Linetsky [25], Li and Mendoza-Arriaga [28] and Li et al. [26] for commodities, Boyarchenko and Levendorskiĭ [29] and Lim et al. [30] for interest rates, Mendoza-Arriaga et al. [31] for credit-equity derivatives and Mendoza-Arriaga and Linetsky [32] for credit derivatives.

The eigenfunction expansion method is applied by for example, Lewis [33], Davydov and Linetsky [34], Gorovoi and Linetsky [35] and Boyarchenko and Levendorskiĭ [29] for pricing European options, and it is extended by Li and Linetsky [25,36,37] and Lim et al. [30] to solve optimal stopping problems and first passage problems. In many cases, the eigenfunctions are orthogonal polynomials, allowing the method to be efficiently implemented based on the recursion for orthogonal polynomials (see [38] for general discussions on Markov processes and orthogonal polynomials). The present paper applies the eigenfunction expansion method to solve more general and complex optimal timing problems in finance, which are usually solved by numerical PDE/PIDE methods in the literature. We will prove that, under some mild conditions, the eigenfunction expansion algorithm converges exponentially in the series truncation level. To our best knowledge, analysis of the computational property of the eigenfunction expansion method in a dynamic programming setting has not been given in the existing literature. Through numerical examples we will show that our method is highly efficient for finding not only the value function but also the boundary points of the switching/exercise regions, and it has significant computational advantages over numerical PDE/PIDE schemes.

The eigenfunction expansion method is analytical in nature. Assuming the payoff functions and the switching cost functions are square-integrable, we are able to obtain analytical solutions to the value function of the optimal switching problem and the optimal multiple stopping problem through eigenfunction expansions, subject to knowing the switching/exercise regions. The knowledge of these regions is also required in methods based on Fourier-cosine expansions and fast Hilbert transform to solve the dynamic programming problem. In many financial applications, these regions are connected. To find them one just needs to locate the boundary points, which can be done by numerically solving globally defined equations. Although connectedness of switching/exercise regions can often be figured out from intuitions, rigorous justification can be difficult due to the complicated nature of these problems. For Bermudan options, Feng and Lin [19] and Li and Linetsky [25] have provided justifications for specific problems considered there using ad hoc arguments. In this paper, we develop easy-to-verify sufficient conditions for general one-dimensional Markov processes under which the switching regions in the optimal switching problem and the exercise regions in the multiple stopping problem are connected. These conditions allow us to make rigorous justification in a variety of classical examples.

The rest of this paper is organized as follows. Sections 2 and 3 study respectively the optimal switching problem and the optimal multiple stopping problem. In each section, we will first develop the dynamic programming procedure, then present sufficient conditions for the connectedness of switching/exercise regions and finally solve the dynamic programming problem by eigenfunction expansions under Assumption 1. Section 4 develops three applications, namely, valuation of combination shipping carriers, interest-rate chooser-flexible caps and commodity swing options. Section 5 presents numerical examples and compares the eigenfunction expansion algorithm to the lattice method and the Crank–Nicolson scheme. The computational advantages of the eigenfunction expansion method are also summarized there. Section 6 concludes and all proofs are collected in the Appendix.

## 2. The optimal switching problem

### 2.1. The Markovian setup and problem formulation

Let $X := (X_t)_{t\geq 0}$ be a conservative time-homogeneous Markov process taking values in an interval $E \subseteq \mathbb{R}$ with left end-point $e_1$ and right end-point $e_2$ ($-\infty \leq e_1 < x < e_2 \leq +\infty$). Our method can be extended to deal with killing, however to simplify the exposition, we do not pursue such extension here. Let $r(x)$ be a real-valued Borel measurable function. The Feynman–Kac (FK) operator, associated with $X$, denoted by $\mathcal{P}_t^r$ is defined as

$$\mathcal{P}_t^r f(x) = \mathbb{E}_x\left[e^{-\int_0^t r(X_u)du} f(X_t)\right],$$

for Borel measurable functions $f$ where the above expectation is finite. In financial terms, $\mathcal{P}_t^r f(x)$ computes the expectation of a time $t$-payoff $f$, discounted at the interest rate $r(X_u)$ to time zero given $X_0 = x$. When $r(x) \equiv 0$, $\mathcal{P}_t^0$ is the transition operator (in the following when we need to use the transition operator, we will simply write it as $\mathcal{P}_t$). If the discount rate is a constant, i.e., $r(x) \equiv r$, then $\mathcal{P}_t^r = e^{-rt}\mathcal{P}_t$. Thus, the FK framework includes constant discounting as a special case. In this paper we present our results under the general FK framework.

We consider a finite horizon $T > 0$. The set of regimes is denoted by $\mathbb{D} := \{0, 1, 2, \ldots, d-1\}$ with $d > 1$. Switching is allowed at a discrete set of times $0 = t_0 < t_1 < \cdots < t_N = T$, including time 0, and at each $t_l$, it can be done at most once. We assume switching takes effect immediately. Without loss of generality, we assume these time points are equally spaced with distance $h$, i.e., $t_l = lh$. At time $t_l$, if the system is switched from regime $i$ to $j$, a cost of $C(X_{t_l}, i, j)$ is incurred, and a payoff of $f(X_{t_l}, j)$ is received. If the system stays in regime $i$ at time $t_l$, a cost $C(X_{t_l}, i, i)$ is incurred and a payoff of $f(X_{t_l}, i)$ is received. Both the cost and payoff functions take finite real value. In general these functions can also depend on time. To simplify the notation, we assume they are time-independent. Generalization to the time-dependent case is straightforward. Denote by $\mathcal{T}_h$ the set of stopping times that take values in $\{t_0, t_1, \ldots, t_N\}$. A strategy $\alpha$ is represented by a sequence of pairs $(\tau_1, \xi_1), \ldots (\tau_n, \xi_n), \ldots$, where each $\tau_n \in \mathcal{T}_h$, $\xi_n \in \mathbb{D}$, $\tau_n < \tau_{n+1}$ and $\xi_n \neq \xi_{n+1}$. In a finite horizon problem this sequence is finite. Under $\alpha$, the regime value process $I_t^\alpha = I_{0-}1_{[0,\tau_1)}1_{\{\tau_1>0\}} + \sum_{n\geq 1}\xi_n 1_{[\tau_n,\tau_{n+1})}$, where $I_{0-}$ is the regime before any switching at time 0. It is clear that $I_t^\alpha$ is a càdlàg process, being constant on each $[\tau_n, \tau_{n+1})$. $I_{t-}^\alpha$ and $I_t^\alpha$ give the index of the regime before and after the switching. If switching occurs at $t$, $I_{t-}^\alpha \neq I_t^\alpha$, otherwise $I_{t-}^\alpha = I_t^\alpha$. Given $X_{t_l} = x$ and $I_{0-} = i$, the expected $t_l$-value of profits received at and after $t_l$ under a strategy $\alpha$ with $\tau_1 \geq t_l$ is given by (note that under $\alpha$, $I_{t_l-}^\alpha = i$)

$$J^l(x, i, \alpha) := \mathbb{E}_x\left[\sum_{n=l}^N e^{-\int_{t_l}^{t_n} r(X_u)du}\left(f(X_{t_n}, I_{t_n}^\alpha) - C(X_{t_n}, I_{t_n-}^\alpha, I_{t_n}^\alpha)\right)\right].$$

Let $\mathcal{A}^l$ denote the set of all $\alpha$ with $\tau_1 \geq t_l$ ($l = 0, 1, \ldots, N$). We wish to find the value function defined as,

$$J^l(x, i) := \sup_{\alpha\in\mathcal{A}^l} J^l(x, i, \alpha),$$

as well as an optimal strategy. We assume for all $n = 1, 2, \ldots, N,\ i, j \in \mathbb{D}$,

$$\mathbb{E}_x\left[e^{-\int_0^{t_n} r(X_u)du}|f(X_{t_n}, i)|\right] < \infty, \qquad \mathbb{E}_x\left[e^{-\int_0^{t_n} r(X_u)du}|C(X_{t_n}, i, j)|\right] < \infty. \tag{1}$$

This implies for each $\alpha$, $J^l(x, i, \alpha)$ has finite value so it is well-defined.

### 2.2. Dynamic programming

The next theorem shows $J^l(x,i)$ can be found by dynamic programming and characterizes the optimal strategy.

**Theorem 1.** *Iteratively define*

$$W^N(x,i,j) := f(x,j) - C(x,i,j), \qquad V^N(x,i) := \max_{j\in\mathbb{D}}\{W^N(x,i,j)\}, \quad i\in\mathbb{D} \tag{2}$$

$$W^l(x,i,j) := f(x,j) - C(x,i,j) + \mathcal{P}_h^r V^{l+1}(x,j), \qquad V^l(x,i) := \max_{j\in\mathbb{D}}\{W^l(x,i,j)\},$$
$$l = N-1, N-2, \ldots, 0, \ i\in\mathbb{D}. \tag{3}$$

*Then $J^l(x,i) = V^l(x,i)$. To characterize the optimal policy, we introduce the following sets: for $i\in\mathbb{D}$, define*

$$\mathcal{R}^l(i,j) := \{x\in E : W^l(x,i,j) > \max_{k\neq j} W^l(x,i,k)\}$$
$$\cup \ \{x : \min\{p : W^l(x,i,p) = \max_{k\neq j}\{W^l(x,i,k)\}\} = j\}. \tag{4}$$

*Then $\bigcup_{j\in\mathbb{D}}\mathcal{R}^l(i,j) = E$, and $\mathcal{R}^l(i,j)\bigcap\mathcal{R}^l(i,k) = \emptyset$ for $j\neq k$. An optimal strategy in $\mathcal{A}^l$ is given by (suppose $I_{t_l-} = i$)*

$$\tau_1^* = \min\{t_m : t_m \geq t_l, X_{t_m}\in \cup_{j\neq i}\mathcal{R}^m(i,j)\}, \qquad \xi_1^* = \{j : j\neq i, X_{\tau_1^*}\in\mathcal{R}^{\tau_1^*}(i,j)\} \tag{5}$$
$$\tau_n^* = \min\{t_m : t_m > \tau_{n-1}^*, X_{t_m}\in\cup_{j\neq\xi_{n-1}^*}\mathcal{R}^m(\xi_{n-1}^*,j)\}, \qquad \xi_n^* = \{j : j\neq\xi_{n-1}^*, X_{\tau_n^*}\in\mathcal{R}^{\tau_n^*}(\xi_{n-1}^*,j)\}$$

*for $n = 2,\ldots$. The sequence continues until some $n'$ for which $\tau_{n'}^* = \infty$ (recall the convention $\min\emptyset = \infty$).*

Theorem 1 shows $W^l(x,i,j)$ gives the value at time $t_l$, after switching from regime $i$ to $j$, while $V^l(x,i)$ gives the value at time $t_l$ if the pre-switching regime is $i$. The dynamic programming procedure can be explained as follows. On the terminal date $T$, the value after switching from $i$ to $j$, is obviously equal to the received payoff minus the switching cost (the first part in Eq. (2)). At an earlier date $t_l$, the value after switching from $i$ to $j$, is equal to the immediate payoff minus the switching cost, plus the expected discounted pre-switch value at time $t_{l+1}$ (the first part in Eq. (3)). At any time, the decision maker chooses the regime with the maximum post-switch value (the second part in Eqs. (2) and (3)).

Intuitively, $\mathcal{R}^l(i,j)$ is the region at time $t_l$ the decision maker would switch from $i$ to $j$. The second part in (4) requires some explanation. For $x$ such that $W^l(x,i,j) = \max_{k\neq j}\{W^l(x,i,k)\}$, it is possible that for some $p\neq j$, $W^l(x,i,j) = W^l(x,i,p)$. We only include $x$ in $\mathcal{R}^l(i,j)$ if $j$ is the regime that has the smallest index compared to all such $p$. We remark that this is just a particular tie-breaking rule which is used to split $E$ into non-overlapping regions. Other rules can also be used, which changes the optimal strategy but does not affect the value function. Note that the notation $\mathcal{R}^{\tau_n^*}(i,j)$ refers to $\mathcal{R}^l(i,j)$ if $\tau_n^* = t_l$.

### 2.3. The structure of switching regions

It is typical in financial applications that each $\mathcal{R}^l(i,j)$ is an interval (which may be empty). This can often be determined from economic intuitions. In the following, we provide sufficient conditions that offer rigorous justification. For convenience, we assume $E = (e_1, e_2)$.

**Theorem 2.** *Suppose the following hold.*

(i) *$\mathcal{P}_h^r g(x)$ is nondecreasing if $g(x)$ is nondecreasing.*
(ii) *$f(x,j)$, $C(x,i,j)$, $W^l(x,i,j)$ are continuous in $x$ for all $i,j\in\mathbb{D}$ and $0\leq l\leq N$.*
(iii) *$f(x,j+1) - C(x,i,j+1) - f(x,j) + C(x,i,j)$ is nondecreasing in $x$ for all $i\in\mathbb{D}$ and $0\leq j\leq d-2$.*
(iv) *$C(x,i,j+1) - C(x,i,j) \geq C(x,i+1,j+1) - C(x,i+1,j)$ for $x\in E$, $0\leq i,j\leq d-2$.*
(v) *$C(x,i,j) - C(x,i+1,j)$ is nondecreasing in $x$ for $0\leq i\leq d-2$ and $j\in\mathbb{D}$.*

*Then for $l = 0,1,\ldots,N$, $W^l(x,i,j) - W^l(x,i,q)$ is continuous and nondecreasing in $x$ for $q<j$, and $\mathcal{R}^l(i,j)$ has the following form (some regions may be empty)*

$$\mathcal{R}^l(i,j) = (x_{i,j}^l, x_{i,j+1}^l], \quad j = 0,1,\ldots,d-2, \qquad \mathcal{R}^l(i,d-1) = (x_{i,d-1}^l, x_{i,d}^l), \tag{6}$$

*with $x_{i,0}^l = e_1$ and $x_{i,d}^l = e_2$. The other points are defined as follows: first define*

$$\tilde{x}_i^l(q,j) := \inf\{x\in E : W^l(x,i,q) < W^l(x,i,j)\}, \quad q<j, \ q,j\in\mathbb{D}.$$

*Here we use the convention $\inf E = e_1$ and $\inf\emptyset = e_2$. For $j = d-1,\ldots,1$, iteratively define*

$$\tilde{x}_{i,j}^l := \max_{q=0,1,\ldots,j-1}\{\tilde{x}_i^l(q,j)\}, \qquad x_{i,j}^l := \min\{x_{i,j+1}^l, \tilde{x}_{i,j}^l\}.$$

*We have $x_{i,0}^l \leq x_{i,1}^l \leq \cdots < x_{i,d-1}^l \leq x_{i,d}^l$, and $x_{i+1,j}^l \leq x_{i,j}^l$ for $0\leq i\leq d-2$ and $1\leq j\leq d-1$.*

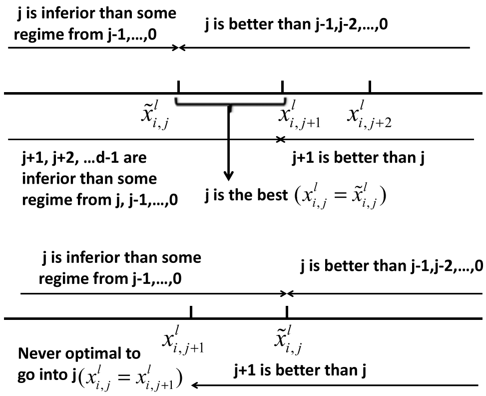



**Fig. 1.** Illustration of the switching region into regime $j$ at time $t_l$ when the current regime is $i$ for the case $\tilde{x}^l_{i,j} < x^l_{i,j+1}$ and the case $\tilde{x}^l_{i,j} > x^l_{i,j+1}$.

Theorem 2 shows the structure of switching regions and also presents a way to find the boundary points. Given $j$, $\tilde{x}^l_i(q, j)$ can be found by some numerical root-finding algorithm such as bisection for $q = 0, 1, \ldots, j-1$. Intuitively, $(\tilde{x}^l_i(q, j), e_2)$ is the region where regime $j$ is preferred over regime $q$ with $q < j$. Then $(\tilde{x}^l_{i,j}, e_2)$ gives the region where $j$ is better than all the regimes with index below it, and for any $x < \tilde{x}^l_{i,j}$ one can always find a regime with index below $j$ that is better than regime $j$. Therefore the region where it is optimal to switch to $j$ must be contained in $(\tilde{x}^l_{i,j}, e_2)$. We already have that for $x > x^l_{i,j+1}$, regime $j+1$ is preferred over $j$, and for $x < x^l_{i,j+1}, j+1, j+2, \ldots, d-1$ are inferior than $j$ or regimes with index below $j$. Therefore, if $x^l_{i,j+1} > \tilde{x}^l_{i,j}$, it is optimal to go into $j$ for $x \in (\tilde{x}^l_{i,j}, x^l_{i,j+1})$ and we set $x^l_{i,j} = \tilde{x}^l_{i,j}$. However if $x^l_{i,j+1} \le \tilde{x}^l_{i,j}$, it is never optimal to switch from $i$ to $j$ and we set $x^l_{i,j} = x^l_{i,j+1}$. These situations are illustrated in Fig. 1.

Condition (i) is a natural property in many stochastic models when the discount rate is constant. (ii) assumes continuity on $f(x, j)$, $C(x, i, j)$ and $W^l(x, i, j)$ (hence by the dynamic programming procedure in Theorem 1, $V^l(x, i)$ is also continuous). This is true in many applications, in particular when $X$ satisfies Assumption 1 and the payoff and cost functions are continuous and square-integrable (see Theorem 3). (iii) states that regime $j+1$ is "better" than $j$ when $x$ is large enough in the sense that the difference between the profit of switching to $j+1$ and to $j$ is nondecreasing. We will apply Theorem 2 to study switching regions in the combination shipping carrier application in Section 5.1. Below we consider the classical three-regime copper mining problem in [3].

**Example 1** (*Operation of a Copper Mine*)**.** There are three regimes of a copper mine: idle, mothballed (temporary suspension) and active (in operation). They are labeled as 0, 1, 2 respectively. In practice, mothballing might be attractive as the cost of reactivating a mine is cheaper than building a new one from scratch ($R < I$ below). However, mothballing incurs maintenance costs. Intuitively one would expect that when the price is high enough, the mine is in operation, and when the price is in some middle range, it is mothballed to enjoy the benefit of cheap reactivation as the price is likely to rise in the future. When the price is low enough, it is closed to stop losses from operation or save maintenance costs.

Let $X_t$ denote the copper spot price at time $t$. Mean-reversion is well-documented in the copper price. So for $X_t$, we consider the classical geometric OU diffusion model in [39] and its subordination extension in [40]. For the payoffs, $f(x, 0) = f(x, 1) = 0$ and $f(x, 2) = \mathbb{E}_x[\int_0^h e^{-rh} X_t dt]$ (assume continuous flow of revenues). It can be shown that conditions (i)–(iii) in Theorem 2 are satisfied. We omit the detailed derivation here.

Following Dixit and Pindyck [3], the switching costs are constant, which are given by

$$\begin{aligned} &C(0,0) = 0, \quad C(0,1) = J + M_h, \quad C(0,2) = I + C_h, \\ &C(1,0) = E_S, \quad C(1,1) = M_h, \quad C(1,2) = R + C_h, \\ &C(2,0) = E, \quad C(2,1) = E_M + M_h, \quad C(2,2) = C_h. \end{aligned}$$

Here $I$ and $J$ are respectively the investment cost to turn the mine from idle into active and mothballed. $E$ is the shut-down cost. $E_M$ is the cost to suspend the mine. $E_S$ is the cost of closing the mine from suspension. We set $E = E_M + E_S$ as in [3]. $R$ is the cost of reactivating the mine. $C_h$ is the operating cost when the mine is active, and $M_h$ is the maintenance cost when the mine is mothballed. Since both operating and maintenance costs are paid out in flows, we put $h$ as a subscript to show the dependence on $h$, and they are understood as the present value of these cost flows at the beginning of a period. $E_S$ and $E_M$ might be negative. We assume $J + E_S \ge 0$ and $R + E_M \ge 0$, which are reasonable assumptions in practice. Clearly condition (v) holds. Condition (iv) holds for $i = j = 0$ and $i = 1, j = 0$ and 1. The discount rate is a constant denoted by $r$.

The mining right expires after $N$ periods. At $t_N$ the mine must be closed, i.e., the regime must be switched to 0. Theorem 2 does not consider such constraints. To fit its setting, we add $e^{-rh}C(i, 0)$ to the switching cost at $t_{N-1}$, where $i$ is the regime before closing the mine. The switching costs at other times remain unchanged. The time-dependence of costs does not affect the applicability of Theorem 2 as long as conditions (iii)–(v) are satisfied at each time. If $I \geq J + R$, one can verify that condition (iv) holds for $i = 0, j = 1$. Theorem 2 can now be applied which implies the switching regions have the form in (6). For $I < J + R$, Theorem 2 cannot be directly applied. However, when $I \leq J + R$, it is never optimal to switch from 0 to 1. Staying in regime 1 has no payoff but incurs maintenance cost, and the cost for turning $0 \to 2$ is always cheaper than the indirect route $0 \to 1 \to 2$. Since $J$ only affects the cost of switching from 0 to 1, which is always suboptimal, the original problem is equivalent to another problem with $J'$ such that $J' + R = I$ ($J' > J$). For this new problem Theorem 2 can be applied. Therefore for $I \leq J + R$, the switching regions are still in the form of (6), but with $R^l(0, 1) = \emptyset$.

### 2.4. The eigenfunction expansion algorithm

Many Markov processes used in financial applications satisfy the following assumption, which allows us to solve the dynamic programming problem through eigenfunction expansions.

**Assumption 1.** The FK semigroup $(\mathcal{P}_t^r)_{t\geq 0}$ under consideration can be defined on $L^2(E, \mathfrak{m}) := \{f \text{ is Borel-measurable } : \int_E f^2(x)\mathfrak{m}(dx) < \infty\}$ for some nonnegative measure $\mathfrak{m}$ on $E$ with full support. For each $t > 0$, $\mathcal{P}_t^r$ is trace-class, and its symmetric kernel $p_t(x, y)$ (its existence is implied by the trace-class condition) is jointly continuous in $x$ and $y$.

Under Assumption 1, we have (see [25] for the proof)

$$\mathcal{P}_t^r f(x) = \sum_{n=1}^{\infty} f_n e^{-\lambda_n t}\varphi_n(x), \quad \text{for any } f \in L^2(E, \mathfrak{m}),\ t > 0,$$

which converges uniformly on compacts in $x$. Here $f_n = \int_E f(x)\varphi_n(x)\mathfrak{m}(dx)$ is the $n$th expansion coefficient. $(\varphi_n(x))_{n\geq 1}$ forms a complete orthonormal basis of $L^2(E, \mathfrak{m})$, and $\varphi_n(x)$ is the $n$th eigenfunction of $\mathcal{P}_t^r$, with eigenvalue $e^{-\lambda_n t}$, i.e., $\mathcal{P}_t^r\varphi_n(x) = e^{-\lambda_n t}\varphi_n(x)$. Each $\varphi_n(x)$ is continuous, $\lambda_1 \leq \lambda_2 \leq \cdots < \infty$, $\lim_{n\to\infty}\lambda_n = \infty$, and

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} < \infty \quad \text{for all } t > 0.$$

The continuity of $\varphi_n(x)$ together with the uniform convergence on compacts of the expansion implies that $\mathcal{P}_t^r f(x)$ is continuous in $x$.

**Remark 1** (*Non-Contractive Semigroups*)**.** In most financial applications, $r(x) \geq 0$ and hence each $\mathcal{P}_t^r$ is a contraction. This is the setting considered in [25]. However for non-contractive semigroups, if they satisfy Assumption 1, the above results regarding eigenfunction expansion also hold. An example is given by the Vasicek short model [21] where $X$ is assumed to be an OU diffusion and $r(x) = x$. In this case, the short rate can become negative and hence $(\mathcal{P}_t^r)_{t\geq 0}$ is not a contraction semigroup (see [35] for the eigenfunction expansion for the FK semigroup of the Vasicek model).

The transition or FK semigroup of many diffusion processes satisfies Assumption 1. To find out the eigenvalues and eigenfunctions, one needs to solve the associated Sturm–Liouville problem. The procedure to solve such problem is presented in detail in [41,42], where explicit expressions for the eigenvalues and eigenfunctions for many diffusions can also be found.

To model jumps which are often needed in financial applications, a particularly useful approach is to apply subordination to diffusion processes. Let $X$ be a time-homogeneous diffusion and $T$ be a Lévy subordinator (a nondecreasing Lévy process taking values in $\mathbb{R}_+$). The Laplace transform of $T$ is given by

$$\mathbb{E}[e^{-\lambda T_t}] = e^{-\phi(\lambda)t}\ (\lambda > 0), \quad \phi(\lambda) = \gamma\lambda + \int_{(0,\infty)} (1 - e^{-\lambda s})\nu(ds),$$

where $\gamma \geq 0$ is the drift and $\nu$ is the Lévy measure satisfying the integrability condition $\int_{(0,\infty)}(s \wedge 1)\nu(ds) < \infty$. One can construct a new Markov process by Bochner's subordination, i.e., $X_t^\phi := X_{T_t}$, which is a time-homogeneous jump–diffusion if $\gamma > 0$ and a pure-jump process if $\gamma = 0$. Suppose the transition semigroup of $X$, $(\mathcal{P}_t)_{t\geq 0}$ admits the following eigenfunction expansion for $f \in L^2(E, \mathfrak{m})$:

$$\mathcal{P}_t f(x) = \sum_{n=1}^{\infty} f_n e^{-\lambda_n t}\varphi_n(x), \quad f_n = (f, \varphi_n).$$

Then for the transition semigroup of $X^\phi$, we have

$$\mathcal{P}_t^\phi f(x) = \sum_{n=1}^{\infty} f_n e^{-\phi(\lambda_n)t}\varphi_n(x), \quad f_n = (f, \varphi_n), \tag{7}$$

where the eigenfunctions remain the same and only the eigenvalues are changed using the Laplace transform of the subordinator. Sufficient conditions for the subordinate semigroup $(\mathcal{P}_t^\phi)_{t\geq 0}$ to satisfy Assumption 1 can be found in [37, Proposition 1]. If $T$ is an additive subordinator (a nondecreasing additive process taking values in $\mathbb{R}_+$), then $X^\phi$ is a time-inhomogeneous jump–diffusion or pure-jump process, and in (7) the Laplace transform of the additive subordinator is used (see [26]). Short rate models with jumps can also be developed by applying subordination to a diffusion FK semigroup and we refer readers to Boyarchenko and Levendorskiĭ [29], Lim et al. [30] for details.

To apply the eigenfunction expansion method to solve the dynamic programming problem, we first introduce some notations. For every Borel subset $A \subseteq E$ define

$$\pi_{m,n}(A) := (1_A\varphi_m, \varphi_n), \quad m, n = 1, 2, \ldots$$

where $1_A(x)$ is the indicator function of the set $A$. For $f(x)$ and $A$, define

$$f_n(A) := (1_A f, \varphi_n), \quad n = 1, 2, \ldots \tag{8}$$

if it is finite. Under Assumption 1, by assuming square-integrable payoffs and switching costs, we are able to develop explicit solutions to the dynamic programming problem.

**Theorem 3.** *Suppose that Assumption 1 holds and $f(x, j)$ and $C(x, i, j)$ are in $L^2(E, \mathfrak{m})$ for any $i, j \in \mathbb{D}$. Then we have*
(i) *$W^l(x, i, j), V^l(x, i) \in L^2(E, \mathfrak{m})$ for all $i, j \in \mathbb{D}$, $l = 0, 1, \ldots, N$.*
(ii) *Each $W^l$ can be represented in the following form:*

$$W^l(x, i, j) = f(x, j) - C(x, i, j) + \sum_{n=1}^{\infty} w_n^l(j)e^{-\lambda_n h}\varphi_n(x), \quad l = N, N-1, \ldots, 0. \tag{9}$$

*The expansion converges uniformly on compacts in $x$, and it is continuous in $x$. The expansion coefficients satisfy:*

$$w_n^N(j) = 0, \tag{10}$$

$$w_n^l(j) = \sum_{k\in\mathbb{D}} \left\{ f_n^k(\mathcal{R}^{l+1}(j, k)) - C_n^{j,k}(\mathcal{R}^{l+1}(j, k)) + \sum_{m=1}^{\infty} w_m^{l+1}(k)e^{-\lambda_m h}\pi_{m,n}(\mathcal{R}^{l+1}(j, k)) \right\} \tag{11}$$

*for $l = N-1, \ldots, 0$. $f_n^k(A)$ and $C_n^{j,k}(A)$ are defined as in (8) using $f(x, k)$ and $C(x, j, k)$.*

Theorem 3 reduces the backward induction for a sequence of functions in Theorem 1 to the backward recursion for its expansion coefficients in the complete orthonormal basis of eigenfunctions of the FK semigroup $(\mathcal{P}_t^r)_{t\geq 0}$. It starts with (10) for the coefficients at time $t_N$. The next step is to determine the switching region $\mathcal{R}^N(j, k)$ for each pair of $j, k$ at time $t_N$. Given $j$, to find $\mathcal{R}^N(j, k)$ for all $k \in \mathbb{D}$, we compare among $W^N(x, j, k)$. Each $W^N(x, j, k)$ is given by (9) with $w_n^N(k) = 0$. Given $\mathcal{R}^N(j, k)$ for all $j, k \in \mathbb{D}$, the coefficients $w_n^{N-1}(j)$ are then determined by (11) for each $j \in \mathbb{D}$, and then $\mathcal{R}^{N-1}(j, k)$ for all $j, k \in \mathbb{D}$ are found. The procedure is continued until time 0 is reached. Finally set the value function $J^0(x, i) = \max_{j\in\mathbb{D}}\{W^0(x, i, j)\}$.

We next discuss the computational implementation of Theorem 3. In computing the infinite expansion in (9) and (11) we truncate the expansion to a given tolerance level. Computing (11) requires computing the quantities $f_n^k(\mathcal{R})$, $C_n^{j,k}(\mathcal{R})$ and $\pi_{m,n}(\mathcal{R})$ ($\mathcal{R}$ is a generic notation for switching regions). In many financial applications the switching region $\mathcal{R}$ is an interval (which may be empty). Suppose the interval is given by $(a, b)$ and recall that $e_1$ and $e_2$ are the left and right end points of the state space $E$. To calculate $\pi_{m,n}(a, b)$, due to the linearity of integrals, $\pi_{m,n}(a, b) = \pi_{m,n}(e_1, b) - \pi_{m,n}(e_1, a)$. Thus we only need to calculate $\pi_{m,n}(e_1, x)$ for $x \in E$. Alternatively we can calculate $\pi_{m,n}(x, e_2)$, since $\pi_{m,n}(e_1, x) = \delta_{m,n} - \pi_{m,n}(x, e_2)$ ($\delta_{m,n}$ is the Kronecker delta) due to the orthonormality of eigenfunctions. When the eigenfunctions are known in closed form, the integral $\int_{e_1}^x \varphi_m(y)\varphi_n(y)m(y)dy$ can often be calculated in closed form as well. Furthermore, for eigenfunctions expressed in terms of orthogonal polynomials, one can obtain computationally efficient recursive algorithms for evaluating $\pi_{m,n}(e_1, x)$.

The coefficients $f_n^k(a, b)$ (and $C_n^{j,k}(a, b)$ alike) can also often be explicitly computed in applications either by first evaluating the expansion coefficients $f_n^k$ of the payoff $f(x, k)$ and then computing $f_n^k(a, b)$ via

$$f_n^k(a, b) = \int_{(a,b)} \sum_{m=1}^{\infty} f_m^k \varphi_m(x)\varphi_n(x)\mathfrak{m}(dx) = \sum_{m=1}^{\infty} f_m^k \pi_{m,n}(a, b),$$

or by directly calculating the integral $\int_a^b f(x, k)\varphi_n(x)m(x)dx$ in closed form. When no closed form solutions are available for the integrals in $\pi_{m,n}(a, b)$ and $f_n^k(a, b)$, they can be computed via numerical integration.

Although in many applications, the switching region is connected as in Theorem 2, in general it can be a union of disjoint of intervals. Our method can handle this case easily. To illustrate, suppose $\mathcal{R} = \cup_{j=1}^J I_j$ where $I_j$ are disjoint intervals. Then we have

$$\pi_{m,n}(\mathcal{R}) = \sum_{j=1}^J \pi_{m,n}(I_j), \qquad f_n(\mathcal{R}) = \sum_{j=1}^J f_n(I_j),$$

which reduces the calculation to single intervals.

## 3. The optimal multiple stopping problem

We consider the same finite horizon discrete time setting as in Section 2. In a multiple stopping problem, the decision maker has $K$ $(1 \leq K \leq N+1)$ number of rights to exercise and on each date only one right can be exercised. When a right is exercised, a payoff is received. We denote the exercise payoff function by $p(x)$. The multiple stopping problem can be formulated as a switching problem with constraints. Let $\mathbb{D} = \{0, 1, \ldots, K\}$, where regime $k \in \mathbb{D}$ refers to the state of having $k$ rights remaining. For all $k \in \mathbb{D}$, set $f(x, k) = 0$. We impose the following constraints. When the state is $k$ $(k = 1, 2, \ldots, K)$, it can only remain in $k$ (i.e., no exercise) or be switched to $k-1$ (i.e., one right exercised). Set $C(x, k, k) = 0$ and $C(x, k, k-1) = -p(x)$. When $k = 0$, it can only remain in 0 (since no rights left) and we set $C(x, 0, 0) = 0$. We assume for all $n = 1, 2, \ldots, N$,

$$\mathbb{E}_x\left[e^{-\int_0^{t_n} r(X_u)du}|p(X_{t_n})|\right] < \infty.$$

This implies for each $\alpha$, $J^l(x, k, \alpha)$ has finite value so it is well-defined.

**Remark 2.** In this paper we distinguish between exercise payoff of an option and its payoff. For example, for a call option with strike $K$ on an asset whose price at time $t$ is denoted by $S_t$, the exercise payoff (which is the payoff if the option is exercised) at time $t$ is $S_t - K$. The payoff, however, considers the possibility of no exercise if exercise is not optimal and thus it is given by $(S_t - K)^+$. We will use the exercise payoff in deriving conditions for the exercise regions to be connected.

### 3.1. Dynamic programming

As a corollary of Theorem 1, we have the following dynamic programming procedure to solve the optimal multiple stopping problem.

**Corollary 1.** *Iteratively define*

$$C^l(x, 0) := 0, \qquad S^l(x, 0) := 0, \qquad V^l(x, 0) := 0, \quad 0 \leq l \leq N,$$
$$C^N(x, k) := 0, \qquad S^N(x, k) := p(x), \qquad V^N(x, k) := \max\{C^N(x, k), S^N(x, k)\}, \quad 1 \leq k \leq K, \tag{12}$$
$$C^l(x, k) := \mathcal{P}_h^r V^{l+1}(x, k), \qquad S^l(x, k) := p(x) + C^l(x, k-1), \qquad V^l(x, k) := \max\{C^l(x, k), S^l(x, k)\},$$
$$l = N-1, N-2, \ldots, 0, \ 1 \leq k \leq K. \tag{13}$$

*Then $J^l(x, k) = V^l(x, k)$. To characterize the optimal strategy, introduce the following sets: for $l = 0, 1, \ldots, N$, $k = 1, 2, \ldots, K$, define*

$$\mathcal{S}^{l,k} := \{x \in E : S^l(x, k) > C^l(x, k)\}, \qquad \mathcal{C}^{l,k} := \{x \in E : S^l(x, k) \leq C^l(x, k)\}. \tag{14}$$

*An optimal strategy in $\mathcal{A}^l$ is given by (assume at $t_l$ there are $k$ remaining rights)*

$$\tau_1^* = \min\{t_m : t_m \geq t_l, X_{t_m} \in \mathcal{S}^{m,k}\}, \qquad \tau_n^* = \min\{t_m : t_m > \tau_{n-1}^*, X_{t_m} \in \mathcal{S}^{m,k-n+1}\}, \quad n = 2, \ldots, k.$$

Corollary 1 shows the following: (1) $V^l(x, k)$ is the value at time $t_l$ of having $k$ number of rights remaining; (2) $C^l(x, k)$ is the value at time $t_l$ of having $k$ number of rights to continue (i.e., no exercise at $t_l$); (3) $S^l(x, k)$ is the value at time $t_l$ of exercising one right and having $k-1$ number of rights to continue. When the decision maker has a total of $k$ rights at time $t_l$, $\mathcal{S}^{l,k}$ is the region she would stop to exercise one right and $\mathcal{C}^{l,k}$ is the region she would continue without exercise.

**Remark 3.** One could also define $\mathcal{S}^{l,k}$ as $\{x \in E : S^l(x, k) \geq C^l(x, k)\}$, which changes the optimal strategy but does not affect the value function. We prefer the definition in (14) as we want to interpret $\mathcal{S}^{l,k}$ as the region where one would actually exercise.

Based on the dynamic programming equations, we observe the following.

**Proposition 1.** *For $l + k > N$ and $k \leq K$, $V^l(x, k) = V^l(x, N-l+1)$, and $\mathcal{S}^{l,k} = \{x : p(x) > 0\}$.*

The implication of Proposition 1 is clear. At time $t_l$, we have a total of $N - l + 1$ dates to exercise the rights on hand. If $k$, the number of rights we have at $t_l$, is greater than or equal to the number of remaining time points $N - l + 1$ (equivalent to $l + k > N$), since we can only exercise $N - l + 1$ of them from $t_l$ to $t_N$, we must have $V^l(x, k) = V^l(x, N - l + 1)$. Furthermore, one right will be exercised immediately at $t_l$ if $p(x) > 0$.

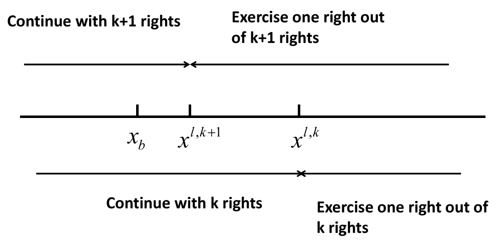



**Fig. 2.** Illustration of the exercise regions at time $t_l$ for a call-type payoff.

### *3.2. The structure of exercise regions*

Our first result considers the inclusion of exercise regions. Intuitively, one becomes more conservative to exercise an option if she has fewer number of options at hand.

**Proposition 2.** *For $l = 1, \ldots, N-1$, $k = 1, \ldots, \min\{N-l, K\}$, $\mathcal{S}^{l,k} \subseteq \mathcal{S}^{l,k+1}$, and $C^l(x, k+1) + C^l(x, k-1) - 2C^l(x, k) \le 0$ for all $x \in E$.*

It is typical in applications that each $\mathcal{S}^{l,k}$ is an interval, which can often be determined from economic intuitions. In the following, we provide sufficient conditions that offer rigorous justification. Our discussion focuses on the call-type payoff (i.e., $p(x)$ is a nondecreasing function). The put-type payoff can be considered similarly. For convenience, we assume $E = (e_1, e_2)$.

**Theorem 4.** *Suppose the following conditions hold.*

(i) *$\mathcal{P}_h^r g(x)$ is nondecreasing if $g(x)$ is nondecreasing.*
(ii) *$p(x)$ and $C^l(x, k)$ are continuous for $1 \le k \le K$ and $0 \le l \le N$.*
(iii) *$p(x)$ is nondecreasing and $x_b := \sup\{x \in E : p(x) = 0\} \in E$.*
(iv) *$p(x) - \mathcal{P}_h^r p(x)$ is nondecreasing in $x$.*

*Then for $l = 0, 1, \ldots, N$, $k = 1, 2, \ldots, K$, $\mathcal{S}^{l,k}$ and $\mathcal{C}^{l,k}$ have the following form*

$$\mathcal{C}^{l,k} = (e_1, x^{l,k}], \qquad \mathcal{S}^{l,k} = (x^{l,k}, e_2), \tag{15}$$

*where for $l + k > N$, $x^{l,k} = x_b$ and for $l + k \le N$, $x^{l,k} = \inf\{x \in E : S^l(x, k) > C^l(x, k)\}$ where $S^l(x, k) - C^l(x, k)$ is continuous and nondecreasing in $x$, and $x_b \le x^{l,k+1} \le x^{l,k}$ $(1 \le k \le K-1)$.*

**Remark 4.** (1) The relation $x_b \le x^{l,k+1} \le x^{l,k}$ shows $[x_b, x^{l,k}]$ is a natural search interval to start with when using bisection to find $x^{l,k+1}$. (2) For put-type payoffs, the conditions become the following: (i) $\mathcal{P}_h^r g(x)$ is nonincreasing if $g(x)$ is nonincreasing; (ii) $p(x)$ and $C^l(x, k)$ are continuous for $1 \le k \le K$ and $0 \le l \le N$; (iii) $p(x)$ is nonincreasing and $x_b := \inf\{x \in E : p(x) = 0\} \in E$; (iv) $p(x) - \mathcal{P}_h^r p(x)$ is nonincreasing in $x$. Under these conditions, $\mathcal{S}^{l,k} = (e_1, x^{l,k})$, $\mathcal{C}^{l,k} = [x^{l,k}, e_2)$, where for $l + k > N$, $x^{l,k} = x_b$ and for $l + k \le N$, $x^{l,k} = \sup\{x \in E : S^l(x, k) > C^l(x, k)\}$ where $S^l(x, k) - C^l(x, k)$ is continuous and nonincreasing in $x$, and $x^{l,k} \le x^{l,k+1} \le x_b$.

Theorem 4 is illustrated in Fig. 2. Condition (i) is a natural property of many stochastic models when the discount rate is constant. (ii) assumes continuity on $p(x)$ and $C^l(x, k)$ (hence by the dynamic programming procedure in Corollary 1, $S^l(x, k)$ and $V^l(x, k)$ are also continuous). This is also true in many stochastic models, in particular in the setting of Corollary 2. In the following, we present some examples where Theorem 4 can be applied.

**Example 2** (*Bermudan Options with Multiple Exercise Rights for Stocks and Futures*). Consider a call option on a dividend paying stock with multiple exercise rights. $r$ denotes the risk-free rate and $q \ge 0$ is the dividend yield. $X_t$ is the stock price process and the exercise payoff is given by $p(x) = x - G$, where $G$ is the strike price. Conditions (i) and (ii) are naturally satisfied in almost all stock models, including in particular exponential Lévy models. Then $p(x) - \mathcal{P}_h^r p(x) = (1 - e^{-qt})x - (1 - e^{-rt})G$, where we used $\mathbb{E}_x[e^{-(r-q)h} X_h] = x$ under the equivalent martingale measure. Hence condition (iv) is satisfied, and by Theorem 4, the exercise regions have the form in (15). When the stock is replaced by a futures contract, $\mathbb{E}_x[X_h] = x$ under the equivalent martingale measure. Hence condition (iv) still holds and Theorem 4 applies. As special cases, the Bermudan-style stock option with only one right in exponential Lévy models considered in [15,19], as well as the Bermudan-style commodity futures option with only one right in the subordinate OU model of Li and Linetsky [40] all have one-sided structure. In those papers, proof is given using ad hoc arguments. Here Theorem 4 can be applied to all of these cases.

**Example 3** (*Commodity Swing Options in Arithmetic Mean-Reverting Models*)**.** Consider a call-type commodity swing option with local volume constraints, that is, upon each exercise, the volume taken cannot exceed $b$ units. It is easy to see that when a right is exercised, the optimal unit to take is the upper bound $b$. Therefore the problem reduces to the optimal multiple stopping problem with exercise payoff $p(x) = b(x - G)$, where $G$ is the strike price and $X_t$ is the spot price process. For many commodities, $X_t$ is mean-reverting. Suppose $X_t$ is modeled as a CIR diffusion [43] or subordinate CIR process [26] or an OU-type process driven by subordinators. Since these processes themselves are positive, one can directly use them to model the spot price and such models are often called arithmetic models in the literature (as opposed to the exponential OU model). Then it can be shown that $\mathbb{E}_x[X_h] = xe^{-a(h)}+$ other terms that do not depend on $x$, where $a(h) > 0$ ($a(h) = \kappa h$ for the CIR where $\kappa$ is the mean-reversion speed). It is easy to see that condition (iv) holds, so Theorem 4 can be applied.

### 3.3. The eigenfunction expansion algorithm

When $X$ satisfies Assumption 1 and $\max\{p(x), 0\} \in L^2(E, \mathfrak{m})$, we have the following eigenfunction expansion algorithm to solve the multiple stopping problem, as a corollary of Theorem 3 and Proposition 1.

**Corollary 2.** *Suppose Assumption 1 holds, and* $\max\{p(x), 0\} \in L^2(E, \mathfrak{m})$. *Then we have*

(i) $C^l(x, k), S^l(x, k), V^l(x, k) \in L^2(E, \mathfrak{m})$ *for all* $l = 0, 1, \dots, N$, $k = 0, 1, \dots, K$.

(ii) *Each* $C^l(x, k)$ *can be represented by an eigenfunction expansion*

$$C^l(x, k) = \sum_{n=1}^{\infty} c_n^{l,k} e^{-\lambda_n h}\varphi_n(x), \quad l = N-1, \dots, 0,\ k = 0, 1, \dots, K$$

*with the expansion which converges uniformly on compacts in* $x$ *and* $C^{l,k}(x)$ *is continuous in* $x$. *The expansion coefficients satisfy*

$$c_n^{l,0} = 0, \quad l = N-1, \dots, 0,$$
$$c_n^{N-1,k} = p_n(\mathcal{S}^{N,k}), \quad k = 1, \dots, K.$$

*For* $l = N-2, \dots, 0$, $k = 1, 2, \dots, K$,

$$c_n^{l,k} = p_n(\mathcal{S}^{l+1,k}) + \sum_{m=1}^{\infty}(c_m^{l+1,k-1} - c_m^{l+1,k})e^{-\lambda_m h}\pi_{m,n}(\mathcal{S}^{l+1,k}) + c_n^{l+1,k}e^{-\lambda_n h}, \quad \text{if } k \le \min\{N-l, K\},$$
$$c_n^{l,k} = c_n^{l,N-l}, \quad \text{if } N-l < k \le K.$$

Implementation of the algorithm in Corollary 2 is similar to Theorem 3, so detailed discussions are omitted.

## 4. Error analysis

Since the eigenfunction expansion and the recursion for the coefficients are infinite series, to compute them numerically one must truncate. We assume all the infinite series are truncated using $M$ terms. For ease of discussion, in this section we only consider the optimal switching problem and assume conditions in Theorem 2 hold so that the switching regions are connected. Results for the optimal multiple stopping problem can be obtained similarly, and extensions can be developed for the case where the regions are disconnected. For the optimal switching problem, the dynamic programming procedure in Theorem 3 is implemented as follows. We use hat to denote approximate values, which depend on the truncation level $M$. However to lighten notations, we do not write out $M$.

(1) $\hat{w}_n^N = 0$ for all $n \in \mathbb{N}$, find $\hat{\mathcal{R}}^N(j, k)$ for all $j, k \in \mathbb{D}$ as in Theorem 2 using $\hat{W}^N(x, j, k) = f(x, k) - C(x, j, k)$.

(2) For $l = N-1, \dots, 0, j \in \mathbb{D}, n = 1, 2, \dots, M$,

$$\hat{w}_n^l(j) = \sum_{k\in\mathbb{D}}\left\{f_n^k(\hat{\mathcal{R}}^{l+1}(j,k)) - C_n^{j,k}(\hat{\mathcal{R}}^{l+1}(j,k)) + \sum_{m=1}^{M}\hat{w}_m^{l+1}(k)e^{-\lambda_m h}\pi_{m,n}(\hat{\mathcal{R}}^{l+1}(j,k))\right\}. \tag{16}$$

Then find $\hat{\mathcal{R}}^l(i, j)$ as in Theorem 2 using the following approximation of $W^l(x, i, j)$ for all $i, j \in \mathbb{D}$,

$$\hat{W}^l(x, i, j) = f(x, j) - C(x, i, j) + \sum_{n=1}^{M}\hat{w}_n^l(j)e^{-\lambda_n h}\varphi_n(x).$$

In (2), to evaluate $\hat{w}_n^l(j)$ for $n = 1, 2, \dots, M, l = N-1, \dots, 0$ and $j \in \mathbb{D}$ requires $O(dNM^2)$ operations. For each $i \in \mathbb{D}$ and $l = N-1, \dots, 0$, to find the region $\hat{\mathcal{R}}^l(i, j)$ for all $j \in \mathbb{D}$, one first needs to find $\tilde{x}_i^l(q, j)$ for each pair of $(q, j)$ with $q < j$. To do this, bisection can be used which takes $O(M)$ operations as in each iteration $\hat{W}^l(x, i, j)$ is evaluated in $M$ operations for given $x$ and bisection terminates in a finite number of steps for a given tolerance level. Since there are

$d(d-1)/2$ pairs to consider for each $i$ and $l$, the total complexity for finding switching regions is $O(d^2(d-1)NM/2)$. Together the complexity for the eigenfunction expansion algorithm is $O(dNM^2 + d^2(d-1)NM/2)$. Since in financial applications, $d$ is often small, the most expensive part lies in computing the expansion coefficients at each time step.

Next we analyze the convergence rate for the value function as $M$ increases to infinity. Under Assumption 1, given $x$, for all $n$, we have (see [25, Proposition 2])

$$|\varphi_n(x)| \le e^{\lambda_n t/2}\sqrt{p_t(x,x)} \quad \text{for all } t > 0. \tag{17}$$

Hence the truncation error for computing the expansion $\mathcal{P}_h^r f(x) = \sum_{n=1}^{\infty} f_n e^{-\lambda_n h}\varphi_n(x)$ for $f \in L^2(E, \mathfrak{m})$ using the first $M$ terms is bounded by

$$\left|\sum_{n=M+1}^{\infty} f_n e^{-\lambda_n h}\varphi_n(x)\right| \le \|f\|\sqrt{p_h(x,x)} \sum_{n=M+1}^{\infty} e^{-\lambda_n h/2},$$

where $\|f\|$ is the $L^2$-norm of the payoff function $f$, and $|f_n| \le \|f\|$ by Cauchy–Schwarz inequality. Therefore, the convergence rate for the expansion for $\mathcal{P}_h^r f(x)$ depends on the growth rate of $\lambda_n$. Below we provide some examples for $\lambda_n$.

(1) If the Sturm–Liouville problem for the diffusion process is regular, then as shown in [44], $\lambda_n \sim O(n^2)$. Examples include but are not limited to the transition semigroup of the Jacobi diffusion and the Geometric Brownian motion with two reflecting barriers (it is used by Dixit and Pindyck [3] as a model for the price in competitive industries with entry and exit).
(2) If the Sturm–Liouville problem for the diffusion is singular, in many cases, $\lambda_n \sim O(n)$. Examples include but are not limited to the transition semigroup of the CEV process, the transition and FK semigroup of the OU process, the CIR process, and the 3/2 process.
(3) Subordination: Suppose for the background diffusion process, $\lambda_n \sim O(n)$ or $O(n^2)$. If the Lévy subordinator has drift $\gamma$, then $\phi(\lambda_n)$ goes to infinity at least as fast as $\gamma\lambda_n$. For Lévy subordinators without drift, consider the tempered stable subordinators which are commonly used in finance. Its Lévy measure is given by $\nu(ds) = Cs^{-p-1}e^{-\eta s}ds$ with $C, \eta > 0$ and $0 < p < 1$. The Laplace exponent $\phi(\lambda) = -C\Gamma(-p)[(\lambda+\eta)^p - \eta^p]$.

In all of the above examples, $\lambda_n \sim O(n^p)$ for some $p > 0$, hence the series $\sum_{n=1}^{\infty} e^{-\lambda_n t}$ converges exponentially for all $t > 0$. In general, a series $\sum_{n=1}^{\infty} a_n$ is said to converge exponentially if $\lim_{n\to\infty} n^\beta a_n = 0$ for any $\beta > 0$ (see [45, Section 2.3] or [14, p. 836]). This condition is clearly verified in the examples as $\lambda_n = n^p$. Furthermore, we have for every $t$ there exist some constant $C > 0$ and $\alpha > 0$ such that

$$\sum_{n=M+1}^{\infty} e^{-\lambda_n t} \le Ce^{-\alpha\lambda_M} \quad \text{for all } M \ge M_0 \text{ for some } M_0 > 0. \tag{18}$$

In view of the above discussions, we make the following assumption on the semigroup $(\mathcal{P}_t^r)_{t\ge 0}$ under consideration.

**Assumption 2.** Suppose Assumption 1 holds. We assume $\sum_{n=1}^{\infty} e^{-\lambda_n t}$ converges exponentially for any $t > 0$ and the inequality (18) holds.

In the following analysis, to simplify the discussion, we also assume that in solving an equation using bisection, the error results from finite termination of the procedure due to error tolerance is negligible. This part of the error can be easily controlled and made arbitrarily small in actual computations. At time $t_N$ there is no error as infinite series are not involved. The error starts to emerge at $t_{N-1}$ and propagates in the backward induction procedure. At time $t_l$, the error for $\hat{w}_n^l(j)$ not only comes from truncation of the series, but also from error of the switching regions and the coefficients at $t_{l+1}$ (see (16)). Our next theorem shows that under some mild conditions, the error after backward induction still converges exponentially in the truncation level $M$ (recall that we have assumed we are in the setting of Theorem 2).

**Theorem 5.** *Suppose Assumption 2 holds and the measure $\mathfrak{m}$ is continuous on $E$. For $l = 0, 1, \dots, N-1$ and all $i, j \in \mathbb{D}$, we assume $W^l(x,i,j) \in C^1(E)$. We also assume when $M$ is sufficiently large, $W^l(x,i,j) - W^l(x,i,k) = 0$ and $\hat{W}^l(x,i,j) - \hat{W}^l(x,i,k) = 0$ have the same number of solutions (at most 1) in $E$ for all $i, j, k \in \mathbb{D}$. Furthermore, for every $x_{i,j}^l$ which is not equal to $x_{i,j+1}^l$ (i.e., $\mathcal{R}^l(i,j)$ is not empty), we assume $\partial_x W^l(x,i,j) - \partial_x W^l(x,i,k) \ne 0$ for $x$ between $x_{i,j}^l$ and $\hat{x}_{i,j}^l$, where at $x_{i,j}^l$, $W^l(x_{i,j}^l, i, j) = W^l(x_{i,j}^l, i, k)$ for some $k < j$.*

*For each $l = N-1, \dots, 0$, let $e_w^l(M) := \max_{1\le n\le M, j\in\mathbb{D}} |\hat{w}^l(j) - w^l(j)|$ and $e_W^l(M) := \max_{x\in\mathcal{C}, i,j\in\mathbb{D}} |\hat{W}^l(x,i,j) - W^l(x,i,j)|$ where $\mathcal{C}$ is a given compact subset of $E$. Then there exist constants $C_w^l, C_W^l > 0$ and $\alpha_w^l, \alpha_W^l > 0$ ($\alpha_W^l \le \alpha_w^l \le \alpha$, where $\alpha$ is given in Assumption 2), independent of $M$, such that for sufficiently large $M$,*

$$e_w^l(M) \le C_w^l e^{-\alpha_w^l \lambda_M}, \qquad e_W^l(M) \le C_W^l e^{-\alpha_W^l \lambda_M}.$$

*Hence the error converges exponentially in the truncation level $M$.*

**Remark 5.** (1) Eigenfunctions are continuously differentiable in many applications. Under Assumption 1, if for any compact interval $J \subseteq E$, $\sum_{n=1}^{\infty} e^{-\lambda_n h}\|\varphi_n'|_J\|_\infty < \infty$ ($\|\cdot\|_\infty$ is the $L^\infty$ norm), then it can be shown that $\mathcal{P}_h^r f(x)$ is continuously differentiable in $E$ for any $f \in L^2(E, \mathfrak{m})$, and

$$\frac{d}{dx}\mathcal{P}_h^r f(x) = \sum_{n=1}^{\infty} f_n e^{-\lambda_n h}\varphi_n'(x). \tag{19}$$

Thus $W^l(x, i, j) \in C^1(E)$.

(2) Bounds tighter than (17) for eigenfunctions are often available. For regular Sturm–Liouville problem, it is shown in [44] that for any given compact set $\mathcal{C}$, $|\varphi_n(x)| \le C$ for all $n$ and all $x \in \mathcal{C}$. For normalized Hermite and generalized Laguerre polynomials, $|\varphi_n(x)| \le C/n^{\frac{1}{4}}$ for all $n$ and all $x \in \mathcal{C}$ (see [40,37]). In these cases, in Theorem 5 we can choose $\alpha_w^l = \alpha_W^l = \alpha$ for all $l$, which are tighter than the original estimates.

## 5. Applications

In this section we apply our algorithm to three important applications. We index eigenfunctions starting from zero, rather than from one as in previous sections. This is more convenient when working with orthogonal polynomials. Starting from one is the standard notation used in theoretical discussions.

### 5.1. Valuation of combination shipping carriers

The world bulk shipping markets consist of two main sectors: the tanker (wet bulk) sector which carries oil and the dry bulk sector which carries dry bulk cargo (e.g., grain, iron ore and coal). Usually ships are designed to operate in only one of the two sectors, but one ship type, the *combination carrier (or combo)*, is designed to carry both wet and dry cargo. Switching from carrying one type of cargo to the other type incurs a cost, mainly due to cleaning the carrier. Clearly combo carriers are more costly than single cargo carriers due to the benefit it provides to switch freely between two sectors to take advantage of the freight rate difference.

A natural question in the shipping business is whether it is worthwhile to spend more money to order combo carriers. To address this question, Sødal et al. [5] developed a real option model for the valuation of combo carriers. The additional value of combo carriers compared to an equal-sized oil tanker (dry cargo carrier) comes from the option to switch to the dry (wet) bulk market and back. By taking the value of an equal-sized oil tanker or dry cargo carrier as given, the valuation problem reduces to a relative one to determine the value of the option to switch between wet and dry bulk markets. Investment decision can then be made by comparing this option value to the actual market price difference between a combo carrier and a single cargo carrier of equal size.

Valuation of the switching option embedded in a combo carrier can be formulated as an optimal switching problem, where the owner of a combo chooses the switching policy optimally based on the freight rate difference. Let $X_t$ denote the freight rate differential defined as the dry bulk rate minus the tanker rate. In [5], $X_t$ is modeled as an OU diffusion, i.e., $dX_t = \kappa(\theta - X_t)dt + \sigma dB_t$, which is shown to capture the movement of the spread. The freight rate is quoted in terms of dollars per unit of time. For tractability purpose, Sødal et al. [5] considered this problem in an infinite horizon continuous time framework. In this paper we determine the switching option value in the more realistic finite horizon discrete time framework.

We follow the setting in Section 2 with $d = 2$. 0 denotes the tanker sector and 1 denotes the dry bulk sector. $f(x, i)$ denotes the additional freight rate earned by carrying cargo in the $i$th sector for a period of length $h$ compared to carrying oil. Thus $f(x, 0) = 0, f(x, 1) = xh$. As in [5], we assume the switching costs do not depend on the freight rate difference, and $C(0, 0) = 0, C(0, 1) = F^+, C(1, 0) = F^-, C(1, 1) = 0$. A constant rate $r$ is used for discounting. The value of the embedded switching option in a combo carrier compared to a tanker carrier is given by $J^0(x, 0)$.

For the OU diffusion $X$, $E = \mathbb{R}$ and the speed density $\mathfrak{m}(x) = \sqrt{\frac{\kappa}{\pi\sigma^2}}e^{-\frac{\kappa(x-\theta)^2}{\sigma^2}}$. The eigenfunction expansion for the OU transition semigroup on $L^2(\mathbb{R}, \mathfrak{m})$ is well-known (see for example [46]), and it satisfies Assumption 1. For $n = 0, 1, \ldots,$

$$\lambda_n = \kappa n, \qquad \varphi_n(x) = \frac{1}{\sqrt{2^n n!}}H_n\left(\frac{\sqrt{\kappa}}{\sigma}(x-\theta)\right), \tag{20}$$

where $H_n(x)$ is the $n$th order Hermite polynomial. $\varphi_n(x)$ can be computed efficiently through the following recursion:

$$\varphi_0(x) = 1, \qquad \varphi_1(x) = \frac{\sqrt{2\kappa}}{\sigma}(x-\theta), \qquad \varphi_n(x) = \sqrt{\frac{2}{n}}\frac{\sqrt{\kappa}}{\sigma}(x-\theta)\varphi_{n-1}(x) - \sqrt{\frac{n-1}{n}}\varphi_{n-2}(x), \quad n \ge 2.$$

It is straightforward to check that $f(x, j)$ and $C(i, j)$ are in $L^2(\mathbb{R}, \mathfrak{m})$, hence Theorem 3 applies.

Intuitively, it is clear that switching from the tanker oil sector to the dry bulk sector only occurs if the freight rate difference $X$ is positive enough, while switching in the other direction only occurs if $X$ is negative enough. This claim can

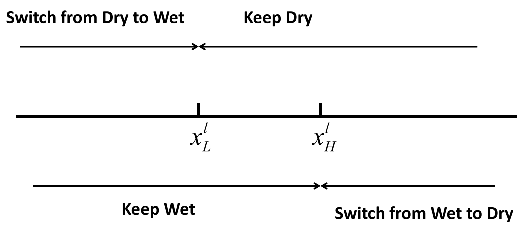



**Fig. 3.** Illustration of the switching regions for the combo carrier at time $t_l$.

be rigorously justified by Theorem 2. The results are summarized in the following proposition with an illustration given in Fig. 3.

**Proposition 3.** *The switching region $R^l(i,j)$ has the following form with $-\infty < x_L^l < x_H^l < \infty$:*

$$\mathcal{R}^l(0,0) = (-\infty, x_H^l], \qquad \mathcal{R}^l(0,1) = (x_H^l, \infty), \qquad \mathcal{R}^l(1,0) = (-\infty, x_L^l], \qquad \mathcal{R}^l(1,1) = (x_L^l, \infty). \tag{21}$$

*$x_H^l$ is found by solving $W^l(x,0,0) = W^l(x,0,1)$, and $x_L^l$ is found by solving $W^l(x,1,0) = W^l(x,1,1)$.*

We next calculate $\pi_{m,n}(\mathcal{R}^l(i,j))$, $f_n^j(\mathcal{R}^l(i,j))$ and $C_n^{i,j}(\mathcal{R}^l(i,j))$ for $i,j \in \{0,1\}$. From (21), we calculate $\pi_{m,n}(-\infty, x)$ and $\pi_{m,n}(x,\infty)$ for generic $x$. Since $\pi_{m,n}(-\infty,x) = \delta_{mn} - \pi_{m,n}(x,\infty)$, only one of them needs to be computed. From Li and Linetsky [25], we have

$$\pi_{m,n}(x,\infty) = \frac{\sqrt{n+1}\varphi_m(x)\varphi_{n+1}(x) - \sqrt{m+1}\varphi_n(x)\varphi_{m+1}(x)}{\sqrt{2\pi}(m-n)} e^{-\frac{\kappa(x-\theta)^2}{\sigma^2}}, \quad m \neq n,\ m \geq 0, n \geq 0,$$

$$\pi_{0,0}(x,\infty) = \Phi\left(-\frac{\sqrt{2\kappa}(x-\theta)}{\sigma}\right), \qquad \pi_{n,n}(x,\infty) = \pi_{n-1,n-1}(x,\infty) + \frac{1}{\sqrt{2\pi n}}\varphi_{n-1}(x)\varphi_n(x)e^{-\frac{\kappa(x-\theta)^2}{\sigma^2}}, \quad n \geq 1,$$

where $\Phi(x)$ is the standard normal CDF. The formulas for $f_n^j(\mathcal{R}^l(i,j))$ and $C_n^{i,j}(\mathcal{R}^l(i,j))$ are summarized in the following proposition (note that $\mathcal{R}^l(i,1)$ has the form $(x,\infty)$ for $i \in \{0,1\}$.)

**Proposition 4.**

$$f_n^0(\mathcal{R}^l(i,0)) = 0, \quad n \geq 0,\ i \in \{0,1\}.$$

$$f_0^1(x,\infty) = \frac{1}{2}\frac{\sigma h}{\sqrt{\pi\kappa}}e^{-\frac{\kappa(x-\theta)^2}{\sigma^2}} + \theta h\Phi(-\sqrt{2}x),$$

$$f_n^1(x,\infty) = \frac{1}{2}\frac{\sigma h}{\sqrt{\pi\kappa}}e^{-\frac{\kappa(x-\theta)^2}{\sigma^2}}\varphi_n(x) + \sigma h\sqrt{\frac{n}{2\kappa}}\pi_{0,n-1}(x,\infty) + \theta h\pi_{0,n}(x,\infty), \quad n \geq 1,$$

$$C_n^{i,j}(\mathcal{R}^l(i,j)) = C(i,j)\pi_{0,n}(\mathcal{R}^l(i,j)), \quad n \geq 0,\ i,j \in \{0,1\}.$$

### *5.2. Chooser flexible caps*

Caps and floors are major derivatives traded in the interest rate markets [47]. We focus our discussion on caps and floors which can be considered similarly. Consider a set of times $0 = t_0 < t_1 < \cdots < t_N < t_{N+1}$. Let $L(t,t')$ be the LIBOR rate observed at $t$ for the maturity $t'$, and the notional amount is assumed to be one. The discounted payoff at time $t_0$ to the holder of a cap spanning from $t_0$ to $t_N$, with strike $G$ is given by

$$\sum_{i=0}^{N} D(0,t_{i+1})\tau_i(L(t_i,t_{i+1}) - G)^+, \tag{22}$$

where $D(0,t)$ is the stochastic factor to discount the cash flow at time $t$ to time 0, and $\tau_i := t_{i+1} - t_i$. Note that the payoff is paid out at time $t_1$ to $t_{N+1}$, with the time $t_{i+1}$-payoff determined by $L(t_i,t_{i+1})$, the LIBOR rate observed at $t_i$ with maturity $t_{i+1}$. From (22), the cap holder is holding $N+1$ European call/put options on the LIBOR. Each option is called a caplet.

The chooser flexible cap is a variant of the standard cap, in which the total number of exercise rights, denoted by $K$, is less than $N+1$. These rights can be exercised from $t_0$ to $t_N$. If a right is exercised at $t_i$, the payoff is paid out at $t_{i+1}$ as in the standard cap contract. Compared to the standard cap, the chooser flexible cap offers investors more flexibility in hedging interest rate risk at a lower cost. In practice, typically $t_i = ih$ and $h$ is often a quarter.

Our method is applicable to many diffusion short rate models, including the Vasicek model [21], the CIR model [22], the 3/2 model [23], Black's model of interest rates as options [35] and the quadratic model [48], as well as their subordinate versions with jumps [30]. Furthermore, it is applicable to time-inhomogeneous extensions of time-homogeneous short rate models by adding a deterministic function of time to $r(X_t)$ to match the initial yield curve. In particular it can be applied to the popular Hull–White model [49] and the CIR++ model [50].

To illustrate our method, we consider the Vasicek model. Under this model, $X_t$ is an OU diffusion, i.e., $dX_t = \kappa(\theta - X_t)dt + \sigma dB_t$, with $\kappa, \theta, \sigma > 0$, and $r(x) = x$. The eigenfunction expansion for the OU FK semigroup on $L^2(\mathbb{R}, \mathfrak{m})$ (recall $\mathfrak{m}$ is the speed measure of the OU diffusion) is obtained in [35], which satisfies Assumption 1. For $n = 0, 1, \ldots$

$$\lambda_n = \theta - \frac{\sigma^2}{2\kappa^2} + n\kappa, \qquad \varphi_n(x) = \frac{e^{-a\xi - a^2/2}}{\sqrt{2^n n!}} H_n(\xi + a),$$

where $H_n(x)$ is the $n$th order Hermite polynomial and $\xi = \frac{\sqrt{\kappa}}{\sigma}(x - \theta)$, $a = \frac{\sigma}{\kappa^{3/2}}$. Based on the classical recursion for Hermite polynomials, it is easy to derive the following recursion for $\varphi_n(x)$, which can be used to compute $\varphi_n(x)$ efficiently:

$$\varphi_0(x) = e^{-a\xi - a^2/2}, \qquad \varphi_1(x) = e^{-a\xi - a^2/2}(\xi + a), \qquad \varphi_n(x) = \sqrt{\frac{2}{n}}(\xi + a)\varphi_{n-1}(x) - \sqrt{\frac{n-1}{n}}\varphi_{n-2}(x).$$

Denote by $Z(x, h)$ the price of a zero-coupon bond with maturity $h$ given the current short rate is $x$, and $L(x, h)$ the current LIBOR with maturity $h$ given the current short rate is $x$. Under the Vasicek model, $Z(x, h)$ and hence $L(x, h)$ are given by

$$\begin{aligned}
&Z(x, h) = A(h)e^{-B(h)x},\\
&B(h) = \frac{1}{\kappa}(1 - e^{-\kappa h}), \qquad A(h) = \exp\left\{\frac{1}{\kappa^2}(B(h) - h)\left(\kappa^2\theta - \frac{\sigma^2}{2}\right) - \frac{\sigma^2 B(h)^2}{4\kappa}\right\}.\\
&L(x, h) = \frac{1}{h}\left(\frac{1}{Z(x, h)} - 1\right) = \frac{1}{h}\left(A^{-1}(h)e^{B(h)x} - 1\right).
\end{aligned}$$

We note that $L(x, h)$ is also the LIBOR with maturity $t + h$ observed at any time $t$, given $X_t = x$, since the Vasicek model is time-homogeneous. For the chooser flexible cap, the exercise payoff $p(x) = Z(x, h)h(L(x, h) - G) = 1 - (1 + hG)A(h)e^{-B(h)x}$. It is easy to see that $\max\{p(x), 0\} \in L^2(E, \mathfrak{m})$, since it is bounded, and $\mathfrak{m}$ is a probability measure. Hence Corollary 2 can be applied.

Note that $p(x)$ is increasing in $x$. Thus intuitively one would exercise one right if $x$ is large enough, i.e., $\mathcal{S}^{l,k} = (x^{l,k}, +\infty)$, where for $l + k > N$, $x^{l,k} = x_b := [\ln(1 + hG) + \ln A(h)]/B(h)$ and for $l + k \le N$, $x^{l,k} = \inf\{x \in E : S^{l,k} > C^{l,k}\}$, and $x_b \le x^{l,k+1} \le x^{l,k}$ $(1 \le k \le K - 1)$. To our regret, we do not have rigorous justification for this result. Theorem 4 cannot be applied because conditions (i) and (iv) do not hold. However, in our examples, we do not detect any violation numerically.

Next we calculate $p_n(x, \infty)$ and $\pi_{m,n}(x, \infty)$ for generic $x$ such that $x \ge x_b$. Define

$$\rho_n(s, x) := \int_x^\infty e^{sy}\varphi_n(y)\mathfrak{m}(dy), \quad s \in \mathbb{R}, \ n = 0, 1, \cdots.$$

Then $p_n(x, \infty)$ can be calculated as follows:

**Proposition 5.** *$p_n(x, \infty) = \rho_n(0, x) - (1 + hG)A(h)\rho_n(-B(h), x)$ for $x \ge x_b$, with*

$$\begin{aligned}
&\rho_0(s, x) = e^{-a^2/2 + s\theta + (a + s\sigma/\sqrt{\kappa})^2/4}\Phi\left(-\frac{1}{\sqrt{2}}\left(2\xi + a - s\frac{\sigma}{\sqrt{\kappa}}\right)\right),\\
&\rho_n(s, x) = \frac{1}{\sqrt{2\pi n}}e^{sx - \xi^2}\varphi_{n-1}(x) + \left(a + s\frac{\sigma}{\sqrt{\kappa}}\right)\frac{1}{\sqrt{2n}}\rho_{n-1}(s, x),
\end{aligned} \tag{23}$$

*where $\Phi(\cdot)$ is the standard normal CDF.*

Using similar derivation as in [25], $\pi_{m,n}(x, \infty)$ can be computed efficiently as follows:

$$\begin{aligned}
&\pi_{m,n}(x, \infty) = \frac{\sqrt{(n+1)}\varphi_m(x)\varphi_{n+1}(x) - \sqrt{(m+1)}\varphi_n(x)\varphi_{m+1}(x)}{\sqrt{2\pi}(m - n)}e^{-\frac{\kappa(x-\theta)^2}{\sigma^2}}, \quad m \ne n, \ m \ge 0, \ n \ge 0.\\
&\pi_{0,0}(x, \infty) = \Phi(-\sqrt{2}(\xi + a)), \qquad \pi_{n,n}(x, \infty) = \pi_{n-1,n-1}(x, \infty) + \frac{1}{\sqrt{\pi n}}e^{-\xi^2}\varphi_n(x)\varphi_{n-1}(x), \quad n \ge 1.
\end{aligned}$$

### 5.3. *Commodity swing options*

Swing options are widely used for managing volume risk in commodity markets. Let $S_t$ denote the commodity spot price at time $t$. We consider the classical Schwartz model [39], i.e., under the pricing measure, $S_t = S_0 e^{X_t}$ where $X_t$ is an OU diffusion with long-run level $\theta$, mean-reversion speed $\kappa$ and volatility $\sigma$. Recently [40,28] improved the Schwartz model by introducing mean-reverting jumps through Lévy and additive subordination. It is shown that the Lévy subordinate model is able to calibrate a variety of volatility smile/skew patterns in commodity markets while the additive subordinate model is able to calibrate the entire implied volatility surface. These models also admit eigenfunction expansions with the diffusion eigenvalue $e^{-\lambda_n t}$ replaced by the Laplace transform of the Lévy/additive subordinator evaluated at $\lambda_n$, which are known in closed-form (see the previously cited two papers for details). Hence the algorithm developed in this paper can also deal with these more complicated models. Below we only consider the Schwartz model.

In practice there are many variants of swing options, and papers in the literature consider different settings (see e.g., [51,52,16]). In this paper we closely follow the setting in [8], however the eigenfunction expansion algorithm can potentially be applied to other settings. We assume exercise can be done on a discrete set of dates $0 = t_0 < t_1 < \cdots < t_N = T$ and on each date only one right can be exercised. There are $K$ rights in total with $1 \le K \le N + 1$. We limit our discussion to a call-type swing option (the put-type can be considered similarly). The exercise payoff depends on both the spot price at the exercise time and the volume taken (denoted by $q$), and is given by $p(x, q) = q(S_0e^x - G)$, where $G$ is the strike price and $q$ is an integer multiple of some basic unit. We assume a constant risk-free rate $r$.

Compared to the standard multiple stopping problem in Section 3, the problem of valuation swing options is more complicated, since one must also decide how much volume to take in addition to timing the exercise. In practice there are two types of constraints on the volume, which leads to different treatment of the problem.

**Local volume constraints**. The local constraints state that when a right is exercised, $q \le b$, where $q$ is the volume taken on the exercise day and $b$ is a positive integer. It has been shown (see for example [8]), in the optimal strategy the volume will be taken in a bang–bang fashion, i.e., $q = b$. Therefore, when only local volume constraints are present, the problem of valuation swing options reduces to the standard multiple stopping problem with exercise payoff $p(x) = b(S_0e^x - G)$. Since $p(x)$ is increasing in $x$, intuitively one would stop to exercise one right if $x$ is sufficiently large, i.e., $\mathcal{S}^{l,k} = (x^{l,k}, +\infty)$, where for $l + k > N$, $x^{l,k} = x_b = \ln(G/S_0)$ and for $l + k \le N$, $x^{l,k} = \inf\{x \in E : S^{l,k} > C^{l,k}\}$, and $x_b \le x^{l,k+1} \le x^{l,k}$ $(1 \le k \le K - 1)$. We do not have rigorous justification for this result. Theorem 4 cannot be applied because condition (iv) does not hold $(\mathbb{E}_h[e^{X_h}] = \exp[xe^{-\kappa h} + \theta(1 - e^{-\kappa h}) + \frac{\sigma^2}{4\kappa}(1 - e^{-2\kappa h})])$. However numerically we do not see any violation in the examples we consider. Although rigorous justification cannot be provided for the geometric mean-reverting models, as shown in Example 3, Theorem 4 can be applied to justify connectedness for arithmetic models.

It is easy to verify that $\max\{p(x), 0\} \in L^2(\mathbb{R}, \mathfrak{m})$. To calculate $p_n(x, \infty)$, define

$$\rho_n(s, x) := \int_x^\infty e^{sy}\varphi_n(y)\mathfrak{m}(dy), \quad s \in \mathbb{R}$$

where $\varphi_n(y)$ is given by (20). Then

$$p_n(x, \infty) = bS_0\rho(1, x) - bG\pi_{0,n}(x, \infty), \quad x \ge \ln(G/S_0).$$

Similar to (23), $\rho_n(s, x)$ can be calculated as

$$\rho_0(s, x) = e^{s\theta + \frac{s^2\sigma^2}{4\kappa}}\Phi\left(-\sqrt{2}\xi + \frac{s\sigma}{\sqrt{2\kappa}}\right), \qquad \rho_n(s, x) = \frac{e^{sx-\xi^2}}{\sqrt{2n\pi}}\varphi_{n-1}(x) + \frac{s\sigma}{\sqrt{2n\kappa}}\rho_{n-1}(s, x),$$

where $\Phi(\cdot)$ is the standard normal CDF, $\xi = \frac{\sqrt{\kappa}}{\sigma}(x - \theta)$.

**Global volume constraints**. In addition to impose the local constraint $q \le b$ for each exercise, the global constraint imposes restrictions on the total volume taken. Let $Q$ be the total volume, and $B$ be the upper bound on the total volume, i.e., $Q \le B$, where $B$ is a positive integer. If $Kb \le B$, the global constraint becomes redundant and the problem reduces to the case only with local constraints. Below we consider the more interesting case $Kb > B$. In this case, the problem is more difficult than a standard optimal multiple stopping problem as the decision maker not only has to consider the number of rights she has but also the remaining volume she can take. To solve the problem, in addition to $x$ and $k$ (remaining number of rights), we add the remaining usage level (equal to $B$ minus the volume already taken), denoted by $u$, to the state variable. The value function can now be found by dynamic programming ($x \wedge y$ denotes $\min\{x, y\}$). Let $p(x) = S_0e^x - G$. Iteratively define

$$\begin{aligned}
&V^l(x, 0, u) := 0, \quad 0 \le l \le N,\ 0 \le u \le B,\\
&V^l(x, k, 0) := 0, \quad 0 \le l \le N,\ 0 \le k \le K,\\
&C^N(x, k, u) := 0, \qquad S^N(x, k, u) := \max_{1\le q\le u\wedge b}\{qp(x)\},\\
&V^N(x, k, u) := \max\{S^N(x, k, u), C^N(x, k, u)\}, \quad 1 \le k \le K,\ 0 \le u \le B.\\
&C^l(x, k, u) := \mathcal{P}_h^r V^{l+1}(x, k, u), \qquad S^l(x, k, u) := \max_{1\le q\le u\wedge b}\{qp(x) + C^l(x, k-1, u-q)\},\\
&V^l(x, k, u) := \max\{S^l(x, k, u), C^l(x, k, u)\}, \quad l = N-1, \ldots, 0,\ 1 \le k \le K,\ 0 \le u \le B.
\end{aligned}$$

Then the value function $J^l(x, k, u) = V^l(x, k, u)$. For $l = N - 1, \ldots, 0$, $k = 1, \ldots, K$, $u = 1, \ldots, B$ and $q = 0, 1, \ldots, u \wedge b$, define

$$\begin{aligned}
\mathcal{R}^{l,k,u}(q) := {}&\{x \in E : qp(x) + C^l(x, k - 1_{\{q>0\}}, u - q) > \max_{j\ne q, 0\le j\le u\wedge b}\{jp(x) + C^l(x, k - 1_{\{j>0\}}, u - j)\}\}\\
&\cup\ \{x : \min\{i : ip(x) + C^l(x, k - 1_{\{i>0\}}, u - i) = \max_{j\ne q, 0\le j\le u\wedge b}\{jp(x) + C^l(x, k - 1_{\{j>0\}}, u - j)\}\} = q\}.
\end{aligned}$$

$\mathcal{R}^{l,k,u}(q)$ is the region at time $t_l$ that the owner of the option should exercise and take $q$ units for $q > 0$ or no exercise for $q = 0$ if he has $k$ number of exercise rights and the remaining usage level is $u$. The second part has the same interpretation as a tie-breaking rule as in Section 2. Intuitively $\mathcal{R}^{l,k,u}(q)$ should have the following form (some of these regions may be empty):

$$\mathcal{R}^{l,k,u}(0) = (-\infty, x^{l,k,u}(0)],\ \mathcal{R}^{l,k,u}(1) = (x^{l,k,u}(0), x^{l,k,u}(1)], \ldots, \mathcal{R}^{l,k,u}(u \wedge b) = (x^{l,k,u}(u \wedge b - 1), +\infty).$$

This is verified numerically in our examples.

Since $\max\{p(x), 0\} \in L^2(\mathbb{R}, \mathfrak{m})$, we can solve the dynamic programming problem by eigenfunction expansions. We have $C^l(x, k, u), S^l(x, k, u), V^l(x, k, u) \in L^2(\mathbb{R}, \mathfrak{m})$ for $0 \le k \le K, 0 \le u \le B, 0 \le l \le N$. Each $C^l(x, k, u)$ can be represented in the following form:

$$C^l(x, k, u) = \sum_{n=0}^{\infty} c_n^{l,k,u} e^{-\lambda_n h} \varphi_n(x), \quad l = N-1, \ldots, 0,\ k = 0, \ldots, K.$$

The expansion converges uniformly on compacts in $x$ and $C^l(x, k, u)$ is continuous in $x$. The coefficients satisfy the following, which can be obtained in a way similar to Theorem 3.

$$\begin{aligned}
c_n^{l,0,u} &= 0, \qquad c_n^{l,k,0} = 0, \quad 0 \le l \le N,\ 0 \le k \le K,\ 0 \le u \le B.\\
c_n^{N-1,k,u} &= (u \wedge b) p_n(\ln(G/S_0), \infty), \quad 1 \le k \le K,\\
c_n^{l,k,u} &= \sum_{j=1}^{u\wedge b} \left\{ j f_n(\mathcal{R}^{l+1,k,u}(j)) + \sum_{m=0}^{\infty} c_m^{l+1,k-1,u-j} e^{-\lambda_m h} \pi_{m,n}(\mathcal{R}^{l+1,k,u}(j)) \right\}\\
&\quad + \sum_{m=0}^{\infty} c_m^{l+1,k,u} e^{-\lambda_m h} \pi_{m,n}(\mathcal{R}^{l+1,k,u}(0)), \quad 1 \le k \le \min\{N-l, K\},\ 0 \le l \le N-2.\\
c_n^{l,k,u} &= c_n^{l,N-l,u}, \quad N-l < k \le K,\ 0 \le l \le N-2.
\end{aligned}$$

**Remark 6** (*Extension to Refraction Period*)**.** Some swing contracts impose a minimum period, called refraction period, between two consecutive exercises. Let $h_R$ denote the refraction period. If $h_R \le h$, then the refraction period constraint becomes redundant. For $h_R > h$, the previous dynamic programming procedure can be extended by adding time left to the next exercisable date to the state variable. Suppose $h_R = Mh$, and let $\tau$ be the number of periods until the next exercisable date. Iteratively define

$$\begin{aligned}
&V^l(x, 0, u, \tau) := 0, \quad 0 \le l \le N,\ 0 \le u \le B,\ 0 \le \tau \le M,\\
&V^l(x, k, 0, \tau) := 0, \quad 0 \le l \le N,\ 0 \le k \le K,\ 0 \le \tau \le M,\\
&C^N(x, k, u, \tau) := 0, \qquad S^N(x, k, u, \tau) := \max_{1 \le q \le u \wedge b} \{q p(x)\} 1_{\{\tau = 0\}},\\
&V^N(x, k, u, \tau) := \max\{S^N(x, k, u, \tau), C^N(x, k, u, \tau)\}, \quad 1 \le k \le K,\ 0 \le u \le B,\ 0 \le \tau \le M.\\
&C^l(x, k, u, \tau) := \mathcal{P}_h^r V^{l+1}(x, k, u, \tau - 1_{\{\tau > 0\}}),\\
&S^l(x, k, u, \tau) := 1_{\{\tau = 0\}} \max_{1 \le q \le u \wedge b} \{q p(x) + C^l(x, k-1, u-q, M)\},\\
&V^l(x, k, u) := \max\{S^l(x, k, u), C^l(x, k, u)\}, \quad 0 \le l \le N-1, 1 \le k \le K,\ 0 \le u \le B,\ 0 \le \tau \le M.
\end{aligned}$$

Then the value function $J^l(x, k, u, \tau) = V^l(x, k, u, \tau)$. The problem can then be solved using eigenfunction expansions, similar to the case without refraction period.

## 6. Numerical examples

In this section, we first discuss how to implement the eigenfunction expansion algorithm and then compare it to several other popular numerical methods.

Since infinite series appear in the eigenfunction expansion algorithm, truncation is needed to implement it. There are two truncation strategies. One is to truncate every infinite series using the same number of terms. The other is an adaptive approach which we employ in the actual implementation. In this approach, we specify a relative error tolerance (denoted by $e_1$) for computing every infinite series, and let the computer decide how many terms to use for each expansion. The adaptive strategy is more efficient than the one using a fixed number of terms everywhere as some expansions converge faster than the others. To find the boundary points for the switching/exercise regions, we solve the equations defining the boundary points using the bisection method with a given absolute error tolerance $e_2$. Alternatively Newton's method can be used with the implementation of the first order derivative of the value functions.

In the following, we compare the eigenfunction function algorithm to some popular methods used in practice. All computations were performed on a Dell workstation with Intel Xeon E5-2687W CPU at 3.10 GHz with 64 GB RAM under

Linux Red Hat 4.4.7-3. Codes were written in C++ and compiled with G++ 4.4.7. All infinite sums were truncated when a given relative error tolerance $e_1$ was reached. The bisection algorithm was used to find the root with a given absolute error tolerance $e_2$.

We first compare the computational performance of the eigenfunction expansion method to the lattice method for the applications developed in Section 5. For the OU diffusion, we use the Hull–White trinomial tree [53], which is perhaps the most commonly used lattice for this process. We will also consider the binomial tree for the OU process which can be built using the method in [54]. In constructing the trinomial and the binomial tree, both the state space and the time between any two exercise dates are discretized. We use the first node on the tree where it becomes optimal to switch/exercise to approximate the boundary. The performance is evaluated by looking at the error of the value function at time 0 for the given starting point as well as the switching/exercise boundaries. Since the boundary is a vector, we measure its accuracy by the root mean squared (RMS) error (if $(e_1, \dots, e_N)$ is the error vector, the RMS error is defined as $\sqrt{\sum_{i=1}^{N} e_i^2/N}$). In all examples, the benchmark is computed by running the eigenfunction expansion algorithm with $e_1 = 10^{-15}$ and $e_2 = 10^{-15}$. To analyze the convergence pattern, we ran the eigenfunction expansion algorithm with $e_1 = 10^{-5}, 10^{-7}, 10^{-9}, 10^{-11}, 10^{-13}$ while fixing $e_2$ at a small level. For the trinomial and the binomial tree, we ran the algorithm for $M = 250, 500, 750, 1000, 2500, 5000, 7500, 10\,000$ where $M$ is the number of time steps between two exercise dates. We next provide details for each numerical example (please refer to Section 5 for more detailed discussions on the model and setting in each example).

(1) **Combo carriers**: The useful life of the carrier is twenty years. The discount rate $r = 10\%$. We assume decisions are made every month, i.e., $h = 1/12$ year (hence there are 240 periods in total). For convenience we denominate money in millions of dollars. Freight rates in the shipping markets are quoted as dollars per day, thus in computing $f(x, 1) = xh$, $h$ is converted from years to days (recall that $f(x, 1)$ is the payoff for shipping dry cargo). Sødal et al. [5] estimated the parameters of the OU diffusion from the freight rate data. We use the same parameters with mean-reversion speed $\kappa = 2.4$, long-run level $\theta = -0.0054$, volatility $\sigma = 0.0226$, and switching cost $F_+ = F_- = 0.04$ (in million). We evaluate the option value at $x_0 = 0$. The relative value of combo shipping carrier to an equal-sized oil tanker is given by $V(x_0, 0) = 5.55599103$ million, rounded to the eighth decimal place. We fix $e_2 = 10^{-10}$.

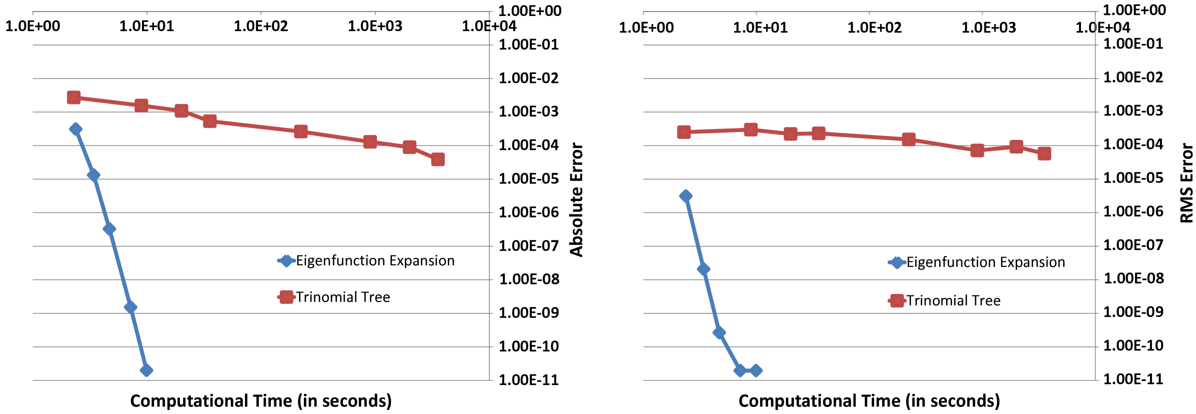
(2) **Chooser flexible caps under the Vasicek short rate model**: We consider a five year chooser flexible cap with one million notional amount. Decisions are made every quarter, i.e., $h = 0.25$ year, and the number of exercise rights $K = 10$. For the Vasicek model, mean-reversion speed $\kappa = 0.3$, volatility $\sigma = 0.01$, long-run level $\theta = 0.05$, and the initial short rate $x_0 = 0.045$. The strike price $G = 0.04$. We use the same value for $\kappa$ and $\sigma$ as Ohnishi and Tamba [7]. The value of this cap, $V^{0,10}(x_0) = 0.02503149$ million, rounded to the eighth decimal place. We fix $e_2 = 10^{-8}$.

(3) **Commodity swing options**: We consider a two year swing option which can be exercised monthly (24 periods in total), with both local and global volume constraints. The owner of the option is entitled with 10 exercise rights, i.e., $K = 10$. In each exercise, the volume taken cannot exceed 2, i.e., $b = 2$ and the total volume taken from all exercises cannot exceed 14, i.e., $B = 14$. The initial spot price $S_0 = 2.5$ and the strike price $G = 2.4$. The discount rate $r = 0.1$. We use the same parameters for the OU diffusion as in [8]. Here mean-reversion speed $\kappa = 3.4$, long-run level $\theta = -0.114$, volatility $\sigma = 0.59$, and starting point $x_0 = 0.0$. The value of the swing option is given by $V(x_0) = 5.23191411$, rounded to the eighth decimal place. We fix $e_2 = 10^{-8}$.

Figs. 4, 5 and 6 show the comparison between the eigenfunction expansion algorithm and the trinomial tree for valuation of combo carriers, chooser flexible caps and commodity swing options, respectively. Fig. 7 gives the comparison between the eigenfunction expansion algorithm and the binomial tree for valuation of combo carriers. In the boundary graphs, the RMS error for the boundary eventually stabilizes in the eigenfunction expansion algorithm as we fix $e_2$. It is clear that in all cases, for a given accuracy level, the eigenfunction expansion algorithm is orders of magnitude faster than the trinomial/binomial tree algorithm for both the value function and the boundary. For example, in the swing case, the CPU time ranges from 21.95 s for $e_1 = 10^{-5}$ to 47.44 s for $e_1 = 10^{-13}$. The absolute error for the option price is $6.62 \times 10^{-5}$ (relative error $1.27 \times 10^{-5}$) at $e_1 = 10^{-5}$ and rapidly decreases to $1.55 \times 10^{-12}$ (relative error $2.96 \times 10^{-13}$) at $e_1 = 10^{-13}$. The RMS error for the exercise boundary is around $5 \times 10^{-6}$ at $e_1 = 10^{-5}$ and rapidly decreases to $2 \times 10^{-9}$ at $e_1 = 10^{-9}$. It then stays at almost the same level as we fix $e_2 = 10^{-8}$ for the bisection. In the trinomial tree algorithm, when $M$ varies from 250 to 10 000, the computation time ranges from 44.31 s to 17336.60 s and errors for the price and the boundary decrease only slightly.

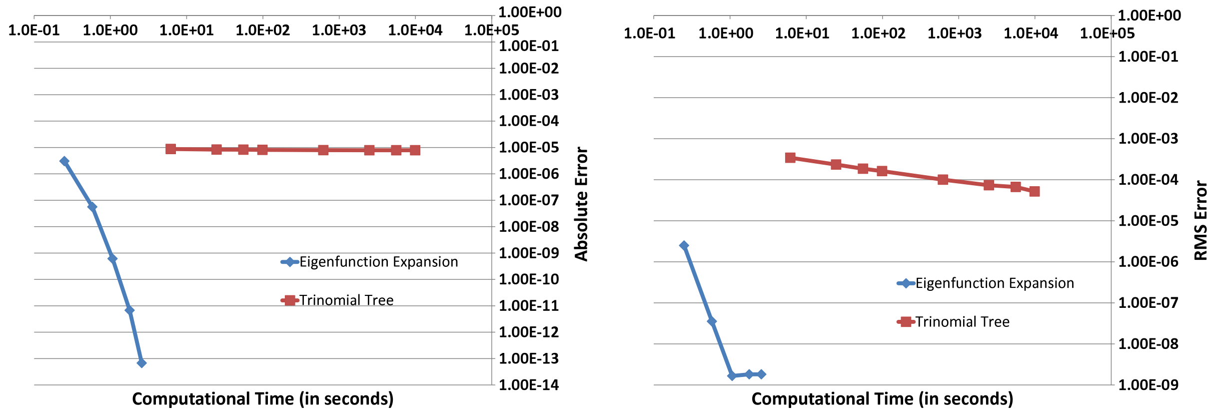
Next we compare the computational performance of the eigenfunction expansion method to the Crank–Nicolson scheme which is a popular numerical PDE scheme in practice and it is more efficient than the lattice method, which corresponds to explicit finite differences. To do this, we revisit the combo shipping carrier problem. To apply the Crank–Nicolson scheme to solve the optimal switching problem, we notice that for the OU diffusion, the continuation value solves a PDE using the pre-switch value function at the next decision date as the payoff function. We use the same model parameters as before, and localize the state space $\mathbb{R}$ to a finite interval $(L, U)$ with $L = -0.2$ and $U = 0.2$. We ran the algorithm for $(M, S) = (250, 560), (500, 792), (750, 970), (1000, 1120), (2500, 1770), (5000, 2504), (7500, 3066), (10\,000, 3540)$, where $M$ is the number of time steps between two exercise dates and $S$ is the number of state steps on $[L, U]$ (a uniform grid is used). Fig. 8 displays the comparison. Again we see the eigenfunction expansion algorithm is orders of magnitude faster.

The rapid convergence we observe for the eigenfunction expansion algorithm can be explained by the exponential convergence rate, as for all these examples the conditions in Theorem 5 are satisfied. The high level of accuracy in the



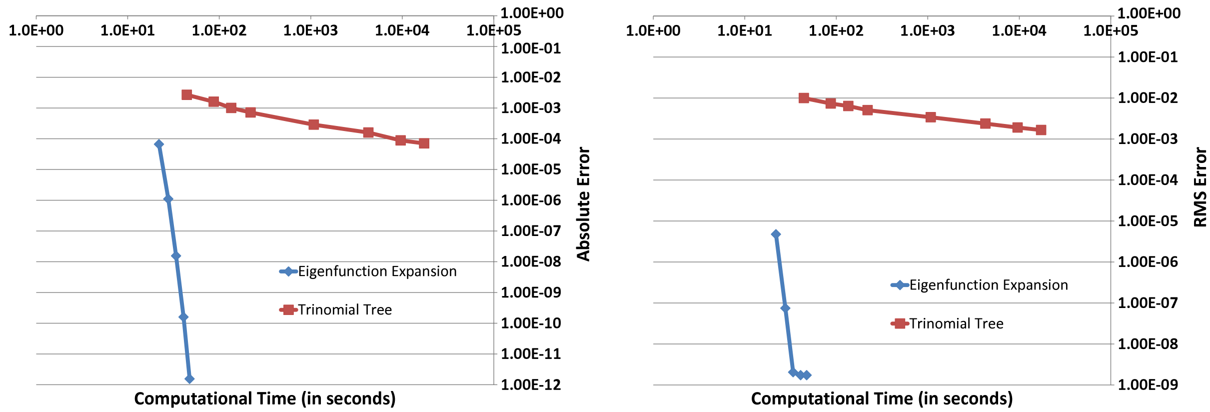

(a) Value function. (b) Boundary.

**Fig. 4.** Eigenfunction expansion vs. trinomial tree for the combo carrier (on log–log scale).




(a) Value function. (b) Boundary.

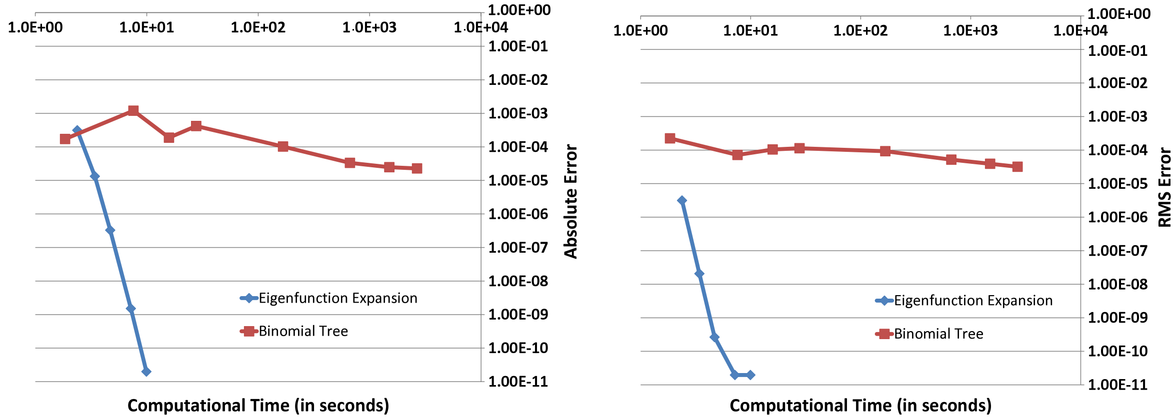
**Fig. 5.** Eigenfunction expansion vs. trinomial tree for the chooser flexible cap (on log–log scale).




(a) Value function. (b) Boundary.

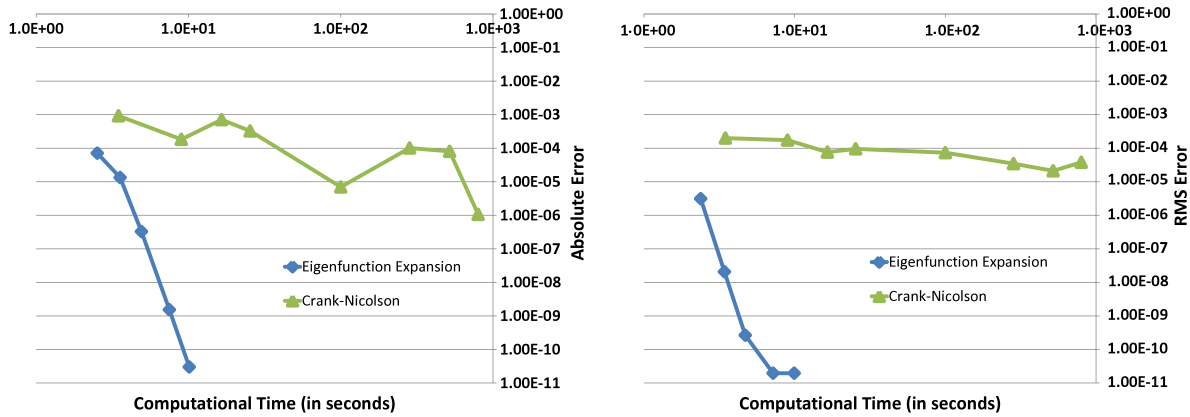
**Fig. 6.** Eigenfunction expansion vs. trinomial tree for the swing option (on log–log scale).

boundary is due to the fact that in the eigenfunction expansion algorithm, the boundary is determined by finding roots of globally defined equations. In contrast, in the lattice or Crank–Nicolson method, the state space is discretized and the error in the boundary is controlled by the grid size. To achieve high precision in the boundary requires exceedingly fine grid.




(a) Value function. (b) Boundary.

**Fig. 7.** Eigenfunction expansion vs. binomial tree for the combo carrier (on log–log scale).




(a) Value function. (b) Boundary.

**Fig. 8.** Eigenfunction expansion vs. Crank–Nicolson for the combo carrier (on log–log scale).

There are several other notable advantages of the eigenfunction expansion algorithm. (1) Running the algorithm once gives us the expansion coefficients, which allows us to determine the value function on the entire state space with high level of accuracy. In comparison, numerical methods that require discretization can only find the value function on the grid, and interpolation must be used to find the value at non-grid points, which cannot be highly accurate. (2) Delta and gamma of the value function can be obtained analytically by differentiating the eigenfunction expansion term-by-term under some mild conditions (see (19) and [25, Proposition 6] for results of this type). The analytical formula allows us to achieve high precision in delta and gamma. (3) The eigenfunction expansion algorithm is generally applicable to Markov processes with discrete spectrum, including not only diffusions, but also jump–diffusions and pure jump processes obtained from diffusions via subordination. Existing numerical PIDE schemes cannot be efficiently applied to subordinate diffusions, as its jump measure is not known in closed-form (see [25, p. 631] the jump density is given by the integral of the diffusion transition probability density integrated with the Lévy measure of the subordinator). To apply them, one has to first compute the jump measure numerically.

**Remark 7.** The characteristic function of an OU diffusion is known in closed-form, hence the method of Fourier-cosine expansions and fast Hilbert transform can also be applied to evaluate combo shipping carriers under the OU freight difference model and commodity swing options under the Schwartz model. We expect these methods to be highly efficient as well and their computational complexity is similar to the eigenfunction expansion method. We noted that Zhang et al. [55] show that by approximating the OU characteristic function properly, the Fourier-cosine expansion method can be implemented with the help of FFT, so for certain parameter values and some target accuracy (say 1 basis point), the Fourier-cosine method can be more efficient than the eigenfunction expansion algorithm, as the latter does not allow FFT to be used. The strength of the eigenfunction expansion algorithm is that it is generally applicable and computationally efficient for Markov processes with discrete spectrum, for which the characteristic function may be unavailable.

## 7. Conclusions

This paper develops an efficient algorithm based on eigenfunction expansions to solve optimal switching and multiple stopping problems in a finite horizon discrete time setting for a rich class of one-dimensional Markov processes that are important in financial applications. This class includes diffusions with purely discrete spectrum, and jump–diffusions and pure jump processes obtained from these diffusions through subordination. We develop a dynamic programming procedure for these problems, and by assuming square-integrable payoffs and switching costs, we show that the dynamic programming problem can be solved explicitly using eigenfunction expansions. We prove that under some mild conditions, our algorithm converges exponentially in the series truncation level. Easy-to-verify conditions are also provided to characterize connectedness of switching/exercise regions. We illustrate the versatility of our method with three applications: valuation of combination carriers, interest-rate chooser flexible caps and commodity swing options. Numerical examples demonstrate the superior computational performance of the eigenfunction expansion algorithm.

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## Appendix. Proofs

**Theorem 1.** For $l = N$, from the definition it is easy to verify that $V^N(x, i) = J^N(x, i)$. We next consider $l < N$. First we shall show $V^l(x, i) \geq J^l(x, i)$. Consider an arbitrary strategy $\alpha \in \mathcal{A}^l$. Using the definition of $V^l(x, i)$,

$$\begin{aligned}
V^l(x,i) &= \max_{j\in\mathbb{D}} \left\{ f(x,j) - C(x,i,j) + \mathcal{P}_h^r V^{l+1}(x,j) \right\} \\
&\geq f(x, I_{t_l}^\alpha) - C(x,i,I_{t_l}^\alpha) + \mathcal{P}_h^r V^{l+1}(x, I_{t_l}^\alpha) \\
&= f(x, I_{t_l}^\alpha) - C(x,i,I_{t_l}^\alpha) + \mathcal{P}_h^r \max_{j\in\mathbb{D}} \left\{ f(x,j) - C(x, I_{t_{l+1}-}^\alpha, j) + \mathcal{P}_h^r V^{l+2}(x,j) \right\} \quad (\text{since } I_{t_l}^\alpha = I_{t_{l+1}-}^\alpha) \\
&\geq f(x, I_{t_l}^\alpha) - C(x,i,I_{t_l}^\alpha) + \mathcal{P}_h^r \{ f(x, I_{t_{l+1}}^\alpha) - C(x, I_{t_{l+1}-}^\alpha, I_{t_{l+1}}^\alpha) \} + \mathcal{P}_h^r \mathcal{P}_h^r V^{l+2}(x, I_{t_{l+1}}^\alpha) \\
&= f(x, I_{t_l}^\alpha) - C(x,i,I_{t_l}^\alpha) + \mathcal{P}_h^r \{ f(x, I_{t_{l+1}}^\alpha) - C(x, I_{t_{l+1}-}^\alpha, I_{t_{l+1}}^\alpha) \} + \mathcal{P}_{2h}^r V^{l+2}(x, I_{t_{l+1}}^\alpha) \quad (\text{since } \mathcal{P}_{2h}^r = \mathcal{P}_h^r \mathcal{P}_h^r) \\
&\;\;\vdots \\
&\geq \mathbb{E}_x \left[ \sum_{n=l}^{N-1} e^{-\int_{t_l}^{t_n} r(X_u)du} \left( f(X_{t_n}, I_{t_n}^\alpha) - C(X_{t_n}, I_{t_n-}^\alpha, I_{t_n}^\alpha) \right) + e^{-\int_{t_l}^{t_N} r(X_u)du} V^N(X_{t_N}, I_{t_N-}^\alpha) \right] \\
&\geq \mathbb{E}_x \left[ \sum_{n=l}^{N} e^{-\int_{t_l}^{t_n} r(X_u)du} \left( f(X_{t_n}, I_{t_n}^\alpha) - C(X_{t_n}, I_{t_n-}^\alpha, I_{t_n}^\alpha) \right) \right] \\
&= J^l(x,i,\alpha).
\end{aligned}$$

In the above, we used the definition of $V^l$ to $V^N$. Since $\alpha$ is arbitrary, $V^l(x, i) \geq J^l(x, i)$. To show the reverse inequality, note that if we use the strategy $\alpha^* = \left((\tau_1^*, \xi_1^*), (\tau_2^*, \xi_2^*), \dots\right)$ defined in (5), $V^l(x, i) = J^l(x, i, \alpha^*)$. This shows $V^l(x, i) \leq J^l(x, i)$. Together we have $V^l(x, i) = J^l(x, i)$ and $\alpha^*$ is an optimal strategy. □

**Theorem 2.** We will use induction to prove the claim of the theorem with another claim that $V^l(x, i + 1) - V^l(x, i)$ is nondecreasing.

Step 1: We verify these claims are true when $l = N$. $W^N(x, i, j+1) - W^N(x, i, j) = f(x, j+1) - C(x, i, j+1) - f(x, j) + C(x, i, j)$ is nondecreasing by condition (iii). From this it is easy to see that $W^N(x, i, j) - W^N(x, i, q)$ is nondecreasing for any $q < j$. Define $\tilde{\mathcal{R}}^N(i, j) := (x_{i,j}^N, x_{i,j+1}^N]$ for $j = 0, 1, d - 2$, $\tilde{\mathcal{R}}^N(i, d - 1) = (x_{i,d-1}^N, x_{i,d}^N)$. We next show that $\tilde{\mathcal{R}}^N(i, j) = \mathcal{R}^N(i, j)$ for $0 \leq j \leq d - 1$.

First from the definition of $\tilde{x}_i^N(q, j)$ and that $W^N(x, i, j) - W^N(x, i, q)$ is nondecreasing, we have for $x > \tilde{x}_i^N(q, j)$, $W^N(x, i, q) < W^N(x, i, j)$, and for $x \leq \tilde{x}_i^N(q, j)$, $W^N(x, i, q) \geq W^N(x, i, j)$. (1) For $d - 1$, we can find $k \leq d - 2$, such that $\tilde{x}_i^N(k, d-1) = \max_{q=0,1,\dots,d-2}\{\tilde{x}_i^N(q, d-1)\}$. Then for any $x \in (e_1, x_{i,d-1}^N]$, $x \leq \tilde{x}_i^N(k, d-1)$, thus $W^N(x, i, k) \geq W^N(x, i, d-1)$, which implies $(e_1, x_{i,d-1}^N] \subseteq E \setminus \mathcal{R}^N(i, d - 1)$. So we have $\mathcal{R}^N(i, d - 1) \subseteq \tilde{\mathcal{R}}^N(i, d - 1)$. Now for any $x \in \tilde{\mathcal{R}}^N(i, d - 1)$, $x > \tilde{x}_i^N(q, d - 1)$ for $0 \leq q \leq d - 2$, thus $W^N(x, i, q) < W^N(x, i, d - 1)$. This implies $x \in \mathcal{R}^N(i, d - 1)$ from the definition of $\mathcal{R}^N(i, d - 1)$. Hence $\tilde{\mathcal{R}}^N(i, d - 1) \subseteq \mathcal{R}^N(i, d - 1)$. Together we have $\tilde{\mathcal{R}}^N(i, d - 1) = \mathcal{R}^N(i, d - 1)$. (2) For $d - 2$, similar to (1), we can show that $\mathcal{R}^N(i, d - 2) \subseteq \bigcup_{j=d-2}^{d-1} \tilde{\mathcal{R}}^N(i, j)$ and $\tilde{\mathcal{R}}^N(i, d - 2) \subseteq \bigcup_{j=d-2}^{d-1} \mathcal{R}^N(i, j)$. Since we already have

$\tilde{\mathcal{R}}^N(i, d-1) = \mathcal{R}^N(i, d-1)$, $\mathcal{R}^N(i, d-2) \bigcap \mathcal{R}^N(i, d-1) = \emptyset$ and $\tilde{\mathcal{R}}^N(i, d-2) \bigcap \tilde{\mathcal{R}}^N(i, d-1) = \emptyset$, we can conclude $\mathcal{R}^N(i, d-2) = \tilde{\mathcal{R}}^N(i, d-2)$. For $j = d-3, \dots, 0$, we can similarly prove $\tilde{\mathcal{R}}^N(i, j) = \mathcal{R}^N(i, j)$.

Next we show that $V^N(x, i+1) - V^N(x, i)$ is nondecreasing, for $i = 0, 1, \dots, d-2$. Note that $E = \bigcup_{q,j=0,1,\dots,d-1} (\mathcal{R}^N(i, q) \bigcap \mathcal{R}^N(i+1, j))$. Below we first show that $V^N(x, i+1) - V^N(x, i)$ is nondecreasing for every interval $\mathcal{R}^N(i, q) \bigcap \mathcal{R}^N(i+1, j)$. Note that for $q < j$ $(1 \le j \le d-1)$,

$$\begin{aligned}
\tilde{x}^N_{i+1}(q, j) &= \inf\{x \in E : W^N(x, i+1, j) - W^N(x, i+1, q) > 0\}, \\
\tilde{x}^N_i(q, j) &= \inf\{x \in E : W^N(x, i, j) - W^N(x, i, q) > 0\} \\
&= \inf\{x : W^N(x, i+1, j) - W^N(x, i+1, q) > -C(x, i+1, j) \\
&\quad + C(x, i+1, q) + C(x, i, j) - C(x, i, q)\}.
\end{aligned}$$

The last equality above is obtained by noting that

$$\begin{aligned}
&W^N(x, i, j) - W^N(x, i, q) = f(x, j) - C(x, i, j) - (f(x, q) - C(x, i, q)) \\
&\quad = W^N(x, i+1, j) + C(x, i+1, j) - C(x, i, j) - (W^N(x, i+1, q) + C(x, i+1, q)) + C(x, i, q) \\
&\quad = W^N(x, i+1, j) - W^N(x, i+1, q) + C(x, i+1, j) - C(x, i+1, q) - C(x, i, j) + C(x, i, q).
\end{aligned}$$

Since $W^N(x, i+1, j) - W^N(x, i+1, q)$ is nondecreasing and $-C(x, i+1, j) + C(x, i+1, q) + C(x, i, j) - C(x, i, q) \ge 0$ by condition (iv), we must have $\tilde{x}^N_{i+1}(q, j) \le \tilde{x}^N_i(q, j)$, which implies $x^N_{i+1,j} \le x^N_{i,j}$. Since $\bigcup_{q=j}^{d-1} \mathcal{R}^N(i, q) = (x^N_{i,j}, e_2)$ and $\bigcup_{q=j}^{d-1} \mathcal{R}^N(i+1, q) = (x^N_{i+1,j}, e_2)$, we have $\bigcup_{q=j}^{d-1} \mathcal{R}^N(i, q) \subseteq \bigcup_{q=j}^{d-1} \mathcal{R}^N(i+1, q)$. For every interval $\mathcal{R}^N(i, q) \bigcap \mathcal{R}^N(i+1, j)$:

(1) If $q > j$, $(\bigcup_{k=j+1}^{d-1} \mathcal{R}^N(i+1, k)) \bigcap \mathcal{R}^N(i+1, j) = \emptyset$, $\bigcup_{k=j+1}^{d-1} \mathcal{R}^N(i, k) \subseteq \bigcup_{k=j+1}^{d-1} \mathcal{R}^N(i+1, k)$, so we have

$$\left( \bigcup_{k=j+1}^{d-1} \mathcal{R}^N(i, k) \right) \bigcap \mathcal{R}^N(i+1, j) = \emptyset.$$

Combined with $\mathcal{R}^N(i, q) \subseteq \bigcup_{k=j+1}^{d-1} \mathcal{R}^N(i, k)$, we get $\mathcal{R}^N(i, q) \bigcap \mathcal{R}^N(i+1, j) = \emptyset$.

(2) If $q = j$, then $V^N(x, i+1) - V^N(x, i) = W^N(x, i+1, j) - W^N(x, i, j) = -C(x, i+1, j) + C(x, i, j)$ is nondecreasing by condition (v).

(3) If $q < j$, then $V^N(x, i+1) - V^N(x, i) = W^N(x, i+1, j) - W^N(x, i, q) = (W^N(x, i+1, j) - W^N(x, i, j)) + (W^N(x, i, j) - W^N(x, i, q))$ is nondecreasing because the first part is nondecreasing by (2) and the second part is also nondecreasing.

Thus $V^N(x, i+1) - V^N(x, i)$ is nondecreasing on each $\mathcal{R}^N(i, q) \bigcap \mathcal{R}^N(i+1, j)$. Since $V^N(x, i+1) - V^N(x, i)$ is continuous, it is nondecreasing on $E$. We now have verified all claims hold at $l = N$.

Step 2: Suppose at $t_{l+1}$, the claim in Theorem 2 is true and $V^{l+1}(x, i+1) - V^{l+1}(x, i)$ is nondecreasing. We verify the claims for $t_l$.

$$\begin{aligned}
W^l(x, i, j+1) - W^l(x, i, j) &= (f(x, j+1) - C(x, i, j+1) + \mathcal{P}^r_h V^{l+1}(x, j+1)) - (f(x, j) - C(x, i, j) + \mathcal{P}^r_h V^{l+1}(x, j)) \\
&= (f(x, j+1) - C(x, i, j+1) - f(x, j) + C(x, i, j)) + \mathcal{P}^r_h (V^{l+1}(x, j+1) - V^{l+1}(x, j)).
\end{aligned}$$

By the induction condition, $V^{l+1}(x, j+1) - V^{l+1}(x, j)$ is nondecreasing, thus $\mathcal{P}^r_h(V^{l+1}(x, j+1) - V^{l+1}(x, j))$ is nondecreasing by condition (i). $f(x, j+1) - C(x, i, j+1) - f(x, j) + C(x, i, j)$ is nondecreasing by condition (iii). Therefore $W^l(x, i, j+1) - W^l(x, i, j)$ is nondecreasing. Then it is easy to see that $W^l(x, i, j) - W^l(x, i, q)$ is nondecreasing for any $q < j$.

Similar to step 1, we define $\tilde{\mathcal{R}}^l(i, j)$. We can then show $\tilde{\mathcal{R}}^l(i, j) = \mathcal{R}^l(i, j)$, and verify $V^l(x, i+1) - V^l(x, i)$ is nondecreasing using the same arguments as in step 1 (replacing $W^N$ in step 1 by $W^l$ and using the definition of $W^l$ in Theorem 1). The details are omitted. By induction, Theorem 2 is proved. □

**Theorem 3.** First, note that Assumption 1 implies (1). Since $|f(x, j)|$ and $|C(x, i, j)|$ are square-integrable, Assumption 1 implies $\mathcal{P}^r_h |f(x, j)|$ and $\mathcal{P}^r_h |C(x, i, j)|$ are continuous functions of $x$ and hence have finite value.

Part (i) follows from (2), (3), and the facts that $\mathcal{P}: L^2(E, \mathfrak{m}) \mapsto L^2(E, \mathfrak{m})$ and $f + g, \max(f, g) \in L^2(E, \mathfrak{m})$ for two Borel measurable functions $f$ and $g$ on $E$ that satisfy $f, g \in L^2(E, \mathfrak{m})$.

Now we consider part (ii). Comparing the terminal condition $W^N(x, i, j) = f(x, j) - C(x, i, j)$ to (9), we have $w^N_n(i, j) = 0$. We assume (9) holds for time $t_{l+1}$, and prove (11) for time $t_l$. By definition, $W^l(x, i, j) = f(x, j) - C(x, i, j) + \mathcal{P}^r_h V^{l+1}(x, j)$, then

$$\begin{aligned}
w^l_n(j) &= \int_E \max_{k \in \mathbb{D}} \{W^{l+1}(x, j, k)\} \varphi_n(x) \mathfrak{m}(dx) \\
&= \sum_{k \in \mathbb{D}} \int_{\mathcal{R}^{l+1}(j,k)} W^{l+1}(x, j, k) \varphi_n(x) \mathfrak{m}(dx)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k\in\mathbb{D}} \int_{\mathcal{R}^{l+1}(j,k)} \left\{ f(x,k) - C(x,j,k) + \sum_{m=1}^{\infty} w_m^{l+1}(k) e^{-\lambda_m h}\varphi_m(x) \right\} \varphi_n(x)\mathfrak{m}(dx) \\
&= \sum_{k\in\mathbb{D}} f_n^k(\mathcal{R}^{l+1}(j,k)) - C_n^{j,k}(\mathcal{R}^{l+1}(j,k)) + \int_{\mathcal{R}^{l+1}(j,k)} \sum_{m=1}^{\infty} w_m^{l+1}(k) e^{-\lambda_m h}\varphi_m(x)\varphi_n(x)\mathfrak{m}(dx) \\
&= \sum_{k\in\mathbb{D}} \left\{ f_n^k(\mathcal{R}^{l+1}(j,k)) - C_n^{j,k}(\mathcal{R}^{l+1}(j,k)) + \sum_{m=1}^{\infty} w_m^{l+1}(k) e^{-\lambda_m h}\pi_{m,n}(\mathcal{R}^{l+1}(j,k)) \right\},
\end{aligned}
$$

where in the last equality we interchanged the order of integration and summation using the continuity of the inner product, i.e., if $g_n \to g$ in $L^2(E,\mathfrak{m})$, then $\lim_{n\to\infty}(g_n,h) = (g,h)$ for any $h \in L^2(E,\mathfrak{m})$. $\square$

**Proposition 1.** When $l = N$, from (12), it is obvious that the claim is true. Now we assume the result holds at time $t_{l+1}$ for all $k > N-l-1$. At time $t_l$, if $k$ satisfies $k > N-l$, we also have $k-1 > N-l-1$. Therefore $V^{l+1}(x,k-1) = V^{l+1}(x,k) = V^{l+1}(x,N-l)$ by the induction hypothesis. From (13) this implies $C^l(x,k-1) = C^l(x,k) = C^l(x,N-l)$. From (13), we also have for $k > N-l$, (i) $S^l(x,k) = p(x) + C^l(x,k-1) = p(x) + C^l(x,k) \ge C^l(x,k)$ since $p(x) \ge 0$; (ii) $S^l(x,k) = p(x) + C^l(x,k-1) = p(x) + C^l(x,N-l) = S^l(x,N-l+1)$. (i) and (ii) imply $V^l(x,k) = V^l(x,N-l+1)$. If $p(x) > 0$, the inequality in (i) is strict. Hence $\mathcal{S}^{l,k} = \{x : p(x) > 0\}$. $\square$

**Proposition 2.** We shall prove the claims by induction. For $l = N-1$, from Proposition 1, $V^N(x,2) = V^N(x,1)$, hence $C^{N-1}(x,2) = C^{N-1}(x,1)$. Thus $C^{N-1}(x,2) + C^{N-1}(x,0) - 2C^{N-1}(x,1) = -C^{N-1}(x,1) \le 0$, where $C^{N-1,1}(x) \ge 0$ is implied by Corollary 1. $\mathcal{S}^{N-1,1} = \{x \in E : p(x) > C^{N-1,1}(x)\}$ and from Proposition 1, $\mathcal{S}^{N-1,2} = \{x \in E : p(x) > 0\}$. Since $C^{N-1,1}(x) \ge 0$, we have $\mathcal{S}^{N-1,1} \subseteq \mathcal{S}^{N-1,2}$. Now assume the claims are true at $l+1$. We verify them for $l$.

$$
C^l(x,k+1) + C^l(x,k-1) - 2C^l(x,k) = \mathcal{P}_h^r(V^{l+1}(x,k+1) + V^{l+1}(x,k-1) - 2V^{l+1}(x,k))
$$

$$
\begin{aligned}
&V^{l+1}(x,k+1) + V^{l+1}(x,k-1) - 2V^{l+1}(x,k) \\
&= \begin{cases}
S^{l+1}(x,k+1) + S^{l+1}(x,k-1) - 2S^{l+1}(x,k), & x \in \mathcal{S}^{l+1,k-1} \\
S^{l+1}(x,k+1) + C^{l+1}(x,k-1) - 2S^{l+1}(x,k), & x \in \mathcal{S}^{l+1,k}/\mathcal{S}^{l+1,k-1} \\
S^{l+1}(x,k+1) + C^{l+1}(x,k-1) - 2C^{l+1}(x,k), & x \in \mathcal{S}^{l+1,k+1}/\mathcal{S}^{l+1,k} \\
C^{l+1}(x,k+1) + C^{l+1}(x,k-1) - 2C^{l+1}(x,k), & x \in \mathcal{C}^{l+1,k+1}
\end{cases} \\
&= \begin{cases}
C^{l+1}(x,k) + C^{l+1}(x,k-2) - 2C^{l+1}(x,k-1), & x \in \mathcal{S}^{l+1,k-1} \\
C^{l+1}(x,k) - S^{l+1}(x,k), & x \in \mathcal{S}^{l+1,k}/\mathcal{S}^{l+1,k-1} \\
S^{l+1}(x,k) - C^{l+1}(x,k), & x \in \mathcal{S}^{l+1,k+1}/\mathcal{S}^{l+1,k} \\
C^{l+1}(x,k+1) + C^{l+1}(x,k-1) - 2C^{l+1}(x,k), & x \in \mathcal{C}^{l+1,k+1}.
\end{cases}
\end{aligned}
$$

For the last equality, we used Corollary 1 to simplify the expression. In the first and the last region, the target is nonpositive by the induction assumption. In the second and the third region, the target is nonpositive due to the definition of $\mathcal{S}^{l+1,k}$. Hence $C^l(x,k+1) + C^l(x,k-1) - 2C^l(x,k) \le 0$. Note that

$$
\begin{aligned}
\mathcal{S}^{l,k} &= \{x \in E : S^l(x,k) - C^l(x,k) > 0\}, \\
\mathcal{S}^{l,k+1} &= \{x \in E : S^l(x,k+1) - C^l(x,k+1) > 0\} \\
&= \{x \in E : S^l(x,k) - C^l(x,k) > C^l(x,k+1) + C^l(x,k-1) - 2C^l(x,k)\}.
\end{aligned}
$$

The third equality is obtained by noting that

$$
\begin{aligned}
S^l(x,k+1) - C^l(x,k+1) &= p(x) + C^l(x,k) - C^l(x,k+1) \\
&= S^l(x,k) - C^l(x,k-1) + C^l(x,k) - C^l(x,k+1).
\end{aligned}
$$

Since $C^l(x,k+1) + C^l(x,k-1) - 2C^l(x,k) \le 0$, we have $\mathcal{S}^{l,k} \subseteq \mathcal{S}^{l,k+1}$. By induction the proposition is proved. $\square$

**Theorem 4.** First we notice that condition (iii) together with Corollary 1 implies that $S^l(x,k)$ and $V^l(x,k)$ are also continuous in $x$ for all $l$ and $k$. We prove (15) with another claim that $S^l(x,k) - C^l(x,k)$ is nondecreasing using induction. At $t_N$, these claims are clearly true. We assume they hold at $t_{l+1}$ and verify them at $t_l$. By definition,

$$
\begin{aligned}
S^l(x,k) - C^l(x,k) &= p(x) + \mathcal{P}_h^r V^{l+1}(x,k-1) - \mathcal{P}_h^r V^{l+1}(x,k) \\
&= (p(x) - \mathcal{P}_h^r p(x)) + \mathcal{P}_h^r(p(x) + V^{l+1}(x,k-1) - V^{l+1}(x,k)).
\end{aligned}
$$

The first part is already nondecreasing by assumption (iv). We also have $\mathcal{C}^{l+1,k-1} = (-\infty, x^{l+1,k-1}]$, $\mathcal{S}^{l+1,k-1} = (x^{l+1,k-1}, \infty)$, $\mathcal{C}^{l+1,k} = (-\infty, x^{l+1,k}]$, $\mathcal{S}^{l+1,k} = (x^{l+1,k}, \infty)$, and $x^{l+1,k} \le x^{l+1,k-1}$. Then

$$
p(x) + V^{l+1}(x,k-1) - V^{l+1}(x,k) = \begin{cases}
p(x) + S^{l+1}(x,k-1) - S^{l+1}(x,k), & x \in (x^{l+1,k-1}, e_2) \\
p(x) + C^{l+1}(x,k-1) - S^{l+1}(x,k), & x \in (x^{l+1,k}, x^{l+1,k-1}] \\
p(x) + C^{l+1}(x,k-1) - C^{l+1}(x,k), & x \in (e_1, x^{l+1,k}]
\end{cases}
$$

$$= \begin{cases} S^{l+1}(x,k-1) - C^{l+1}(x,k-1), & x \in (x^{l+1,k-1}, e_2) \\ 0, & x \in (x^{l+1,k}, x^{l+1,k-1}] \\ S^{l+1}(x,k) - C^{l+1}(x,k), & x \in (e_1, x^{l+1,k}] \end{cases}$$

is nondecreasing by the induction assumption. Thus the second part of $S^l(x,k) - C^l(x,k)$ is nondecreasing by condition (i). Together $S^l(x,k) - C^l(x,k)$ is nondecreasing. This, combined with the continuity of $S^l(x,k) - C^l(x,k)$, the definition of $x^{l,k}$ and $\mathcal{S}^{l,k} = \{x \in E : S^l(x,k) - C^l(x,k) > 0\}$, implies that $\mathcal{S}^{l,k} = (x^{l,k}, e_2)$. Since $S^l(x_b,k) - C^l(x_b,k) = p(x_b) + C^l(x_b,k-1) - C^l(x_b,k) = C^l(x_b,k-1) - C^l(x_b,k) \le 0$, we have $x^{l,k} \ge x_b$. From Proposition 2, $\mathcal{S}^{l,k} \subseteq \mathcal{S}^{l,k+1}$, hence we also have $x^{l,k+1} \le x^{l,k}$. To prove for $l+k > N$, $x^{l,k} = x_b$, note that from Proposition 1, $\mathcal{S}^{l,k} = \{x \in E : p(x) > 0\} = (x_b, e_2)$ due to condition (iii). □

**Theorem 5.** Obviously, the claim is true at time $t_N$ as there is no error. We assume the claim holds at time $t_{l+1}$ and prove the claim for time $t_l$. We have

$$\begin{aligned} \left|\hat{w}_n^l(j) - w_n^l(j)\right| \le & \sum_{k\in\mathbb{D}} \left\{\left|f_n^k(\hat{\mathcal{R}}^{l+1}(j,k)) - f_n^k(\mathcal{R}^{l+1}(j,k))\right| + \left|C_n^{j,k}(\hat{\mathcal{R}}^{l+1}(j,k)) - C_n^{j,k}(\mathcal{R}^{l+1}(j,k))\right|\right\} \\ & + \sum_{k\in\mathbb{D}} \left\{\sum_{m=1}^{M} \left|w_m^{l+1}(k)\right| e^{-\lambda_m h} \left|\pi_{m,n}(\hat{\mathcal{R}}^{l+1}(j,k)) - \pi_{m,n}(\mathcal{R}^{l+1}(j,k))\right|\right\} \\ & + \sum_{k\in\mathbb{D}} \left\{\sum_{m=1}^{M} \left|\hat{w}_m^{l+1}(k) - w_m^{l+1}(k)\right| e^{-\lambda_m h} \pi_{m,n}(\mathcal{R}^{l+1}(j,k))\right\} \\ & + \sum_{k\in\mathbb{D}} \left\{\sum_{m=1}^{M} \left|\hat{w}_m^{l+1}(k) - w_m^{l+1}(k)\right| e^{-\lambda_m h} \left|\pi_{m,n}(\hat{\mathcal{R}}^{l+1}(j,k)) - \pi_{m,n}(\mathcal{R}^{l+1}(j,k))\right|\right\} \\ & + \sum_{k\in\mathbb{D}} \left\{\sum_{m=M+1}^{\infty} \left|w_m^{l+1}(k)\right| e^{-\lambda_m h} \left|\pi_{m,n}(\mathcal{R}^{l+1}(j,k))\right|\right\}. \end{aligned}$$

In the following we analyze each part. For the first part, we have for any $t > 0$

$$\begin{aligned} & \max_{1\le n\le M} \sum_{k\in\mathbb{D}} \left\{\left|f_n^k(\hat{\mathcal{R}}^{l+1}(j,k)) - f_n^k(\mathcal{R}^{l+1}(j,k))\right| + \left|C_n^{j,k}(\hat{\mathcal{R}}^{l+1}(j,k)) - C_n^{j,k}(\mathcal{R}^{l+1}(j,k))\right|\right\} \\ & \le d \max_{k\in\mathbb{D}} \bar{f}_t^{l+1}(j,k) e^{\lambda_M t/2} \mathrm{Leb}(\hat{\mathcal{R}}^{l+1}(j,k) \triangle \mathcal{R}^{l+1}(j,k)) \end{aligned}$$

where $\mathrm{Leb}(A)$ is the Lebesgue measure of a set A and $A \triangle B := (A \bigcup B) \backslash (A \bigcap B)$. $\bar{f}_t^{l+1}(j,k) = \max_{x\in\hat{\mathcal{R}}^{l+1}(j,k)\triangle\mathcal{R}^{l+1}(j,k)} (|f(x,k)| + |C(x,j,k)|)\sqrt{p_t(x,x)}m(x)$ if $\hat{\mathcal{R}}^{l+1}(j,k) \triangle \mathcal{R}^{l+1}(j,k) \ne \emptyset$ and $\bar{f}_t^{l+1}(j,k) = 0$ if $\hat{\mathcal{R}}^{l+1}(j,k) \triangle \mathcal{R}^{l+1}(j,k) = \emptyset$. The inequality follows that for all $1 \le n \le M$, $|\varphi_n(x)| \le e^{\lambda_n t/2}\sqrt{p_t(x,x)} \le e^{\lambda_M t/2}\sqrt{p_t(x,x)}$ for any $t > 0$. We specify the choice of $t$ later.

Now consider $\mathrm{Leb}(\hat{\mathcal{R}}^{l+1}(j,k) \triangle \mathcal{R}^{l+1}(j,k))$ where $\mathcal{R}^{l+1}(j,k)$, $\hat{R}^{l+1}(j,k)$ are determined by the intersection points of value functions as in Theorem 2. By our assumption on identical number of solutions, we have either (a) $\mathcal{R}^{l+1}(j,k) = \hat{R}^{l+1}(j,k) = \emptyset$ or $\mathcal{R}^{l+1}(j,k) = \hat{R}^{l+1}(j,k) = E$ or (b) $\mathcal{R}^{l+1}(j,k) = (x_l, x_r]$, $\hat{R}^{l+1}(j,k) = (\hat{x}_l, \hat{x}_r]$, which are both non-empty subsets of $E$ with finite end-points, or $\mathcal{R}^{l+1}(j,k) = (x_l, e_2)$, $\hat{R}^{l+1}(j,k) = (\hat{x}_l, e_2)$ or $\mathcal{R}^{l+1}(j,k) = (e_1, x_r]$, $\hat{R}^{l+1}(j,k) = (e_1, \hat{x}_r]$. For case (a) there is no error. For case (b), $\mathrm{Leb}(\hat{\mathcal{R}}^{l+1}(j,k) \triangle \mathcal{R}^{l+1}(j,k)) = |\hat{x}_l - x_l| + |\hat{x}_r - x_r|$ or $|\hat{x}_l - x_l|$ or $|\hat{x}_r - x_r|$. Next we estimate $|\hat{x}_l - x_l|$ assuming the left-end points are not $e_1$. The other part can be done in the same way.

Suppose $x_l$ is the intersection point of $W^{l+1}(x,j,p)$ and $W^{l+1}(x,j,q)$. It is not difficult to see that, when $M$ is large enough, $\hat{x}_l$ is the intersection point of $\hat{W}^{l+1}(x,j,p)$ and $\hat{W}^{l+1}(x,j,q)$. There exists some $\xi$ between $x_l$ and $\hat{x}_l$ such that,

$$\begin{aligned} 0 &= \hat{W}^{l+1}(\hat{x}_l,j,p) - \hat{W}^{l+1}(\hat{x}_l,j,q) \\ &= W^{l+1}(x_l,j,p) - W^{l+1}(x_l,j,q) \\ &= W^{l+1}(\hat{x}_l,j,p) - W^{l+1}(\hat{x}_l,j,q) + (\partial_x W^{l+1}(\xi,j,p) - \partial_x W^{l+1}(\xi,j,q))(x_l - \hat{x}_l). \end{aligned}$$

Recall that by assumption (2) in Theorem 5, $(\partial_x W^{l+1}(\xi,j,p) - \partial_x W^{l+1}(\xi,j,q)) \ne 0$. Hence

$$|\hat{x}_l - x_l| \le \frac{|\hat{W}^{l+1}(\hat{x}_l,j,p) - W^{l+1}(\hat{x}_l,j,p)| + |\hat{W}^{l+1}(\hat{x}_l,j,q) - W^{l+1}(\hat{x}_l,j,q)|}{|\partial_x W^{l+1}(\xi,j,p) - \partial_x W^{l+1}(\xi,j,q)|}.$$

For $M$ sufficiently large, $\bigcup_{j,k\in\mathbb{D}} \hat{\mathcal{R}}^{l+1}(j,k) \triangle \mathcal{R}^{l+1}(j,k)$ is a compact subset of $E$. Hence we can bound $1/|\partial_x W^{l+1}(\xi,j,p) - \partial_x W^{l+1}(\xi,j,q)|$. This together with our induction hypothesis on the error for the value function, implies that there exists $C > 0$ independent of $M$ such that for all $j,k \in \mathbb{D}$, $\mathrm{Leb}(\hat{\mathcal{R}}^{l+1}(j,k) \triangle \mathcal{R}^{l+1}(j,k)) \le Ce^{-\alpha_W^{l+1}\lambda_M}$. Now choosing $t$ such that $0 < t < 2\alpha_W^{l+1}$, we come to the conclusion that, there exist $C_1, \alpha_1 > 0$ ($\alpha_1 = \alpha_W^{l+1} - t/2$) independent of $M$ such that,

$$\max_{1\le n\le M} \sum_{k\in\mathbb{D}} \left\{\left|f_n^k(\hat{\mathcal{R}}^{l+1}(j,k)) - f_n^k(\mathcal{R}^{l+1}(j,k))\right| + \left|C_n^{j,k}(\hat{\mathcal{R}}^{l+1}(j,k)) - C_n^{j,k}(\mathcal{R}^{l+1}(j,k))\right|\right\} \le C_1 e^{-\alpha_1\lambda_M}.$$

Next we consider the second part of the error. For any $t > 0$,

$$\begin{aligned}
&\max_{1\le n\le M}\sum_{k\in\mathbb{D}}\left\{\sum_{m=1}^{M}\left|w_m^{l+1}(k)\right|e^{-\lambda_m h}\left|\pi_{m,n}(\hat{\mathcal{R}}^{l+1}(j,k))-\pi_{m,n}(\mathcal{R}^{l+1}(j,k))\right|\right\}\\
&\le \sum_{k\in\mathbb{D}} e^{\lambda_M t/2}e^{\lambda_M t/2}\bar{\varphi}_t^{l+1}(j,k)\mathrm{Leb}(\hat{\mathcal{R}}^{l+1}(j,k)\vartriangle \mathcal{R}^{l+1}(j,k))\left(\sum_{m=1}^{\infty}\left|w_m^{l+1}(k)\right|e^{-\lambda_m h}\right)\\
&\le d\max_{k\in\mathbb{D}}\left[e^{\lambda_M t}\bar{\varphi}_t^{l+1}(j,k)\mathrm{Leb}(\hat{\mathcal{R}}^{l+1}(j,k)\vartriangle \mathcal{R}^{l+1}(j,k))\right]\max_{k\in\mathbb{D}}\sum_{m=1}^{\infty}\left|w_m^{l+1}(k)\right|e^{-\lambda_m h},
\end{aligned}$$

where $\bar{\varphi}_t^{l+1}(j,k)=\max_{x\in\hat{\mathcal{R}}^{l+1}(j,k)\vartriangle\mathcal{R}^{l+1}(j,k)}p_t(x,x)m(x)$ if $\hat{\mathcal{R}}^{l+1}(j,k)\vartriangle\mathcal{R}^{l+1}(j,k)\neq\emptyset$ and $\bar{\varphi}_t^{l+1}(j,k)=0$ if $\hat{\mathcal{R}}^{l+1}(j,k)\vartriangle\mathcal{R}^{l+1}(j,k)=\emptyset$. As shown just before, there exists $C>0$ independent of $M$ such that $\mathrm{Leb}(\hat{\mathcal{R}}^{l+1}(j,k)\vartriangle\mathcal{R}^{l+1}(j,k))\le Ce^{-\alpha_W^{l+1}\lambda_M}$. Notice that by Cauchy–Schwarz inequality, $\max_{n\ge1,0\le l\le N,k\in\mathbb{D}}|w_n^{l+1}(k)|$ is bounded by a constant $C$, so $\sum_{m=1}^{\infty}\left|w_m^{l+1}(k)\right|e^{-\lambda_m h}\le C\sum_{m=1}^{\infty}e^{-\lambda_m h}<\infty$. Now choosing $t$ such that $0<t<\alpha_W^{l+1}$, there exist $C_2,\alpha_2>0$ ($\alpha_2=\alpha_W^{l+1}-t$) independent of $M$ such that,

$$\max_{1\le n\le M}\sum_{k\in\mathbb{D}}\left\{\sum_{m=1}^{M}\left|w_m^{l+1}(k)\right|e^{-\lambda_m h}\left|\pi_{m,n}(\hat{\mathcal{R}}^{l+1}(j,k))-\pi_{m,n}(\mathcal{R}^{l+1}(j,k))\right|\right\}\le C_2e^{-\alpha_2\lambda_M}.$$

Next we look at the third part of the error. By Cauchy–Schwarz inequality, $|\pi_{m,n}(A)|\le1$ for all $A$. Hence we have

$$\begin{aligned}
\max_{1\le n\le M}\sum_{k\in\mathbb{D}}\left\{\sum_{m=1}^{M}\left|\hat{w}_m^{l+1}(k)-w_m^{l+1}(k)\right|e^{-\lambda_m h}\left|\pi_{m,n}(\mathcal{R}^{l+1}(j,k))\right|\right\}&\le d\max_{1\le p\le M,k\in\mathbb{D}}|\hat{w}_p^{l+1}(k)-w_p^{l+1}(k)|\sum_{m=1}^{\infty}e^{-\lambda_m h}\\
&\le C_3e^{-\alpha_w^{l+1}\lambda_M},
\end{aligned}$$

for some constant $C_3>0$ independent of $M$, by the induction hypothesis for the error at $t_{l+1}$.

For the fourth part of the error, we have for any $t>0$,

$$\begin{aligned}
&\max_{1\le n\le M}\sum_{k\in\mathbb{D}}\left\{\sum_{m=1}^{M}\left|\hat{w}_m^{l+1}(k)-w_m^{l+1}(k)\right|e^{-\lambda_m h}\left|\pi_{m,n}(\hat{\mathcal{R}}^{l+1}(j,k))-\pi_{m,n}(\mathcal{R}^{l+1}(j,k))\right|\right\}\\
&\le 2d\max_{1\le m\le M,k\in\mathbb{D}}\left|\hat{w}_m^{l+1}(k)-w_m^{l+1}(k)\right|\sum_{m=1}^{\infty}e^{-\lambda_m h},\\
&\le C_4e^{-\alpha_w^{l+1}\lambda_M},
\end{aligned}$$

for some constant $C_4>0$ independent of $M$, again by the induction hypothesis.

For the fifth part of the error,

$$\begin{aligned}
\max_{1\le n\le M}\sum_{k\in\mathbb{D}}\left\{\sum_{m=M+1}^{\infty}\left|w_m^{l+1}(k)\right|e^{-\lambda_m h}\left|\pi_{m,n}(\mathcal{R}^{l+1}(j,k))\right|\right\}&\le Cd\max_{k\in\mathbb{D}}\sum_{m=M+1}^{\infty}e^{-\lambda_m h}\\
&\le C_5e^{-\alpha\lambda_M},
\end{aligned}$$

for some $C, C_5>0$ independent of $M$. The last inequality follows from Assumption 2.

Putting the five parts together, there exist $C_w^l>0$ and $0<\alpha_w^l\le\alpha$ which are independent of $M$ such that

$$\max_{1\le n\le M,j\in\mathbb{D}}\left|\hat{w}_n^l(j)-w_n^l(j)\right|\le C_w^le^{-\alpha_w^l\lambda_M}.$$

Finally consider the error in the value functions. Given a compact subset $\mathcal{C}$ of $E$, for any $0<t<2h$,

$$\begin{aligned}
&\max_{x\in\mathcal{C},i,j\in\mathbb{D}}\left|\hat{W}^l(x,i,j)-W^l(x,i,j)\right|\\
&\le\max_{x\in\mathcal{C},i,j\in\mathbb{D}}\left[\sum_{n=1}^{M}\left|\hat{w}_n^l(j)-w_n^l(j)\right|e^{-\lambda_n h}|\varphi_n(x)|+\sum_{n=M+1}^{\infty}\left|w_n^l(j)\right|e^{-\lambda_n h}|\varphi_n(x)|\right]\\
&\le\max_{1\le n\le M,j\in\mathbb{D}}\left|\hat{w}_n^l(j)-w_n^l(j)\right|\sum_{m=1}^{\infty}e^{-\lambda_m(h-t/2)}\max_{x\in\mathcal{C}}\sqrt{p_t(x,x)}+C\max_{x\in\mathcal{C}}\sqrt{p_t(x,x)}\sum_{n=M+1}^{\infty}e^{-\lambda_n(h-t/2)}\\
&\le C'e^{-\alpha_w^l\lambda_M}+C''e^{-\alpha\lambda_M}\quad\text{(by the error for coefficients and Assumption 2)}\\
&\le C_W^le^{-\alpha_W^l\lambda_M},
\end{aligned}$$

for some $C, C', C'', C_W^l, \alpha_W^l>0$ ($\alpha_W^l\le\alpha_w^l\le\alpha$) independent of $M$. This concludes the proof of the theorem. □

**Proposition 3.** The OU transition semigroup satisfies condition (i) (see proof of Proposition 3 in [25]). Since $f(x, j) \in L^2(\mathbb{R}, \mathfrak{m})$ for all $j$, from Theorem 3, condition (ii) is also satisfied. Conditions (iii)–(v) clearly hold. Hence Theorem 2 implies (21) with $x_H^l \geq x_L^l$. Note that $x_L^l = \inf\{x : W^l(x, 1, 1) - W^l(x, 1, 0) > 0\}$, $x_H^l = \inf\{x : W^l(x, 0, 1) - W^l(x, 0, 0) > 0\} = \inf\{x : W^l(x, 1, 1) - W^l(x, 1, 0) > -C(1, 1) + C(1, 0) + C(0, 1) - C(0, 0)\}$. $W^l(x, i, 1) - W^l(x, i, 0)$ $(i = 0, 1)$ is increasing and tend to $\infty$ $(-\infty)$ as $x \to \infty$ $(-\infty)$ because $f(x, 1) - C(i, 1) - f(x, 0) + C(i, 0)$ has such property. Hence $x_L^l$ and $x_H^l$ are finite. We have $x_H^l > x_L^l$ because $-C(1, 1) + C(1, 0) + C(0, 1) - C(0, 0) > 0$. □

**Proposition 4.** Since $f(x, 0) = 0$, we have $f_n^0(\mathcal{R}^l(i, 0)) = 0$ for $i \in \{0, 1\}$.

$$f_n^1(R^l(i, 1)) = \int_{R^l(i,1)} xh\varphi_n(x)\mathfrak{m}(dx) = \int_{R^l(i,1)} (x - \theta)h\varphi_n(x)\mathfrak{m}(dx) + \int_{R^l(i,1)} \theta h\varphi_n(x)\mathfrak{m}(dx).$$

In the first part,

$$\int_{R^l(i,1)} \theta h\varphi_n(x)m(dx) = \theta h \int_{R^l(i,1)} \varphi_0(x)\varphi_n(x)m(dx) = \theta h\pi_{0,n}(R^l(i, 1)).$$

In the second part, since $R^l(0, 1) = (x_H^l, \infty)$, $R^l(1, 1) = (x_L^l, \infty)$, we just need to consider $\int_x^\infty (y - \theta)h\varphi_n(y)m(dy)$. We first recall for Hermite polynomials, $\frac{d}{dx}H_n(x) = 2nH_{n-1}(x)(n \geq 1)$ and $\frac{d}{dx}H_0(x) = 0$. Using the expression for $\varphi_n(x)$, we obtain $\frac{d}{dx}\varphi_n(x) = \frac{\sqrt{2n\kappa}}{\sigma}\varphi_{n-1}(x)(n \geq 1)$ and $\frac{d}{dx}\varphi_0(x) = 0$. We also have $\frac{d}{dx}\mathfrak{m}(x) = -\frac{2\kappa(x-\theta)}{\sigma^2}\mathfrak{m}(x)$. Thus

$$\int_x^\infty (y - \theta)h\varphi_n(y)\mathfrak{m}(dy) = \int_x^\infty -\frac{h\sigma^2}{2\kappa}\varphi_n(y)d\mathfrak{m}(y) = \frac{h\sigma^2}{2\kappa}\varphi_n(x)\mathfrak{m}(x) + \int_x^\infty \frac{h\sigma^2}{2\kappa}\mathfrak{m}(y)d\varphi_n(y).$$

For $n = 0$, $\int_x^\infty (y - \theta)h\varphi_0(y)\mathfrak{m}(dy) = \frac{h\sigma^2}{2\kappa}\varphi_0(y)\mathfrak{m}(y)$. For $n \geq 1$:

$$\begin{aligned}\int_x^\infty (y - \theta)h\varphi_n(y)m(dy) &= \frac{h\sigma^2}{2\kappa}\varphi_n(x)\mathfrak{m}(x) + \int_x^\infty \frac{h\sigma^2}{2\kappa}\frac{\sqrt{2n\kappa}}{\sigma}\varphi_{n-1}(y)\mathfrak{m}(dy) \\ &= \frac{h\sigma^2}{2\kappa}\varphi_n(x)\mathfrak{m}(x) + \sigma h\sqrt{\frac{n}{2\kappa}}\pi_{0,n-1}(x, \infty).\end{aligned}$$

Combining the result for the first and the second part we get the formula for $f_n(1, \mathcal{R}^l(i, 1))$. For $C_n^{i,j}(\mathcal{R}^l(i, j))$:

$$C_n^{i,j}(\mathcal{R}^l(i, j)) = \int_{\mathcal{R}^l(i,j)} C(i, j)\varphi_n(x)\mathfrak{m}(dx) = C(i, j)\int_{\mathcal{R}^l(i,j)} \varphi_0(x)\varphi_n(x)\mathfrak{m}(dx) = C(i, j)\pi_{0,n}(\mathcal{R}^l(i, j)).\quad\square$$

**Proposition 5.** Since $x > x_0$,

$$\begin{aligned}p_n(x, \infty) &= \int_x^\infty (1 - (1 + hG)A(h)e^{-B(h)y})\varphi_n(y)m(dy) \\ &= \int_x^\infty \varphi_n(y)m(dy) - (1 + hG)A(h)\int_x^\infty e^{-B(h)y}\varphi_n(y)m(dy) \\ &= \rho_n(0, x) - (1 + hG)A(h)\rho_n(-B(h), x).\end{aligned}$$

The formula for $\rho_0(s, x)$ can be directly obtained by integrating with the Gaussian density. For $n \geq 1$, by change of variable, we obtain

$$\rho_n(s, x) = \frac{e^{s\left(\theta - \frac{\sigma a}{\sqrt{\kappa}}\right) - \frac{a^2}{2}}}{\sqrt{\pi 2^n n!}} b_n\left(s\frac{\sigma}{\sqrt{\kappa}} + a, \frac{\sqrt{\kappa}}{\sigma}(x - \theta) + a\right), \quad b_n(s, x) := \int_x^\infty e^{sy - y^2}H_n(y)dy. \tag{A.1}$$

Using $\frac{d}{dx}\left(e^{-x^2}H_n(x)\right) = -e^{-x^2}H_{n+1}(x)$, we have

$$b_n(s, x) = \int_x^\infty e^{sy}d(-e^{-y^2}H_{n-1}(y)) = e^{sx - x^2}H_{n-1}(x) + sb_{n-1}(s, x).$$

Using (A.1), we obtain the recursion for $\rho_n(s, x)$ which is (23). □

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