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PII: S0377-0427(17)30494-6
DOI: https://doi.org/10.1016/j.cam.2017.10.004
Reference: CAM 11333

To appear in: *Journal of Computational and Applied Mathematics*

Received date : 30 September 2016
Revised date : 4 August 2017

Please cite this article as: B. Bylina, The block WZ factorization, *Journal of Computational and Applied Mathematics* (2017), https://doi.org/10.1016/j.cam.2017.10.004

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The Block WZ Factorization

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Abstract

In the paper the author presents a novel kind of the WZ factorization algorithm, namely a block WZ factorization algorithm. The aim of this new algorithm is to utilize the computational power of contemporary computers with hierarchical memory. In the paper, some properties of the matrix Z are given and analyzed. Next, a version of the block WZ factorization is presented. The author shows that such a block WZ factorization exists for strictly diagonally dominant matrices. The computational cost of this block algorithm is presented. The time and the accuracy of proposed block WZ factorization algorithm for random dense square diagonally dominant matrices are reported. The block algorithm turned out to be faster even up to 300 times than the original WZ factorization.

Keywords: WZ factorization, block algorithm, solution of linear systems, Quadrant Interlocking Factorization, memory hierarchy

1. Introduction

The solution of linear systems of the form:

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \qquad \text{where} \quad \mathbf{A} \in \mathbb{R}^{n\times n}, \quad \mathbf{b} \in \mathbb{R}^{n}, \tag{1}$$

is an important and common problem in engineering and scientific computations.

The matrix $\mathbf{A}$ is dense, non-singular, square, and (for our consideration) of an even size (without loss of generality).

One of the direct methods of solving a dense linear system (1) is to factorize the matrix $\mathbf{A}$ into some simpler matrices — it is its decomposition

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into factor matrices of a simpler structure or of some specific properties — and then solving simpler linear systems. The most known factorization is the LU factorization. In this work we study another form of the factorization, namely the WZ factorization.

The WZ factorization was first introduced in [10]. It was a novel method for solving linear systems in parallel, for SIMD (*Single Instruction, Multiple Data*, [11]) computers. It was an alternative factorization to the LU one and was originally named Quadrant Interlocking Factorization method (QIF). The method was not implemented for real parallel machines. However, a theorem that strictly diagonally dominant matrices do not require pivoting was proven. In [6] the existence and uniqueness of the WZ factorization are proven and some examples (and proofs) of matrices which do not require any pivoting are given. A theorem about the WZ factorization existence without pivoting for symmetric positive-defined matrices and for diagonally dominant matrices is also given.

In [12], an adaptation and implementation of the WZ algorithm for SIMD computers with the hypercube topology and the distributed memory is described. The results presented in that article show that the WZ factorization is faster than the LU factorization. In [14], a parallel WZ factorization is presented and a couple of pivoting techniques are considered. In that article, it is shown that a parallel implementation of the WZ factorization can be faster than the LU factorization. In [9], four parallel implementations for solving linear systems are studied with the use of BSP (*Bulk Synchronous Parallel*) programming model. One of the algorithms is the WZ factorization for matrices of even sizes. The result of that paper is (among others) the information that the implementation of the WZ factorization with the use of the BSP model for a computer cluster is about 30%–40% faster comparing with the LU factorization. In [2, 3, 4] the authors showed that there are matrices for which applying the incomplete WZ preconditioning gives better results than the incomplete LU factorization and they also presented the use of the WZ factorization for solving Markovian models.

The authors of [8] describe a block partition of the WZ factorization for tridiagonal matrices. Some theoretical statements about its existence are given. A parallel algorithm is also presented, however without any implementation or experimental results.

In this paper we present a block version of the WZ factorization for general dense matrices. Through the reorganization of the computation sequence we can obtain a block algorithm, the implementation of which (although more

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complex than the traditional one) can be much faster. In the block algorithm, the basic scalar operations were changed to matrix operations, for instance: $\alpha + \beta \to \mathbf{A} + \mathbf{B}$, $\alpha \cdot \beta \to \mathbf{A} \cdot \mathbf{B}$, $\alpha/\beta \to \mathbf{A} \cdot \mathbf{B}^{-1}$. The block LU factorization [5] is one of the practical ways to use the block techniques.

The efficiency of a computer depends on an effective use of the memory hierarchy because nowadays the cost (that is, the time) of the computations depends on both the number of arithmetic operations and the time of transferring the data between different memory levels. We want to perform as many computations as possible on the data present in the fastest available memory to reduce the cost of transmitting them between different levels of memory.

Without a sufficiently fast memory, the processor will be waiting for the data through the most of time (instead of computing), and its efficiency will decrease considerably. Nowadays, the struggle of numerical optimization is to ensure that the processor has enough data to operate on them all the time. It is particularly apparent in algorithms with a lot of operations conducted on abundant data — and takes place in numerical algebra algorithms.

Here we want to modify the WZ solver to use the potential of contemporary computers equipped with hierarchical memory. The aim of the paper is to build a block version of the WZ factorization and show the computational cost of such a block algorithm, as well as the examples of matrices for which such a factorization exists. We also conduct some experiments to compare our block WZ factorization algorithm with the original one.

The contents of the paper are as follows. In Section 2, we describe the idea of the WZ factorization [6, 14] and the way the matrix $\mathbf{A}$ is factorized to a product of matrices $\mathbf{W}$ and $\mathbf{Z}$. Such a factorization exists for every nonsingular matrix (with pivoting) which was shown in [6]. Additionally, some properties of the matrix $\mathbf{Z}$ of an even size are presented. Section 3 describes the block WZ factorization (called here BWZ) with the division of the matrix $\mathbf{A}$ into r^2 blocks. Section 4 shows the existence of the factorization for diagonally dominant matrices. Section 5 describes the computational cost of the proposed algorithm. In Section 6, we present the results of our experiments. We analyze the time of the algorithms WZ and BWZ and study the influence of the size of the matrix on the achieved numerical accuracy. Section 7 is the summary of our research.

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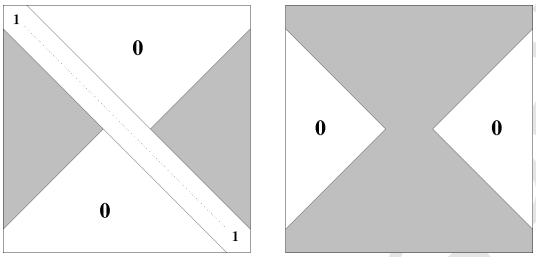


Figure 1: Structures of the matrices $\mathbf{W}$ (left) and $\mathbf{Z}$ (right)

2. WZ factorization

In this section we describe shortly the usage of the WZ factorization to solve (1). The WZ factorization is described in [6, 10, 14]. Assume that $\mathbf{A}$ is an $n \times n$ nonsingular matrix (and n is even). We want to find matrices $\mathbf{W}$ and $\mathbf{Z}$ that fulfill $\mathbf{WZ} = \mathbf{A}$ and the matrices $\mathbf{W}$ and $\mathbf{Z}$ have the structure shown in Figure 1 where, gray fields are non-zeros. The matrix $\mathbf{W}$ is non-singular with $\det \mathbf{W} = 1$. Thus, $\det \mathbf{A} = \det \mathbf{Z}$.

After the factorization we can solve two linear systems:

$$\begin{aligned} \mathbf{Wc} &= \mathbf{b}, \\ \mathbf{Zx} &= \mathbf{c} \end{aligned} \tag{2}$$

(where $\mathbf{c}$ is an auxiliary intermediate vector) instead of one (1).

In this paper we are only interested in obtaining the matrices $\mathbf{Z}$ and $\mathbf{W}$. The first part of the algorithm consists of setting successive parts of columns of the matrix $\mathbf{A}$ to zeros. In the first step we do that with the elements in columns 1st and nth — from the 2nd row to the $n-1$st row. Next we update the matrix $\mathbf{A}$.

More formally we can describe the first step of the algorithm in the following way.

1. For every $i = 2, \ldots, n-1$ we compute w_{i1} and w_{in} from the system:

$$\left\{ \begin{array}{rcrcl} a_{11}w_{i1} & + & a_{n1}w_{in} & = & a_{i1} \\ a_{1n}w_{i1} & + & a_{nn}w_{in} & = & a_{in} \end{array} \right.$$

and we put them in a matrix of the form:

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$$\mathbf{W}^{(1)} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -w_{21} & 1 & \ddots & \vdots & -w_{2n} \\ \vdots & 0 & \ddots & 0 & \vdots \\ -w_{n-1,1} & \vdots & \ddots & 1 & -w_{n-1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

The inverse of this matrix is:

$$\left(\mathbf{W}^{(1)}\right)^{-1} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ w_{21} & 1 & \ddots & \vdots & w_{2n} \\ \vdots & 0 & \ddots & 0 & \vdots \\ w_{n-1,1} & \vdots & \ddots & 1 & w_{n-1,n} \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

2. We compute:

$$\mathbf{A}^{(1)} = \mathbf{W}^{(1)}\mathbf{A}.$$

After the first step we get a matrix of the form:

$$\mathbf{A}^{(1)} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1,n-1} & a_{1n} \\ 0 & a_{22}^{(1)} & \cdots & a_{2,n-1}^{(1)} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{n-1,2}^{(1)} & \cdots & a_{n-1,n-1}^{(1)} & 0 \\ a_{n1} & a_{n2} & \cdots & a_{n,n-1} & a_{nn} \end{bmatrix}, \tag{3}$$

where (for $i, j = 2, \ldots, n-1$):

$$a_{ij}^{(1)} = a_{ij} - w_{i1}a_{1j} - w_{in}a_{nj}. \tag{4}$$

Then, we proceed analogously — but for the inner $(n-2)\times(n-2)$ square matrix — $\mathbf{A}^{(1)}$ — and so on.

So, the whole algorithm is as follows:

For $k = 1, 2, \ldots, \frac{n}{2} - 1$:

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$$\mathbf{W}^{(k)} = \begin{bmatrix} 1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & 1 & & & & & & \\ & & -w_{k+1,k} & \ddots & & & -w_{k+1,n-k+1} & & \\ & & \vdots & & \ddots & & \vdots & & \\ & & -w_{n-k,k} & & & \ddots & -w_{n-k,n-k+1} & & \\ & & & & & & 1 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \end{bmatrix}$$

Figure 2: The matrix $\mathbf{W}^{(k)}$

1. For every $i = k+1, \ldots, n-k$ we compute w_{ik} and $w_{i,n-k+1}$ from the system:

$$\begin{cases} a_{kk}^{(k-1)} w_{ik} + a_{n-k+1,k}^{(k-1)} w_{i,n-k+1} = a_{ik}^{(k-1)} \\ a_{k,n-k+1}^{(k-1)} w_{ik} + a_{n-k+1,n-k+1}^{(k-1)} w_{i,n-k+1} = a_{i,n-k+1}^{(k-1)} \end{cases}$$

   and we put them in a matrix of the form shown in Figure 2.
2. We compute:

$$\mathbf{A}^{(k)} = \mathbf{W}^{(k)} \mathbf{A}^{(k-1)}.$$

After $(\frac{n}{2} - 1)$ steps we get the matrix

$$\mathbf{Z} = \mathbf{A}^{(\frac{n}{2}-1)}.$$

Moreover, we know that:

$$(\mathbf{W}^{(\frac{n}{2}-1)}) \cdot \ldots \cdot (\mathbf{W}^{(1)}) \cdot \mathbf{A} = \mathbf{Z},$$

so we get:

$$\mathbf{A} = (\mathbf{W}^{(1)})^{-1} \cdot \ldots \cdot (\mathbf{W}^{(\frac{n}{2}-1)})^{-1} \cdot \mathbf{Z} = \mathbf{W}\mathbf{Z}.$$

The computational cost of the WZ factorization algorithm can be described with a formula:

$$\sum_{k=1}^{\frac{n}{2}-1} (3 + \sum_{i=k+1}^{n-k} (8 + \sum_{j=k+1}^{n-k} 4)) = \frac{4n^3 - 7n - 18}{6}. \tag{5}$$

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2.1. Properties of the matrix $\mathbf{Z}$

Let $\mathbf{Z}$ be a $n \times n$ matrix (for an even n) of the form:

$$\mathbf{Z} = \begin{bmatrix} f_1 & * & \dots & * & * & \dots & * & c_1 \\ & f_2 & \dots & * & * & \dots & c_2 & \\ & & \ddots & * & * & \iddots & & \\ & & & f_{\frac{n}{2}} & c_{\frac{n}{2}} & & & \\ & & & b_{\frac{n}{2}} & d_{\frac{n}{2}} & & & \\ & & \iddots & * & * & \ddots & & \\ & b_2 & \dots & * & * & \dots & d_2 & \\ b_1 & * & \dots & * & * & \dots & * & d_1 \end{bmatrix} \tag{6}$$

where $*$ denotes nonzeros.

Theorem 1. *The determinant of the matrix* $\mathbf{Z}$ *is given by the formula:*

$$\det \mathbf{Z} = \prod_{i=1}^{\frac{n}{2}} (f_i d_i - c_i b_i).$$

Proof. We use the Laplace formula to find the determinant of the matrix $\mathbf{Z}$. We do the Laplace expansion along the first column and we get determinants of $(n-1) \times (n-1)$ matrices. Then we expand all of these

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determinants along the last column:

$$\det \mathbf{Z} = \det \begin{bmatrix} f_1 & * & \dots & * & * & \dots & * & c_1 \\ & f_2 & \dots & * & * & \dots & c_2 & \\ & & \ddots & * & * & \iddots & & \\ & & & f_{\frac{n}{2}} & c_{\frac{n}{2}} & & & \\ & & & b_{\frac{n}{2}} & d_{\frac{n}{2}} & & & \\ & & \iddots & * & * & \ddots & & \\ & b_2 & \dots & * & * & \dots & d_2 & \\ b_1 & * & \dots & * & * & \dots & * & d_1 \end{bmatrix} =$$

$$= (f_1 d_1 - c_1 b_1) \cdot \det \begin{bmatrix} f_2 & \dots & * & * & \dots & c_2 \\ & \ddots & * & * & \iddots & \\ & & f_{\frac{n}{2}} & c_{\frac{n}{2}} & & \\ & & b_{\frac{n}{2}} & d_{\frac{n}{2}} & & \\ & \iddots & * & * & \ddots & \\ b_2 & \dots & * & * & \dots & d_2 \end{bmatrix} = \tag{7}$$

$$= (f_1 d_1 - c_1 b_1)(f_2 d_2 - c_2 b_2) \cdot \ldots \cdot (f_{\frac{n}{2}} d_{\frac{n}{2}} - c_{\frac{n}{2}} b_{\frac{n}{2}}) =$$

$$= \prod_{i=1}^{\frac{n}{2}} (f_i d_i - c_i b_i).$$

■

Corollary 1. *If* $\forall_{i=1,\dots,\frac{n}{2}} (f_i d_i - c_i b_i) \neq 0$*, then the matrix* $\mathbf{Z}$ *is nonsingular.*

Divide the matrix $\mathbf{Z}$ into 4 square $\frac{n}{2} \times \frac{n}{2}$ blocks in the following manner:

$$\mathbf{Z} = \begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} \\ \mathbf{Z}_{21} & \mathbf{Z}_{22} \end{bmatrix}, \tag{8}$$

where $\mathbf{Z}_{11}, \mathbf{Z}_{12}, \mathbf{Z}_{21}, \mathbf{Z}_{22}$ are these $\frac{n}{2} \times \frac{n}{2}$ matrices. For the matrix $\mathbf{Z}$ define the Schur complement [13].

Definition 1. *If the matrix* $\mathbf{Z}_{11}$ *is invertible, the Schur complement of the block* $\mathbf{Z}_{11}$ *of the matrix* $\mathbf{Z}$ *is the matrix of the form:*

$$\mathbf{Z}_{22} - \mathbf{Z}_{21} \mathbf{Z}_{11}^{-1} \mathbf{Z}_{12}$$

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and denote:

$$[\mathbf{Z}/\mathbf{Z}_{11}].$$

For the matrix $\mathbf{Z}$ we can write (using [15]):

$$\det \mathbf{Z} = \det \mathbf{Z}_{11} \det[\mathbf{Z}/\mathbf{Z}_{11}] = \det \mathbf{Z}_{11} \det(\mathbf{Z}_{22} - \mathbf{Z}_{21}\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12}). \tag{9}$$

The properties of the aforementioned (8) submatrices of the matrix $\mathbf{Z}$ are:

1. $\det \mathbf{Z}_{11} = f_1 f_2 \cdot \ldots \cdot f_{\frac{n}{2}}$.
2. If $\forall_{i=1,\ldots,\frac{n}{2}} f_i \neq 0$ then there exists an inverse of the matrix $\mathbf{Z}_{11}$ and it has the following form:

$$\mathbf{Z}_{11}^{-1} = \begin{bmatrix} \frac{1}{f_1} & * & \ldots & * \\ 0 & \frac{1}{f_2} & \ddots & * \\ \vdots & \ddots & \ddots & * \\ 0 & \ldots & 0 & \frac{1}{f_{\frac{n}{2}}} \end{bmatrix}. \tag{10}$$

Theorem 2. *If the matrix* $\mathbf{Z}$ *and the matrix* $\mathbf{Z}_{11}$ *are invertible then the matrix* $(\mathbf{Z}_{22} - \mathbf{Z}_{21}\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12})$ *is a lower triangular invertible matrix.*

Proof. With the use of the formula (10) which gives the form of the inverse of the matrix $\mathbf{Z}_{11}$ we can see that the matrix product $\mathbf{Z}_{21}\mathbf{Z}_{11}^{-1}$ is a triangular matrix of the form:

$$\mathbf{Z}_{21}\mathbf{Z}_{11}^{-1} = \begin{bmatrix} 0 & \ldots & 0 & \frac{b_{\frac{n}{2}}}{f_{\frac{n}{2}}} \\ \vdots & \iddots & \iddots & * \\ 0 & \frac{b_2}{f_2} & \iddots & * \\ \frac{b_1}{f_1} & * & \ldots & * \end{bmatrix}. \tag{11}$$

The product $\mathbf{Z}_{21}\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12}$ is a lower triangular matrix of the form:

$$\mathbf{Z}_{21}\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12} = \begin{bmatrix} \frac{b_{\frac{n}{2}} c_{\frac{n}{2}}}{f_{\frac{n}{2}}} & 0 & \ldots & 0 \\ & \ddots & \ddots & \vdots \\ & \ldots & \frac{b_2 c_2}{f_2} & 0 \\ & \ldots & * & \frac{b_1 c_1}{f_1} \end{bmatrix}. \tag{12}$$

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The difference of the lower triangular matrix $\mathbf{Z}_{22}$ and the lower triangular matrix of the form (12) — that is $\mathbf{Z}_{22}-\mathbf{Z}_{21}\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12}$ — gives a lower triangular matrix of the form:

$$\mathbf{Z}_{22}-\mathbf{Z}_{21}\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12}=\begin{bmatrix} d_{\frac{n}{2}}-\frac{b_{\frac{n}{2}}c_{\frac{n}{2}}}{f_{\frac{n}{2}}} & 0 & \dots & 0 \\ & \ddots & \ddots & \vdots \\ & \dots & d_2-\frac{b_2c_2}{f_2} & 0 \\ & \dots & * & d_1-\frac{b_1c_1}{f_1} \end{bmatrix}. \tag{13}$$

The determinant of the matrix $\mathbf{Z}_{22}-\mathbf{Z}_{21}\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12}$ is given by a formula:

$$\begin{aligned}\det(\mathbf{Z}_{22}-\mathbf{Z}_{21}\mathbf{Z}_{11}^{-1}\mathbf{Z}_{12}) &= \left(d_1-\frac{b_1c_1}{f_1}\right)\left(d_2-\frac{b_2c_2}{f_2}\right)\cdot\ldots\cdot\left(d_{\frac{n}{2}}-\frac{b_{\frac{n}{2}}c_{\frac{n}{2}}}{f_{\frac{n}{2}}}\right)\\ &= \frac{\det\mathbf{Z}}{\det\mathbf{Z}_{11}}. \end{aligned}\tag{14}$$

The determinant (14) is not equal to zero if $\forall_{i=1,\dots,\frac{n}{2}}(d_if_i-b_ic_i\neq 0)$ and $\forall_{i=1,\dots,\frac{n}{2}}(f_i\neq 0)$ — and it is true because the matrices $\mathbf{Z}$ and $\mathbf{Z}_{11}$ are invertible (see Corollary 1). It is also in line with (9). ■

3. Block WZ factorization (BWZ)

Let $\mathbf{A}$ be an $n\times n$ nonsingular matrix where n is a multiple of an even integer r. The elements of the matrix $\mathbf{A}$ are divided into r^2 blocks in the following manner:

$$\mathbf{A}=\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \dots & \mathbf{A}_{1r} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \dots & \mathbf{A}_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{A}_{r1} & \mathbf{A}_{r2} & \dots & \mathbf{A}_{rr} \end{bmatrix}. \tag{15}$$

The matrices $\mathbf{A}_{ij}$ (where $i,j=1,\dots,r$) are $s\times s$ matrices and $n=r\cdot s$. Thus:

$$\mathbf{A}_{ij}=\begin{bmatrix} a_{(i-1)s+1,(j-1)s+1} & \dots & a_{(i-1)s+1,js} \\ \dots & \vdots & \dots \\ a_{is,(j-1)s+1} & \dots & a_{is,js} \end{bmatrix}. \tag{16}$$

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The matrix $\mathbf{W}$ divided into r^2 blocks (each of the size $s \times s$) is of the following form:

$$\mathbf{W} = \begin{bmatrix} \mathbf{W}_{11} & \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{W}_{1r} \\ \vdots & \ddots & \mathbf{0} & \mathbf{0} & \ddots & \dots \\ \mathbf{W}_{\frac{r}{2},1} & \dots & \mathbf{W}_{\frac{r}{2},\frac{r}{2}} & \mathbf{W}_{\frac{r}{2},\frac{r}{2}+1} & \dots & \mathbf{W}_{\frac{r}{2},r} \\ \mathbf{W}_{\frac{r}{2}+1,1} & \dots & \mathbf{W}_{\frac{r}{2}+1,\frac{r}{2}} & \mathbf{W}_{\frac{r}{2}+1,\frac{r}{2}+1} & \dots & \mathbf{W}_{\frac{r}{2}+1,r} \\ \vdots & \iddots & \mathbf{0} & \mathbf{0} & \ddots & \dots \\ \mathbf{W}_{r1} & \mathbf{0} & \dots & \dots & \mathbf{0} & \mathbf{W}_{rr} \end{bmatrix}. \tag{17}$$

The matrices $\mathbf{W}_{ii}$ (for $i = 1, \dots, \frac{r}{2}$) which are on the main diagonal of $\mathbf{W}$ are lower triangular matrices with ones on their main diagonals and with $\det \mathbf{W}_{ii} = 1$ and they have the following form:

$$\mathbf{W}_{ii} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ w_{(i-1)s+2,(i-1)s+1} & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ w_{is,(i-1)s+1} & \dots & w_{is,is-1} & 1 \end{bmatrix}. \tag{18}$$

The matrices $\mathbf{W}_{ii}$ (for $i = (\frac{r}{2}+1), \dots, r$) which are on the main diagonal of $\mathbf{W}$ are upper triangular matrices with ones on their main diagonals and with $\det \mathbf{W}_{ii} = 1$ and they have the following form:

$$\mathbf{W}_{ii} = \begin{bmatrix} 1 & w_{(i-1)s+1,(i-1)s+2} & \dots & w_{(i-1)s+2,is} \\ 0 & 1 & \dots & w_{(i-1)s+3,is} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 \end{bmatrix}. \tag{19}$$

The matrices $\mathbf{W}_{ij}$ (for $i+j = (r+1)$ and $i = 1, \dots, \frac{r}{2}$ and $j = r, \dots, (\frac{r}{2}+1)$) which are on the secondary diagonal of $\mathbf{W}$ are triangular matrices with zeros on their secondary diagonals and with $\det \mathbf{W}_{ij} = 0$ and they have the following form:

$$\mathbf{W}_{ij} = \begin{bmatrix} 0 & \dots & \dots & 0 \\ 0 & \dots & 0 & w_{(i-1)s+2,js} \\ \vdots & \iddots & \iddots & \vdots \\ 0 & w_{is,(j-1)s+2} & \dots & w_{is,js} \end{bmatrix}. \tag{20}$$

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The matrices $\mathbf{W}_{ij}$ (for $i+j=(r+1)$ and $i=(\frac{r}{2}+1),\ldots,r$ and $j=\frac{r}{2},\ldots,1$) which are on the secondary diagonal of $\mathbf{W}$ are triangular matrices with zeros on their secondary diagonals and with $\det \mathbf{W}_{ij}=0$ and they have the following form:

$$\mathbf{W}_{ij}=\begin{bmatrix} w_{(i-1)s+1,(j-1)s+1} & \cdots & w_{(i-1)s+1,js-1} & 0 \\ \vdots & \iddots & \iddots & \vdots \\ & & & \vdots \\ w_{is-1,(j-1)s+1} & 0 & \cdots & \vdots \\ 0 & \cdots & \cdots & 0 \end{bmatrix}. \tag{21}$$

The matrices $\mathbf{W}_{ij}$ (for $(i>j)$ and $(i<r-j+1)$ or $(i<j)$ and $(i>r-j+1)$ where $i,j=1,\ldots,r$) are dense matrices with the following form:

$$\mathbf{W}_{ij}=\begin{bmatrix} w_{(i-1)s+1,(j-1)s+1} & \cdots & \cdots & w_{(i-1)s+1,js} \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ w_{is,(j-1)s+1} & \cdots & \cdots & w_{is,js} \end{bmatrix}. \tag{22}$$

The matrix $\mathbf{Z}$ is divided into r^2 blocks — each of the size $s\times s$ — and it has the following form:

$$\mathbf{Z}=\begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} & \cdots & \cdots & \mathbf{Z}_{1,(r-1)} & \mathbf{Z}_{1r} \\ \mathbf{0} & \ddots & \cdots & \cdots & \iddots & \mathbf{0} \\ \vdots & \vdots & \mathbf{Z}_{\frac{r}{2},\frac{r}{2}} & \mathbf{Z}_{\frac{r}{2},\frac{r}{2}+1} & \vdots & \vdots \\ \vdots & \vdots & \mathbf{Z}_{\frac{r}{2}+1,\frac{r}{2}} & \mathbf{Z}_{\frac{r}{2}+1,\frac{r}{2}+1} & \vdots & \vdots \\ \mathbf{0} & \iddots & \cdots & \cdots & \ddots & \mathbf{0} \\ \mathbf{Z}_{r1} & \mathbf{Z}_{r2} & \cdots & \cdots & \mathbf{Z}_{r,(r-1)} & \mathbf{Z}_{rr} \end{bmatrix}. \tag{23}$$

The matrices $\mathbf{Z}_{ii}$ (for $i=1,\ldots,\frac{r}{2}$) which are on the main diagonal of the matrix $\mathbf{Z}$ are upper triangular matrices and have the following form:

$$\mathbf{Z}_{ii}=\begin{bmatrix} z_{(i-1)s+1,(i-1)s+1} & z_{(i-1)s+1,(i-1)s+2} & \cdots & z_{(i-1)s+1,is} \\ 0 & z_{(i-1)s+2,(i-1)s+2} & \ddots & z_{(i-1)s+2,is} \\ 0 & 0 & \ddots & \vdots \\ 0 & \cdots & 0 & z_{is,is} \end{bmatrix}. \tag{24}$$

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The matrices $\mathbf{Z}_{ii}$ (for $i = \frac{r}{2}+1, \ldots, r$) which are on the main diagonal of the matrix $\mathbf{Z}$ are lower triangular matrices and have the following form:

$$\mathbf{Z}_{ii} = \begin{bmatrix} z_{(i-1)s+1,(i-1)s+1} & 0 & \cdots & 0 \\ z_{(i-1)s+2,(i-1)s+1} & z_{(i-1)s+2,(i-1)s+2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ z_{is,(i-1)s+1} & z_{is,(i-1)s+2} & \cdots & z_{is,is} \end{bmatrix}. \tag{25}$$

The matrices $\mathbf{Z}_{ij}$ (for $i+j = r+1$ and $i = 1, \ldots, \frac{r}{2}$ and $j = r, \ldots, (\frac{r}{2}+1)$) which are on the secondary diagonal of $\mathbf{Z}$ are triangular matrices of the following form:

$$\mathbf{Z}_{ij} = \begin{bmatrix} z_{(i-1)s+1,(j-1)s+1} & \cdots & z_{(i-1)s+1,js-1} & z_{(i-1)s+1,js} \\ \vdots & \vdots & \iddots & 0 \\ z_{is-1,(j-1)s+1} & z_{is-1,(j-1)s+2} & 0 & \vdots \\ z_{is,(j-1)s+1} & 0 & \cdots & 0 \end{bmatrix}. \tag{26}$$

The matrices $\mathbf{Z}_{ij}$ (for $i+j = r+1$ and $i = (\frac{r}{2}+1), \ldots, r$ and $j = \frac{r}{2}, \ldots, 1$) which are on the secondary diagonal of $\mathbf{Z}$ are triangular matrices of the following form:

$$\mathbf{Z}_{ij} = \begin{bmatrix} 0 & \cdots & 0 & z_{(i-1)s+1,js} \\ 0 & \cdots & \iddots & \vdots \\ \vdots & \iddots & \vdots & z_{is-1,js} \\ z_{is,(j-1)s+1} & z_{is,(j-1)s+2} & \cdots & z_{is,js} \end{bmatrix}. \tag{27}$$

The matrices $\mathbf{Z}_{ij}$ (for $(i < j)$ and $(i < r-j+1)$ or $(i > j)$ and $(i > r-j+1)$ where $i, j = 1, \ldots, r$) are dense matrices of the following form:

$$\mathbf{Z}_{ij} = \begin{bmatrix} z_{(i-1)s+1,(j-1)s+1} & \cdots & \cdots & z_{(i-1)s+1,js} \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ z_{is,(j-1)s+1} & \cdots & \cdots & z_{is,js} \end{bmatrix}. \tag{28}$$

If the matrix $\mathbf{A}$ is a nonsingular one of an even size and there exists its WZ factorization:

$$\mathbf{A} = \mathbf{WZ}, \tag{29}$$

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then there also exists its block WZ factorization shown below.

Here we present the kth step of an algorithm which finds such a block WZ factorization. We perform the below shown steps for $k = 1, \ldots, \frac{r}{2}$ — each step consists of four stages which can be generally described in the following manner:

3.1. Stage 1

In the first stage of the kth step of the block WZ factorization of the matrix $\mathbf{A}$, we create a new matrix $\widetilde{\mathbf{A}} \in \mathbb{R}^{2s \times 2s}$ from the corner submatrices of the matrix $\mathbf{A}$. Let $k_2 = r - k + 1$. Then:

$$\widetilde{\mathbf{A}} = \begin{bmatrix} \mathbf{A}_{kk} & \mathbf{A}_{kk_2} \\ \mathbf{A}_{k_2k} & \mathbf{A}_{k_2k_2} \end{bmatrix}. \tag{30}$$

From the Factorization Theorem by Rao [6] we know that this matrix is nonsingular and thus we know that $\widetilde{\mathbf{A}}$ has a WZ factorization. For such a matrix we perform the original WZ factorization and obtain the matrices $\widetilde{\mathbf{W}}$, $\widetilde{\mathbf{Z}}$ of the form:

$$\widetilde{\mathbf{W}} = \begin{bmatrix} \mathbf{W}_{kk} & \mathbf{W}_{kk_2} \\ \mathbf{W}_{k_2k} & \mathbf{W}_{k_2k_2} \end{bmatrix}, \tag{31}$$

$$\widetilde{\mathbf{Z}} = \begin{bmatrix} \mathbf{Z}_{kk} & \mathbf{Z}_{kk_2} \\ \mathbf{Z}_{k_2k} & \mathbf{Z}_{k_2k_2} \end{bmatrix}. \tag{32}$$

Of course, the blocks of the matrices $\widetilde{\mathbf{Z}}$ and $\widetilde{\mathbf{W}}$ are corresponding blocks of the matrices $\mathbf{Z}$ and $\mathbf{W}$ in the WZ factorization (29).

3.2. Stage 2

Having corner blocks of the matrix $\mathbf{Z}$, we can compute two block columns of the matrix $\mathbf{W}$ — kth and k_2nd columns — that is blocks $\mathbf{W}_{ik}$ and $\mathbf{W}_{ik_2}$ (for $i = (k+1), \ldots, (k_2 - 1)$) from a linear system:

$$\begin{cases} \mathbf{W}_{ik}\mathbf{Z}_{kk} + \mathbf{W}_{ik_2}\mathbf{Z}_{k_2k} = \mathbf{A}_{ik}, \\ \mathbf{W}_{ik}\mathbf{Z}_{kk_2} + \mathbf{W}_{ik_2}\mathbf{Z}_{k_2k_2} = \mathbf{A}_{ik_2}. \end{cases} \tag{33}$$

Thus, we obtain the matrices $\mathbf{W}_{ik}$ and $\mathbf{W}_{ik_2}$:

$$\begin{cases} \mathbf{W}_{ik_2} = (\mathbf{A}_{ik_2} - \mathbf{A}_{ik}\mathbf{Z}_{kk}^{-1}\mathbf{Z}_{kk_2})(-\mathbf{Z}_{k_2k}\mathbf{Z}_{kk}^{-1}\mathbf{Z}_{kk_2} + \mathbf{Z}_{k_2k_2})^{-1}, \\ \mathbf{W}_{ik} = (\mathbf{A}_{ik} - \mathbf{W}_{ik_2}\mathbf{Z}_{k_2k})\mathbf{Z}_{kk}^{-1}. \end{cases} \tag{34}$$

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To solve (33) we used Gaussian elimination and obtained the Schur complement of the block $\mathbf{Z}_{kk}$ of the matrix $\widetilde{\mathbf{Z}}$.

Its worth noting that to obtain the matrices $\mathbf{W}_{ik}$ and $\mathbf{W}_{ik_2}$ it is necessary to find an inverse of the matrix $\mathbf{Z}_{kk}$ (that is, $\mathbf{Z}_{kk}^{-1}$), but this matrix can be found if the main diagonal of $\mathbf{Z}_{kk}$ does not contain any zeros.

Theorem 3. *If $\mathbf{Z}_{kk}$ is an invertible matrix, then the matrix $(-\mathbf{Z}_{k_2k}\mathbf{Z}_{kk}^{-1}\mathbf{Z}_{kk_2}+\mathbf{Z}_{k_2k_2})$ is a lower triangular matrix for $k=1,\ldots,\frac{r}{2}$ and $k_2=r-k+1$.*

Proof. Because the matrix $\widetilde{\mathbf{Z}}$ is invertible, thus, we obtain Theorem 3 from Theorem 2. ■

3.3. Stage 3

At the same time (as Stage 2), having corner blocks of the matrix $\mathbf{W}$, we can compute two block rows of the matrix $\mathbf{Z}$ — kth and k_2nd row — that is blocks $\mathbf{Z}_{ki}$ and $\mathbf{Z}_{k_2i}$ (for $i=(k+1),\ldots,(k_2-1)$) from a linear system:

$$\begin{cases} \mathbf{W}_{kk}\mathbf{Z}_{ki}+\mathbf{W}_{kk_2}\mathbf{Z}_{k_2i}=\mathbf{A}_{ki}, \\ \mathbf{W}_{k_2k}\mathbf{Z}_{ki}+\mathbf{W}_{k_2k_2}\mathbf{Z}_{k_2i}=\mathbf{A}_{k_2i}. \end{cases} \tag{35}$$

Thus, we obtain the matrices $\mathbf{Z}_{ki}$ and $\mathbf{Z}_{k_2i}$:

$$\begin{cases} \mathbf{Z}_{k_2i}=(-\mathbf{W}_{k_2k}\mathbf{W}_{kk}^{-1}\mathbf{W}_{kk_2}+\mathbf{W}_{k_2k_2})^{-1}(\mathbf{A}_{k_2i}-\mathbf{W}_{k_2k}\mathbf{W}_{kk}^{-1}\mathbf{A}_{ki}), \\ \mathbf{Z}_{ki}=\mathbf{W}_{kk}^{-1}(\mathbf{A}_{ki}-\mathbf{W}_{kk_2}\mathbf{Z}_{k_2i}). \end{cases} \tag{36}$$

To solve (35) we used Gaussian elimination and obtained the Schur complement of the block $\mathbf{W}_{kk}$ of the matrix $\widetilde{\mathbf{W}}$.

We should notice that to obtain the matrices $\mathbf{Z}_{ki}$ and $\mathbf{Z}_{k_2i}$ we are to find the inverse $\mathbf{W}_{kk}^{-1}$ which can be achieved because all diagonal elements of $\mathbf{W}_{kk}$ are ones (so $\det\mathbf{W}_{kk}=1$ and thus, $\mathbf{W}_{kk}$ is nonsingular).

Theorem 4. *The matrix $(-\mathbf{W}_{k_2k}\mathbf{W}_{kk}^{-1}\mathbf{W}_{kk_2}+\mathbf{W}_{k_2k_2})$ is an invertible upper triangular matrix with 1s on its diagonal for $k=1,\ldots,\frac{r}{2}$ and $k_2=r-k+1$.*

Proof. The matrix $\mathbf{W}_{kk}^{-1}$ is a lower triangular matrix with ones on its main diagonal — just like (18). The matrix $\mathbf{W}_{k_2k}$ has the form (21). Thus, the product $(\mathbf{W}_{k_2k}\mathbf{W}_{kk}^{-1})$ has the form of $\mathbf{W}_{k_2k}$ — that is (21).

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The matrix $\mathbf{W}_{kk_2}$ has the form (20). The product $(-\mathbf{W}_{k_2k}\mathbf{W}_{kk}^{-1}\mathbf{W}_{kk_2})$ is an upper triangular matrix with zeros on the main diagonal and directly above the main diagonal — of the following form:

$$\begin{bmatrix} 0 & 0 & * & * & \dots & * \\ 0 & 0 & 0 & * & \dots & * \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \dots & \dots & 0 \end{bmatrix}, \tag{37}$$

where $*$ denotes arbitrary elements.

This matrix is singular, with two zero rows (the last one and the next-to-last one) and two zero columns (the 1st one and the 2nd one).

The matrix $\mathbf{W}_{k_2k_2}$ is an upper triangular matrix with ones on the main diagonal — (19).

The sum of the matrices of the form (37) and the form (19) is an upper triangular matrix with ones on the main diagonal:

$$\begin{bmatrix} 1 & * & * & * & \dots & * \\ 0 & 1 & * & * & \dots & * \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & * \\ \vdots & \vdots & \vdots & \ddots & \ddots & * \\ 0 & 0 & 0 & \dots & \dots & 1 \end{bmatrix}. \tag{38}$$

Of course, such a matrix has an inverse which is also an upper triangular matrix with ones on the main diagonal. Its determinant is also one. ■

3.4. Stage 4

The last stage is updating the elements of the middle submatrix of $\mathbf{A}$ using the obtained earlier blocks of the matrices $\mathbf{W}$ and $\mathbf{Z}$. These elements can be computed from:

$$\mathbf{A}_{ij} \leftarrow \mathbf{A}_{ij} - \mathbf{W}_{ik}\mathbf{Z}_{kj} - \mathbf{W}_{ik_2}\mathbf{Z}_{k_2j}, \tag{39}$$

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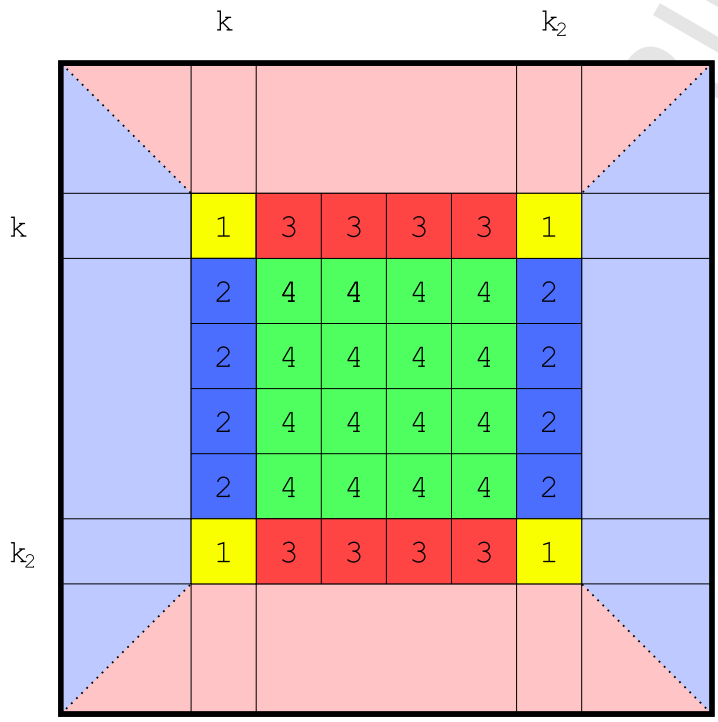


Figure 3: The order of computing blocks in BWZ

where

$$i, j = (k+1), \ldots, (k_2 - 1).$$

Algorithm 1 presents the block WZ factorization (called BWZ). Figure 3 shows the blocks used in computations (in one step of the algorithm) and the order of computing (stages of one step).

4. Matrix with a dominant diagonal

The block WZ factorization exists if the matrix $\mathbf{A}$ is nonsingular and if in each iteration of the Stage 2 loop, the matrices $\mathbf{Z}_{kk}$ (for $k = 1, \ldots, \frac{n}{2}$) are nonsingular.

We can also formulate a theorem:

Theorem 5. *The block WZ factorization exists if the matrix* $\mathbf{A}$ *has a strict dominant diagonal.*

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Algorithm 1 Block WZ factorization for an even r (called BWZ)

Require: $\mathbf{A}$, r
Ensure: $\mathbf{W}$, $\mathbf{Z}$

for $k \leftarrow 1, r/2$ **do**
  $k_2 \leftarrow r - k + 1$
  The WZ factorization for corner blocks of $\mathbf{A}$ *— Stage 1*
  $\widetilde{\mathbf{A}} \leftarrow \begin{bmatrix} \mathbf{A}_{kk} & \mathbf{A}_{kk_2} \\ \mathbf{A}_{k_2k} & \mathbf{A}_{k_2k_2} \end{bmatrix}$
  $[\widetilde{\mathbf{W}}, \widetilde{\mathbf{Z}}] \leftarrow wz(\widetilde{\mathbf{A}})$
  $\begin{bmatrix} \mathbf{W}_{kk} & \mathbf{W}_{kk_2} \\ \mathbf{W}_{kk_2} & \mathbf{W}_{k_2k_2} \end{bmatrix} \leftarrow \widetilde{\mathbf{W}}$
  $\begin{bmatrix} \mathbf{Z}_{kk} & \mathbf{Z}_{kk_2} \\ \mathbf{Z}_{kk_2} & \mathbf{Z}_{k_2k_2} \end{bmatrix} \leftarrow \widetilde{\mathbf{Z}}$
  Obtaining k*th and* k_2*nd block columns of* $\mathbf{W}$ *— Stage 2*
  for $i \leftarrow k+1, k_2-1$ **do**
    $\mathbf{W}_{ik_2} \leftarrow (\mathbf{A}_{ik_2} - \mathbf{A}_{ik}\mathbf{Z}_{kk}^{-1}\mathbf{Z}_{kk_2})(-\mathbf{Z}_{k_2k}\mathbf{Z}_{kk}^{-1}\mathbf{Z}_{kk_2} + \mathbf{Z}_{k_2k_2})^{-1}$
    $\mathbf{W}_{ik} \leftarrow (\mathbf{A}_{ik} - \mathbf{W}_{ik_2}\mathbf{Z}_{k_2k})\mathbf{Z}_{kk}^{-1}$
  end for
  Obtaining k*th and* k_2*nd block rows of* $\mathbf{Z}$ *— Stage 3*
  for $i \leftarrow k+1, k_2-1$ **do**
    $\mathbf{Z}_{k_2i} \leftarrow (-\mathbf{W}_{k_2k}\mathbf{W}_{kk}^{-1}\mathbf{W}_{kk_2} + \mathbf{W}_{k_2k_2})^{-1}(\mathbf{A}_{k_2i} - \mathbf{W}_{k_2k}\mathbf{W}_{kk}^{-1}\mathbf{A}_{ki})$
    $\mathbf{Z}_{ki} \leftarrow \mathbf{W}_{kk}^{-1}(\mathbf{A}_{ki} - \mathbf{W}_{kk_2}\mathbf{Z}_{k_2i})$
  end for
  Updating the matrix $\mathbf{A}$ *— Stage 4*
  for $i \leftarrow k+1, k_2-1$ **do**
    for $j \leftarrow k+1, k_2-1$ **do**
      $\mathbf{A}_{ij} \leftarrow \mathbf{A}_{ij} - \mathbf{W}_{ik}\mathbf{Z}_{kj} - \mathbf{W}_{ik_2}\mathbf{Z}_{k_2j}$
    end for
  end for
end for

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Proof. Let the matrix $\mathbf{A}$ have a strict dominant diagonal, that is:

$$\forall_{i=1,\ldots,n}|a_{ii}| > \sum_{j=1,j\neq i}^{n} |a_{ij}|. \tag{40}$$

We consider only the first step of the WZ factorization. All other steps are analogous. From the definition of $\mathbf{W}_{i1}$ and $\mathbf{W}_{in}$ we have (analogously to [14]) that

$$a_{ij}^{(1)} = a_{ij} - \frac{a_{i1}a_{nn} - a_{in}a_{n1}}{a_{11}a_{nn} - a_{1n}a_{n1}}a_{1j} - \frac{a_{11}a_{in} - a_{1n}a_{i1}}{a_{11}a_{nn} - a_{1n}a_{n1}}a_{nj} = a_{ij} + u_j a_{in} + v_j a_{i1}, \tag{41}$$

where

$$u_j = \frac{a_{1j}a_{n1} - a_{nj}a_{11}}{a_{11}a_{nn} - a_{1n}a_{n1}} \tag{42}$$

and

$$v_j = \frac{a_{nj}a_{1n} - a_{1j}a_{nn}}{a_{11}a_{nn} - a_{1n}a_{n1}}. \tag{43}$$

Using (40) and $\sum_{j=2}^{n-1}|u_j| \leq 1$ [14], we have:

$$|u_j| \leq 1 \tag{44}$$

and similarly for v_j — from $\sum_{j=2}^{n-1}|v_j| \leq 1$ [14], we have:

$$|v_j| \leq 1. \tag{45}$$

From (41) we have:

$$\forall_{i=1,\ldots,n} \quad a_{ii}^{(1)} = a_{ii} + u_i a_{in} + v_i a_{i1}. \tag{46}$$

Using (44) and (45) we have:

$$|u_i a_{in} + v_i a_{i1}| \leq |u_i||a_{in}| + |v_i||a_{i1}| \leq |a_{in}| + |a_{i1}|. \tag{47}$$

After multiplying (47) by -1 we obtain:

$$-|u_i a_{in} + v_i a_{i1}| \geq -|a_{in}| - |a_{i1}|. \tag{48}$$

Transforming (40) — which defines the dominant diagonal — we obtain:

$$|a_{ii}| > \sum_{j=1}^{n}|a_{ij}| = |a_{in}| + |a_{i1}| + \sum_{j=2}^{n-1}|a_{ij}|. \tag{49}$$

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Adding (48) and (49) we have:

$$|a_{ii}| - |u_i a_{in} + v_i a_{i1}| > \sum_{j=2}^{n-1} |a_{ij}|. \tag{50}$$

Using (46) we can write:

$$|a_{ii}^{(1)}| \geq |a_{ii}| - |u_i a_{in} + v_i a_{i1}| > \sum_{j=2}^{n-1} |a_{ij}| \geq 0, \tag{51}$$

and this implies:

$$\forall_{i=1,\ldots,n} |a_{ii}^{(1)}| > 0. \tag{52}$$

Because $a_{ii}^{(1)} = z_{11}$, we know that $\det \mathbf{Z_{kk}} \neq 0$ for $k = 1, \ldots, \frac{n}{2}$, and this allows conducting the WZ factorization. ■

5. Cost of the block algorithm

We can show that the cost of the block computations is of the same order as of the scalar algorithm.

The elimination phase of the WZ factorization has the cost of $O(n^3)$.

To find the cost of the block algorithm observe that:

- the cost of the WZ factorization is $C_{WZ}(n) = \frac{2}{3}n^3 - \frac{7}{6}n - 3$;
- the cost of the matrix-matrix multiplication is $2n^3$;
- the cost of the solution of a matrix linear system with a triangular matrix is n^3.

Thus, we can give formulas for obtaining the cost of the stages of the block WZ factorization algorithm:

1. Stage 1 is a WZ factorization of a matrix of $2s$ rows. The cost of Stage 1 is:

$$C_1 = C_{WZ}(2s) = \frac{16}{3}s^3 - \frac{7}{3}s - 3.$$

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2. Stage 2 is finding some blocks of the matrix $\mathbf{W}$. Here we have a couple of operations: a matrix-matrix multiplication and finding an inverse of the matrix (by solving triangular systems). The cost of Stage 2 is:

$$C_2(k) = 3s^3 + s^2 + \sum_{i=k+1}^{r-k} (6s^3 + 2s^2) = 3s^3 + s^2 + (6s^3 + 2s^2)(r - 2k).$$

3. Stage 3 is finding some blocks of the matrix $\mathbf{Z}$. Its cost is precisely the same as the cost of Stage 2:

$$C_3(k) = C_2(k).$$

4. Stage 4 is two matrix-matrix multiplications and two matrix-matrix additions — for $(r - 2k)^2$ blocks. The cost of the Stage 4 is:

$$C_4(k) = \sum_{i=k+1}^{r-k} \sum_{j=k+1}^{r-k} (4s^3 + 2s^2) = (4s^3 + 2s^2)(r - 2k)^2.$$

The cost of the whole block WZ factorization is:

$$\sum_{k=1}^{\frac{r}{2}} (C_1 + 2C_2(k) + C_4(k)) = \frac{n^3(4s + 2) + 6n^2s^2 + n(6s^3 - 2s^2 - 7s - 9)}{6s}.$$

Thus, the block WZ factorization has the cost of $O(n^3)$. Asymptotically, it is the same as the cost of the original WZ factorization (5).

6. Numerical experiments

In this section, we tested the time, the speedup and the accuracy of the block WZ factorization. Our intention was to investigate our block WZ factorization (BWZ, from Section 3) and to compare two sequential implementations — our implementation of the original WZ factorization (WZ, from Section 2) with our block implementation (BWZ). The block and original WZ factorizations were implemented with the use of a high level language, namely MATLAB [16].

The input matrices were generated by the author. They were random matrices, with a dominant diagonal of an even size of 500, 1000, ..., 7000. Various numbers of submatrices were tested, namely

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Table 1: Hardware used in the experiments

CPU	AMD FX-8120 3.1 GHz
cores	4 modules × 2 cores
Level 1 instruction cache	2-way, 64 KB per module
Level 1 data cache	4-way, 16 KB per module
Level 2 cache	2 MB per module
Level 3 cache	8 MB common
Host memory	16 GB

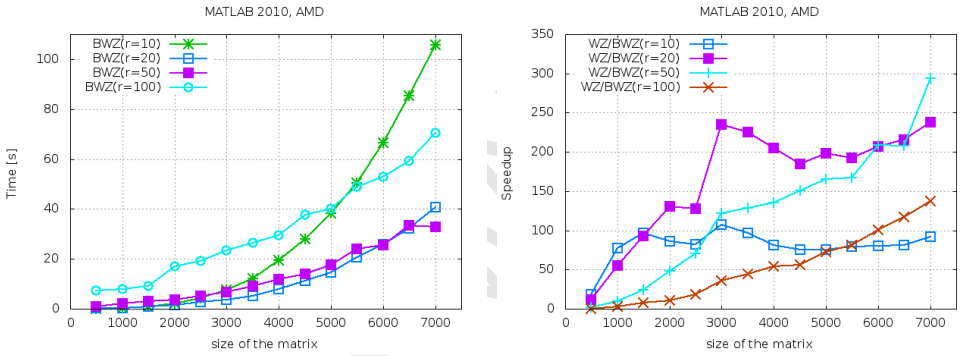


Figure 4: Run-time in seconds (left) and speedup (right) of the block WZ factorization for four different $r = 10, 20, 50, 100$

The MATLAB implementation was compiled and tested under MS Windows. Table 1 shows details of the specification of the hardware used in the experiment. It also shows that the architecture used in the tests of WZ and BWZ had a highly hierarchical memory (which is significant for block algorithms). The performance times were measured with the use of standard MATLAB functions, namely: `tic` and `toc`. We measured the difference between $\mathbf{A}$ and $\mathbf{WZ}$ by absolute error $||\mathbf{A} - \mathbf{WZ}||_2$ — to compute the norm, a standard MATLAB function (`norm`) was used.

Figure 4 (left) presents the time (in seconds) of the block WZ factorization (BWZ) for 4 different $r = 10, 20, 50, 100$. The tests show that for $r = 20$ the algorithm is the fastest. Figure 4 (right) shows the speed up of the block WZ factorization with respect to the original WZ factorization for 4 various $r = 10, 20, 50, 100$. We can see that the block algorithm for $r = 50$ can be even 300 times faster in comparison with the original WZ factorization.

Table 2 illustrates the norm for the WZ factorization in MATLAB. The

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Table 2: Norm for the WZ factorization

matrix size	WZ	BWZ($r = 10$)	BWZ($r = 20$)	BWZ($r = 50$)	BWZ($r = 100$)
500	$1.2506 \cdot 10^{-12}$	$5.6846 \cdot 10^{-13}$	$5.1165 \cdot 10^{-13}$	$5.1166 \cdot 10^{-13}$	$7.3900 \cdot 10^{-13}$
1000	$3.1833 \cdot 10^{-12}$	$1.2510 \cdot 10^{-12}$	$1.4780 \cdot 10^{-12}$	$1.2506 \cdot 10^{-12}$	$1.3643 \cdot 10^{-12}$
1500	$8.8676 \cdot 10^{-12}$	$2.9562 \cdot 10^{-12}$	$2.5012 \cdot 10^{-12}$	$3.1833 \cdot 10^{-12}$	$2.5012 \cdot 10^{-12}$
2000	$9.0951 \cdot 10^{-12}$	$4.7749 \cdot 10^{-12}$	$3.1833 \cdot 10^{-12}$	$2.7286 \cdot 10^{-12}$	$2.9562 \cdot 10^{-12}$
2500	$1.7280 \cdot 10^{-11}$	$9.5497 \cdot 10^{-12}$	$5.9118 \cdot 10^{-12}$	$5.0023 \cdot 10^{-12}$	$5.4572 \cdot 10^{-12}$
3000	$2.0464 \cdot 10^{-11}$	$1.0914 \cdot 10^{-11}$	$1.0005 \cdot 10^{-11}$	$1.0004 \cdot 10^{-11}$	$8.1855 \cdot 10^{-12}$
3500	$2.5011 \cdot 10^{-11}$	$1.0914 \cdot 10^{-11}$	$7.2760 \cdot 10^{-12}$	$5.9117 \cdot 10^{-12}$	$5.9122 \cdot 10^{-12}$
4000	$2.9559 \cdot 10^{-11}$	$1.2278 \cdot 10^{-11}$	$8.1855 \cdot 10^{-12}$	$5.0026 \cdot 10^{-12}$	$5.9118 \cdot 10^{-12}$
4500	$7.0031 \cdot 10^{-11}$	$2.2737 \cdot 10^{-11}$	$1.8190 \cdot 10^{-11}$	$1.2733 \cdot 10^{-11}$	$1.2733 \cdot 10^{-11}$
5000	$5.0932 \cdot 10^{-11}$	$3.0923 \cdot 10^{-11}$	$2.5466 \cdot 10^{-11}$	$2.3647 \cdot 10^{-11}$	$2.3647 \cdot 10^{-11}$
5500	$6.0027 \cdot 10^{-11}$	$3.4561 \cdot 10^{-11}$	$3.3651 \cdot 10^{-11}$	$2.5466 \cdot 10^{-11}$	$3.3651 \cdot 10^{-11}$
6000	$6.3665 \cdot 10^{-11}$	$4.6384 \cdot 10^{-11}$	$2.5466 \cdot 10^{-11}$	$2.3647 \cdot 10^{-11}$	$1.9100 \cdot 10^{-11}$
6500	$7.0031 \cdot 10^{-11}$	$3.8199 \cdot 10^{-11}$	$2.3647 \cdot 10^{-11}$	$1.5461 \cdot 10^{-11}$	$1.3642 \cdot 10^{-11}$
7000	$7.0031 \cdot 10^{-11}$	$3.7289 \cdot 10^{-11}$	$2.2737 \cdot 10^{-11}$	$2.3647 \cdot 10^{-11}$	$2.2738 \cdot 10^{-11}$

results show that the block algorithms give the same or sometimes even better accuracy.

7. Conclusion

In this article we considered the decomposition of a matrix $\mathbf{A}$ into factors $\mathbf{W}$ and $\mathbf{Z}$ — for a square, dense, nonsingular matrix of an even size. Such a problem has a computationally intensive nature. To reduce computing time significantly and to use the contemporary computers architectures, the author proposed a partition of the matrix $\mathbf{A}$ into blocks.

It was shown that the block WZ factorization exists for matrices with strict dominant diagonal. The author also presented that the block algorithms are as costly as the original one. We compared two implementations — the original WZ factorization and our block WZ factorization for different block sizes. From the results, we can conclude that the block WZ factorization increases the speed of the factorization even up to 300 times. The implementation had no impact on the accuracy of the factorization — the accuracy depended on the size of the matrix only, which is quite self-evident.

In future works, the author plans to implement a block WZ factorization with the use of block routines from LAPACK [1].

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