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Optimal weighted upwind finite volume method for convection–diffusion equations in 2D

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ABSTRACT

This paper develops and analyzes the optimal weighted upwind finite volume method (OWUFVM) for convection–diffusion equations on nonuniform rectangular meshes. The dual meshes in this method are associated with the local Peclet numbers, which are determined by the convection and diffusion coefficients and the mesh size. We first prove the stability and H^1-norm error estimate. Then we prove a weak error estimate, which plays a key role in deriving the optimal second-order error estimate in the L^2 norm. The L^2-norm error estimate is the most important result of this paper. We also briefly examine the discrete maximum principle. Numerical experiments are presented to illustrate the theoretical analysis.

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1. Introduction

The convection–diffusion equations are widely used in many fields of science and engineering such as fluid mechanics, petroleum reservoir simulation, groundwater prediction and environmental protection [1–4]. In order to simulate the problems accurately, various numerical methods, e.g., the finite difference method (FDM) and finite element method (FEM), have been used. However, these methods with standard treatment for the case of convection-dominated often exhibit severe non-physical oscillations, since the schemes are unstable and non-monotone [5,6].

Due to the property of local conservation, the finite volume method(FVM) has attracted wide attention for solving the partial differential equations [7–16] and the authors presented an overview of the progress of the finite volume methods in [12]. To satisfy the stability and avoid the spurious oscillations, upwinding technique combined with FVM has been studied in [4,8,17–21]. However, for standard upwind finite volume method(SUFVM), the numerical accuracy in L^2 norm is only first order. Thus, several modified upwind FVMs have been investigated [7,22–27] to improve the accuracy. In [22], Lazarov R.D. et al. presented some strategies of upwind for cell-centered FVM to solve the convection–diffusion problems. Recently, S. Wang et al. [23] used a nonlinear approximation for both diffusion and advection fluxes giving a second-order scheme. Additionally, a kind of finite volume-complete flux scheme(CFS) for the advection–diffusion–reaction equation is studied by J. ten Thije Boonkkamp et al. [26], the scheme has second-order accuracy.

In [25], Liang and Zhao proposed an optimal weighted upwind finite volume method (OWUFVM) for improving the numerical accuracy in L^2 norm. To define the method, a non-standard dual partition was considered, which is determined

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by the local Peclet's numbers. Therefore, they provided some strategies to derive the Peclet's numbers for constructing the dual mesh. The authors numerically showed this method has the optimal convergent order in L^2 norm. However, the analysis was given only for 1D case, there is no theoretical proof in 2D. Since it is a difficult task to prove the stability and L^2-norm error estimate with the method for convection–diffusion problems in two dimensions, the proof of the L^2-norm error estimates for 2D is the main purpose of this paper.

In this article, we give the theoretical analysis of the OWUFVM, for two dimensions convection–diffusion equations, with constant coefficients. Firstly, due to the non-standard dual partition, the elemental analysis technique plays a key role to prove the stability of which is one of the main contributions in this work. Secondly, we give the optimal error estimates in L^2 norm which is the most important contribution. We would like to point out that the proof of optimal L^2-norm error estimate usually employs the Aubin–Nitsche technique, however it may be not proper for current non-standard dual mesh. Thus, the weak error estimate of the first type is proposed to analysis the L^2-norm convergence, where the Taylor's expansion technique is applied. Thirdly, under some restrictions on grid, we present the discrete maximum principle for the OWUFVM, which guarantees the numerical solution not to produce spurious oscillations.

The rest of the paper is organized as follows. In Section 2, we introduce the definition of the optimal weighted upwind bilinear finite volume method on rectangular meshes. In Section 3, the stability and the optimal H^1-norm estimate is discussed. For Section 4, the two-order weak estimate of the first type and optimal error estimates in L^2 norm are derived, respectively. Then we provide the discrete maximum principle and some numerical experiments in Sections 5 and 6. Finally, we address a brief conclusion in the end.

2. Optimal weighted upwind finite volume method

Consider the convection–diffusion equation

$$\begin{cases} -\nabla\cdot(D\nabla u)+\nabla\cdot(\mathbf{b}u)=f, & in\ \ \Omega,\\ u=0, & on\ \ \partial\Omega, \end{cases} \tag{1}$$

where $\Omega\subset R^2$ is a bounded domain with boundary $\partial\Omega$, diffusion coefficient $D=D(x,y)$ satisfies $0<d_0\le D\le d_1$ in $\overline{\Omega}$, $\mathbf{b}=(b_1(x,y),b_2(x,y))^T$ is the convection velocity satisfying $\frac{1}{2}div\mathbf{b}\ge 0$, and $f\in L^2(\Omega)$ represents the source term. We assume that $|D|\ll\|\mathbf{b}\|_\infty$.

For simplicity, let $\Omega=[a_x,b_x]\times[a_y,b_y]$ be a rectangular domain, and $N_x,\ N_y$ denote the number of grid intervals on each direction. Set $\mathbb{N}_x^m=\{m,\dots,N_x|m=0,1\}$ and $\mathbb{N}_y^m=\{m,\dots,N_y|m=0,1\}$. Then define $\Omega_h=\{P_{i,j}|P_{i,j}=(x_i,y_j),i\in\mathbb{N}_x^0,j\in\mathbb{N}_y^0\}$ be the set of the nodes, where $(x_0,y_0)=(a_x,b_x)$ and $(x_{N_x},y_{N_y})=(a_y,b_y)$, and let $\dot{\Omega}_h=\Omega_h\backslash\partial\Omega$ be the set of interior nodes. Introduce the primary partition $\mathcal{T}_h=\left\{K_{i,j}|K_{i,j}=[x_{i-1},x_i]\times[y_{j-1},y_j],\ i\in\mathbb{N}_x^1,\ j\in\mathbb{N}_y^1\right\}$, which can be a nonuniformly rectangular mesh. Denote $h_x^{(i)}=(x_i-x_{i-1})$ for $i\in\mathbb{N}_x^1$ and $h_y^{(j)}=(y_j-y_{j-1})$ for $j\in\mathbb{N}_y^1$, and set $h=\max\{h_x^{(i)},h_y^{(j)}\}$. In addition, assume the primary mesh be regular and quasi-uniform, that is, there exist positive constants $C_1\le C_2$ and $C_3\le C_4$ such that

$$C_1\le\frac{h_x^{(i)}}{h_y^{(j)}}\le C_2,\ for\ any\ i\in\mathbb{N}_x^1\ and\ j\in\mathbb{N}_y^1, \tag{2}$$

$$C_3h^2\le S_{K_{i,j}}\le C_4h^2,\quad for\ any\ K_{i,j}\in\mathcal{T}_h. \tag{3}$$

where $S_{K_{i,j}}$ represents the area of element $K_{i,j}\in\mathcal{T}_h$.

Take the piecewise bilinear functions as the trial space U_h, that is

$$U_h=\left\{u_h\in C^0(\overline{\Omega}):u_h|_{K_{i,j}}is\ a\ bilinear\ function\ for\ any\ K_{i,j}\in\mathcal{T}_h,u_h|_{\partial\Omega}=0\right\}.$$

In order to define the finite volume scheme, one should introduce the dual mesh which is non-standard (see [25]). As shown in Fig. 1, we set weighted points $M_{i-\frac{1}{2},j}=(x^*_{i-1/2,j},y_j)$ on $\overline{P_{i-1,j}P_{i,j}}$ and $M_{i,j-\frac{1}{2}}=(x_i,y^*_{i,j-1/2})$ on $\overline{P_{i,j-1}P_{i,j}}$, where

$$\begin{cases} x^*_{i-1/2,j}=x_{i-1}+\alpha_{i-1/2,j}h_x^{(i)}, & i\in\mathbb{N}_x^1,j\in\mathbb{N}_y^0,\\ y^*_{i,j-1/2}=y_{j-1}+\beta_{i,j-1/2}h_y^{(j)}, & i\in\mathbb{N}_x^0,j\in\mathbb{N}_y^1. \end{cases}$$

Here, local weighted factors $\alpha_{i-1/2,j}$ and $\beta_{i,j-1/2}$ are defined as

$$\begin{cases} \alpha_{i-1/2,j}=\left(\frac{1}{2}+\frac{1}{Pe_x(M_{i-\frac{1}{2},j})}\right)\pm\sqrt{\frac{1}{4}+\frac{1}{Pe_x^2(M_{i-\frac{1}{2},j})}}, & i\in\mathbb{N}_x^1,j\in\mathbb{N}_y^0,\\ \beta_{i,j-1/2}=\left(\frac{1}{2}+\frac{1}{Pe_y(M_{i,j-\frac{1}{2}})}\right)\pm\sqrt{\frac{1}{4}+\frac{1}{Pe_y^2(M_{i,j-\frac{1}{2}})}}, & i\in\mathbb{N}_x^0,j\in\mathbb{N}_y^1, \end{cases} \tag{4}$$

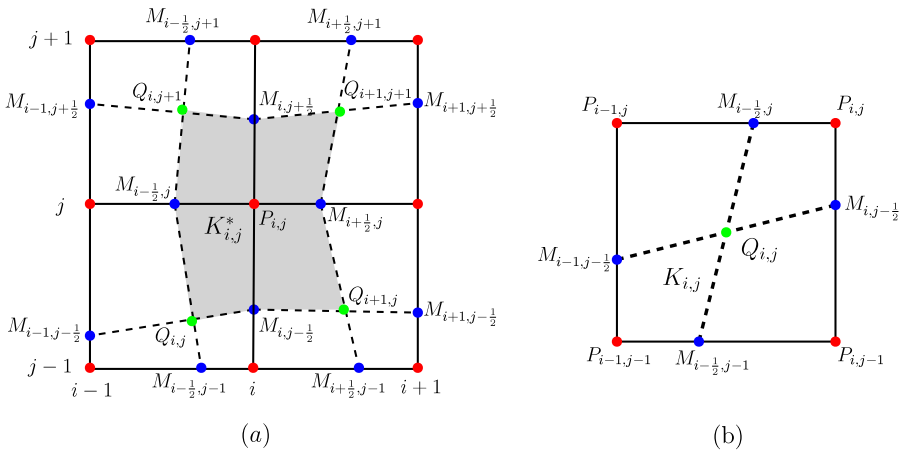


Fig. 1. Primary and dual partitions.

where

$$Pe_x(M_{i-\frac{1}{2},j}) = \frac{b_1(M_{i-\frac{1}{2},j})h_x^{(i)}}{D(M_{i-\frac{1}{2},j})}, \quad Pe_y(M_{i,j-\frac{1}{2}}) = \frac{b_2(M_{i,j-\frac{1}{2}})h_y^{(j)}}{D(M_{i-\frac{1}{2},j})}, \tag{5}$$

are the local Peclet's numbers with respect to the weighted points $M_{i-\frac{1}{2},j}$ and $M_{i,j-\frac{1}{2}}$. If $b_1(M_{i-\frac{1}{2},j}) = 0$ or $b_2(M_{i,j-\frac{1}{2}}) = 0$, then $\alpha_{i-1/2,j} = \frac{1}{2}$ or $\beta_{i,j-1/2} = \frac{1}{2}$, correspondingly.

Remark 1. Here, the expressions of local weighted factors are implicit. To obtain the factor $\alpha_{i-1/2,j}$ or $\beta_{i,j-1/2}$, we need to substitute the definition (5) into (4) to form a nonlinear equation about $\alpha_{i-1/2,j}$ or $\beta_{i,j-1/2}$. The nonlinear equation is defined like (63) in Section 4. Furthermore, we can get the weighted point and local Peclet number.

Let $Q_{i,j} \in K_{i,j}$ be the intersection point of $\overline{M_{i-\frac{1}{2},j-1}M_{i-\frac{1}{2},j}}$ and $\overline{M_{i-1,j-\frac{1}{2}}M_{i,j-\frac{1}{2}}}$. Set the dual partition $\mathcal{T}_h^* = \left\{K_{i,j}^* | i \in \mathbb{N}_x^0, j \in \mathbb{N}_y^0\right\}$, where $K_{i,j}^*$ is a dual element surrounding the node $P_{i,j} \in \Omega_h$ and $K_{i,j}^*$ is a polygon formed by successively connecting points $\left\{Q_{i,j}., M_{i-\frac{1}{2},j}, Q_{i,j+1}, M_{i,j+\frac{1}{2}}, Q_{i+1,j+1}, M_{i+\frac{1}{2},j}, Q_{i+1,j}, M_{i,j-\frac{1}{2}}, Q_{i,j}\right\}$. Similarly, we can also give the definition of $K_{i,j}^*$ for $P_{i,j} \in \partial\Omega$.

Remark 2. It is necessary to explain the plus or minus sign $'\pm'$ in (4). Taking $\alpha_{i-1/2,j}$ as an example, if $b_1(M_{i-\frac{1}{2},j}) > 0$, then the sign takes $'-'$, i.e. the local weighted factor $\alpha_{i-1/2,j}$ is uniquely determined in $(0, \frac{1}{2})$. If $b_1(M_{i-\frac{1}{2},j}) < 0$, then it takes $'+'$ and $\alpha_{i-1/2,j} \in (\frac{1}{2}, 1)$. As for $\beta_{i,j-1/2}$, it depends on $b_2(M_{i,j-\frac{1}{2}})$ in a similar way. This demonstrates that the determination of weighted factors embodies the upwinding property.

Define the test function space V_h as:

$$V_h = \left\{v_h \in L^2(\Omega) : v_h|_{K_{i,j}^*} = constant, P_{i,j} \in \dot{\Omega}_h, v_h|_{K_{i,j}^*} = 0, P_{i,j} \in \partial\Omega\right\}.$$

Let $\psi_{P_{i,j}}$ denote the characteristic function on $K_{i,j}^*$ and

$$v_h = \sum_{P_{i,j} \in \dot{\Omega}_h} v_h(P_{i,j})\psi_{P_{i,j}}, \quad \forall v_h \in V_h. \tag{6}$$

For $u_h \in U_h$ and $v_h \in V_h$, we have the following bilinear forms:

$$\begin{cases} a(u_h, v_h) = \sum\limits_{K_{i,j}^*} \int_{\partial K_{i,j}^*} -D\dfrac{\partial u_h}{\partial \mathbf{n}} v_h ds, \\ b(u_h, v_h) = \sum\limits_{K_{i,j}^*} \int_{\partial K_{i,j}^*} (\mathbf{b} \cdot \mathbf{n}) u_h v_h ds, \end{cases} \tag{7}$$

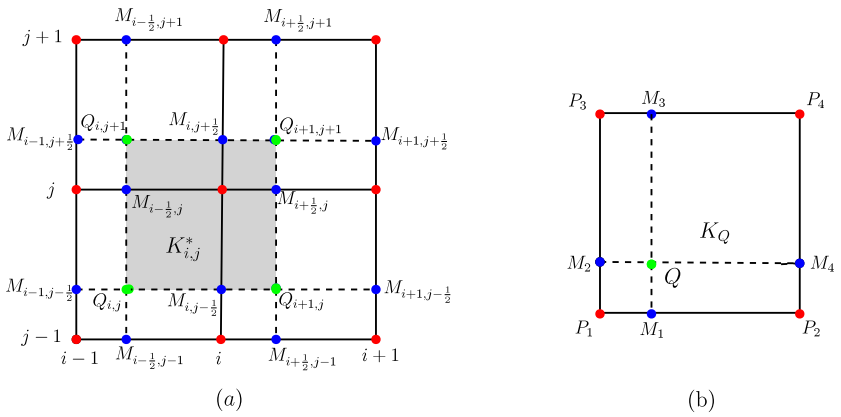


Fig. 2. Primary and dual partitions for the case of constant coefficients.

and

$$(f, v_h) = \sum_{K^*_{i,j}} \int_{K^*_{i,j}} f v_h dxdy. \tag{8}$$

where $\mathbf{n}$ is the unit outer normal vector on $\partial K^*_{i,j}$.

Thus, the weighted upwind finite volume method is to find $u_h \in U_h$, such that

$$A(u_h, v_h) = a(u_h, v_h) + b(u_h, v_h) = (f, v_h), \quad \forall v_h \in V_h, \tag{9}$$

Furthermore, let Π_h be the interpolation projection from $U = H^1_0(\Omega) \bigcap H^2(\Omega)$ onto U_h, and Π^*_h be the interpolation projection from U_h onto V_h, there hold (*cf.* Chap.3 in [28], Chap.4 in [29])

$$\|u - \Pi_h u\|_m \le Ch^{2-m}|u|_2, \ m = 0, 1, 2, \tag{10}$$

$$\|u - \Pi^*_h u\|_0 \le Ch|u|_1, \tag{11}$$

where $\|\cdot\|_m$ is standard *Sobolev* H^m norm [30].

3. Stability and error estimate in the H^1-norm

In this section, we provide the proof of the stability and error estimate in H^1 norm for the OWUFVM. The stability analysis is the basis to get the error estimates in H^1 and L^2 norm. The key point in our analysis is to prove the positive definiteness of the bilinear form corresponding to the diffusion term, which is obtained through an element-by-element analysis strategy.

For simplicity, assume that the diffusion and convection coefficients are constant. Under these assumptions, the partitions are implemented using definitions, as shown in Fig. 2. For any $K_Q \in \mathcal{T}_h$ presented in Fig. 2-(b), denote the vertices by $P_i = (x_i, y_i)$ for $i = 1, 2, 3, 4$ and the stepsize on each direction by h^Q_x and h^Q_y. Then we introduce the notations as follows:

$$\begin{cases} Pe^Q_x = \dfrac{b_1 h^Q_x}{D}, \quad Pe^Q_y = \dfrac{b_2 h^Q_y}{D}, \\ \alpha_Q = \alpha_{M_1} = \alpha_{M_3}, \quad \beta_Q = \beta_{M_2} = \beta_{M_4}, \end{cases} \tag{12}$$

where α_{M_1}, α_{M_3}, β_{M_2}, β_{M_4} are the local weighted factors associated with points M_1, M_3, M_2, M_4.

In addition, by defining the parameter transformation on K_Q

$$\xi = \frac{x - x_1}{h^Q_x}, \quad \eta = \frac{y - y_1}{h^Q_y}, \tag{13}$$

and $u_h(P_i) = u_i$ for $i = 1, 2, 3, 4$, it holds that

$$u_h|_{K_Q} = (1-\xi)(1-\eta)u_1 + \xi(1-\eta)u_2 + (1-\xi)\eta u_3 + \xi\eta u_4. \tag{14}$$

Introduce the discrete norms $|\cdot|_{1,h}$ for U_h, such as

$$|u_h|_{1,h} = \Big(\sum_{K\in\mathcal{T}_h} |u_h|^2_{1,K,h}\Big)^{1/2}, \quad \forall u_h \in U_h, \tag{15}$$

where

$$|u_h|^2_{1,K,h} = (u_2 - u_1)^2 + (u_4 - u_3)^2 + (u_3 - u_1)^2 + (u_4 - u_2)^2. \tag{16}$$

Proposition 1. *The semi-norms $|u_h|_{1,h}$ and $|u_h|_1$ are equivalent in U_h, i.e. there exist constants $0 < c_1 \le c_2$ independent of h such that(cf. [9])*

$$c_1|u_h|_{1,h} \le |u_h|_1 \le c_2|u_h|_{1,h}. \tag{17}$$

In terms of the descriptions, we give the following analysis for stability.

Theorem 1. *Assume the partition $\mathcal{T}_h$ satisfy regular condition (2), the convection and diffusion coefficients are constant, and all weighted factors $\alpha_Q, \beta_Q \in \left(\frac{3-\sqrt{6}}{6}, \frac{3+\sqrt{6}}{6}\right)$, then there exist a constant $C > 0$ for $0 < h < h_0$, such that*

$$A(u_h, \Pi_h^* u_h) = a(u_h, \Pi_h^* u_h) + b(u_h, \Pi_h^* u_h) \ge C|u_h|_1^2, \quad \forall u_h \in U_h. \tag{18}$$

Proof. Firstly, the bilinear form of diffusion term $a(u_h, \Pi_h^* u_h)$ will be discussed. Rearranging the terms by all elements gives

$$a(u_h, \Pi_h^* u_h) = \sum_{P \in \dot{\Omega}_h} u_h(P) a(u_h, \psi_P) = \sum_{K_Q \in \mathcal{T}_h} I_Q(u_h, \Pi_h^* u_h), \tag{19}$$

where $Z = [z_1, z_2, z_3, z_4]^T = [u_2 - u_1, u_4 - u_3, u_3 - u_1, u_4 - u_2]^T$ and

$$\begin{aligned} I_Q(u_h, \Pi_h^* u_h) &= \sum_{P \in \dot{\Omega}_h} u_h(P) \int_{\partial K_P^* \bigcap K_Q} D\left(\frac{\partial u_h}{\partial y}dx - \frac{\partial u_h}{\partial x}dy\right) \\ &= D\left(z_1 \int_{\overline{M_1Q}} \frac{\partial u_h}{\partial x}dy + z_2 \int_{\overline{QM_3}} \frac{\partial u_h}{\partial x}dy + z_3 \int_{\overline{M_2Q}} \frac{\partial u_h}{\partial y}dx + z_4 \int_{\overline{QM_4}} \frac{\partial u_h}{\partial y}dx\right). \end{aligned} \tag{20}$$

Considering the first term in bracket of the right side of (20), it holds that

$$\begin{aligned} z_1 \int_{\overline{M_1Q}} \frac{\partial u_h}{\partial x}dy &= z_1 \int_{y_{M_1}}^{y_Q} \frac{1}{h_x^Q}\left[(u_2 - u_1) + (u_4 - u_3 - u_2 + u_1)\frac{y - y_1}{h_y^Q}\right]dy \\ &= k_1^Q \beta_Q z_1 \left[z_1 + \frac{1}{2}\beta_Q(z_2 - z_1)\right], \end{aligned} \tag{21}$$

where $k_1^Q = \frac{h_y^Q}{h_x^Q}$, $k_2^Q = \frac{1}{k_1^Q}$. Similarly, there hold

$$\begin{cases} z_2 \int_{\overline{QM_3}} \frac{\partial u_h}{\partial x}dy = k_1^Q(1 - \beta_Q)z_2\left[z_1 + \frac{1 + \beta_Q}{2}(z_2 - z_1)\right], \\ z_3 \int_{\overline{M_2Q}} \frac{\partial u_h}{\partial y}dx = k_2^Q \alpha_Q z_3\left[z_3 + \frac{\alpha_Q}{2}(z_4 - z_3)\right], \\ z_4 \int_{\overline{QM_4}} \frac{\partial u_h}{\partial y}dx = k_2^Q(1 - \alpha_Q)z_4\left[z_3 + \frac{1 + \alpha_Q}{2}(z_4 - z_3)\right]. \end{cases} \tag{22}$$

The combination of (21) and (22) yields

$$I_Q(u_h, \Pi_h^* u_h) = DZ^T WZ = DZ^T W_0 Z, \tag{23}$$

where

$$W = \frac{1}{2}\begin{bmatrix} k_1^Q\beta_Q(2 - \beta_Q) & k_1^Q(1 - \beta_Q)^2 & 0 & 0 \\ k_1^Q\beta_Q^2 & k_1^Q(1 - \beta_Q^2) & 0 & 0 \\ 0 & 0 & k_2^Q\alpha_Q(2 - \alpha_Q) & k_2^Q(1 - \alpha_Q)^2 \\ 0 & 0 & k_2^Q\alpha_Q^2 & k_2^Q(1 - \alpha_Q^2) \end{bmatrix}, \tag{24}$$

and $W_0 = \frac{(W + W^T)}{2}$ has the following expression

$$W_0 = \frac{1}{4}\begin{bmatrix} 2k_1^Q\beta_Q(2 - \beta_Q) & k_1^Q\left[(1 - \beta_Q)^2 + \beta_Q^2\right] & 0 & 0 \\ k_1^Q\left[(1 - \beta_Q)^2 + \beta_Q^2\right] & 2k_1^Q(1 - \beta_Q^2) & 0 & 0 \\ 0 & 0 & 2k_2^Q\alpha_Q(2 - \alpha_Q) & k_2^Q\left[(1 - \alpha_Q)^2 + \alpha_Q^2\right] \\ 0 & 0 & k_2^Q\left[(1 - \alpha_Q)^2 + \alpha_Q^2\right] & 2k_2^Q(1 - \alpha_Q^2) \end{bmatrix}.$$

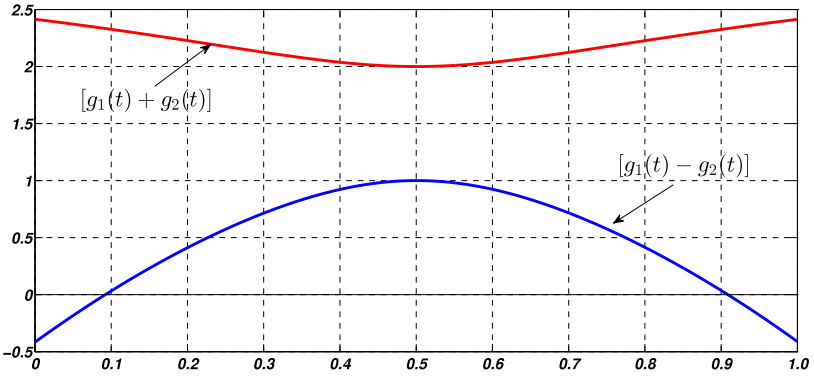


Fig. 3. Graphs of functions $[g_1(t) \pm g_2(t)]$.

By directly computing, one can get the four eigenvalues of W_0 are as follows.

$$\lambda_1 = \frac{k_1^Q}{4}[g_1(\beta_Q) + g_2(\beta_Q)], \quad \lambda_2 = \frac{k_1^Q}{4}[g_1(\beta_Q) - g_2(\beta_Q)],$$
$$\lambda_3 = \frac{k_2^Q}{4}[g_1(\alpha_Q) + g_2(\alpha_Q)], \quad \lambda_4 = \frac{k_2^Q}{4}[g_1(\alpha_Q) - g_2(\alpha_Q)],$$

where

$$\begin{cases} g_1(t) = \dfrac{1}{2}[3 - (2t-1)^2] > 1, \ t \in (0,1), \\ g_2(t) = \sqrt{4t^2(t-1)^2 + 2(2t-1)^2} \geq \dfrac{1}{2}, \ t \in (0,1). \end{cases}$$

Noting that k_1^Q and k_2^Q are positive, then we need only to find $t \in (0, 1)$ such that

$$[g_1(t) \pm g_2(t)] > 0. \tag{25}$$

In Fig. 3, the graphs of functions $[g_1(t) \pm g_2(2)]$ is provided.

By solving (25), we can see that if $\alpha_Q, \ \beta_Q \in \left(\frac{3-\sqrt{6}}{6}, \frac{3+\sqrt{6}}{6}\right) \approx (0.09175, 0.90825)$, then W_0 is a symmetric positive definite matrix. Thus, there exists a positive constant $\widetilde{C}_0$ such that

$$I_Q(u_h, \Pi_h^* u_h) \geq \widetilde{C}_0 \|Z\|_2^2 = \widetilde{C}_0 |u_h|_{1,K,h}^2. \tag{26}$$

For any K_Q and $\alpha_Q, \beta_Q \in \left(\frac{3-\sqrt{6}}{6}, \frac{3+\sqrt{6}}{6}\right)$, substituting (26) into (19) and using (17), we obtain

$$a(u_h, \Pi_h^* u_h) \geq C_0 |u_h|_1^2, \quad \forall u_h \in U_h, \tag{27}$$

where $C_0 > 0$ is a constant independent of h.

Recalling the bilinear of the convection term $b(u_h, \Pi_h^* u_h)$, we have

$$\begin{aligned} b(u_h, \Pi_h^* u_h) &= \sum_{K_{i,j}^*} \int_{\partial K_{i,j}^*} (\mathbf{b} \cdot \mathbf{n}) u_h \Pi_h^* u_h ds \\ &= \int_\Omega \nabla \cdot (\mathbf{b} u_h) \Pi_h^* u_h dxdy = \int_\Omega \nabla \cdot (\mathbf{b} u_h) u_h dxdy + \int_\Omega \nabla \cdot (\mathbf{b} u_h)(\Pi_h^* u_h - u_h) dxdy \\ &= \frac{1}{2} \int_\Omega (\nabla \cdot \mathbf{b}) u_h^2 dxdy + \int_\Omega [(\nabla \cdot \mathbf{b}) u_h + \mathbf{b} \cdot \nabla u_h](\Pi_h^* u_h - u_h) dxdy, \end{aligned} \tag{28}$$

where the Green's formula and boundary value condition are employed.

By Cauchy–Schwarz inequality, (11) and constant velocity $\mathbf{b}$, there hold

$$\frac{1}{2} \int_\Omega (\nabla \cdot \mathbf{b}) u_h^2 dxdy = 0, \tag{29}$$

$$\left| \int_\Omega \left[(\nabla \cdot \mathbf{b}) u_h + \mathbf{b} \cdot \nabla u_h \right] (\Pi_h^* u_h - u_h) dxdy \right| \leq Cbh |u_h|_1^2, \tag{30}$$

where $\nabla \cdot \mathbf{b} = 0$ and $b = \max\{|b_1|, |b_2|\}$.

Finally, it follows from (27)–(30) that (18).

Remark 3. Since the dual partition is non-standard, an analysis on the element stiffness matrix plays an important role in proving the stability. The technique involves calculation of all eigenvalues of the element stiffness matrix. This leads to the limit for all weighted factors $\alpha_Q, \beta_Q \in \left(\frac{3-\sqrt{6}}{6}, \frac{3+\sqrt{6}}{6}\right)$. Furthermore, the conditions are sufficient but not necessary, as shown in the numerical experiments in Section 6. For practical applications, the conditions may not be restrictive.

Corollary 1 (*The Existence Theorem*)**.** *Based on the stability, the optimal weighted upwind finite volume method (9) exists a unique solution $u_h \in U_h$.*

Theorem 2 (H^1*-norm Error Estimate*)**.** *Let $u \in H_0^1(\Omega) \cap H^2(\Omega)$ and $u_h \in U_h$ are the solution to (1) and (9), respectively. Assuming the stability holds, then there exists a constant $C > 0$, such that*

$$|u - u_h|_1 \le Ch|u|_2. \tag{31}$$

Proof. Obviously, the solution u and $u_h \in U_h$ satisfy (9), it holds that

$$A(u - u_h, v_h) = 0, \quad \forall v_h \in V_h. \tag{32}$$

By (18), we have

$$|u_h - \Pi_h u|_1^2 \le \frac{1}{C} A(u_h - \Pi_h u, \Pi_h^*(u_h - \Pi_h u)) = \frac{1}{C} A(u - \Pi_h u, \Pi_h^*(u_h - \Pi_h u)),$$

this gives

$$|u_h - \Pi_h u|_1 \le \frac{1}{C} \sup_{w_h \in U_h} \frac{|A(u - \Pi_h u, \Pi_h^* w_h)|}{|w_h|_1}, \tag{33}$$

where

$$A(u - \Pi_h u, \Pi_h^* w_h) = \sum_{K_Q \in \mathcal{T}_h} \left[w_{12}E_1 + w_{31}E_2 + w_{34}E_3 + w_{42}E_4 \right], \tag{34}$$

and

$$\begin{cases} w_{ij} \equiv w_i - w_j, \quad i, j = 1, 2, 3, 4, \\ |E_l| = \left| \int_{\overline{M_l Q}} \left[D \dfrac{\partial(u - \Pi_h u)}{\partial \mathbf{n}} - (\mathbf{b} \cdot \mathbf{n})(u - \Pi_h u) \right] ds \right|, \ l = 1, 2, 3, 4. \end{cases} \tag{35}$$

From (15)–(17), it is easy to see that $w_{ij} \le C|w_h|_{1,K}$. Let

$$\varphi_0 = u - \Pi_h u, \quad \varphi_1 = \frac{\partial(u - \Pi_h u)}{\partial x}, \quad \varphi_2 = \frac{\partial(u - \Pi_h u)}{\partial y}. \tag{36}$$

Invoking the Cauchy–Schwarz inequality yields

$$|E_1| \le Ch^{1/2} \left[\left(\int_{\overline{M_1 Q}} |\varphi_1|^2 dy \right)^{1/2} + \left(\int_{\overline{M_1 Q}} |\varphi_0|^2 dy \right)^{1/2} \right]. \tag{37}$$

It follows from the trace theorem and *Sobolev* semi-norm transformation relations that

$$\int_{\overline{M_1 Q}} |\varphi_i|^2 dy \le C \left(h|\varphi_i|_{1,K}^2 + h^{-1}\|\varphi_i\|_{0,K}^2 \right), \ i = 0, 1, 2. \tag{38}$$

Substituting (38) into (37) and using (10), we get

$$|E_1| \le Ch|u|_{2,K}.$$

Similarly, we can derive the estimations of $E_l (i = 2, 3, 4)$. There holds

$$\left[w_{12}E_1 + w_{31}E_2 + w_{34}E_3 + w_{42}E_4 \right] \le Ch|w_h|_{1,K}|u|_{2,K}. \tag{39}$$

With the help of Proposition 1, we obtain

$$A(u - \Pi_h u, \Pi_h^* w_h) \le Ch|u|_2|w_h|_1, \tag{40}$$

which implies

$$|u_h - \Pi_h u|_1 \le Ch|u|_2. \tag{41}$$

It follows from triangular inequality, (10) and (41) that (31).

4. Optimal L^2-norm error estimate

As the most important result in this paper, an optimal error estimate in L^2 norm is provided. To achieve the proof, the Aubin–Nitsche technique is usually considered. However, the model problem with OWUFVM leads a non-standard dual partition, it may be not proper with the common method. Thus, we prove the L^2-norm convergence based on an interpolated weak estimate of the first type. In order to prove this weak estimate, we apply Taylor's expansion technique to the bilinear forms corresponding to the convection and diffusion terms in (7), incorporate the first-order terms in these expansions and make them to be vanishing by means of the local Peclet's numbers which are used to determine the optimal dual mesh for OWUFVM. By doing so, we get a key second-order weak estimate. As a corollary, we can directly obtain the superconvergence of the method. Finally, utilizing the results above, we provide the proof of optimal L^2-norm error estimate.

For expression (14), denote the local basis on $K_Q \in \mathcal{T}_h$ by $N_i = N_i(\xi, \eta)(i = 1, 2, 3, 4)$.

$$N_1 = (1-\xi)(1-\eta), \quad N_2 = \xi(1-\eta), \quad N_3 = (1-\xi)\eta, \quad N_4 = \xi\eta. \tag{42}$$

By direct computation, we get Lemma 1.

Lemma 1. *For any rectangular element $K_Q \in \mathcal{T}_h$, there hold*

$$\sum_{i=1}^{4} N_i = 1, \quad \sum_{i=1}^{4} N_i x_i = x, \quad \sum_{i=1}^{4} N_i y_i = y, \quad \sum_{i=1}^{4} N_i x_i y_i = xy, \tag{43}$$

$$\sum_{i=1}^{4} N_i x_i^2 = x^2 + \xi(1-\xi)(h_x^Q)^2, \quad \sum_{i=1}^{4} N_i y_i^2 = y^2 + \eta(1-\eta)(h_y^Q)^2. \tag{44}$$

Moreover, we have the following lemmas.

Lemma 2. *Assume $u \in W^{3,\infty}(\Omega)$, then one can get*

$$(u - \Pi_h u)\Big|_{K_Q} = \frac{1}{2}\xi(\xi-1)(h_x^Q)^2 \frac{\partial^2 u}{\partial x^2}(Q) + \frac{1}{2}\eta(\eta-1)(h_y^Q)^2 \frac{\partial^2 u}{\partial y^2}(Q) + R_2(Q), \tag{45}$$

$$\frac{\partial(u - \Pi_h u)}{\partial x}\Big|_{K_Q} = \frac{1}{2}(2\xi-1)h_x^Q \frac{\partial^2 u}{\partial x^2}(Q) + R_1^x(Q), \tag{46}$$

$$\frac{\partial(u - \Pi_h u)}{\partial y}\Big|_{K_Q} = \frac{1}{2}(2\eta-1)h_y^Q \frac{\partial^2 u}{\partial y^2}(Q) + R_1^y(Q), \tag{47}$$

where

$$\begin{cases} R_2(Q) = r_2(Q) - \sum_{i=1}^{4} N_i r_2^{(i)}(Q), \\ R_1^x(Q) = r_1^x(Q) - \frac{1}{h_x^Q}\left\{(1-\eta)[r_2^{(2)}(Q) - r_2^{(1)}(Q)] + \eta[r_2^{(4)}(Q) - r_2^{(3)}(Q)]\right\}, \\ R_1^y(Q) = r_1^y(Q) - \frac{1}{h_y^Q}\left\{(1-\xi)[r_2^{(3)}(Q) - r_2^{(1)}(Q)] + \xi[r_2^{(4)}(Q) - r_2^{(2)}(Q)]\right\}. \end{cases} \tag{48}$$

and the remainders

$$r_2(Q) = \frac{1}{3!}(h_0\frac{\partial}{\partial x} + k_0\frac{\partial}{\partial y})^3 u(x_Q + \theta_0 h_0, y_Q + \theta_0 k_0), \ \theta_0 \in (0,1), \tag{49}$$

$$r_2^{(i)}(Q) = \frac{1}{3!}(h_i\frac{\partial}{\partial x} + k_i\frac{\partial}{\partial y})^3 u(x_Q + \theta_i h_i, y_Q + \theta_i k_i), \ \theta_i \in (0,1), \ i = 1,2,3,4, \tag{50}$$

$$r_1^x(Q) = \frac{1}{2}(h_0\frac{\partial}{\partial x} + k_0\frac{\partial}{\partial y})^2 \frac{\partial u}{\partial x}(x_Q + \theta_x h_0, y_Q + \theta_x k_0), \ \theta_x \in (0,1), \tag{51}$$

$$r_1^y(Q) = \frac{1}{2}(h_0\frac{\partial}{\partial x} + k_0\frac{\partial}{\partial y})^2 \frac{\partial u}{\partial y}(x_Q + \theta_y h_0, y_Q + \theta_y k_0), \ \theta_y \in (0,1). \tag{52}$$

Here, $h_0 = (x - x_Q)$, $k_0 = (y - y_Q)$, $h_i = (x_i - x_Q)$, $k_i = (y_i - y_Q)$, and $Q = (x_Q, y_Q) \in K_Q$.

Proof. By Taylor's formula, we obtain

$$u(x,y) = \sum_{m=0}^{2} \frac{1}{m!}\left(h_0\frac{\partial}{\partial x} + k_0\frac{\partial}{\partial y}\right)^m u(Q) + r_2(Q), \tag{53}$$

$$u(x_i, y_i) = \sum_{m=0}^{2} \frac{1}{m!} \left(h_i \frac{\partial}{\partial x} + k_i \frac{\partial}{\partial y} \right)^m u(Q) + r_2^{(i)}(Q), \ \ i = 1, 2, 3, 4, \tag{54}$$

Substituting (53)–(54) into the left-hand side of (45) yields

$$\begin{aligned}
&(u - \Pi_h u)|_{K_Q} = u(x, y) - \sum_{i=1}^{4} N_i u(x_i, y_i) \\
=&(1 - \sum_{i=1}^{4} N_i) u(Q) + \left[\left(h_0 - \sum_{i=1}^{4} N_i h_i \right) \frac{\partial u}{\partial x}(Q) + \left(k_0 - \sum_{i=1}^{4} N_i k_i \right) \frac{\partial u}{\partial y}(Q) \right] \\
&+ \frac{1}{2} \left[\left(h_0^2 - \sum_{i=1}^{4} N_i h_i^2 \right) \frac{\partial^2 u}{\partial x^2}(Q) + \left(k_0^2 - \sum_{i=1}^{4} N_i k_i^2 \right) \frac{\partial^2 u}{\partial y^2}(Q) \right. \\
&\left. + 2 \left(h_0 k_0 - \sum_{i=1}^{4} N_i h_i k_i \right) \frac{\partial^2 u}{\partial x \partial y}(Q) \right] + \left[r_2^{(0)}(Q) - \sum_{i=1}^{4} N_i r_2^{(i)}(Q) \right]. \\
=&\frac{1}{2} \xi(\xi - 1)(h_x^Q)^2 \frac{\partial^2 u}{\partial x^2}(Q) + \frac{1}{2} \eta(\eta - 1)(h_y^Q)^2 \frac{\partial^2 u}{\partial y^2}(Q) + R_2(Q),
\end{aligned} \tag{55}$$

where the results in Lemma 1 was employed.

Similarly, (46) and (47) can be obtained. The proof is completed.

Lemma 3. *For the remainders $r_2(Q)$, $r_2^{(i)}(Q)$, $r_1^x(Q)$ and $r_1^y(Q)$ in Lemma 2, suppose $u \in W^{3,\infty}(K_Q)$, there hold*

$$\left| \int_L r_2(Q) ds \right| \le C h^4 |u|_{3,K,\infty}, \tag{56}$$

$$\left| \int_L r_2^{(i)}(Q) ds \right| \le C h^4 |u|_{3,K,\infty}, \ \ i = 1, 2, 3, 4, \tag{57}$$

$$\left| \int_L r_1^x(Q) ds \right| \le C h^3 |u|_{3,K,\infty}, \tag{58}$$

$$\left| \int_L r_1^y(Q) ds \right| \le C h^3 |u|_{3,K,\infty}, \tag{59}$$

where $L = \overline{M_l Q}$, $(l = 1, 2, 3, 4)$.

With the help of Lemmas 2 and 3, we provide the following theorem.

Theorem 3. *Let $u \in W^{3,\infty}(\Omega)$. Assume the partition $\mathcal{T}_h$ satisfy quasi-uniform condition (3), and the dual mesh is defined by (4) with constant convection and diffusion coefficients, then we have*

$$A(u - \Pi_h u, \Pi_h^* w_h) \le C h^2 \|u\|_{3,\infty} |w_h|_1, \ \ \forall w_h \in U_h, \tag{60}$$

where C is a positive constant independent of h.

Proof. Recalling the proof of H^1-norm error estimates, one can get

$$A(u - \Pi_h u, \Pi_h^* w_h) = \sum_{K_Q \in \mathcal{T}_h} \left[w_{12} E_1 + w_{31} E_2 + w_{34} E_3 + w_{42} E_4 \right]. \tag{61}$$

Substitute (45) and (46) into E_1 to give

$$\begin{aligned}
E_1 =& \int_{\overline{M_1 Q}} \left[-D \frac{\partial (u - \Pi_h u)}{\partial x} + b_1 (u - \Pi_h u) \right] dy \\
=& \int_{y_1}^{y_Q} \left\{ -D \left[\frac{1}{2} (2\alpha_Q - 1) h_x^Q \frac{\partial^2 u}{\partial x^2}(Q) + R_1^x(Q) \right] \right. \\
&\left. + b_1 \left[\frac{1}{2} \alpha_Q (\alpha_Q - 1)(h_x^Q)^2 \frac{\partial^2 u}{\partial x^2}(Q) + \frac{1}{2} \eta(\eta - 1)(h_y^Q)^2 \frac{\partial^2 u}{\partial y^2}(Q) + R_2(Q) \right] \right\} dy
\end{aligned}$$

$$= \int_{y_1}^{y_Q} \frac{Dh_x^Q}{2} \Phi(Pe_x^Q, \alpha_Q) \frac{\partial^2 u}{\partial x^2}(Q)dy + b_1 \int_{y_1}^{y_Q} \frac{1}{2}\eta(\eta-1)(h_y^Q)^2 \frac{\partial^2 u}{\partial y^2}(Q)dy$$
$$+ \int_{y_1}^{y_Q} \left[b_1 R_2(Q) - DR_1^x(Q)\right] dy, \tag{62}$$

where

$$\Phi(Pe_x^Q, \alpha_Q) = \frac{b_1 h_x^Q}{D}\alpha_Q(\alpha_Q - 1) - (2\alpha_Q - 1) = Pe_x^Q \alpha_Q^2 - (Pe_x^Q + 2)\alpha_Q + 1. \tag{63}$$

We note that α_Q defined in (4) is just the solution to $\Phi(Pe_x^Q, \alpha_Q) = 0$ in the case of constant coefficients. Thus, while taking the α_Q such that $\Phi(Pe_x^Q, \alpha_Q) = 0$, the first term in (62) will be vanishing.

Considering the second term of the right side in (62), it holds that

$$\left| b_1 \int_{y_1}^{y_Q} \frac{1}{2}\eta(\eta-1)(h_y^Q)^2 \frac{\partial^2 u}{\partial y^2}(Q)dy \right| \le Ch^3 |u|_{2,K,\infty}, \tag{64}$$

For the last term in (62), by (48) and Lemma 3, this gives

$$\left| \int_{\overline{M_1 Q}} \left[b_1 R_2(Q) - DR_1^x(Q)\right] dy \right| \le Ch^3 |u|_{3,K,\infty}. \tag{65}$$

Combining the above analysis, there holds

$$|E_1| \le Ch^3 \left(|u|_{2,K,\infty} + |u|_{3,K,\infty}\right) \le Ch^3 \|u\|_{3,K,\infty}. \tag{66}$$

With the same procedure, we can obtain the estimates of E_i for $i = 2, 3, 4$. Similar to estimates for E_1 and E_3, a proper β_Q should be chosen such that $\Phi(Pe_y, \beta_Q) = 0$ for the estimations E_2 and E_4. Note that such β_Q is just the second expression that defined in (4). Therefore, we get

$$|w_{12}E_1 + w_{31}E_2 + w_{34}E_3 + w_{42}E_4| \le Ch^3 \|u\|_{3,K,\infty} |w_h|_{1,K}.$$

With the discrete Cauchy–Schwarz inequality, we have

$$A(u - \Pi_h u, \Pi_h^* w_h) \le Ch^3 \sum_{K\in\mathcal{T}_h} \left(\|u\|_{3,K,\infty} |w_h|_{1,K}\right)$$
$$\le Ch^3 \|u\|_{3,\infty} \left(\sum_{K\in\mathcal{T}_h} 1\right)^{\frac{1}{2}} \left(\sum_{K\in\mathcal{T}_h} |w_h|_{1,K}^2\right)^{\frac{1}{2}} \le Ch^2 \|u\|_{3,\infty} |w_h|_1, \tag{67}$$

where (15), (17) and the assumption (3) were employed.

Corollary 2. *[Superconvergence] Under the assumptions of Theorem 3, there exists a positive constant C independent of h such that*

$$\|u_h - \Pi_h u\|_1 \le Ch^2 \|u\|_{3,\infty}. \tag{68}$$

Proof. Using (33) and (60), we have

$$|u_h - \Pi_h u|_1 \le \frac{1}{C} \sup_{w_h \in U_h} \frac{|A(u - \Pi_h u, \Pi_h^* w_h)|}{|w_h|_1} \le Ch^2 \|u\|_{3,\infty}, \tag{69}$$

it follows that (68).

Theorem 4. *Assuming $u \in H_0^1(\Omega) \bigcap W^{3,\infty}(\Omega)$ and $u_h \in U_h$ are the solution to (1) and WUFVM (9), respectively. If the primary partition are regular (2) and quasi-uniform (3), and the diffusion and convection coefficients are constant, then there exists a constant $C > 0$ independent of h, such that*

$$\|u - u_h\|_0 \le Ch^2 \left(|u|_2 + \|u\|_{3,\infty}\right). \tag{70}$$

Proof. By the triangular inequality, (10) and (68), it holds

$$\|u - u_h\|_0 = \|u - \Pi_h u + \Pi_h u - u_h\|_0$$
$$\le \|u - \Pi_h u\|_0 + \|\Pi_h u - u_h\|_1 \le Ch^2 \left(|u|_2 + \|u\|_{3,\infty}\right). \tag{71}$$

The proof is completed.

Remark 4. Although the optimal L^2-norm convergence is established for the OWUFVM for 2D convection–diffusion problems with constant coefficients, it may hold for variable coefficients also, as demonstrated by the numerical results in Section 6.

Remark 5. The OWUFVM and analysis techniques developed in this paper can be extended to time-dependent convection–diffusion problems. See [31] for similar results for 1D.

5. Discrete maximum principle

In this section, we briefly discuss the discrete maximum principle for the case of constant diffusion and convection coefficients. Firstly, we assume the primary partition $\mathcal{T}_h$ is uniform, i.e. $h_x^{(i)} = h_x$ and $h_y^{(j)} = h_y$ for $i \in \mathbb{N}_x^1$ and $j \in \mathbb{N}_y^1$. By (9), we have

$$A(u_h, \psi_{P_{i,j}}) = a(u_h, \psi_{P_{i,j}}) + b(u_h, \psi_{P_{i,j}}) = (f, \psi_{P_{i,j}}), \quad \forall P_{i,j} \in \dot{\Omega}_h. \tag{72}$$

The left side of (72) results in a nine-points scheme as follows:

$$\begin{aligned} &A_{i-1,j-1}u_{i-1,j-1} + A_{i+1,j-1}u_{i+1,j-1} + A_{i-1,j+1}u_{i-1,j+1} + A_{i+1,j+1}u_{i+1,j+1} \\ &\quad + A_{i,j}u_{i,j} + A_{i,j-1}u_{i,j-1} + A_{i-1,j}u_{i-1,j} + A_{i+1,j}u_{i+1,j} + A_{i,j+1}u_{i,j+1} = (f, \psi_{P_{i,j}}) \end{aligned} \tag{73}$$

where

$$\begin{cases} A_{i-1,j-1} = -\dfrac{D(1-\alpha)(1-\beta)}{2\alpha\beta}\big[k_1\beta(1-\beta) + k_2\alpha(1-\alpha)\big], \\ A_{i+1,j-1} = -\dfrac{D\alpha(1-\beta)}{2(1-\alpha)\beta}\big[k_1\beta(1-\beta) + k_2\alpha(1-\alpha)\big], \\ A_{i-1,j+1} = -\dfrac{D(1-\alpha)\beta}{2\alpha(1-\beta)}\big[k_1\beta(1-\beta) + k_2\alpha(1-\alpha)\big], \\ A_{i+1,j+1} = -\dfrac{D\alpha\beta}{2(1-\alpha)(1-\beta)}\big[k_1\beta(1-\beta) + k_2\alpha(1-\alpha)\big], \end{cases} \tag{74}$$

$$A_{i,j} = D\left[\frac{k_1(2\alpha^2 - 2\alpha + 1)(-2\beta^2 + 2\beta + 1)}{2\alpha(1-\alpha)} + \frac{k_2(-2\alpha^2 + 2\alpha + 1)(2\beta^2 - 2\beta + 1)}{2\beta(1-\beta)}\right], \tag{75}$$

$$\begin{cases} A_{i,j-1} = \dfrac{D(1-\beta)}{2\alpha(1-\alpha)\beta}\big[k_1\beta(1-\beta)(2\alpha^2 - 2\alpha + 1) + k_2\alpha(1-\alpha)(2\alpha^2 - 2\alpha - 1)\big], \\ A_{i-1,j} = \dfrac{D(1-\alpha)}{2\alpha\beta(1-\beta)}\big[k_1\beta(1-\beta)(2\beta^2 - 2\beta - 1) + k_2\alpha(1-\alpha)(2\beta^2 - 2\beta + 1)\big], \\ A_{i+1,j} = \dfrac{D\alpha}{2(1-\alpha)\beta(1-\beta)}\big[k_1\beta(1-\beta)(2\beta^2 - 2\beta - 1) + k_2\alpha(1-\alpha)(2\beta^2 - 2\beta + 1)\big], \\ A_{i,j+1} = \dfrac{D\beta}{2\alpha(1-\alpha)(1-\beta)}\big[k_1\beta(1-\beta)(2\alpha^2 - 2\alpha + 1) + k_2\alpha(1-\alpha)(2\alpha^2 - 2\alpha - 1)\big]. \end{cases} \tag{76}$$

Here $k_1 = \frac{h_y}{h_x}$ and $k_2 = \frac{h_x}{h_y}$.

Let $g_3(t) = 2t^2 - 2t + 1$, and $g_4(t) = -2t^2 + 2t + 1$. If $t \in (0, 1)$, then we have

$$g_3(t) \in [\frac{1}{2}, 1), \quad g_4(t) \in (1, \frac{3}{2}], \quad \frac{g_3(t)}{g_4(t)} \in [\frac{1}{3}, 1), \quad \frac{g_4(t)}{g_3(t)} \in (1, 3]. \tag{77}$$

Noting $\alpha, \beta \in (0, 1)$ and (77), we can obtain

(I). $\sum A_{l,m} = 0$, for $l = i-1, i, i+1$ and $m = j-1, j, j+1$,
(II). $A_{i,j} > 0$, and $A_{l,m} < 0$ for $l = i-1, i+1$ and $m = j-1, j+1$,
(III). $A_{i,m} \le 0$ for $m = j-1, j+1$ and $A_{l,j} \le 0$ for $l = i-1, i+1$, while k_1 satisfy

$$\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\frac{g_3(\beta)}{g_4(\beta)} \le k_1^2 \le \frac{\alpha(1-\alpha)}{\beta(1-\beta)}\frac{g_4(\alpha)}{g_3(\alpha)}. \tag{78}$$

The constraint condition (78) is reasonable, since $\frac{g_4(\alpha)}{g_3(\alpha)} > \frac{g_3(\beta)}{g_4(\beta)} > 0$ is always right. By (I)–(III), we have the following maximum principle.

Theorem 5. *Assume that $\mathcal{T}_h$ is a uniform partition and the relation (78) holds. Let the diffusion and convection coefficients be constant, then the optimal weighted upwind finite volume method (9) satisfies the discrete maximum principle.*

6. Numerical experiments

In this section, some numerical experiments will be provided to confirm the theoretical results. We compare three numerical methods and compute their errors in L^2 and H^1 norm. For the optimal weighted upwind finite volume method (OWUFVM) and standard upwind finite volume method [17](SUFVM), we denote the standard *Sobolev* L^2 norm and H^1 semi-norm of errors by $\|u-u_h\|_0$ and $|u-u_h|_1$, respectively. For the modified upwind finite volume method [6](MUFVM), the discrete norms, $\|u-u_h\|_{0,h}$ and $|u-u_h|_{1,h}$, are used which have defined in [6]. In addition, denote the rate of convergence by *Rate*.

Consider the convection–diffusion problems:

$$\begin{cases} -D\Delta u + \nabla\cdot(\mathbf{b}u) = f, & in\ \Omega = (0,1)\times(0,1) \\ u = g, & on\ \Gamma = \partial\Omega \end{cases} \tag{79}$$

where $\mathbf{b} = (b_1(x,y), b_2(x,y))^T$ is convection velocity and D is a diffusion constant. In all examples, we take $D = 1, 10^{-3}$ and 10^{-7}, and the step size of primary mesh satisfies $h_x^{(i)} = h_y^{(j)} = h$ for $i \in \mathbb{N}_x^1$ and $j \in \mathbb{N}_y^1$, which is chosen from $h = \frac{1}{16}$ to $h = \frac{1}{256}$ by refining the mesh uniformly.

Example. 1. Taking the constant convection velocity $\mathbf{b} = (2,1)^T$ and the exact solution of (79)

$$u = 1 - e^{2x+y-3/D} + e^{x+y}. \tag{80}$$

The boundary value can be given by $u|_\Gamma$, and the right-side function is

$$f(x,y) = 5(D-1)e^{2x+y-3/D} + (3-2D)e^{x+y}.$$

In terms of the definitions in Section 2, we have

$$Pe_x = \frac{b_1 h}{D}, \quad Pe_y = \frac{b_2 h}{D}, \tag{81}$$

and $\alpha_{i-\frac{1}{2},j} = \alpha$ and $\beta_{i,j-\frac{1}{2}} = \beta$ with the following expressions

$$\alpha = \left(\frac{1}{2} + \frac{1}{Pe_x}\right) - \sqrt{\frac{1}{4} + \frac{1}{Pe_x^2}}, \quad \beta = \left(\frac{1}{2} + \frac{1}{Pe_y}\right) - \sqrt{\frac{1}{4} + \frac{1}{Pe_y^2}}. \tag{82}$$

Example. 2. Taking the variable convection velocity $\mathbf{b} = (2 - x^2 y,\ 1 - xy^2)^T$ and the exact solution (80) also. Then the boundary condition is $u|_\Gamma$ and right-side function is

$$f = [(5D + 4xy) - (2b_1 + b_2)]e^{2x+y-3/D} + [(b_1 + b_2) - 2(D + 2xy)]e^{x+y} - 4xy. \tag{83}$$

Moreover, the Peclet's numbers and weighted factors can be derived from (5) and (4).

Example. 3. In this example, we consider the following exact solution

$$u(x,y) = \frac{1}{5 + exp(-80(\sqrt{x^2+y^2} - 0.8))}, \tag{84}$$

with the variable convective velocity $\mathbf{b} = (2 - x^3y^2,\ 1 + x^2y^3)^T$ and the plot of u shown in Fig. 4. The derivations of right-side function, boundary value, the Peclet's numbers and weighted factors are similar to Example 2.

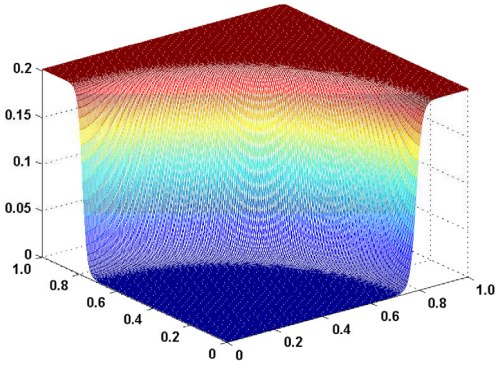


Fig. 4. Plot for the exact solution u in (84).

Table 1
Errors and rates of convergence in L^2 norm for Example 1.

$D=$	h	SUFVM		MUFVM		OWUFVM	
		$\|u-u_h\|_0$	Rate	$\|u-u_h\|_{0,h}$	Rate	$\|u-u_h\|_0$	Rate
1	1/16	8.810e−03	–	2.734e−04	–	1.627e−03	–
	1/32	4.149e−03	1.08	7.100e−05	1.95	4.067e−04	2.00
	1/64	2.011e−03	1.04	1.808e−05	1.97	1.017e−04	2.00
	1/128	9.899e−04	1.02	4.561e−06	1.99	2.542e−05	2.00
	1/256	4.911e−04	1.01	1.145e−06	1.99	6.355e−06	2.00
10^{-3}	1/16	5.315e−02	–	5.143e−02	–	1.737e−03	–
	1/32	2.800e−02	0.93	2.674e−02	0.94	4.240e−04	2.03
	1/64	1.435e−02	0.96	1.323e−02	1.02	1.046e−04	2.02
	1/128	7.269e−03	0.98	6.230e−03	1.09	2.597e−05	2.01
	1/256	3.657e−03	0.99	2.751e−03	1.18	6.471e−06	2.00
10^{-7}	1/16	5.332e−02	–	5.255e−02	–	1.737e−03	–
	1/32	2.808e−02	0.93	2.789e−02	0.91	4.238e−04	2.04
	1/64	1.441e−02	0.96	1.436e−02	0.96	1.046e−04	2.02
	1/128	7.295e−03	0.98	7.282e−03	0.98	2.596e−05	2.01
	1/256	3.670e−03	0.99	3.667e−03	0.99	6.467e−06	2.01

Table 2
Errors and rates of convergence in H^1 norm for Example 1.

$D=$	h	SUFVM		MUFVM		OWUFVM	
		$\|u-u_h\|_{1,h}$	Rate	$\|u-u_h\|_1$	Rate	$\|u-u_h\|_1$	Rate
1	1/16	7.195e−02	–	1.296e−03	–	6.275e−02	–
	1/32	3.622e−02	0.99	3.378e−04	1.94	3.137e−02	1.00
	1/64	1.817e−02	1.00	8.609e−05	1.97	1.569e−02	1.00
	1/128	9.100e−03	1.00	2.172e−05	1.99	7.844e−03	1.00
	1/256	4.554e−03	1.00	5.455e−06	1.99	3.922e−03	1.00
10^{-3}	1/16	5.219e−01	–	5.132e−01	–	8.170e−02	–
	1/32	3.832e−01	0.45	3.743e−01	0.46	4.081e−02	1.00
	1/64	2.718e−01	0.50	2.602e−01	0.52	2.039e−02	1.00
	1/128	1.870e−01	0.54	1.712e−01	0.60	1.019e−02	1.00
	1/256	1.242e−01	0.59	1.034e−01	0.73	5.095e−03	1.00
10^{-7}	1/16	5.280e−01	–	5.245e−01	–	8.170e−02	–
	1/32	3.920e−01	0.43	3.909e−01	0.42	4.081e−02	1.00
	1/64	2.839e−01	0.47	2.835e−01	0.46	2.039e−02	1.00
	1/128	2.032e−01	0.48	2.030e−01	0.48	1.019e−02	1.00
	1/256	1.445e−01	0.49	1.445e−01	0.49	5.095e−03	1.00

Table 3
Errors and rates of convergence in L^2 norm for Example 2.

$D=$	h	SUFVM		MUFVM		OWUFVM	
		$\|u-u_h\|_0$	Rate	$\|u-u_h\|_{0,h}$	Rate	$\|u-u_h\|_0$	Rate
1	1/16	4.582e−03	–	1.772e−04	–	1.663e−03	–
	1/32	2.034e−03	1.17	4.439e−05	2.00	4.159e−04	2.00
	1/64	9.589e−04	1.08	1.109e−05	2.00	1.040e−04	2.00
	1/128	4.659e−04	1.04	2.772e−06	2.00	2.600e−05	2.00
	1/256	2.297e−04	1.02	6.928e−07	2.00	6.499e−06	2.00
10^{-3}	1/16	3.756e−02	–	3.725e−02	–	1.842e−03	–
	1/32	2.004e−02	0.91	1.890e−02	0.98	4.550e−04	2.02
	1/64	1.034e−02	0.95	9.067e−03	1.06	1.130e−04	2.01
	1/128	5.255e−03	0.98	4.091e−03	1.15	2.814e−05	2.01
	1/256	2.649e−03	0.99	1.707e−03	1.26	7.020e−06	2.00
10^{-7}	1/16	3.773e−02	–	3.879e−02	–	1.839e−03	–
	1/32	2.014e−02	0.91	2.042e−02	0.93	4.538e−04	2.02
	1/64	1.040e−02	0.95	1.047e−02	0.96	1.127e−04	2.01
	1/128	5.283e−03	0.98	5.302e−03	0.98	2.807e−05	2.00
	1/256	2.663e−03	0.99	2.667e−03	0.99	7.007e−06	2.00

As shown in Tables 1–6, the OWUFVM always has the optimal convergence in L^2 and H^1 norm with second and first order, respectively. Comparing the results in Tables 1, 3 and 5, we note that SUFVM and MUFVM just have first-order rate of convergence in L^2 norm with very small diffusion coefficient $D = 10^{-7}$. However, the OWUFVM has fully second order

Table 4
Errors and rates of convergence in H^1 norm for Example 2.

$D=$	h	SUFVM		MUFVM		OWUFVM	
		$\lvert u-u_h\rvert_1$	Rate	$\lvert u-u_h\rvert_{1,h}$	Rate	$\lvert u-u_h\rvert_1$	Rate
	1/16	6.480e−02	–	8.083e−04	–	6.275e−02	–
	1/32	3.249e−02	1.00	2.030e−04	2.00	3.137e−02	1.00
1	1/64	1.627e−02	1.00	5.079e−05	2.00	1.569e−02	1.00
	1/128	8.140e−03	1.00	1.270e−05	2.00	7.844e−03	1.00
	1/256	4.071e−03	1.00	3.173e−06	2.00	3.922e−03	1.00
	1/16	2.998e−01	–	3.073e−01	–	8.160e−02	–
	1/32	2.156e−01	0.48	2.073e−01	0.57	4.077e−02	1.00
10^{-3}	1/64	1.499e−01	0.52	1.334e−01	0.64	2.038e−02	1.00
	1/128	1.008e−01	0.57	7.996e−02	0.74	1.019e−02	1.00
	1/256	6.534e−03	0.63	4.310e−02	0.89	5.095e−03	1.00
	1/16	3.046e−01	–	3.246e−01	–	8.160e−02	–
	1/32	2.231e−01	0.45	2.307e−01	0.49	4.077e−02	1.00
10^{-7}	1/64	1.604e−01	0.48	1.632e−01	0.50	2.038e−02	1.00
	1/128	1.143e−01	0.49	1.153e−01	0.50	1.019e−02	1.00
	1/256	8.114e−02	0.49	8.148e−02	0.50	5.095e−03	1.00

Table 5
Errors and rates of convergence in L^2 norm for Example 3.

$D=$	h	SUFVM		MUFVM		OWUFVM	
		$\lVert u-u_h\rVert_0$	Rate	$\lVert u-u_h\rVert_{0,h}$	Rate	$\lVert u-u_h\rVert_0$	Rate
	1/16	1.202e−02	–	3.423e−02	–	6.042e−03	–
	1/32	2.255e−03	2.41	2.708e−03	3.66	1.567e−03	1.95
1	1/64	7.315e−04	1.62	4.534e−04	2.58	3.459e−04	2.18
	1/128	3.163e−04	1.21	1.093e−04	2.05	8.422e−05	2.04
	1/256	1.491e−04	1.08	2.711e−05	2.01	2.092e−05	2.01
	1/16	1.822e−02	–	2.193e−02	–	8.480e−03	–
	1/32	9.754e−03	0.90	9.752e−03	1.17	1.915e−03	2.15
10^{-3}	1/64	5.173e−03	0.91	4.840e−03	1.01	4.617e−04	2.05
	1/128	2.685e−03	0.95	2.305e−03	1.07	1.168e−04	1.98
	1/256	1.380e−03	0.96	1.030e−03	1.16	2.929e−05	2.00
	1/16	1.834e−02	–	2.236e−02	–	9.304e−03	–
	1/32	9.816e−03	0.90	1.017e−02	1.14	2.107e−03	2.14
10^{-7}	1/64	5.213e−03	0.91	5.274e−03	0.95	4.619e−04	2.19
	1/128	2.714e−03	0.94	2.723e−03	0.95	1.169e−04	1.98
	1/256	1.403e−03	0.95	1.405e−03	0.95	2.931e−05	2.00

Table 6
Errors and rates of convergence in H^1 norm for Example 3.

$D=$	h	SUFVM		MUFVM		OWUFVM	
		$\lvert u-u_h\rvert_1$	Rate	$\lvert u-u_h\rvert_{1,h}$	Rate	$\lvert u-u_h\rvert_1$	Rate
	1/16	4.185e−01	–	6.135e−01	–	3.996e−01	–
	1/32	2.253e−01	0.89	1.279e−01	2.26	2.248e−01	0.83
1	1/64	1.126e−01	1.00	3.036e−02	2.07	1.127e−01	1.00
	1/128	5.651e−02	0.99	7.532e−03	2.01	5.647e−02	1.00
	1/256	2.830e−02	1.00	1.880e−03	2.00	2.825e−02	1.00
	1/16	5.494e−01	–	5.463e−01	–	5.066e−01	–
	1/32	3.584e−01	0.62	3.113e−01	0.81	2.325e−01	1.12
10^{-3}	1/64	2.058e−01	0.80	1.675e−01	0.89	1.119e−01	1.06
	1/128	1.105e−01	0.90	8.329e−02	1.01	5.636e−02	0.99
	1/256	5.856e−02	0.92	3.932e−02	1.08	2.824e−02	1.00
	1/16	5.542e−01	–	5.566e−01	–	5.523e−01	–
	1/32	3.625e−01	0.61	3.237e−01	0.78	2.597e−01	1.09
10^{-7}	1/64	2.088e−01	0.80	1.823e−01	0.83	1.119e−01	1.21
	1/128	1.127e−01	0.89	9.846e−02	0.89	5.636e−02	0.99
	1/256	6.114e−02	0.88	5.437e−02	0.86	2.824e−02	1.00

rate of convergence in L^2 norm. Moreover, when the convection is strongly dominant, the convergent rate in H^1 norm, by using SUFVM and MUFVM, is only close to half order shown in Tables 2 and 4. While OWUFVM keeps a fully first-order accuracy, which is consistent with the theoretical results.

7. Concluding remarks

In this paper, we have developed an optimal weighted upwind finite volume method for the convection–diffusion equation on rectangular meshes. Rigorous analysis is established for the case of constant coefficients. This includes

- Optimal order error estimate in the H^1-norm;
- Optimal estimate in the L^2-norm, which is the most important result in this article;
- Discrete maximum principle for a uniform partition with condition (78), which ensures that OWUFVM will not produce spurious oscillations.

These theoretical results are confirmed by various numerical experiments.

Further theoretical analysis for the case of variable coefficients will be reported in our future work.

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