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A hybridizable discontinuous Galerkin method for a class of fractional boundary value problems

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Abstract

In this paper, we present a hybridizable discontinuous Galerkin (HDG) method for solving a class of fractional boundary value problems involving Caputo derivatives. The HDG methods have the computational advantage of eliminating all internal degrees of freedom and the only globally coupled unknowns are those at the element interfaces. Furthermore, the global stiffness matrix is tridiagonal, symmetric, and positive definite. Internal degrees of freedom are recovered at an element-by-element postprocessing step. We carry out a series of numerical experiments to ascertain the performance of the proposed method.

Keywords: Hybridizable discontinuous Galerkin methods, fractional boundary value problems, Caputo derivative.

1. Introduction

Fractional differential equations find applications in a wide range of areas including but not limited to physics, chemistry, biology, economics, medicine, and engineering. They provide an excellent tool for the description of memory and hereditary properties of various materials and processes. This provides a major advantage to fractional order models over integer models. Re-

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cently, the interest in fractional differential equations have increased in terms of their theory, existence–uniqueness results, and numerical approximation techniques. There has been an extensive collection in fractional differential equations such as Miller and Ross [1], Samko et al. [2], Podlubny [3], Kilbas et al. [4] and Zhou [5]. The theory and approximation of FBVPs are in their early stages and many aspects of them remain to be explored. Unlike classical differential equations, there is no systematic method to find the exact solution of these problems.

Aleroev et al. [6] proved existence and uniqueness results for several two–point FBVPs with Caputo and Riemann–Liouville derivatives. They also designed shooting methods to approximate solutions of these problems. Among these problems, we are interested in the following family of FBVPs in this paper

$${}_a^C D_x^\alpha u(x) + g(x, u(x), {}_a^C D_x^\beta u(x)) = 0 \quad x \in \Omega = (a, b), \quad (1.1a)$$

$$u(x) = u_D(x) \quad x \in \partial\Omega = \{a, b\}, \quad (1.1b)$$

where $u : [a, b] \rightarrow \mathbb{R}$ and $g : [a, b] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are continuous functions, and g satisfies the Lipschitz condition

$$|g(x, u_2, v_2) - g(x, u_1, v_1)| \leq K_g |u_2 - u_1| + L_g |v_2 - v_1| \quad (1.2)$$

for all $x \in \Omega$ and $u_1, u_2, v_1, v_2 \in \mathbb{R}$ with $K_g, L_g > 0$. Here and throughout the paper $1 < \alpha \leq 2$ and $0 < \beta \leq 1$, and the Caputo fractional derivatives are defined as

$${}_a^C D_x^\alpha u(x) = \frac{1}{\Gamma(2-\alpha)} \int_a^x u''(t)(x-t)^{1-\alpha} dt,$$

$${}_a^C D_x^\beta u(x) = \frac{1}{\Gamma(1-\beta)} \int_a^x u'(t)(x-t)^{-\beta} dt.$$

Rehman and Khan [7] also studied (1.1) and proved the existence and uniqueness of its solution using the Arzela–Ascoli theorem, Schauder fixed point theorem, and a contraction theorem. They also designed a numerical scheme based on Haar wavelets.

In this paper, we devise and implement an HDG method for a special case of (1.1), namely,

$${}_a^C D_x^\alpha u + c_1 u + c_2 {}_a^C D_x^\beta u = f \quad \text{in } \Omega, \quad (1.3a)$$

$$u = u_D \quad \text{on } \partial\Omega. \quad (1.3b)$$

where c_1 and c_2 are bounded functions and $f : [a, b] \rightarrow \mathbb{R}$ is a continuous function. It is easy to see that the Lipschitz condition (1.2) is satisfied if we choose K_g and L_g to be any two positive constants greater than or equal to $\sup_{x \in \Omega} |c_1(x)|$ and $\sup_{x \in \Omega} |c_2(x)|$, respectively. Observe that for $\alpha = 2$, ${}_a^C D_x^\alpha u = u''$ and for $\beta = 1$, ${}_a^C D_x^\beta u = u'$. Hence, the FBVP (1.3) reduces to a classical two-point boundary value problem for this special case. It is worthwhile to note that the HDG methods we introduce in this paper are robust with respect to both of these parameters, see Sec. 3.

We quote the following existence and uniqueness theorems for (1.3) from [6] and [7].

Theorem 1.1. *Let g be the function in (1.1a) and suppose that there exists a function $\mu : [a, b] \rightarrow [0, \infty]$ such that*

$$|g(x, u, v)| \leq \mu(x) + a_1|u| + a_2|v| \quad \text{for all } x \in [a, b]$$

for some $a_1, a_2 \geq 0$ with $a_1 + a_2 \leq m$ where

$$m = \min \left\{ \frac{\Gamma(\alpha + 1)}{8}, \frac{\Gamma(\alpha)\Gamma(2 - \beta) + \Gamma(\alpha - \beta + 1)}{4\Gamma(\alpha - \beta + 1)\Gamma(\alpha)\Gamma(2 - \beta)} \right\}.$$

Then, the boundary value problem (1.1) has a solution.

Proof. See Theorem 3.2 in [7].

Theorem 1.2. *Let g be the function in (1.1a) and suppose that there exists a positive constant*

$$r < \left(\frac{3 + \Gamma(\alpha + 1)}{\Gamma(\alpha + 1)\Gamma(\alpha - \beta + 1)} \right)^{-1/2}$$

such that

$$|f(x, u, v) - f(x, \tilde{u}, \tilde{v})| \leq r(|u - \tilde{u}| + |v - \tilde{v}|)$$

for all $x \in [a, b]$ and all $u, \tilde{u}, v, \tilde{v} \in \mathbb{R}$. Then, the boundary value problem (1.1) has a unique solution.

Proof. See Theorem 3.3 in [7].

Theorem 1.3. *Let g be the function in (1.1a) and suppose that it satisfies the Lipschitz condition (1.2). If*

$$K_g \frac{2(b - a)^\alpha}{\Gamma(\alpha + 1)} + L_g \left[\frac{(b - a)^{\alpha - \beta}}{\Gamma(\alpha + 1)\Gamma(2 - \beta)} + \frac{(b - a)^{\alpha - \beta}}{\Gamma(\alpha + 1 - \beta)} \right] < 1$$

then the boundary value problem (1.1) has a unique solution.

Proof. See Theorem 3.1 in [6].

The rest of the paper is organized as follows. In Section 2, we introduce the HDG methods for the FBVP (1.3) and state a theorem which shows that these methods are efficiently implementable. In Section 3, we investigate the performance of these methods in a series of numerical examples. We end in Section 4 with a few concluding remarks.

2. The HDG Methods

In this section, we introduce the HDG methods for the FBVP (1.3). HDG methods were first introduced in [8] in the framework of second order elliptic problems. Since then they have been successfully applied to a wide range of partial differential equations arising in various areas of science, technology, and engineering. Recently, the present authors applied HDG methods to approximate the solution of the Bagley-Torvik equation which is fractional differential equation [9]. Therein, certain advantages of applying these methods to fractional problems have been pointed out. To the best of our knowledge, this is the first time that these methods are applied for solving a FBVP of type (1.3). This is the main contribution of this paper.

We begin with introducing two auxiliary unknowns, p and q , as follows

$$q := -u', \quad p := -q' = u'' \quad \text{in } \Omega.$$

Consequently, we define the operators T_α and T_β , acting on p and q , respectively, as

$$T_\alpha[p(x)] := {}^C D_x^\alpha u(x) \quad \text{and} \quad T_\beta[q(x)] := -{}^C D_x^\beta u(x).$$

Thus, we rewrite (1.3) as the following system of equations

$$\begin{aligned} q + u' &= 0 & \text{in } \Omega, \\ p + q' &= 0 & \text{in } \Omega, \\ T_\alpha[p] + c_1 u - c_2 T_\beta[q] &= f & \text{in } \Omega. \end{aligned} \tag{2.1}$$

This process of introducing the additional unknowns p and q may be criticized in the sense that it increases the computational cost of the method. However, we will show at the end of this section that this can be compensated by far by eliminating all internal degrees of freedom and obtaining a global linear system that only involves unknowns associated with the original variable u

at the element interfaces. This is one of the main characteristics of all HDG methods and it is to this property that the methods owe their popularity since it renders them computationally competitive.

Given a positive integer N , we partition the domain Ω into N elements by setting

$$\Omega_h := \{I_j = (x_{j-1}, x_j) : a = x_0 < x_1 < \dots < x_{N-1} < x_N = b\}.$$

We associate the set of nodes, $\mathcal{E}_h := \{x_0, x_1, \dots, x_N\}$, and the set of interior nodes $\mathcal{E}_h^\circ := \mathcal{E}_h \setminus \partial\Omega$; we also set $\partial\Omega_h := \{\partial K : K \in \Omega_h\}$. For each element $K \in \Omega_h$, let h_K denote the length of K , and set $h := \max_{K \in \Omega_h} \{h_K\}$. Finally, for any given polynomial degree $k \geq 0$ and an element $K \in \Omega_h$, we define $\mathcal{P}^k(K)$ as the set of polynomials of degree less than or equal to k on K . The space of piecewise polynomials of degree k on Ω is defined accordingly as

$$V_h^k := \{v : \Omega_h \mapsto \mathbb{R} : v|_K \in \mathcal{P}^k(K) \text{ for all } K \in \Omega_h\}.$$

We also set

$$L_0^2(\mathcal{E}_h) := \{m \in L^2(\mathcal{E}_h) : m = 0 \text{ on } \partial\Omega\}$$

where $L^2(\mathcal{E}_h)$ is simply a copy of the Euclidean space \mathbb{R}^{N+1} .

The HDG methods seek an approximation $(q_h, p_h, u_h, \hat{u}_h)$ to the exact solution $(q, p, u, u|_{\mathcal{E}_h})$, of (2.1), in the finite dimensional space $V_h^k \times V_h^k \times V_h^k \times L^2(\mathcal{E}_h)$. It is determined by requiring that

$$\begin{aligned} (q_h, v) - (u_h, v') + \langle \hat{u}_h, v n \rangle &= 0, \\ (p_h, w) - (q_h, w') + \langle \hat{q}_h, w n \rangle &= 0, \\ (T_\alpha[p_h], z) + (c_1 u_h, z) - (c_2 T_\beta[q_h], z) &= (f, z), \\ \langle \hat{q}_h n, \mu \rangle &= 0, \end{aligned} \tag{2.2}$$

hold for all

$$(v, w, z, \mu) \in V_h^k \times V_h^k \times V_h^k \times L_0^2(\mathcal{E}_h).$$

Here, the outward unit normal vectors are $n(x^\mp) := \pm 1$ for $x \in \mathcal{E}_h$. The “volume” inner product is defined as

$$(u, v) := \sum_{K \in \Omega_h} (u, v)_K \text{ where } (u, v)_K := \int_K u(x)v(x) dx,$$

and the boundary inner product is defined as

$$\langle u, v n \rangle := \sum_{K \in \Omega_h} \langle u, v n \rangle_{\partial K} \text{ where } \langle u, v \rangle_{\partial K} := u(x_j^-)v(x_j^-) + u(x_{j-1}^+)v(x_{j-1}^+),$$

when $K = (x_{j-1}, x_j)$, and $u(x^\pm) := \lim_{\epsilon \downarrow 0} u(x \pm \epsilon)$ for $x \in \mathcal{E}_h$. The boundary condition (1.3b) is imposed by requiring that

$$\hat{u}_h = u_D \quad \text{on } \partial\Omega.$$

The numerical trace \hat{q}_h is defined as

$$\hat{q}_h = q_h + \tau(u_h - \hat{u}_h) n \quad (2.3)$$

where τ is a nonnegative function defined on $\partial\Omega_h$. This completes the definition of the HDG method.

Proving the existence and uniqueness of the HDG approximation defined by (2.2) and (2.3) is a delicate and nontrivial task. To see why this is so we refer to the proof of existence of uniqueness of the corresponding FBVP in [6] which requires various fractional calculus properties and contraction properties in suitably defined normed spaces. In addition to these ingredients, in the present context, we are dealing with a weak formulation and the solution space allows discontinuities which further complicates the analysis significantly. However, it is well-known [8] that for most HDG methods for classical BVPs and partial differential equations the existence and uniqueness of the approximate solution follows from the assumption that $\tau \geq 0$. This is also the case for HDG methods studied in [12] and [13] for specific types of fractional diffusion problems. This was also the case for all the numerical experiments we carried out, see Sec. 3.

Next, we show that the *only* globally coupled unknowns of the HDG method defined by the weak formulation (2.2), and the formula (2.3) for the numerical traces are the approximations at the nodes \hat{u}_h . We also show that the remaining components of the approximate solution can be expressed solely in terms of element-by-element-defined operators acting on \hat{u}_h . To do this, we follow the framework provided in [8, 10]. To this end, we define the *local solvers*. They are called “*local*” because they are defined on a single element $K \in \Omega_h$ and they are called “*solvers*” because they form an approximate solution to the original problem on the element K .

The first local solver is defined on the element $K \in \Omega_h$ as the mapping

$$\mu \in L^2(\partial K) \mapsto (\mathbf{Q}\mu, \mathbf{P}\mu, \mathbf{U}\mu) \in [\mathcal{P}^k(K)]^3$$

where

$$\begin{aligned} (\mathbf{Q}\mu, v)_K - (\mathbf{U}\mu, v')_K &= -\langle \mu, vn \rangle_{\partial K}, \\ (\mathbf{P}\mu, w)_K - (\mathbf{Q}\mu, w')_K + \langle \hat{\mathbf{Q}}\mu, wn \rangle_{\partial K} &= 0, \\ (T_\alpha[\mathbf{P}\mu], z)_K + (c_1 \mathbf{U}\mu, z)_K - (c_2 T_\beta[\mathbf{Q}\mu], z)_K &= 0, \end{aligned} \quad (2.4)$$

for all $v, w, z \in \mathcal{P}^k(K)$ where

$$\widehat{Q}\mu = Q\mu + \tau(U\mu - \mu)n \quad \text{on } \partial K.$$

The second local solver is defined on the element $K \in \Omega_h$ as the mapping

$$f \in L^2(K) \mapsto (Qf, Pf, Uf) \in [\mathcal{P}^k(K)]^3$$

where

$$\begin{aligned} (Qf, v)_K - (Uf, v')_K &= 0, \\ (Pf, w)_K - (Qf, w')_K + \langle \widehat{Q}f, wn \rangle_{\partial K} &= 0, \\ (T_\alpha[Pf], z)_K + (c_1 Uf, z)_K - (c_2 T_\beta[Qf], z)_K &= (f, z)_K, \end{aligned} \quad (2.5)$$

for all $v, w, z \in \mathcal{P}^k(K)$ where

$$\widehat{Q}f = Qf + \tau(Uf)n \quad \text{on } \partial K.$$

It can be readily observed that (2.4) defines an approximation $(Q\mu, P\mu, U\mu)$ to the solution (q, p, u) of the problem

$$\begin{aligned} q + u' &= 0 & \text{in } K, \\ p + q' &= 0 & \text{in } K, \\ T_\alpha[p] + c_1 u - c_2 T_\beta[q] &= 0 & \text{in } K, \end{aligned} \quad (2.6)$$

subject to the boundary conditions

$$u = \mu \quad \text{on } \partial K,$$

whereas (2.5) defines an approximation (Qf, Pf, Uf) to the solution (q, p, u) of the problem

$$\begin{aligned} q + u' &= 0 & \text{in } K, \\ p + q' &= 0 & \text{in } K, \\ T_\alpha[p] + c_1 u - c_2 T_\beta[q] &= f & \text{in } K, \end{aligned} \quad (2.7)$$

subject to the homogeneous boundary conditions

$$u = 0 \quad \text{on } \partial K.$$

Observe that (2.6) is a restriction to K of (2.1) with $f = 0$ and (2.7) is a restriction to K of (2.1) equipped with homogeneous boundary conditions.

Let us note that the solution of the problems (2.4) and (2.5) require the solution of a linear system of size $(k + 1) \times (k + 1)$ and hence it is computationally extremely inexpensive. Furthermore, since the definition is local in the sense that (2.4) and (2.5) are defined on a single element $K \in \Omega_h$, the computation is trivially parallelizable.

The function u_D , as well as any other function defined only on $\partial\Omega$ is extended to \mathcal{E}_h by zero. We also set

$$\lambda_h := \begin{cases} \hat{u}_h & \text{on } \partial\Omega_h \setminus \partial\Omega \\ 0 & \text{on } \partial\Omega, \end{cases}$$

so that we have that $\hat{u}_h = \lambda_h + u_D$ where $\lambda_h \in L_0^2(\mathcal{E}_h)$. We can now state a theorem that characterizes the HDG approximate solution in terms of the local solvers.

Theorem 2.1. *The HDG solution $(q_h, p_h, u_h, \lambda_h) \in [V_h^k]^3 \times L_0^2(\mathcal{E}_h)$ given by the HDG method (2.2) can be expressed in terms of the local solvers as*

$$\begin{aligned} q_h &= \mathbf{Q}\lambda_h + \mathbf{Q}u_D + \mathbf{Q}f, \\ p_h &= \mathbf{P}\lambda_h + \mathbf{P}u_D + \mathbf{P}f, \\ u_h &= \mathbf{U}\lambda_h + \mathbf{U}u_D + \mathbf{U}f, \end{aligned}$$

where λ_h satisfies

$$a_h(\lambda_h, \mu) = b_h(\mu), \quad \text{for all } \mu \in L_0^2(\mathcal{E}_h), \quad (2.8)$$

where we have defined

$$\begin{aligned} a_h(\lambda_h, \mu) &= -\langle \hat{\mathbf{Q}}\lambda_h, \mu n \rangle_{\partial\Omega_h}, \\ b_h(\mu) &= \langle \hat{\mathbf{Q}}u_D + \hat{\mathbf{Q}}f, \mu n \rangle_{\partial\Omega_h}. \end{aligned}$$

The proof of this theorem is straightforward since (2.8) is nothing but a rewriting of the last equation (namely, the conservativity condition) in (2.2). For details we refer to [8, 10].

Note that the total number of globally coupled unknowns in the equation (2.8) is $N - 1$ where N is the number of elements in Ω_h . This should be contrasted with the total number of globally coupled unknowns in the original

form of the equation (2.2) that defines the HDG method, namely, $3N(k + 1) + N - 1$. Thus, the result in Theorem 2.1 shows that the total number of globally coupled unknowns can be significantly reduced. This is what we mean when we say that HDG methods are efficiently implementable. It is also worth noting that the global linear system resulting from most numerical methods for fractional differential equations are not sparse. This is due to the presence of the fractional derivative which is defined via an integral over the interval $[a, x]$. However, the global linear system that results from the method we propose here is tridiagonal. This is evident from (2.8) since for any given node of the mesh, the contribution to the stiffness matrix is nonzero only from the node itself and two neighboring nodes. This gives a striking computational advantage to HDG methods over other numerical methods for such problems. As an indication of the existence and uniqueness of the HDG approximation we display the condition number of this global stiffness matrix in Sec. 3. We observe that the behavior of the condition number is very similar to that of finite element methods for second order elliptic problems.

As far as the storage requirements of the HDG methods are concerned one has to first compute and store the local solvers (2.4) for each basis function $\mu \in L_0^2(\mathcal{E}_h)$ and the local solver (2.5) only for the right hand-side function f given in (1.3a). We also have to store the global stiffness matrix of size $(N - 1) \times (N - 1)$ and the load vector of length $N - 1$. However, since the stiffness matrix is tridiagonal we only have to store the diagonal vector (of size $N - 1$) and the sub- and super-diagonal vectors (of size $N - 2$). Hence, the storage requirements are minimal for HDG methods.

3. Numerical Examples

In this section, we display the results of some numerical experiments to assess the performance of the HDG methods described in the previous section. In Tables 1–6, for $\varphi = u, q, p$, $\|\varphi - \varphi_h\|_0$ denotes the L^2 -error computed by

$$\|\varphi - \varphi_h\|_0 = \left(\sum_{K \in \Omega_h} \int_K (\varphi - \varphi_h)^2 \right)^{\frac{1}{2}}.$$

We display these quantities in the “error” column in our tables. The approximate convergence order of the error is then displayed in the “order” column next to the error column, and it contains the logarithm of the ratio of the

two consecutive errors corresponding to two HDG solutions with two consecutive meshes. Therefore, for instance, if the numbers in the order column approaches κ , it indicates that the error satisfies a formula of the form

$$\|\varphi - \varphi_h\|_0 \leq Ch^\kappa$$

for some constant C that is independent of the mesh size h , that is, the method converges with order κ with respect to h . We also compute and display the error in the numerical trace \hat{u}_h which is defined as

$$\|u - \hat{u}_h\|_\infty = \max_{e \in \mathcal{E}_h} |(u - \hat{u}_h)(e)|.$$

In all of our examples, the function τ in the definition (2.3) of the HDG method is taken to be $\tau = 0.1$ on $\partial\Omega_h$.

Example 3.1. Consider the fractionally damped mechanical oscillator equation [6] on $\Omega = (0, 1)$

$$\begin{aligned} {}^C_0 D_x^{1.5} u + \frac{1}{3}u + \frac{1}{4} {}^C_0 D_x^{0.5} u &= -\frac{7}{2\sqrt{\pi}}x^{0.5} + \frac{x}{3} - \frac{2}{3\sqrt{\pi}}x^{1.5} - \frac{x^2}{3} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned}$$

The unique solution of this problem is $u(x) = x(1 - x)$. The errors and convergence orders are presented in Table 1. We observe that for $k = 1$, u_h , q_h , and p_h converge with the optimal order $k + 1 = 2$. The numerical trace \hat{u}_h also converges with the same order. For $k = 2$, we observe that the HDG method captures the exact solution up to the machine accuracy.

Example 3.2. Consider the FBVP

$$\begin{aligned} {}^C_0 D_x^\alpha u - {}^C_0 D_x^\beta u &= -e^{x-1} - 1 \quad \text{in } \Omega = (0, 1), \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

which was studied in [11]. Although there exists a unique solution to this problem, it is, in general, not known explicitly. If we take the integer order derivatives $\alpha = 2$ and $\beta = 1$, then the exact solution is $u(x) = x(1 - e^{x-1})$. In this case, the equation governs the motion of a damped harmonic oscillator. In Table 2, we display the errors and convergence orders of the HDG method for this case, namely, $\alpha = 2$ and $\beta = 1$. In Table 3, we compare the values of

Table 1: History of convergence for Example 3.1.

k	N	$\ u - u_h\ _0$		$\ q - q_h\ _0$		$\ p - p_h\ _0$		$\ u - \hat{u}_h\ _\infty$	
		error	order	error	order	error	order	error	order
1	8	1.16e-03	2.00	1.49e-04	1.99	5.50e-05	2.61	3.14e-06	2.46
	16	2.91e-04	2.00	3.75e-05	1.99	1.20e-05	2.20	7.38e-07	2.09
	32	7.28e-05	2.00	9.39e-06	2.00	2.85e-06	2.07	1.82e-07	2.02
	64	1.82e-05	2.00	2.35e-06	2.00	6.98e-07	2.03	4.51e-08	2.01
2	4	6.18e-15	—	1.70e-15	—	2.24e-14	—	3.05e-16	—
	8	6.34e-15	—	2.00e-14	—	9.70e-14	—	8.77e-15	—
	16	1.19e-14	—	4.15e-14	—	5.48e-13	—	1.66e-14	—
	32	7.00e-14	—	2.77e-13	—	2.45e-12	—	9.98e-14	—

the approximate HDG solution at the nodes of the mesh for various values of α with $\beta = 1$ to those of the known solution above. These results are obtained with $N = 32$. We observe that as α approaches to 2, the corresponding solution to the FBVP approaches to the solution of integer order BVP. In Table 4, we carry out a similar study in which we fix $\alpha = 2$ and let β approach the limiting value of 1. Once again, we see that the solution of the FBVP approaches to that of the integer order BVP.

Example 3.3. Consider the FBVP

$$\begin{aligned}
 {}^C_0 D_x^\alpha u + \frac{1}{19}u &= x + \frac{1}{19} \frac{x^{\alpha+1}}{\Gamma(\alpha+2)} \quad \text{in } \Omega = (0, 1), \\
 u(0) &= 0, \quad u(1) = \frac{1}{\Gamma(\alpha+2)},
 \end{aligned}$$

which was studied in [11]. The exact solution of this problem is $u(x) = \frac{x^{\alpha+1}}{\Gamma(\alpha+2)}$. In Table 5, we compare the values of the HDG approximation at the nodes of the mesh (with $N = 32$) to those of the exact solution given above. We observe that the method produces better approximations with increasing values of α .

In Table 6, we display the errors and the convergence orders for Example 3.3 for which the exact solution is not smooth. More specifically, we take

Table 2: History of convergence for Example 3.2 with $\alpha = 2$, $\beta = 1$.

k	N	$\ u - u_h\ _0$		$\ q - q_h\ _0$		$\ p - p_h\ _0$		$\ u - \hat{u}_h\ _\infty$	
		error	order	error	order	error	order	error	order
1	8	1.81e-02	1.99	1.40e-03	1.99	1.79e-03	1.99	5.89e-07	2.55
	16	4.53e-03	2.00	3.52e-04	2.00	4.47e-04	2.00	8.26e-08	2.83
	32	1.13e-03	2.00	8.81e-05	2.00	1.12e-04	2.00	1.09e-08	2.92
	64	2.83e-04	2.00	2.20e-05	2.00	2.80e-05	2.00	1.40e-09	2.96
2	8	2.23e-04	2.99	1.89e-05	2.99	2.29e-05	2.99	1.28e-10	4.73
	16	2.79e-05	3.00	2.37e-06	3.00	2.87e-06	3.00	4.29e-12	4.89
	32	3.49e-06	3.00	2.96e-07	3.00	3.59e-07	3.00	1.41e-13	4.93
	64	4.37e-07	3.00	3.70e-08	3.00	4.49e-08	3.00	2.22e-14	—
3	8	2.04e-06	3.99	1.81e-07	3.99	2.12e-07	3.99	1.27e-14	6.82
	16	1.28e-07	4.00	1.13e-08	4.00	1.33e-08	4.00	5.27e-16	4.59
	32	8.01e-09	4.00	7.07e-10	4.00	8.31e-10	4.00	4.50e-15	—
	64	5.00e-10	4.00	4.42e-11	4.00	5.20e-11	4.00	7.85e-15	—

Table 3: $\|u - \hat{u}_h\|_\infty$ for various values of α for Example 3.2 with $\beta = 1$.

α	$k = 1$	$k = 2$	$k = 3$
1.7	5.71e-02	4.99e-02	5.04e-02
1.8	3.63e-02	3.13e-02	3.18e-02
1.9	1.74e-02	1.47e-02	1.51e-02
1.999	1.68e-04	1.39e-04	1.44e-04
1.99999	1.69e-06	1.39e-06	1.44e-06
2.0	1.09e-08	1.41e-13	4.50e-15

Table 4: $\|u - \hat{u}_h\|_\infty$ for various values of β for Example 3.2 with $\alpha = 2$.

β	$k = 1$	$k = 2$	$k = 3$
0.7	1.83e-02	1.41e-02	1.51e-02
0.8	1.40e-02	1.06e-02	1.14e-02
0.9	8.12e-03	6.04e-03	6.48e-03
0.999	9.46e-05	6.91e-05	7.41e-05
0.99999	9.37e-07	6.92e-07	7.42e-07
1.0	1.09e-08	1.41e-13	4.50e-15

Table 5: $\|u - \hat{u}_h\|_\infty$ for various values of α for Example 3.3.

α	$k = 1$	$k = 2$	$k = 3$
1.2	3.88e-05	9.49e-06	4.04e-06
1.4	1.18e-04	2.23e-05	8.62e-06
1.6	1.91e-04	2.45e-05	8.39e-06
1.8	2.32e-04	1.53e-04	4.51e-06
2.0	1.44e-10	1.14e-15	1.20e-15

$\alpha = 1.2$ so that the exact solution is $u(x) = \frac{x^{2.2}}{\Gamma(3.2)}$ whose first and second derivatives are in $L^2(\Omega)$ but not its third derivative. As is typical of all finite element, DG, as well as HDG methods, this affects the convergence orders of the method. That is, the convergence order is limited by the smoothness of the exact solution no matter how high the polynomial degree of the approximation is. Indeed, we see that u_h converges with order 1.7 and p_h converges with order 0.7 for all $k \geq 1$. For q_h and \hat{u}_h the errors seem to stagnate around a certain accuracy that's why we cannot extract a clear convergence order for these two variable from this table.

We see from Table 7 that the condition number of the global stiffness matrix is in the order of h^{-2} independent of the polynomial degree. This is precisely the behavior of the condition number of the stiffness matrix for practically all finite element methods for second order elliptic problems. Our numerical experiments which we do not report here indicate that the condition numbers exhibit a similar behavior for other values of $\tau > 0$.

Table 6: History of convergence for Example 3.3 with $\alpha = 1.2$.

k	N	$\ u - u_h\ _0$		$\ q - q_h\ _0$		$\ p - p_h\ _0$		$\ u - \hat{u}_h\ _\infty$	
		error	order	error	order	error	order	error	order
1	8	7.69e-03	1.67	5.66e-04	1.67	7.89e-03	0.70	3.75e-05	-
	16	2.40e-03	1.68	1.95e-04	1.54	4.87e-03	0.70	3.83e-05	-
	32	7.44e-04	1.69	1.06e-04	0.88	3.01e-03	0.69	3.88e-05	-
	64	2.32e-04	1.68	9.30e-05	0.18	1.88e-03	0.68	3.90e-05	-
2	8	1.35e-03	1.70	1.01e-04	1.66	3.79e-03	0.70	9.42e-06	-
	16	4.15e-04	1.70	3.76e-05	1.42	2.34e-03	0.70	9.46e-06	-
	32	1.28e-04	1.70	2.42e-05	0.64	1.44e-03	0.70	9.49e-06	-
	64	3.99e-05	1.68	2.25e-05	0.10	8.89e-04	0.69	9.49e-06	-
3	8	4.87e-04	1.70	3.85e-05	1.66	2.40e-03	0.70	4.00e-06	-
	16	1.50e-04	1.70	1.49e-05	1.37	1.48e-03	0.70	4.03e-06	-
	32	4.62e-05	1.70	1.01e-05	0.55	9.11e-04	0.70	4.04e-06	-
	64	1.45e-05	1.67	9.57e-06	0.08	5.62e-04	0.70	4.05e-06	-

Table 7: Condition numbers of the global stiffness matrix for $\alpha = 1.5$ and $\beta = 0.5$.

N	$k = 1$	$k = 2$	$k = 3$
8	3.08e+01	3.08e+01	3.08e+01
16	1.06e+02	1.05e+02	1.05e+02
32	4.25e+02	4.24e+02	4.24e+02
64	1.70e+03	1.70e+03	1.70e+03

4. Conclusion

We have introduced an HDG method for a class of fractional boundary value problems involving Caputo derivatives. We showed that the HDG method is efficiently implementable since the only globally coupled unknowns are those associated with the numerical trace \hat{u}_h of the primal variable u_h . The interior degrees freedom for u_h and the remaining variables can then be recovered from \hat{u}_h by an element-by-element computation. We have displayed numerical results to assess the convergence behavior of the methods. We observed that when polynomials of degree at most k are used for all unknowns, u_h converges with the optimal order $k + 1$, p_h and q_h seem to converge optimal order $k + 1$ or suboptimally with order k **when the exact solution is sufficiently smooth. For non-smooth solutions the convergence orders are limited by the smoothness level of exact solution.** A theoretical study of these convergence properties is the subject of ongoing work.

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