Solutions to matrix equations $X - AXB = CY + R$ and $X - A\widehat{X}B = CY + R$

Caiqin Song, Guoliang Chen

PII: S0377-0427(18)30273-5
DOI: https://doi.org/10.1016/j.cam.2018.05.003
Reference: CAM 11667

To appear in: *Journal of Computational and Applied Mathematics*

Received date : 16 February 2016
Revised date : 19 January 2018

Solutions to matrix equations $X - AXB = CY + R$ and $X - A\widehat{X}B = CY + R$ *

Caiqin Song[a]†, Guoliang Chen[b]

[a] *School of Mathematical Sciences, University of Jinan, Jinan 250022, P. R. China*

[b] *Department of Mathematics, East China Normal University, Shanghai 200241, P.R. China*

Abstract

The present work proposed an alternative approach to find the closed-form solutions of the nonhomogeneous Yakubovich matrix equation $X - AXB = CY + R$. Based on the derived closed-form solution to the nonhomogeneous Yakubovich matrix equation, the solutions to the nonhomogeneous Yakubovich quaternion j-conjugate matrix equation $X - A\widehat{X}B = CY + R$ are obtained by the use of the real representation of a quaternion matrix. The existing complex representation method requires the coefficient matrix A to be a block diagonal matrix over complex field. In contrast in this publication we allow a quaternion matrix of any dimension. As an application, eigenstructure assignment problem for descriptor linear systems is considered.

Keywords: Closed-form solution; quaternion matrix equation; real representation

1. Introduction

The generalized Sylvester matrix equation

$$AX - EXF = CY, \tag{1.1}$$

is closely related with many problems in control theory, such as pole/eigenstructure assignment

*This project is granted financial support from NSFC(No.11501246), NSFC(No.11471122) and Postdoctoral Science Foundation of China (No.244) .

†Corresponding author E-mail addresses: songcaiqin@gmail.com(C.Song), glchen@math.ecnu.edu.cn (G.Chen)

design [1-2], Luenberger-type observer design [3-5], robust fault detection [6-11] and so on. Finding the complete parametric solutions of the matrix equation (1.1), i.e. parametric solutions consisting of the maximum number of free parameters, allows for a robust control system design for a larger class of design problems. The generalized Sylvester matrix equation (1.1) is equivalent to the generalized Sylvester matrix equation

$$X - AXB = CY, \tag{1.2}$$

with their coefficient matrices satisfying some relations. Due to the above mentioned applications, the generalized Sylvester matrix equation (1.1) and (1.2) has been studied by many authors. Assuming F in matrix equation (1.1) is in Jordan form, Duan [12, 13] has obtained a parametric solution of the generalized Sylvester matrix equation through the use of the right coprime factorization of the input-state transfer function $(sE - A)^{-1}B$. However, this solution is not in a direct, explicit form but in a recursive form. In [14-16], an explicit, analytical and complete solution to the matrix equation (1.2) is obtained. In [14], the explicit solution can be obtained by the use of the Kronecker map [52]. In [15], the proposed solution is in a very compact form and can be immediately obtained with a series of matrices $D_i, i = 0, 1, \cdots, \omega$. Obviously, the following nonhomogeneous Yakubovich matrix equation

$$X - AXB = CY + R, \tag{1.3}$$

where $A \in R^{n\times n}$, $B \in R^{p\times p}$ and $C \in R^{n\times r}$ are the given real matrices, $X \in R^{n\times p}$ and $Y \in R^{r\times p}$ are the matrices to be determined, is the generalized form of the matrix equation (1.2).This nonhomogeneous Yakubovich matrix equation plays an important role in output regulator design for time-invariant linear systems and eigenstructure assignment for matrix second-order linear systems [17-20]. However, there is little in the literature considering such type of matrix equation [21].

For two complex square matrices A and B, there exists a nonsingular complex matrix P such that $A = P^{-1}B\overline{P}$ [22]. Then A and B is called consimilarity. This is another type of similarity in the field of linear algebra. The consimilarity theory of complex matrices plays an important role in the research of modern quantum theory [23]. For this reason, the matrix equation $X - AXB = CY$ in complex field has been extended to complex conjugate matrix equation $X - A\overline{X}B = CY$ by an application of the consimilarity concept [18]. Moreover, the consimilarity of quaternion matrix is also defined in [24]. Similarly, the quaternion matrix equation $X - AXB = CY$ has also been extended to the quaternion j-conjugate matrix equation $X - A\widehat{X}B = CY$ by an application of the consimilarity concept over quaternion field [25]. Due to its compact notation, moderate computational requirements and avoidance of singularities associated to 3×3 rotation matrices [26], these quaternion matrix equations have been extensively used in computer graphics, vector-sensor processing, and aerospace problems [27-30].

Let us point out that the least squares solution of the quaternion j-conjugate matrix equation $X - A\widehat{X}B = C$ (where $\widehat{X}$ denotes the j-conjugate of quaternion matrix X) with the least squares norm [31] has been obtained by the use of the complex representation of quaternion matrix, the Moore-Penrose generalized inverse and the Kronecker product. Some necessary and sufficient conditions for the existence of the solution to the quaternion matrix equations $AXB + CYD = E, (AX, XC) = (B, D), AXB = C$, have been proposed and the solutions have been derived in the references [32-34]. The explicit solutions to the quaternion j-conjugate matrix equation $X - A\widehat{X}B = C$ and $X - A\widehat{X}B = CY$ [25, 35, 36], where the coefficient quaternion matrix A is a block diagonal form over complex field, have been studied. The general solution, parameterize general solution and explicit solution to some quaternion matrix equations are obtained in [37-41]. Some necessary and sufficient conditions, the maximal and minimal ranks of the solutions are presented in these paper.

In this paper, we are concerned with the explicit solutions to two kinds of nonhomogeneous Yakubovich matrix equations, which are considered in real field and in quaternion field. The class of matrix equations which are studied in the current paper include many linear matrix equations as special cases, such as discrete Sylvester matrix equation, generalized Sylvester matrix equation, discrete Lyapunov matrix equation, and so on [18-22, 31, 35-36]. The motivation for this paper comes from the references [15-18, 20, 21, 25, 35, 36, 39-46, 48-55]. This study mainly includes two parts. Firstly, we present an alternative approach to obtain the explicit solution of the nonhomogeneous Yakubovich matrix equation in real field. It is also shown that the explicit solution to the considered matrix equation is expressed as the coefficient matrices A, B, C, the free parameter matrix Z and the coefficient matrix of polynomial matrix $adj(I - sA)$. Secondly, based on the derived solutions to the nonhomogeneous Yakubovich matrix equation in real field, the explicit solutions to the j-conjugate matrix equation in quaternion field are discussed by means of real representation of a quaternion matrix.

The rest of this paper is organized as follows. In Section 2, the real matrix equation is studied. First of all, a method is presented for obtaining the solution to the nonhomogeneous Yakubovich matrix equation. Section 3 is devoted to introducing the real representation of a quaternion matrix and studying the solution to the quaternion j-conjugate matrix equation. In Section 4, some numerical examples and application examples are provided to show the effectiveness of the obtained results. Finally, the paper is ended with a brief conclusion in Section 5.

Throughout this paper, we use the following notations. Let R, C and $Q = R \bigoplus Ri \bigoplus Rj \bigoplus Rk$ denote the real number field, the complex number field and the quaternion field, respectively, where $i^2 = j^2 = k^2 = -1, ij = -ji = k$. $R^{m\times n}(C^{m\times n}$ or $Q^{m\times n})$ denotes the set of all $m \times n$ matrices on R (C or Q). For any matrix $A \in C^{m\times n}$, A^T, $\overline{A}$, A^H, $detA$ and $adj(A)$ represent the transpose, conjugate, conjugate transpose, determinant and adjoint of A, respectively. In addition, symbol A_σ is the real representation of a quaternion matrix A. $A \otimes B = (a_{ij}B)$ denotes the Kronecker product of two matrices A and B. If $A \in Q^{m\times n}$, let $A = A_1 + A_2 i + A_3 j + A_4 k$, where $A_t \in R^{m\times n}, t = 1, \cdots, 4$, and define $\widehat{A} = A_1 - A_2 i + A_3 j - A_4 k$ to be the j-conjugate of the quaternion matrix A. For $A \in C^{m\times n}$, $vec(A)$ is defined as $vec(A) = \begin{bmatrix} a_1^T & a_2^T & \cdots & a_n^T \end{bmatrix}^T$, where $a_i (i = 1, 2, \cdots, n)$ are some complex numbers. We denote the $n \times n$ identity matrix by I_n. We also write it as I . Furthermore, let $A \in R^{n\times n}$,$B \in R^{n\times r}$, and $C \in R^{m\times n}$, we have the following notations associated with these matrices:

$$Ctr_n(A, B) = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}, Obs_k(A, C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{k-1} \end{bmatrix},$$

$$f_{(I,A)}(s) = det(I - sA) = \alpha_n s^n + \alpha_{n-1} s^{n-1} + \cdots + \alpha_1 s + 1,$$

$$S_r(I, A) = \begin{bmatrix} I_r & \alpha_1 I_r & \alpha_2 I_r & \cdots & \alpha_{n-1} I_r \\ & I_r & \alpha_1 I_r & \cdots & \alpha_{n-2} I_r \\ & & & \cdots & \cdots \\ & & & I_r & \alpha_1 I_r \\ & & & & I_r \end{bmatrix}.$$

Thus, $Ctr_n(A, B)$ is the controllability matrix of the matrix pair (A, B), $Obs_k(A, C)$ is the observability matrix of the matrix pair (A, C), $S_r(I, A)$ is a symmetric operator matrix.

2. Real matrix equation $X - AXB = CY + R$

In this section, we will propose an alternative approach to solve the nonhomogeneous Yakubovich matrix equation (1.3). Before proceeding, we need the following lemma on the matrix equation

$$X_0 - AX_0B = R, \tag{2.1}$$

where $A \in R^{n\times n}$, $B \in R^{p\times p}$ and $R \in R^{n\times p}$ are known matrices, and X_0 is the matrix to be determined.

Lemma 1 [47]. Given matrices $A \in R^{n\times n}$, $B \in R^{p\times p}$ and $R \in R^{n\times p}$, suppose that $\{s|det(I - sA) = 0\} \bigcap \lambda(B) = \phi$, and let

$$f_{(I,A)}(s) = det(I - sA) = \alpha_n s^n + \cdots + \alpha_1 s + \alpha_0, \alpha_0 = 1,$$

and

$$adj(I - sA) = \sum_{i=0}^{n-1} R_i s^i. \tag{2.2}$$

Then the unique solution to the matrix equation (2.1) can be characterized by

$$X_0 = \left(\sum_{i=0}^{n-1} R_i R B^i\right) \left[f_{(I,A)}(B)\right]^{-1}.$$

Theorem 1. Given matrices $A \in R^{n\times n}$, $B \in R^{p\times p}$, $C \in R^{n\times r}$ and $R \in R^{n\times p}$, suppose that $\{s|det(I - sA) = 0\} \bigcap \lambda(B) = \phi$, and (2.2) holds. Then all solutions to the nonhomogeneous Yakubovich matrix equation (1.3) can be expressed as

$$\begin{cases} X = & \sum_{i=0}^{n-1} R_i CZB^i + \left(\sum_{i=0}^{n-1} R_i R B^i\right) \left[f_{(I,A)}(B)\right]^{-1}, \\ Y = & Zf_{(I,A)}(B), \end{cases} \tag{2.3}$$

where $Z \in R^{r\times p}$ is an arbitrary chosen free parameter matrix.

Proof. Regarding the solution of $\alpha_i, i \in I[0, n]$, and $R_i, i \in I[0, n - 1]$ in Lemma 1, the so-called generalized Leverrier algorithm [45] can be stated as the following relation:

$$\begin{cases} R_i = AR_{i-1} + \alpha_i I, & R_0 = I, i \in I[1, n]. \\ \alpha_i = \frac{trace(AR_{i-1})}{i}, & i \in I[1, n]. \end{cases}$$

Suppose that a solution to the nonhomogeneous Yakubovich matrix equation (1.3) can be expressed as $X = T + \widetilde{X}, Y = \widetilde{Y}$ with $(\widetilde{X}, \widetilde{Y})$ being a solution to matrix equation (1.2). Then one has

$$(T + \widetilde{X}) - A(T + \widetilde{X})B - C\widetilde{Y} - R = \widetilde{X} - A\widetilde{X}B + T - ATB - C\widetilde{Y} - R = T - ATB - R.$$

Under this condition, when $X_0 = T$ is a solution to (2.1), $X = T + \widetilde{X}, Y = \widetilde{Y}$ is a solution to (1.3). Combining this fact and Theorem 2 in [14], now we can obtain the solution to the nonhomogeneous Yakubovich matrix equation (1.3).

Corollary 1.Given matrices $A \in R^{n\times n}$, $B \in R^{p\times p}$, $C \in R^{n\times r}$ and $R \in R^{n\times p}$, suppose that $\{s|det(I - sA) = 0\} \bigcap \lambda(B) = \phi$, and (2.2) holds, then the relation

$$\begin{cases} X = \left(\sum_{i=0}^{n-1} R_i R B^i\right) \left[f_{(I,A)}(B)\right]^{-1}, \\ Y = 0, \end{cases} \tag{2.4}$$

is also a special solution to the matrix equation (1.3), where $Z \in R^{r\times p}$ is an arbitrary chosen free parameter matrix.

Proof. If we let $Z = 0$ in relation (2.3), we can obtain the relation (2.4). This complete the proof of this theorem.

Next we have the following equivalent forms of the solution in Theorem 1.

Theorem 2. Given matrices $A \in R^{n\times n}$, $B \in R^{p\times p}$, $C \in R^{n\times r}$ and $R \in R^{n\times p}$, suppose that $\{s|det(I - sA) = 0\} \bigcap \lambda(B) = \phi$, and (2.2) holds. Then all solutions to the nonhomogeneous Yakubovich matrix equation (1.3) can be expressed as

$$\begin{cases} X f_{(I,A)}(B) = \sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \alpha_k A^{i-k} C Z B^i f_{(I,A)}(B) + \sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \alpha_k A^{i-k} R B^i, \\ Y = Z f_{(I,A)}(B), \end{cases} \tag{2.5}$$

where $Z \in R^{r\times p}$ is an arbitrary chosen free parameter matrix.

Proof. According to the relation (2.3), we have

$$\begin{cases} R_0 = I_n, \\ R_1 = \alpha_1 I_n + A, \\ R_2 = \alpha_2 I_n + \alpha_1 A + A^2, \\ \cdots \\ R_{n-1} = \alpha_{n-1} I_n + \alpha_{n-2} A + \cdots + A^{n-1}. \end{cases}$$

This relation can be compactly expressed as

$$R_i = \sum_{k=0}^{i} \alpha_k A^{i-k}, \alpha_0 = 1, i = 1, 2, \cdots, n-1.$$

Hence, we can obtain

$$\sum_{i=0}^{n-1} R_i CZB^i = \sum_{i=0}^{n-1} \sum_{k=0}^{i} \alpha_k A^{i-k} CZB^i = \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \alpha_k A^{i-k} CZB^i,$$

and

$$\sum_{i=0}^{n-1} R_i RB^i = \sum_{i=0}^{n-1} \sum_{k=0}^{i} \alpha_k A^{i-k} RB^i = \sum_{k=0}^{n-1} \sum_{i=0}^{n-1} \alpha_k A^{i-k} RB^i.$$

By Theorem 1, we can derive the conclusion. $\square$

Corollary 2. Let $A \in R^{n\times n}$, $B \in R^{p\times p}$, $C \in R^{n\times r}$ and $R \in R^{n\times p}$, suppose that $\{s|det(I - sA) = 0\} \bigcap \lambda(B) = \phi$, and (2.2) holds. Then all solutions to the nonhomogeneous Yakubovich matrix equation (1.3) can be expressed as

$$\begin{cases} X = \sum_{i=0}^{n-1} R_i[CZf_{(I,A)}(B) + R)B^i[f_{(I,A)}(B)]^{-1}], \\ Y = Zf_{(I,A)}(B), \end{cases}$$

where $Z \in R^{r\times p}$ is an arbitrary chosen free parameter matrix.

Proof. By considering (2.3), we can get

$$\begin{aligned} Xf_{(I,A)}(B) &= \sum_{i=0}^{n-1} R_i CZB^i f_{(I,A)}(B) + \sum_{i=0}^{n-1} R_i RB^i \\ &= \sum_{i=0}^{n-1} R_i CZf_{(I,A)}(B)B^i + \sum_{i=0}^{n-1} R_i RB^i \\ &= \sum_{i=0}^{n-1} R_i(CZf_{(I,A)}(B) + R)B^i \end{aligned}$$

Now Post-multiplying the two sides of the above relation by $(f_{(I,A)}(B))^{-1}$, we can obtain

$$X = \sum_{i=0}^{n-1} R_i(CZf_{(I,A)}(B) + R)B^i[f_{(I,A)}(B)]^{-1}$$

The proof is finished. □

Corollary 3. Given matrices $A \in R^{n\times n}$, $B \in R^{p\times p}$, $C \in R^{n\times r}$ and $R \in R^{n\times p}$, suppose that $\{s|det(I - sA) = 0\} \bigcap \lambda(B) = \phi$, and (2.2) holds. Then all the solutions to the nonhomogeneous Yakubovich matrix equation (1.3) can be characterized by

$$\begin{cases} X = & Ctr_n(A, CZ)S_r(I, A)Obs_n(B, f_{(I,A)}(B)) + Ctr_n(A, R)S_r(I, A)Obs_n(B, I_n) \\ Y = & Zf_{(I,A)}(B), \end{cases}$$

where $Z \in R^{r\times p}$ is an arbitrary chosen free parameter matrix.

Proof. By the direct computation, it follows from Theorem 2 that we can get

$$\begin{array}{ll} \sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \alpha_k(A)^{i-k}CZB^i f_{(I,A)}(B) & = \sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \alpha_k(A)^{i-k}CZf_{(I,A)}(B)B^i \\ & = Ctr_n(A, CZ)S_r(I, A)Obs_n(B, f_{(I,A)}(B)) \end{array},$$

and

$$\sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \alpha_k A^{i-k}RB^i = Ctr_n(A, R)S_r(I, A)Obs_n(B, I_n).$$

Hence, we have

$$\begin{array}{l} \sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \alpha_k(A)^{i-k}CZB^i f_{(I,A)}(B) + \sum_{k=0}^{n-1}\sum_{i=0}^{n-1} \alpha_k A^{i-k}RB^i \\ = Ctr_n(A, CZ)S_r(I, A)Obs_n(B, f_{(I,A)}(B)) + Ctr_n(A, R)S_r(I, A)Obs_n(B, I_n) \end{array}.$$

The proof is finished. □

3. Quaternion j-conjugate matrix equation $X - A\widehat{X}B = CY + R$

In this section, we consider the solution to the quaternion j-conjugate matrix equation by applying of the real representation of a quaternion matrix. First of all, we introduce the definition and some properties of the real representation of a quaternion matrix.

3.1. *Real representation of a quaternion matrix*

For any quaternion matrix $A = A_1 + A_2 i + A_3 j + A_4 k \in Q^{m\times n}, A_l \in R^{m\times n}(l = 1, 2, 3, 4)$, the real representation matrix of quaternion matrix A can be defined as

$$A_\sigma = \begin{bmatrix} A_1 & A_2 & -A_3 & A_4 \\ A_2 & -A_1 & -A_4 & -A_3 \\ A_3 & -A_4 & A_1 & A_2 \\ A_4 & A_3 & A_2 & -A_1 \end{bmatrix} \in R^{4m\times 4n}.$$

For a $m \times n$ quaternion matrix A, we define $A^t_\sigma = (A_\sigma)^t$. In addition, if we let

$$P_t = \begin{bmatrix} I_t & 0 & 0 & 0 \\ 0 & -I_t & 0 & 0 \\ 0 & 0 & I_t & 0 \\ 0 & 0 & 0 & -I_t \end{bmatrix}, Q_t = \begin{bmatrix} 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \\ 0 & 0 & 0 & I_t \\ 0 & 0 & -I_t & 0 \end{bmatrix},$$
$$S_t = \begin{bmatrix} 0 & 0 & 0 & -I_t \\ 0 & 0 & I_t & 0 \\ 0 & -I_t & 0 & 0 \\ I_t & 0 & 0 & 0 \end{bmatrix}, R_t = \begin{bmatrix} 0 & 0 & I_t & 0 \\ 0 & 0 & 0 & I_t \\ -I_t & 0 & 0 & 0 \\ 0 & -I_t & 0 & 0 \end{bmatrix}.$$

in which I_t is a $t \times t$ identity matrix, then P_t, Q_t, S_t, R_t are unitary matrices. Next, the properties of the real representation can be expressed as the following, which are given in [43].

Proposition 1 [43]. Assume that the quaternion matrices $A, B \in Q^{m\times n}, C \in Q^{n\times s}, a \in R$, then

(1) $(A + B)_\sigma = A_\sigma + B_\sigma, (aA)_\sigma = aA_\sigma, (AC)_\sigma = A_\sigma P_n C_\sigma = A_\sigma(\widehat{C})_\sigma P_s$;

(2) $A = B \Leftrightarrow A_\sigma = B_\sigma$;

(3) $Q^{-1}_m A_\sigma Q_n = -A_\sigma, R^{-1}_m A_\sigma R_n = A_\sigma, S^{-1}_m A_\sigma S_n = -A_\sigma, P^{-1}_m A_\sigma P_n = (\widehat{A})_\sigma$;

(4) The quaternion matrix A is nonsingular if and only if A_σ is nonsingular, and the quaternion matrix A is an unitary matrix if and only if A_σ is an orthogonal matrix;

(5) If $A \in Q^{m\times m}$, then $A^{2k}_\sigma = ((A\widehat{A})^k)_\sigma P_m$;

(6) $A \in Q^{m\times m}, B \in Q^{n\times n}, C \in Q^{m\times n}$ and $k + l$ is even, then

$$A^k_\sigma C_\sigma B^l_\sigma = \begin{cases} ((A\widehat{A})^s (A\widehat{C}B)(\widehat{B}B)^t)_\sigma, & k = 2s + 1, l = 2t + 1, \\ ((A\widehat{A})^s C(\widehat{B}B)^t)_\sigma, & k = 2s, l = 2t. \end{cases}$$

Proposition 2 [**43**]. If λ is a characteristic value of A_σ, then so are $\pm\lambda, \pm\overline{\lambda}$.

For any $A \in Q^{m\times m}$, let the characteristic polynomial of the real representation matrix A_σ be $f_{(I,A_\sigma)}(\lambda) = det(I_{4m} - \lambda A_\sigma) = \sum_{k=0}^{2m} a_{2k}\lambda^{2k}$, and define $h_{A_\sigma}(\lambda) = \lambda^{4m} f_{(I,A_\sigma)}(\lambda^{-1}) = \sum_{k=0}^{2m} a_{2k}\lambda^{2(2m-k)}$. So by Proposition 1 and Proposition 2 we have the following Proposition 3.

Proposition 3. Let $A \in Q^{m\times m}, B \in Q^{n\times n}$. Then
(1) $f_{(I,A_\sigma)}(\lambda)$ is a real polynomial, and $f_{(I,A_\sigma)}(\lambda) = \sum_{k=0}^{2m} a_{2k}\lambda^{2k}$;
(2) $h_{A_\sigma}(\lambda)$ is a real polynomial, and $h_{A_\sigma}(\lambda) = \sum_{k=0}^{2m} a_{2k}\lambda^{2(2m-k)}$;
(3) $h_{A_\sigma}(B_\sigma) = (g_{A_\sigma}(B\widetilde{B}))_\sigma P_n, f_{(I,A_\sigma)}(B_\sigma) = (p_{A_\sigma}(B\widetilde{B}))_\sigma P_n$,
in which $g_{A_\sigma}(\lambda) = \sum_{k=0}^{2m} a_{2k}\lambda^{2m-k}$, $p_{A_\sigma}(\lambda) = \sum_{k=0}^{2m} a_{2k}\lambda^{k}$ are real polynomials.
Proof. By Proposition 2, we easily know that a_k is a real number, and $a_{2k+1} = 0$. For any k, by Proposition 1, we have $B_\sigma^{2k} = ((B\widetilde{B})^k)_\sigma P_n$, so we can obtain the result (3).

3.2. *On solutions to the quaternion j-conjugate matrix equation* $X - A\widehat{X}B = CY + R$

In this subsection, the solution to the quaternion j-conjugate matrix equation

$$X - A\widehat{X}B = CY + R, \tag{3.1}$$

is presented by the use of real representation of a quaternion matrix, in which $A \in Q^{n\times n}$, $B \in Q^{p\times p}$, $C \in Q^{n\times r}$ and $R \in Q^{n\times p}$ are given quaternion matrices, $X \in Q^{n\times p}$ and $Y \in Q^{r\times p}$ are the determined quaternion matrices. Moreover, the real representation matrix equation of quaternion matrix equation (3.1) is defined as

$$V - A_\sigma V B_\sigma = C_\sigma P_r W + R_\sigma. \tag{3.2}$$

According to (1) in Proposition 1, the quaternion matrix equation (3.1) is equivalent to the matrix equation $(X - A\widehat{X}B)_\sigma = X_\sigma - A_\sigma X_\sigma B_\sigma$. Hence, the matrix equation (3.1) can be converted into $X_\sigma - A_\sigma X_\sigma B_\sigma = C_\sigma P_r Y_\sigma + R_\sigma$. Therefore, we have the following conclusions.

Theorem 3. Suppose that $A \in Q^{n\times n}$, $B \in Q^{p\times p}$, $C \in Q^{n\times r}$ and $R \in Q^{n\times p}$, the quaternion matrix equation (3.1) has a solution (X, Y) if and only if the real representation matrix equation (3.2) has a solution $(V, W) = (X_\sigma, Y_\sigma)$. Furthermore, if (V, W) is a solution to (3.2), the following quaternion matrices are solutions to the quaternion matrix equation (3.1)

$$\begin{cases} X = \dfrac{1}{16} \begin{bmatrix} I_n & iI_n & jI_n & kI_n \end{bmatrix} (V - Q_n^{-1} V Q_p + R_n^{-1} V R_p - S_n^{-1} V S_p) \begin{bmatrix} I_p \\ -iI_p \\ -jI_p \\ -kI_p \end{bmatrix}, \\ Y = \dfrac{1}{16} \begin{bmatrix} I_r & iI_r & jI_r & kI_r \end{bmatrix} (W - Q_n^{-1} W Q_p + R_n^{-1} W R_p - S_n^{-1} W S_p) \begin{bmatrix} I_p \\ -iI_p \\ -jI_p \\ -kI_p \end{bmatrix}. \end{cases} \tag{3.3}$$

Proof. By (3) of Proposition 1, the matrix equation (3.2) is equivalent to

$$V - R_n^{-1} A_\sigma R_n V R_p^{-1} B_\sigma R_p = R_n^{-1} C_\sigma R_r P_r W + R_\sigma. \tag{3.4}$$

Post-multiplying the two sides of the quaternion matrix equation (3.4) by R_p^{-1}, we can obtain

$$V R_p^{-1} - R_n^{-1} A_\sigma R_n V R_p^{-1} B_\sigma = R_n^{-1} C_\sigma R_r P_r W R_p^{-1} + R_\sigma R_p^{-1}. \tag{3.5}$$

Pre-multiplying the two sides of (3.5) by R_n and $R_p^{-1} = -R_p, R_r P_r = P_r R_r$, we can get

$$R_n^{-1} V R_p - A_\sigma R_n^{-1} V R_p B_\sigma = C_\sigma P_r R_r^{-1} W R_p + R_\sigma. \tag{3.6}$$

This shows that if (V, W) is a real solution of matrix equation (3.2), $(R_n^{-1} V R_p, R_r^{-1} W R_p)$ is also a real solution of matrix equation (3.2). In addition, according to (3) of Proposition 1, the matrix equation (3.2) is also equivalent to

$$V - Q_n A_\sigma Q_n V Q_p B_\sigma Q_p = Q_n C_\sigma Q_r P_r W + Q_n R_\sigma Q_p. \tag{3.7}$$

Post-multiplying the two sides of the quaternion matrix equation (3.7) by Q_p^{-1}, we have

$$V Q_p^{-1} - Q_n A_\sigma Q_n V Q_p B_\sigma = Q_n C_\sigma Q_r P_r W Q_p^{-1} + Q_n R_\sigma. \tag{3.8}$$

Note $Q_p^{-1} = -Q_p, Q_r P_r = -P_r Q_r$, pre-multiplying the two sides of the quaternion matrix equation (3.8) by Q_n^{-1}, we conclude that

$$(-Q_n^{-1} V Q_p) - A_\sigma(-Q_n^{-1} V Q_p) B_\sigma = C_\sigma P_p(-Q_r^{-1} W Q_p) + R_\sigma.$$

This means $(-Q_n^{-1}VQ_p, -Q_r^{-1}WQ_p)$ is also a real solution of matrix equation (3.2) if (V, W) is a real solution of matrix equation (3.2). Similarly, we can prove $(-S_n^{-1}VS_p, -S_r^{-1}WS_p)$ is also a real solution of quaternion matrix equation (3.2). In this case, the conclusion can be obtained along the line of the proof of Theorem 4.2 in [43]. $\square$

Theorem 4. Given quaternion matrices $A \in Q^{n\times n}$, $B \in Q^{p\times p}$, $C \in Q^{n\times r}$, $R \in Q^{n\times p}$, let

$$f_{(I,A_\sigma)}(s) = det(I_{4n} - sA_\sigma) = \sum_{k=0}^{2n} a_{2k}s^{2k}, \qquad p_{A_\sigma}(s) = \sum_{k=0}^{2n} a_{2k}s^k,$$

then the solutions to the quaternion j-conjugate matrix equation (3.1) can be characterized by

$$\begin{cases} Xp_{A_\sigma}(\widehat{B}B) = \sum\limits_{k=0}^{2n-1}\sum\limits_{s=0}^{2n-1} \alpha_{2k}(A\widehat{A})^{s-k}(CZ + A\widehat{C}\widehat{Z}B)(\widehat{B}B)^s p_{A_\sigma}(\widehat{B}B) \\ \qquad\qquad + \sum\limits_{k=0}^{2n-1}\sum\limits_{s=0}^{2n-1} \alpha_{2k}(A\widehat{A})^{s-k}(R + A\widehat{R}B)(\widehat{B}B)^s, \\ Y = Zp_{A_\sigma}(\widehat{B}B), \end{cases}$$

in which Z is an arbitrary quaternion matrix.

Proof. If the Yakubovich quaternion j-conjugate matrix equation (3.1) has solution (X, Y), then its real representation matrix equation (3.2) has solution $(V, W) = (X_\sigma, Y_\sigma)$ with the free parameter Z_σ. Now according to Theorem 2 and 3, we can conclude that

$$X_\sigma f_{(I,A_\sigma)}(B_\sigma) = \sum_{k=0}^{2n-1}\sum_{j=0}^{4n-1} \alpha_{2k}A_\sigma^{j-2k}C_\sigma P_r Z_\sigma B_\sigma^j f_{(I,A_\sigma)}(B_\sigma) + \sum_{k=0}^{2n-1}\sum_{j=0}^{4n-1} \alpha_{2k}A_\sigma^{j-2k}R_\sigma B_\sigma^j,$$

and

$$Y_\sigma = Z_\sigma f_{(I,A_\sigma)}(B_\sigma).$$

Moreover, by Proposition 1 and Proposition 3, we can get

$$f_{(I,A_\sigma)}(B_\sigma) = (p_{A_\sigma}(B\widehat{B}))_\sigma P_p.$$

Hence, we can obtain

$$X_\sigma f_{(I,A_\sigma)}(B_\sigma) = X_\sigma[p_{A_\sigma}(B\widehat{B})]_\sigma P_p = [Xp_{A_\sigma}(\widehat{B}B)]_\sigma,$$

and

$$Y_\sigma = Z_\sigma f_{(I,A_\sigma)}(B_\sigma) = Z_\sigma (p_{A_\sigma}(B\widehat{B}))_\sigma P_p = (Z p_{A_\sigma}(\widehat{B}B))_\sigma.$$

Furthermore, by the use of Proposition 1 we have

$$\begin{aligned}
&\sum_{k=0}^{2n-1}\sum_{j=0}^{4n-1} \alpha_{2k} A_\sigma^{j-2k} C_\sigma P_r Z_\sigma B_\sigma^j f_{(I,A_\sigma)}(B_\sigma) \\
&= \sum_{k=0}^{2n-1} \alpha_{2k} \left[\sum_{s=0}^{2n-1} A_\sigma^{2s-2k} C_\sigma P_r Z_\sigma B_\sigma^{2s} + \sum_{s=1}^{2n} A_\sigma^{2s-2k+1} C_\sigma P_r Z_\sigma B_\sigma^{2s+1} \right] f_{(I,A_\sigma)}(B_\sigma) \\
&= \sum_{k=0}^{2n-1}\sum_{s=0}^{2n-1} \alpha_{2k} \left[((A\widehat{A})^{s-k})_\sigma P_n C_\sigma P_r Z_\sigma ((B\widehat{B})^s)_\sigma P_p + ((A\widehat{A})^{s-k})_\sigma P_n A_\sigma C_\sigma P_r Z_\sigma B_\sigma ((B\widehat{B})^s)_\sigma P_p \right] f_{(I,A_\sigma)}(\\
&= \sum_{k=0}^{2n-1}\sum_{s=0}^{2n-1} \alpha_{2k} \left[\left((A\widehat{A})^{s-k} C Z (\widehat{B}B)^s \right)_\sigma + \left((A\widehat{A})^{s-k} A\widehat{C}\widehat{Z} B (\widehat{B}B)^s \right)_\sigma \right] f_{(I,A_\sigma)}(B_\sigma) \\
&= \sum_{k=0}^{2n-1}\sum_{s=0}^{2n-1} \alpha_{2k} \left[\left((A\widehat{A})^{s-k} C Z (\widehat{B}B)^s p_{A_\sigma}(\widehat{B}B) \right)_\sigma + \left((A\widehat{A})^{s-k} A\widehat{C}\widehat{Z} B (\widehat{B}B)^s p_{A_\sigma}(\widehat{B}B) \right)_\sigma \right],
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{2n-1}\sum_{j=0}^{4n-1} \alpha_{2k} A_\sigma^{j-2k} R_\sigma B_\sigma^j = \sum_{k=0}^{2n-1} \alpha_{2k} \left[\sum_{s=0}^{2n-1} A_\sigma^{2s-2k} R_\sigma B_\sigma^{2s} + \sum_{s=1}^{2n} A_\sigma^{2s+1-2k} R_\sigma B_\sigma^{2s+1} \right] \\
&= \sum_{k=0}^{2n-1} \alpha_{2k} \left[\sum_{s=0}^{2n-1} ((A\widehat{A})^{s-k})_\sigma P_n R_\sigma ((\widehat{B}B)^s)_\sigma P_p + \sum_{s=1}^{2n} ((A\widehat{A})^{s-k})_\sigma P_n A_\sigma R_\sigma B_\sigma ((B\widehat{B})^s)_\sigma P_p \right] \\
&= \sum_{k=0}^{2n-1} \alpha_{2k} \left[\sum_{s=0}^{2n-1} \left((A\widehat{A})^{s-k} R (\widehat{B}B)^s \right)_\sigma + \sum_{s=0}^{2n-1} \left((A\widehat{A})^{s-k} A\widehat{R} B (\widehat{B}B)^s \right)_\sigma \right] \\
&= \sum_{k=0}^{2n-1}\sum_{s=0}^{2n-1} \alpha_{2k} \left[\left((A\widehat{A})^{s-k} R (\widehat{B}B)^s \right)_\sigma + \left((A\widehat{A})^{s-k} A\widehat{R} B (\widehat{B}B)^s \right)_\sigma \right].
\end{aligned}$$

The proof is finished. $\square$

In the following, we provide an equivalent statement of the above Theorem 4.

Theorem 5. Given quaternion matrices $A \in Q^{n\times n}$, $B \in Q^{p\times p}$, $C \in Q^{n\times r}$ and $R \in Q^{n\times p}$, let

$$f_{(I,A_\sigma)}(s) = det(I_{4n} - sA_\sigma) = \sum_{k=0}^{2n} a_{2k} s^{2k}, \qquad p_{A_\sigma}(s) = \sum_{k=0}^{2n} a_{2k} s^k,$$

then the solutions to the quaternion j-conjugate matrix equation (3.1) can be characterized as

$$\begin{cases} Xp_{A_\sigma}(\widehat{B}B) = Ctr_{2n}(A\widehat{A}, C)S_r(I, A_\sigma)Obs_{2n}(\widehat{B}B, Zp_{A_\sigma}(\widehat{B}B)) + Ctr_{2n}(A\widehat{A}, A\widehat{C})S_r(I, A_\sigma) \\ \quad Obs_{2n}(\widehat{B}B, \widehat{Z}Bp_{A_\sigma}(\widehat{B}B)) + Ctr_{2n}(A\widehat{A}, R)S_r(I, A_\sigma)Obs_{2n}(\widehat{B}B, I_r) \\ \quad + Ctr_{2n}(A\widehat{A}, A, 2n)S_r(I, A_\sigma)Obs_{2n}(\widehat{B}B, \widehat{R}B), \\ Y = Zp_{A_\sigma}(\widehat{B}B), \end{cases}$$

in which Z is an arbitrary quaternion matrix.

Proof. By the direct computation, we can get

$$\sum_{k=0}^{2n-1}\sum_{s=0}^{2n-1} \alpha_{2k}(A\widehat{A})^{s-k}CZ(\widehat{B}B)^s = Ctr_{2n}(A\widehat{A}, C)S_r(I, A_\sigma)Obs_{2n}(\widehat{B}B, Zp_{A_\sigma}(\widehat{B}B)),$$

and

$$\sum_{k=0}^{2n-1}\sum_{s=0}^{2n-1} \alpha_{2k}(A\widehat{A})^{s-k}A\widehat{C}\widehat{Z}B(\widehat{B}B)^s = Ctr_{2n}(A\widehat{A}, A\widehat{C})S_r(I, A_\sigma)Obs_{2n}(\widehat{B}B, \widehat{Z}Bp_{A_\sigma}(\widehat{B}B)).$$

Moreover, we can obtain

$$\sum_{k=0}^{2n-1}\sum_{s=0}^{2n-1} \alpha_{2k}(A\widehat{A})^{s-k}R(\widehat{B}B)^s = Ctr_{2n}(A\widehat{A}, R)S_r(I, A_\sigma)Obs_{2n}(\widehat{B}B, I_r),$$

and

$$\sum_{k=0}^{2n-1}\sum_{s=0}^{2n-1} \alpha_{2k}(A\widehat{A})^{s-k}A\widehat{R}B(\widehat{B}B)^s = Ctr_{2n}(A\widehat{A}, A)S_r(I, A_\sigma)Obs_{2n}(\widehat{B}B, \widehat{R}B).$$

Thus, the first conclusion has been proved. With this the second conclusion is obviously true. $\square$

4. Examples

4.1. Numerical Examples

In the following, we give two numerical examples and application examples to show the effectiveness of the obtained results.

Example 1 . Consider the nonhomogeneous Yakubovich matrix equation in the form of (1.3) with the following parameters:

$$A = \begin{pmatrix} 1 & 2 & 4 & -6 & -1 & -2 \\ -8 & -1 & 0 & -8 & 0 & 1 \\ 1 & 9 & 6 & -12 & 4 & 0 \\ -5 & -2 & 1 & -9 & 0 & 0 \\ 0 & 1 & 2 & -3 & -1 & 0 \\ 1 & 0 & 0 & 1 & -2 & 3 \end{pmatrix}, B = \begin{pmatrix} -11 & 1 & 0 & 7 & 2 & 1 & 0 & -11 \\ -9 & -8 & 1 & 19 & 21 & 23 & 24 & 2 \\ 1 & -2 & 3 & 9 & 11 & 5 & -4 & 3 \\ 18 & 1 & 4 & 6 & 7 & 9 & -1 & 0 \\ 9 & 8 & -7 & 6 & 5 & 4 & 3 & -2 \\ 0 & 1 & 11 & -2 & -3 & -4 & 5 & 4 \\ 11 & -1 & -2 & -3 & 4 & 5 & 0 & -5 \\ -4 & 2 & 3 & 6 & -7 & 9 & -19 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} -6.8367 & 6.9493 & -9.0781 & 2.5301 & -3.9528 & -5.5645 & -1.0500 & 2.2414 \\ 19.3891 & -7.2607 & 13.7305 & -8.0050 & 6.7793 & 7.7253 & -9.2575 & -1.0242 \\ -12.8797 & -8.9994 & -17.5988 & 9.5575 & 2.0448 & -11.2606 & 5.3451 & 11.1267 \\ 14.6108 & -3.0322 & 10.1333 & -5.9000 & 1.7579 & 7.7094 & -7.7738 & -1.9275 \\ 0.5285 & -1.7823 & -2.0056 & 0.7015 & -1.2497 & -0.5900 & -0.8977 & 0.7600 \\ 9.4432 & -17.0285 & 5.1795 & 0.1731 & -0.6505 & 6.7242 & 0.9935 & -1.7142 \end{pmatrix},$$

$$R = \begin{pmatrix} 1 & 2 & 3 & -11 & 4 & -3 & 9 & 9 \\ 4 & 3 & 7 & -12 & 3 & 4 & 10 & 7 \\ -1 & 4 & 6 & 13 & 2 & 5 & 11 & 6 \\ -12 & -9 & 5 & 14 & -1 & 6 & 12 & 5 \\ 1 & 2 & -4 & 6 & -1 & 7 & 11 & 4 \\ 6 & 7 & 3 & 7 & -2 & 8 & 13 & 3 \end{pmatrix}.$$

It is easy to check that $\{s|det(I - sA) = 0\} \bigcap \lambda(B) = \phi$. By simple computations, we have

$$f_{(I,A)}(s) = det(I - sA) = 752s^6 - 2648s^5 + 110s^4 + 529s^3 - 92s^2 + s + 1,$$

$$\alpha_6 = 752, \alpha_5 = -2648, \alpha_4 = 110, \alpha_3 = 529, \alpha_2 = -92, \alpha_1 = 1, \alpha_0 = 1,$$

$$[f_{(I,A)}(B)]^{-1}$$
$$= 10^{-4} \times \begin{pmatrix} -0.0117 & 0.0064 & 0.0121 & -0.0416 & 0.0177 & 0.0315 & 0.0466 & 0.0111 \\ 0.0261 & -0.0143 & -0.0269 & 0.0928 & -0.0396 & -0.0702 & -0.1040 & -0.0247 \\ 0.0046 & -0.0025 & -0.0047 & 0.0162 & -0.0069 & -0.0123 & -0.0182 & -0.0043 \\ -0.0223 & 0.0122 & 0.0230 & -0.0791 & 0.0337 & 0.0599 & 0.0887 & 0.0210 \\ 0.0154 & -0.0084 & -0.0159 & 0.0549 & -0.0234 & -0.0415 & -0.0615 & -0.0146 \\ 0.0133 & -0.0073 & -0.0137 & 0.0473 & -0.0202 & -0.0358 & -0.0531 & -0.0126 \\ -0.0017 & 0.0009 & 0.0017 & -0.0060 & 0.0026 & 0.0045 & 0.0067 & 0.0016 \\ 0.0073 & -0.0040 & -0.0075 & 0.0259 & -0.0110 & -0.0196 & -0.0290 & -0.0069 \end{pmatrix},$$

$$R_4 = \begin{pmatrix} 71511 & -250742 & -151984 & 219259 & -510594 & -884097 \\ -310094 & 995853 & 901646 & -452672 & 1219872 & 1634013 \\ -617957 & 1092862 & 1659610 & 306572 & -67287 & -2376366 \\ 51196 & 319077 & 658978 & 689135 & -879342 & -2518257 \\ -15284 & 57010 & 50659 & 28685 & 24113 & -166172 \\ -189958 & 122836 & 126710 & -397128 & 742797 & 1698921 \end{pmatrix},$$

$$R_3 = \begin{pmatrix} 40 & -4938 & -7886 & -4554 & 6482 & 13963 \\ 1005 & 4570 & 17506 & 20766 & -31423 & -73745 \\ 13188 & -15327 & -15015 & 29613 & -57452 & -134461 \\ 578 & -29546 & -15099 & 10990 & -37314 & -40491 \\ 3721 & 504 & -1626 & 1667 & -5308 & -5002 \\ -8426 & 7928 & 15041 & 222 & 2923 & -12609 \end{pmatrix},$$

$$R_2 = \begin{pmatrix} 26 & 289 & 38 & -125 & 207 & 444 \\ -128 & -934 & -430 & 190 & -747 & -969 \\ 714 & -414 & -1265 & -136 & -72 & 1257 \\ -510 & -319 & -364 & -638 & 1227 & 1350 \\ -81 & -119 & -22 & 15 & 115 & 227 \\ 68 & -46 & -79 & 211 & -405 & -1278 \end{pmatrix}.$$

$$R_1 = \begin{pmatrix} 2 & 2 & 4 & 6 & -1 & -2 \\ -8 & 0 & -11 & -18 & 29 & 31 \\ 1 & 9 & 7 & -12 & 34 & 98 \\ 25 & 42 & 1 & -8 & 0 & 0 \\ 0 & 1 & 2 & -3 & -10 & 0 \\ 11 & 0 & -12 & 1 & -2 & 4 \end{pmatrix}, R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_5 = \begin{pmatrix} -2320777 & 7724192 & 11939537 & 5440584 & -5125757 & -25462747 \\ -717929 & -11296227 & -24405967 & -28557225 & 43159177 & 124754978 \\ -26176969 & 25416451 & 24150092 & -48228231 & 94230584 & 190627318 \\ -12314894 & 33778445 & 29798340 & -19429040 & 46316565 & 66814068 \\ -1531472 & 1597236 & 1686683 & -2222468 & 3455433 & 6263944 \\ 7713995 & -15298941 & -20649354 & -1826634 & -3508267 & 21699527 \end{pmatrix},$$

$$R_6 = 10^9 \begin{pmatrix} -0.1962 & 0.3185 & 0.2781 & -0.3553 & 0.7396 & 1.3378 \\ 0.7236 & -1.3660 & -1.4643 & 0.7442 & -1.8809 & -2.3663 \\ 0.6858 & -1.7918 & -2.3867 & -0.5624 & 0.1666 & 3.7789 \\ -0.0035 & -0.5599 & -0.9706 & -0.9368 & 1.3619 & 4.1924 \\ 0.0007 & -0.0794 & -0.0841 & -0.0423 & 0.0547 & 0.2367 \\ 0.3025 & -0.2353 & -0.1940 & 0.6181 & -1.1583 & -2.4482 \end{pmatrix},$$

Choose $Z = \begin{pmatrix} 1 & 2 & 19 & 2 & 2 & 21 & 0 & -1 \\ 4 & 3 & 21 & 0 & 3 & 12 & 2 & 0 \\ -1 & -13 & 22 & 1 & 4 & -13 & 3 & 1 \\ 1 & -14 & 23 & -1 & -5 & 6 & 4 & -9 \\ 9 & 1 & 2 & 3 & 4 & 0 & 1 & -1 \\ 2 & 3 & -1 & 0 & 0 & 9 & 6 & 2 \\ 9 & -1 & 2 & -9 & 8 & 7 & 2 & -1 \\ -10 & -9 & 8 & 7 & 0 & 10 & 11 & 13 \end{pmatrix}$. So from Theorem 1 we have

$$X = \begin{pmatrix} 2 & 3 & 36 & 16 & 9 & 4 & 1 & 9 \\ 3 & 4 & 48 & 1 & 7 & 9 & 2 & -1 \\ 4 & 2 & 1 & 2 & 6 & 2 & -3 & -2 \\ -5 & 0 & -1 & 3 & -2 & 7 & 0 & 0 \\ -6 & 11 & 0 & 4 & -3 & 8 & -9 & -3 \\ 7 & 29 & 15 & -4 & -2 & 9 & -8 & 0 \end{pmatrix},$$

$$Y = 10^{12} \times \begin{pmatrix} 0.9967 & 0.1219 & 0.1325 & 0.7507 & 1.2366 & 0.7164 & 1.2608 & -0.2614 \\ 1.4717 & 0.1579 & 0.1926 & 0.4968 & 1.2267 & 0.5219 & 1.5579 & 0.0043 \\ 0.1493 & 0.2854 & -0.1894 & 0.1609 & -0.1677 & 0.2949 & -0.9352 & -0.5724 \\ -0.6657 & 0.0236 & -0.2209 & 0.2364 & -0.2277 & 0.2602 & -0.9383 & -0.5034 \\ 0.3622 & 0.1181 & 0.0215 & 0.1922 & 0.3992 & 0.1250 & 0.6353 & -0.1945 \\ 0.2367 & 0.0378 & 0.0426 & 0.1431 & 0.2769 & 0.1123 & 0.3676 & -0.0511 \\ -0.7892 & -0.0015 & -0.0096 & 0.2866 & 0.0100 & -0.0059 & 0.1746 & -0.3507 \\ -0.2435 & 0.3674 & -0.2141 & 0.6507 & -0.0918 & 0.7113 & -1.5539 & -1.0505 \end{pmatrix}.$$

4.2. Application Examples

Lemma 2 [15]. The generalized Sylvester matrix equation (1.1), where $A, E \in R^{n\times n}$, $B \in R^{n\times r}$, and $F \in R^{p\times p}$ are known and the matrix pair (E, A) is regular, is equivalent to the generalized discrete Sylvester matrix equation (1.2) with

$$M = (\gamma E - A)^{-1}E, \quad N = \gamma I - F, \quad T = (\gamma E - A)^{-1}B,$$

where γ is an arbitrary scalar such that $(\gamma E - A)$ is nonsingular.

Proof. Since the matrix pair (E, A) is regular, there exists a scalar γ such that $(\gamma E - A)$ is nonsingular. Premultiplying (1.2) by $(\gamma E - A)^{-1}$ produces

$$(\gamma E - A)^{-1}AX - (\gamma E - A)^{-1}EXF = (\gamma E - A)^{-1}BY, \tag{4.1}$$

Let $M = (\gamma E - A)^{-1}E$ and note that

$$\begin{aligned} \gamma M - (\gamma E - A)^{-1}A &= \gamma(\gamma E - A)^{-1}E - (\gamma E - A)^{-1}A \\ &= (\gamma E - A)^{-1}(\gamma E - A) \\ &= I, \end{aligned}$$

We have $(\gamma E - A)^{-1}A = \gamma M - I$. So (4.1) is equivalent to

$$(\gamma M - I)X - MXF = TY$$

or

$$MX(\gamma I - F)X - X = TY.$$

Let $N = \gamma I - F$, then the above equation is reduced to (1.1).

Example 2. Consider the following linear system:

$$E\dot{x} = Ax + Bu, \tag{4.2}$$

where $A, E \in R^{n\times n}$ and $B \in R^{n\times r}$ are known coefficient matrices. If the following state feedback controller

$$u = -K_p x - K_d \dot{x}, \tag{4.3}$$

is applied to the above system (4.2), the closed-loop system becomes

$$(E + BK_d)\dot{x} = (A - BK_p)x, \tag{4.4}$$

The eigenstructure assignment problem is to determine the matrix K_p such that the closed-loop system matrix $A - BK_p$ has desired eigenvalues and struture, i.e., determine the matrix K_p such that

$$A - BK_p = EXFX^{-1}, \tag{4.5}$$

where F is the desired Jordan form of the closed-loop system and X is the corresponding eigenvector matrix. If we choose $K_pX + K_dXF = Y$, the equation (4.5) is equivalent to the generalized Sylvester matrix equation $AX - EXF = BY$. By Lemma 2, the generalized Sylvester matrix equation $AX - EXF = BY$ is equivelent into the Yakubovich matrix equation $X - MXF = NY$. By Corollary 2, we can obtain the solution to Yakubovich matrix equation $X - MXF = NY$.

Example 3 . Consider the following linear system:

$$A\dot{x} = x + Bu, \tag{4.6}$$

where $A \in R^{n\times n}$ and $B \in R^{n\times r}$ are known coefficient matrices. If the following state feedback controller

$$u = -K_p x, \tag{4.7}$$

is applied to the above system (4.6), the closed-loop system becomes

$$A\dot{x} = (I - BK_p)x, \tag{4.8}$$

The eigenstructure assignment problem is to determine the matrix K_p such that the closed-loop system matrix $I - BK_p$ has desired eigenvalues and struture, i.e., determine the matrix K_p such that

$$AXF = X - BK_pX, \tag{4.9}$$

where F is the desired Jordan form of the closed-loop system and X is the corresponding eigenvector matrix. If we choose $K_pX = Y$, the equation (4.9) is equivalent to the generalized Sylvester matrix equation $X - AXF = BY$. We choose the following parametric matrices.

$$A = \begin{pmatrix} 1 & 21 & 3 & 4 \\ 2 & 3 & 11 & 132 \\ 2 & 3 & 4 & 0 \\ 3 & 4 & 0 & 45 \end{pmatrix}, B = \begin{pmatrix} 10 & 0 & 9 & 19 \\ 21 & 2 & 7 & 20 \\ 2 & 33 & 8 & 10 \\ 3 & 4 & 1 & 0 \end{pmatrix}.$$

$$F = \begin{pmatrix} 6 & -11 & 14 & 12 & 1 & -29 \\ 1 & 129 & 118 & 11 & -2 & -8 \\ -6 & -4 & -2 & -131 & 2 & -6 \\ 9 & 7 & -6 & 5 & 14 & -3 \\ 21 & 23 & 24 & 126 & 115 & 119 \\ 26 & -11 & 16 & -71 & -9 & -112 \end{pmatrix},$$

By computation, one can obtain

$$f_A(s) = 43069s^4 - 3107s^3 - 242s^2 - 53s + 1,$$

$$\alpha_4 = 43069, \alpha_3 = -3107, \alpha_2 = -242, \alpha_1 = -53, \alpha_0 = 1.$$

By (2.10), we have

$$R_3 = \begin{pmatrix} 3057 & 3311 & -11398 & -9984 \\ -2214 & 138 & 1281 & -208 \\ 132 & -1759 & -6029 & 5148 \\ -7 & -233 & 646 & -273 \end{pmatrix}, R_2 = \begin{pmatrix} -234 & -1004 & 87 & 2744 \\ 320 & 211 & -500 & -652 \\ -90 & -96 & -399 & 404 \\ -13 & 43 & 53 & -62 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} -52 & 21 & 3 & 4 \\ 2 & -50 & 11 & 132 \\ 2 & 3 & -49 & 0 \\ 3 & 4 & 0 & -8 \end{pmatrix}, R_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

If we choose $Z = 10^{-}3 * \begin{pmatrix} 0.1078 & 0.0030 & 0.0277 & 0.1043 & -0.0177 & -0.0525 \\ -0.0108 & -0.0003 & -0.0028 & -0.0104 & 0.0018 & 0.0053 \\ 0.2041 & 0.0057 & 0.0528 & 0.1972 & -0.0335 & -0.0996 \\ -0.1479 & -0.0041 & -0.0382 & -0.1429 & 0.0243 & 0.0721 \end{pmatrix}$,
then by (2.9) we can derive

$$Y = \begin{pmatrix} -83150 & 19690 & -57530 & 286620 & 59350 & 400540 \\ -5200 & 880 & -3340 & 17640 & 3060 & 23820 \\ -26860 & 10120 & -20370 & 97890 & 25940 & 142270 \\ 55140 & -14770 & 38330 & -19173 & -41460 & -269840 \end{pmatrix},$$

$$X = \begin{pmatrix} 1 & 0 & 7 & -11 & -25 & 129 \\ -2 & 2 & -6 & -29 & -2 & 11 \\ -21 & 4 & -5 & -39 & -1 & -21 \\ -98 & 9 & -9 & -48 & -9 & 291 \end{pmatrix}.$$

5. Conclusions

In this paper we mainly consider the explicit solutions to two types of matrix equations. Firstly, we provide an alternative approach to solve the explicit solution to the nonhomogeneous Yakubovich matrix equation. These obtained solutions can provide all the degrees of freedom, which is represented by an arbitrarily chosen parameter matrix Z. All the coefficient matrices are not restricted to be in any canonical form. The matrix B explicitly appears in the solutions, thus can be unknown a priori. Secondly, we present the solutions to the nonhomogeneous Yakubovich j-conjugate matrix equation in quaternion field by the real representation of a quaternion matrix. We generalize our previous results in [25, 35, 32].

References

[1] B. H. Kwon, M. J. Youn. Eigenvalue-generalized eigenvector assignment by output feedback. IEEE Transactions Automatic Control, 32(1987) 417-421.

[2] M. M. FahmAy, J. Oreilly.Eigenstructure assignment in linear multivariable systemsa parametric solution. IEEE Transactions Automatic Control, 28(1983) 990-994.

[3] D. G. Luenberger. An introduction to observers. IEEE Transactions Automatic Control, 16 (1971)596-602.

[4] J. Chen, R. Patton, H. Zhang. Design unknown input observers and robust fault detection filters. Int. J. Control, 63 (1996) 85-105.

[5] C. C. Tsui. New approach to robust observer design. International Journal of Control, 47(1988)745-751.

[6] G. R. Duan, R. J. Patton. Robust fault detection using Luenberger-type unknown input observers-a parametric approach. International Journal Systerms Sciences, 32(2001)533-540.

[7] J. Park, G. Rizzoni, An eigenstructure assignment algorithm for the design of fault detection filters. IEEE Transactions Automatic Control, 39(1994)1521-1524.

[8] V. Simoncini. Computational methods for linear matrix equations. SIAM Review, 58(2016) 377-441.

[9] Z. G Jia, M.X Zhao. A structured condition number for self-adjoint polynomial matrix equations with applications in linear control. Journal of Computational and Applied Mathematics, 331(2018)208-216.

[10] N. L. Bihan, J. Mars. Singular value decomposition of quaternion matrices:A new tool for vector-sensor signal processing. Signal Processing, 84(7)(2004) 1177-1199.

[11] A.Dmytryshyn, B.Kagstrom. Coupled Sylvester-type matrix equations and block diagonalization. SIAM Journal on Matrix Analysis and Applications, 36(2)(2015)580-593.

[12] G. R. Duan. Solution to matrix equation $AV + BW = EVF$ and eigenstructure assignment for descriptor systems. Automatica, 28(1992)639-643.

[13] G. R. Duan. On the solution to Sylvester matrix equation $AV + BW = EVF$. IEEE Transactions Automatic Control, 41 (1996) 612-614.

[14] A. G. Wu, Y. M. Yan, G. R. Duan. On solutions of matrix equations $V - AVF = BW$ and $V - A\overline{V}F = BW$. Mathematical and Computer Modelling, 47(2008)1181-1197.

[15] B. Zhou, G. R. Duan. Parametric solutions to the generalized discrete Sylvester matrix equation $MXN-X = TY$ and their applications,IMA Journal of Mathematical Control and Information, 26(1)(2009)59-78.

[16] C. C. Tsui. A complete analytical solution to the matrix equation $TA - FT = LC$ and its applications. IEEE Transaction Automatica Control, 32(1987) 742-744.

[17] A. Saberi, A. A. Stoorvogel, P. Sannuti. Control of Linear Systems with Regulation and Input Constraints, in series of Communications and Control Engineering, Spring-Verlag, New York (1997).

[18] G. R. Duan, G. P. Liu. Complete Parametric Approach for Eigenstructure Assignment in a Class of Second-Order Linear Systems. Automatica, 38(2002)725-729.

[19] G. R. Duan. Solution to matrix equation $AV + BW = VF$ and their application to eigenstructure assignment in linear systems. IEEE Transaction Attomatica Control, 38(2)(1993)276-280.

[20] B. Zhou, G. R. Duan. An explicit solution to the matrix equation $AX - XF = BY$. Linear Algebra and its Applications, 402(2005)345-366.

[21] G. R. Duan. On Solution to the matrix equation AV +BW = EVJ + R. Applied Mathematics Letters, 17(2004)1197-1202.

[22] J. H. Bevis, F. J. Hall, R. E. Hartwing. The matrix equation $A\overline{X} - XB = C$ and its special cases. SIAM Journal Matrix Analysis and its Appliations, 9(3)(1998)348-359.

[23] T. S. Jiang, X. H. Cheng, L. Chen. An algebraic relation between consimilarity and similarity of complex matrices and its applications. Journal of Physics A: Mathematical and General, 39 (2006) 9215-9222.

[24] L. P. Huang. Consimilarity and quaternion matrices and complex matrices. Linear algebra and its applications, 331(2001)21-30.

[25] C. Q. Song, J. Feng, X. D. Wang, J. L. Zhao. A Real Representation Method for Solving Yakubovich-j-Conjugate Quaternion Matrix Equation, Abstract and Applied Analysis, Volume 2014, Article ID 285086, 9 pages http://dx.doi.org/10.1155/2014/285086

[26] J. B. Kuipers. Quaternions and Rotation Sequences: A Primer with Applications to Orbits, Aerospace and Virtual Reality. Princeton, NJ: Princeton Univ. Press, 2002.

[27] S. Miron, N. LeBihan, J. Mars. Quaternion-MUSIC for vector-sensor array processing. IEEE Transactions on Signal Processing, 54(4) (2006)1218-1229.

[28] J. P. Ward. Quaternions and Cayley Numbers: Algebra and Applications. Dordrecht, The Netherlands: Kluwer, 1997.

[29] A. J. Hanson. Visualizing Quaternions. San Mateo, CA: Morgan Kaufmann, 2005.

[30] L. Fortuna, G. Muscato, M. Xibilia. A comparison between HMLP and HRBF for attitude control. IEEE Transactions Neural Network, 12(2)(2001)318-328.

[31] S. F. Yuan, A. P. Liao. Least squares solution of the quaternion matrix equation$X - A\hat{X}B = C$ with the least norm. Linear and Multilinear Algebra, 59(9)(2011)985-998.

[32] Q. W. Wang. H. S. Zhang, S. W. Yu. On solutions to the quaternion matrix equation $AXB + CYD = E$. Electronic Journal of Linear Algebra, 17(2008)343-358.

[33] Q. W. Wang. Bisymmetric and centrosymmetric solutions to system of real quaternion matrix equation. Computer Mathematics and its Applications, 49(2005)641-650.

[34] Q. W. Wang. The general solution to a system of real quaternion matrix equations. Computer Mathematics and its Applications, 49(2005)665-675.

[35] C. Q. Song, G.L Chen, Q.B.Liu. Explicit solutions to the quaternion matrix equations $X - AXF = C$ and $X - A\widetilde{X}F = C$. International Journal of Computer Mathematics, 89(2012)890-900.

[36] C. Q. Song, X. D. Wang, X. Y. Zhang. On solutions of the generalized Stein quaternion matrix equation, Journal of Applied Mathematics and Computing, 43(2013) 115-131.

[37] Q. W. Wang, J. W. van der Woude, H. X. Chang. A system of real quaternion matrix equations with applications. Linear Algebra and its Applications, 431 (2009) 2291-2303.

[38] Q. W. Wang, G. J. Song, C. Y. Lin, Extreme ranks of the solution to a consistent system of linear quaternion matrix equations with an application. Applied Mathematics and Computation, 189(2007)1517-1532.

[39] C. Q. Song, G. L. Chen. On solutions of matrix equations $XF - AX = C$ and $XF - A\widetilde{X} = C$ over quaternion field. Journal of Applied Mathematics and Computing, 37(2011)57-68.

[40] C. Q. Song, G. L. Chen, X. D. Wang. On solutions of quaternion matrix equations $XF - AX = BY$ and $XF - A\widetilde{X} = BY$. Acta Mathematica Scientia, 32B(5)(2012)1967-1982.

[41] C.Q.Song, J.Feng. On solutions to the matrix equations $XB - AX = CY$ and $XB - A\widehat{X} = CY$, Journal of the Franklin Institute, 353(1016)1075-1088.

[42] B. Zhou, G. R. Duan. Parametric solutions to the generalized Sylvester matrix equation $AX - XF = BY$ and the regulator equation $AX - XF = BY + R$. Asian Journal of Control, 9(4)(2007)475-483.

[43] T. S. Jiang, M.S. Wei. On a solution of the quaternion matrix equation $X - A\widetilde{X}B = C$ and its application. Acta Mathematica Sinica English Series, 21(3)(2005)483-490.

[44] A. G. Wu, G. Feng, J. Q. Hu, G. R. Duan, Closed-form solutions to the nonhomogeneous Yakubovich-conjugate matrix equations, Applied Mathematics and Computation, 214(2009)442-450.

[45] F. L. Lewis. Further remarks on the Cayley-Hamiiton theorem and Leverrier's method for matrix pencil$(sI - A)$. IEEE Transactions on Automatic Control, 31(9)(1986)869-870.

[46] B. Zhou, G. R. Duan. An explicit solution to polynomial matrix right corprime factorization with applications in eigenstructure assignment. Journal of Control and its Applications. 2(2006)447-454.

[47] A. G. Wu, H. Q. Wang, G. R, Duan. on matrix equations $X-AXF = C$ and $X-A\overline{X}F = C$. Journal of Computational and Applied Mathematics, 214(2009)442-450.

[48] G. R. Duan, R. J. Patton, Eigenstructure assignment in descriptor systems via proportional derivative feedback. Internationa Journal of Control, 68(1997)1147-1162.

[49] A. Bradshwa,B. Proter, Design of Linear Multivariable Continuous-Time Tracking Systems Incorporating Feedforward and Feedback Controllers. International Journal Systems Sciences, 6 (1975)233-245.

[50] H. C. Chen, F. R. Chang, Chained eigenstructure assignment for constraint ratio proportional and derivative (CRPD) control law in controllable singular systems, Systems Control Letters, 21(1993) 405-411.

[51] T. S. Jiang, M. S. Wei, On solutions of the matrix equation $X - AXB = C$ and $X - A\overline{X}B = C$. Linear Algebra and its Applications, 367(2003)225-233.

[52] A. G. Wu, G. R. Duan, Solution to the generalized Sylvester matrix equation $AV + BW = EVF$, IET Control Theory and Its Application, 31(1)(2007)402-408.

[53] Z. H. He , Q.W Wang, Y. Zhang. A system of quaternary coupled Sylvester-type real quaternion matrix equations. Automatica, 87 (2018) 25-31.

[54] D. Palitta , V. Simoncini. Computationally enhanced projection methods for symmetric Sylvester and Lyapunov matrix equations. Journal of Computational and Applied Mathematics, 330 (2018) 648-659.

[55] Z. G. Jia, M.S. Wei, S.T Ling. A new structure-preserving method for quaternion Hermitian eigenvalure problems. Journal of Computational and Applied Mathematics, 239(2013)12-24.