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On the nonlinear matrix equation $X^p = A + M^T(X\#B)M$

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ABSTRACT

The nonlinear matrix equation $X^p = A + M^T(X\#B)M$, where $p \geq 1$ is a positive integer, M is an $n \times n$ nonsingular matrix, A is a positive semidefinite matrix and B is a positive definite matrix, is considered. We denote by $C\#D$ the geometric mean of positive definite matrices C and D. Based on the properties of the Thompson metric, we prove that this nonlinear matrix equation always has a unique positive definite solution and that the fixed-point iteration method can be efficiently employed to compute it. In addition, estimates of the positive definite solution and perturbation analysis are investigated. Numerical experiments are given to confirm the theoretical analysis.

1. Introduction

We consider the nonlinear matrix equation

$$X^p = A + M^T(X\#B)M, \tag{1.1}$$

where $p \geq 1$ is a positive integer, M is an $n \times n$ nonsingular matrix, A is a positive semidefinite matrix and B is a positive definite matrix. We denote by $C\#D$ the geometric mean of positive definite matrices C and D, which is extended by Ando [1] from the case of two positive scalars to the case of positive semidefinite operators. For positive definite operators C and D, the geometric mean is defined as

$$C\#D := C^{1/2}(C^{-1/2}DC^{-1/2})^{1/2}C^{1/2},$$

see [2,3], and this definition can be extended to positive semidefinite operators, see [4, P. 107] for example.

The geometric mean is of great importance in the theory of matrix inequalities [1,5,6], semidefinite programming [7] and geometry [3,8]. It also appears in solving matrix equations. In [3], it was proved that for positive definite matrices A and Q, the Riccati equation $XA^{-1}X = Q$ has a unique positive definite solution $A\#Q$. For positive definite matrices Q and C, Lim [9] computed the unique positive definite solution of the matrix equation $X = Q + CX^{-1}C$ explicitly in terms of the geometric mean, which is $X = \frac{1}{2}(Q + Q\#(Q + 4CQ^{-1}C))$.

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The matrix equation of the type

$$X = B\#(A + X), \tag{1.2}$$

which is a special case of Eq. (1.1), first appeared in [10]. It was proved that (1.2) has a unique positive definite solution $X = \frac{1}{2}(B + B\#(B + 4A))$, which served as a tool for finding the explicit solution of a special matrix equation. We found that if A is a positive definite matrix and the nonlinear matrix equation

$$X = A(X\#A^{-3})A \tag{1.3}$$

has a positive definite solution X, then X also solves equation $AXA = XAX$, a special case of the Yang–Baxter equation which plays an important role in topics of statistical mechanics, braid groups and knot theory, see [11–14] and the references therein. In general, an explicit formula of the solutions of equations of the type (1.2) is nontrivial, efficient numerical algorithms for the computation of the numerical solutions can be developed instead. Here, we give a comprehensive study of the matrix equations of the type (1.1), including numerical algorithms, lower and upper bounds of the solution and perturbation bound of the solution with respect to small perturbations on the coefficient matrices.

It seems that Eq. (1.1) has a similar form as the nonlinear matrix equation

$$X^p = A + M^T(X + B)M, \tag{1.4}$$

which has been well studied, see [10,15–17]. It was proved by Jung, Kim and Lim in [10] that the unique positive definite solution of (1.4) always exists providing that $p \geq 2$ and both A and B are positive semidefinite matrices. For the case $p = 1$, it is the well-known Stein equation and has a unique positive definite solution if the spectral radius $\rho(M^TM) < 1$.

In this paper, the relationship of the solutions of Eqs. (1.1) and (1.4) is studied. We prove that the nonlinear matrix equation (1.1) always has a unique positive definite solution X_+, and $X_+ \leq X_{**}$, where X_{**} is the unique positive definite solution of Eq. (1.4). A fixed-point iteration for finding the unique positive definite solution of Eq. (1.1) is proposed. By using the harmonic–geometric–arithmetic mean inequality, some easy-to-compute elegant estimates of the unique positive definite solution are given. Perturbation analysis of the solution with respect to small perturbations on the coefficient matrices is presented.

This paper is organized as follows. In Section 2, based on the elegant properties of the Thompson metric, we show that Eq. (1.1) always has a unique positive definite solution and a fixed-point iteration is proposed to compute it. In Section 3, elegant estimates of the positive definite solution are given. In Section 4, we derive a sharp perturbation bound for the positive definite solution with respect to small perturbations on the coefficient matrices. In Section 5, numerical examples are given to confirm the theoretical analysis.

We begin with the notation used throughout this paper. $\mathbb{R}^{n\times n}$ and $\mathbf{P}(n)$ represent, respectively, the set of $n \times n$ matrices with elements on field $\mathbb{R}$ and the set of $n \times n$ symmetric positive definite matrices. $\|\cdot\|$ and $\|\cdot\|_F$ are the spectral norm and the Frobenius norm, respectively. For a matrix H, $\lambda_{\max}(H)$ ($\lambda_{\min}(H)$) denotes the maximal (minimal) eigenvalue of H, and $\sigma_{\max}(H)$ ($\sigma_{\min}(H)$) represents the maximal (minimal) singular value. For a matrix $A = (a_1, a_2, \cdots, a_n) = (a_{ij}) \in \mathbb{R}^{n\times n}$ and a matrix B, vec(A) is a vector defined by vec$(A) = (a_1^T, \cdots, a_n^T)^T$; $A \otimes B = (a_{ij}B)$ is the Kronecker product. For Hermitian matrices X and Y, $X \geq Y$ ($X > Y$) means that $X - Y$ is positive semidefinite (definite). I represents the identity matrix of size implied by context. $[\alpha I, \beta I]$ denotes the matrix set $\{X : X - \alpha I \geq 0$ and $\beta I - X \geq 0\}$.

2. Solvability of the matrix equation

In this section, we show that matrix equation (1.1) always has a unique positive definite solution. A fixed-point iteration for computing the positive definite solution is proposed. We start this section by recalling some well-known results.

Lemma 2.1 (*Löwner–Heinz Inequality, [18, Theorem 1.1]*). *If $A \geq B \geq 0$ and $0 \leq r \leq 1$, then $A^r \geq B^r$.*

The Thompson metric on $\mathbf{P}(n)$ is defined by

$$d(A, B) = \max\{\log M(A/B), \log M(B/A)\},$$

where $M(A/B) = \inf\{\lambda > 0 : A \leq \lambda B\} = \lambda_{\max}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$.

It is known that $\mathbf{P}(n)$ is a complete metric space in the Thompson metric [19].

Lemma 2.2 (*[20]*). *For any $X, Y \in \mathbf{P}(n)$ and any $n \times n$ nonsingular matrix M, it holds*

$$d(X, Y) = d(X^{-1}, Y^{-1}) = d(M^TXM, M^TYM),$$

and

$$d(X^r, Y^r) \leq |r|d(X, Y), \quad r \in [-1, 1].$$

The following properties of the Thompson metric are proved by Lim in [21, Lemma 2.1].

Lemma 2.3. *For any* $A, B, C, D \in \mathbf{P}(n)$,

$$d(A+B, C+D) \leq \max\{d(A,C), d(B,D)\}.$$

Especially,

$$d(A+B, A+C) \leq d(B,C).$$

It is shown in [2,22,23] that

$$d(A\#B, C\#D) \leq \frac{1}{2}\big(d(A,C) + d(B,D)\big). \tag{2.1}$$

Lemma 2.4. *[Harmonic–Geometric–Arithmetic Mean Inequality, [3]] For two positive definite matrices A and B, the harmonic–geometric–arithmetic mean inequality holds*

$$2(A^{-1}+B^{-1})^{-1} \leq A\#B \leq (A+B)/2.$$

Theorem 2.5. *Matrix equation (1.1) always has a unique positive definite solution* X_+. *The matrix sequence* $\{X_k\}$ *generated by the iteration*

$$X_0 \in \mathbf{P}(n), \quad X_{k+1} = \big(A + M^T(X_k\#B)M\big)^{\frac{1}{p}} \tag{2.2}$$

converges to X_+.

Proof. Define a map $g : \mathbf{P}(n) \to \mathbf{P}(n)$ by

$$g(X) = \big(A + M^T(X\#B)M\big)^{\frac{1}{p}}. \tag{2.3}$$

For any $X, Y \in \mathbf{P}(n)$, under Thompson metric and according to Lemmas 2.2 and 2.3, we have

$$\begin{aligned} d(g(X), g(Y)) &= d\big((A + M^T(X\#B)M)^{\frac{1}{p}}, (A + M^T(Y\#B)M)^{\frac{1}{p}}\big) \\ &\leq \frac{1}{p} d\big(M^T(X\#B)M, M^T(Y\#B)M\big) \\ &\leq \frac{1}{p} d\big(X\#B, Y\#B\big) \\ &\leq \frac{1}{2p} d(X,Y), \end{aligned}$$

which shows that the map g is a strict contraction in the Thompson metric with the contraction constant $\frac{1}{2p}$. In view of the Banach fixed point theorem, there is a unique $X_+ \in \mathbf{P}(n)$ such that $X_+ = g(X_+)$, that is, X_+ is the unique positive definite solution of Eq. (1.1), and for every $X_0 \in \mathbf{P}(n)$, the sequence $\{X_k\}$ generated by (2.2) converges to X_+. □

3. Lower and upper bounds

In this section, we further investigate the properties of the positive definite solution of Eq. (1.1). Sharp lower and upper bounds on the solution are obtained. Consider the following two auxiliary nonlinear matrix equations:

$$X^p = A + M^T(X^{-1}+B^{-1})^{-1}M \tag{3.1}$$

and

$$X^p = A + M^TBM + M^TXM, \tag{3.2}$$

where p, A, M and B are defined the same as in Eq. (1.1).

For the case $p = 1$, Eqs. (3.1) and (3.2) are, respectively, the discrete algebraic Riccati equation and the Stein matrix equation. Eq. (3.1) has a unique positive definite solution when A and B are positive definite matrices while Eq. (3.2) has a unique positive definite solution if $M^TM < I$. If $p \geq 2$ is a positive integer, Eqs. (3.1) and (3.2) are studied in [24] and both of them have a unique positive definite solution. We show that the unique positive definite solution X_+ of Eq. (1.1) lies between the positive definite solutions of Eqs. (3.1) and (3.2).

Theorem 3.1. *Let* X_* *and* X_{**} *be the unique positive definite solutions of Eqs.* (3.1) and (3.2), *respectively, then the unique positive definite solution* X_+ *of Eq.* (1.1) *satisfies*

$$X_* \leq X_+ \leq X_{**}. \tag{3.3}$$

Proof. Define a map $f:\mathbf{P}(n)\to\mathbf{P}(n)$ by

$$f(X)=\left(A+M^T(X^{-1}+B^{-1})^{-1}M\right)^{\frac{1}{p}}.$$

It follows from Lemma 2.4 that for any $X\in\mathbf{P}(n)$

$$X\#B\geq 2(X^{-1}+B^{-1})^{-1}\geq (X^{-1}+B^{-1})^{-1}.$$

For any $p\geq 1$, applying Lemma 2.1 yields

$$\left(A+M^T(X\#B)M\right)^{\frac{1}{p}}\geq\left(A+M^T(X^{-1}+B^{-1})^{-1}M\right)^{\frac{1}{p}},$$

that is, $f(X)\leq g(X)$ for any $X\in\mathbf{P}(n)$, where g is defined by (2.3). Thus, $X_*=f(X_*)\leq g(X_*)$.

Also, it has been proved in [3] that for $C,D,C',D'\in\mathbf{P}(n)$, $C'\#D'\leq C\#D$ whenever $C'\leq C$ and $D'\leq D$. Thus, the map g is monotonically increasing. Since $X_*=f(X_*)\leq g(X_*)$, it follows that $X_*\leq g(X_*)\leq g^2(X_*)\leq\ldots\leq g^m(X_*)\leq\ldots$. According to Theorem 2.5, the sequence $\{g^m(X_*)\}$ is convergent and $\lim_{m\to\infty}g^m(X_*)=X_+$, which shows that $X_*\leq X_+$.

Applying Lemma 2.4 again yields

$$X\#B\leq\frac{X+B}{2}\leq X+B,$$

which, together with Lemma 2.1, leads to

$$\left(A+M^T(X\#B)M\right)^{\frac{1}{p}}\leq\left(A+M^TBM+M^TXM\right)^{\frac{1}{p}}. \tag{3.4}$$

Set $Q=A+M^TBM$ and let $h:\mathbf{P}(n)\to\mathbf{P}(n)$ be the map defined by

$$h(X)=(Q+M^TXM)^{\frac{1}{p}}.$$

Then, $X_{**}=h(X_{**})$ and it follows from (3.4) that $g(X)\leq h(X)$ for any $X\in\mathbf{P}(n)$. Analogously, we get $X_+\leq X_{**}$. □

Although inequality (3.3) provides a lower and an upper bound on the unique positive definite solution X_+ of Eq. (1.1), the computation of X_* and X_{**} is not trivial. Hence, finding a more explicit lower and upper bound which is easy to compute is of interest.

In [24], it was showed that if $p\geq 2$, the unique positive definite solution X_* of Eq. (3.1) is bounded below by a diagonal matrix $\alpha_1 I$, where α_1 is the unique positive zero of the function

$$f_1(x)=x^p-\lambda_{\min}(M^TM)(x^{-1}+\lambda_{\min}^{-1}(B))^{-1}-\lambda_{\min}(A).$$

On the other hand, it was proved in [16] that the unique positive definite solution X_{**} of Eq. (3.2) is bounded above by $\alpha_2 I$, where $\alpha_2>0$ is the unique positive zero of the function

$$h_2(x)=x^p-\lambda_{\max}(M^TM)x-\lambda_{\max}(A+M^TBM). \tag{3.5}$$

A direct application of Theorem 3.1 yields the following theorem.

Theorem 3.2. *Suppose $p\geq 2$ is a positive integer, then the unique positive definite solution X_+ of Eq. (1.1) satisfies*

$$\alpha_1 I\leq X_+\leq\alpha_2 I, \tag{3.6}$$

where α_1 and α_2 are, respectively, the unique positive zeros of functions $f_1(x)$ and $h_2(x)$.

Remark 3.3. For the case $p=1$, if both A and B are positive definite matrices, Meng and Kim [24] proved that $X_*\geq\alpha_1 I$ still holds true. It follows that $X_+\geq\alpha_1 I$ holds true for $p=1$ if both A and B are positive definite matrices.

Remark 3.4. For the case $p=1$, if $\lambda_{\max}(M^TM)<1$, then function $h_2(x)$ also has a unique positive zero α_2. It can be proved analogously to the proof of Theorem 7 in [16] that $X_{**}\leq\alpha_2 I$. It follows that $X_+\leq\alpha_2 I$ holds true for $p=1$ if $\lambda_{\max}(M^TM)<1$.

Example 3.5. Set $p=3$ and $M=\mathtt{diag}(\alpha*\mathtt{rand}(\mathtt{n},1))$ with $\alpha=1,10,30$ and $n=10,30,60,80$. Let $A=Q_1^*\Lambda_1 Q_1$, $B=Q_2^*\Lambda_2 Q_2$, where, in MATLAB commands, $\Lambda_i=\mathtt{diag}(1+\mathtt{rand}(\mathtt{n},1))\in\mathbb{R}^{n\times n}$, $Q_i=\mathtt{orth}(\mathtt{rand}(\mathtt{n}))\in\mathbb{R}^{n\times n}$, $i=1,2$. We compute the lower bound $\alpha_1 I$ and the upper bound $\alpha_2 I$. The results are shown in Table 1.

In the above example, the largest eigenvalue of the matrix M^TM is controlled by a parameter α, where $\lambda_{\max}(M^TM)\approx\alpha^2$. It can be seen from Table 1 that the largest eigenvalue of the matrix M^TM makes a difference to the value of α_2. When $\lambda_{\max}(M^TM)$ becomes large, $\alpha_2 I$ becomes a rather loose upper bound. Actually, note that α_2 is the unique positive zero of the function $h_2(x)$, which is defined by (3.5). Since $h_2'(x)=px^{p-1}-\lambda_{\max}(M^TM)$, it implies that $\alpha_2>\left(\frac{\lambda_{\max}(M^TM)}{p}\right)^{\frac{1}{p-1}}$ for

Table 1
Lower and upper bounds (3.6) of the unique positive definite solution of Eq. (1.1).

n	$\alpha_1 I$	$\lambda_{\min}(X_+)I$	$\lambda_{\max}(X_+)I$	$\alpha_2 I$
	$\alpha = 1$			
10	1.0030I	1.0666I	1.4115I	1.4115I
30	1.0066I	1.0578I	1.4313I	1.6631I
60	1.0001I	1.0660I	1.4588I	1.6875I
80	1.0007I	1.0439I	1.4596I	1.6952I
	$\alpha = 10$			
10	1.0582I	1.1090I	6.1949I	9.5424I
30	1.0975I	1.3049I	6.9752I	10.7031I
60	1.0081I	1.1195I	6.8818I	10.6199I
80	1.0001I	1.1368I	6.8693I	10.6840I
	$\alpha = 30$			
10	1.4330I	1.8833I	15.5767I	28.4264I
30	1.0176I	1.1161I	16.2378I	29.8444I
60	1.0062I	1.1745I	16.3280I	29.8836I
80	1.0492I	1.2177I	16.4255I	30.7172I

$p > 1$. If $p = 1$, a direct computation yields $\alpha_2 = \frac{\lambda_{\max}(A+M^TBM)}{1-\lambda_{\max}(M^TM)}$. In both cases, the upper bound $\alpha_2 I$ may become loose when $\lambda_{\max}(M^TM)$ is very large.

According to the idea developed in [24, Theorem 2.3], we turn to the following real functions

$$k_1(x) = x^p - \lambda_{\min}(M^TM)\lambda_{\min}^{\frac{1}{2}}(B)x^{1/2} - \lambda_{\min}(A),$$
$$k_2(x) = x^p - \lambda_{\max}(M^TM)\lambda_{\max}^{\frac{1}{2}}(B)x^{1/2} - \lambda_{\max}(A).$$

Consider the real function $k(x) = x^p - bx^{\frac{1}{2}} - a,\ \ a > 0, b > 0$. It can be seen that $k(x)$ has only one positive stationary point $x_0 = (\frac{b}{2p})^{\frac{2}{2p-1}}$. Since $p \geq 1$, we deduce that $k(x)$ is monotonically decreasing on $(0, (\frac{b}{2p})^{\frac{2}{2p-1}})$ and monotonically increasing on $((\frac{b}{2p})^{\frac{2}{2p-1}}, +\infty)$. Moreover, since $k(0) = -a < 0$ and $k(x) \to +\infty$ as $x \to +\infty$, it implies that $k(x)$ has a unique positive zero in $(0, +\infty)$. Using the same technique, we find that $k_1(x)$ has a unique positive zero β_1 and $k_2(x)$ has a unique positive zero β_2. Moreover, $\beta_1 \leq \beta_2$ since $k_1(x) \geq k_2(x)$ for $x \in (0, +\infty)$. We show that $\beta_1 I$ and $\beta_2 I$ are lower and upper bounds on X_+ which are tighter than $\alpha_1 I$ and $\alpha_2 I$.

Theorem 3.6. *Suppose $p \geq 1$ is a positive integer, then the unique positive definite solution X_+ of Eq. (1.1) satisfies*

$$\beta_1 I \leq X_+ \leq \beta_2 I. \tag{3.7}$$

Moreover, $\alpha_1 I \leq \beta_1 I$ and if $\lambda_{\max}(A) + \frac{1}{4}\lambda_{\max}(M^TM)\lambda_{\max}(B) \leq \lambda_{\max}(A + M^TBM)$, then it holds that $\beta_2 I \leq \alpha_2 I$.

Proof. Note that

$$\sqrt{\lambda_{\min}(X_+)\ \lambda_{\min}(B)}\ I\ \leq X_+\#B \leq \sqrt{\lambda_{\max}(X_+)\ \lambda_{\max}(B)}\ I.$$

It can be proved as in Theorem 2.3 in [24] that $\beta_1 I \leq X_+ \leq \beta_2 I$.

Applying the harmonic–geometric–arithmetic mean inequality, it holds that $k_1(x) \leq f_1(x)$ for all $x > 0$. This implies that $\alpha_1 I \leq \beta_1 I$.

Set $q(x) = h_2(x) - k_2(x)$, then

$$q(x) = \lambda_{\max}(M^TM)\lambda_{\max}^{\frac{1}{2}}(B)x^{1/2} + \lambda_{\max}(A) - \lambda_{\max}(M^TM)x - \lambda_{\max}(A + M^TBM).$$

With the help of the first derivative of $q(x)$ in $(0, +\infty)$, we know that $q(x)$ has only one positive stationary point $x_0 = \frac{1}{4}\lambda_{\max}(B)$ and $\max\limits_{x\in(0,+\infty)} q(x) = q(x_0)$. It can be seen that if

$$\lambda_{\max}(A) + \frac{1}{4}\lambda_{\max}(M^TM)\lambda_{\max}(B) \leq \lambda_{\max}(A + M^TBM),$$

then $q(x) = h_2(x) - k_2(x) \leq q(x_0) \leq 0$ for all $x > 0$. This leads to $\beta_2 I \leq \alpha_2 I$. $\square$

4. Perturbation analysis

In this section, we consider the perturbed matrix equation

$$\hat{X}^p = \hat{A} + \hat{M}^T(\hat{X}\#\hat{B})\hat{M}, \tag{4.1}$$

where $\hat{A} = A + \Delta A$, $\hat{M} = M + \Delta M$ and $\hat{B} = B + \Delta B$. In the rest of this section, we suppose that $M + \Delta M$ is nonsingular, $A + \Delta A$ is symmetric positive semidefinite and $B + \Delta B$ is symmetric positive definite. Then, according to Theorem 2.5, matrix equation (4.1) has a unique positive definite solution $\hat{X}$.

Lemma 4.1 (*[25, Lemma 2.1]*)**.** *For symmetric positive definite matrices A and B,*

$$\|A - B\| \le (e^{d(A,B)} - 1)\|A\|.$$

Lemma 4.2 (*[26]*)**.** *If $0 < \theta \le 1$, and P and Q are positive definite matrices of the same order with $P, Q \ge bI > 0$, then $\|P^\theta - Q^\theta\|_u \le \theta b^{\theta-1}\|P - Q\|_u$ and $\|P^{-\theta} - Q^{-\theta}\|_u \le \theta b^{-(\theta+1)}\|P - Q\|_u$. Here $\|\cdot\|_u$ stands for any unitarily invariant matrix norm.*

Consider the case where only the matrix A in Eq. (1.1) is perturbed by ΔA. Theorem 2.5 implies that there is a unique positive definite matrix Y such that

$$Y^p = \hat{A} + M^T(Y\#B)M.$$

Suppose X is the unique positive definite solution of Eq. (1.1). Applying Lemmas 2.2 and 2.3, we have

$$\begin{aligned} d(X, Y) &= d\Big((A + M^T(X\#B)M)^{\frac{1}{p}}, (\hat{A} + M^T(Y\#B)M)^{\frac{1}{p}}\Big) \\ &\le \frac{1}{p} d(A + M^T(X\#B)M, \hat{A} + M^T(Y\#B)M) \\ &\le \frac{1}{p} \max\Big\{d(A, \hat{A}), \frac{1}{2} d(X, Y)\Big\}. \end{aligned} \tag{4.2}$$

The last inequality follows from Lemma 2.3. In fact, we can deduce from (4.2) that

$$d(X, Y) \le \frac{1}{p} d(A, \hat{A}). \tag{4.3}$$

Indeed, suppose $0 < d(A, \hat{A}) < \frac{1}{2} d(X, Y)$, we have from (4.2) that

$$d(X, Y) \le \frac{1}{2p} d(X, Y),$$

which indicates that $d(A, \hat{A}) < \frac{1}{2} d(X, Y) = 0$, which is a contradiction. Hence, $d(A, \hat{A}) \ge \frac{1}{2} d(X, Y)$ and inequality (4.3) follows immediately from (4.2).

According to Lemma 4.1, we have

$$\frac{\|X - Y\|}{\|X\|} \le e^{\frac{1}{p} d(\hat{A},\ A)} - 1.$$

For the case where only the matrix B in Eq. (1.1) is perturbed by ΔB, Theorem 2.5 shows that there is a unique positive definite matrix Z such that

$$Z^p = A + M^T(Z\#\hat{B})M.$$

Then, from Lemma 2.3 and (2.1), we have

$$\begin{aligned} d(X, Z) &\le \frac{1}{p} d(A + M^T(X\#B)M, A + M^T(Z\#\hat{B})M) \\ &\le \frac{1}{p} d(M^T(X\#B)M, M^T(Z\#\hat{B})M) \\ &= \frac{1}{p} d(X\#B, Z\#\hat{B}) \\ &\le \frac{1}{2p}\Big(d(X, Z) + d(B, \hat{B})\Big). \end{aligned}$$

It follows that

$$d(X, Z) \le \frac{1}{2p - 1} d(B, \hat{B}). \tag{4.4}$$

According to Lemma 4.1, we have

$$\frac{\|X - Z\|}{\|X\|} \le e^{\frac{1}{2p-1} d(B,\hat{B})} - 1.$$

For the case when both A and B are perturbed by ΔA and ΔB, respectively, we have the following theorem.

Theorem 4.3. *Suppose the matrices A and B are perturbed by ΔA and ΔB, respectively, where $A+\Delta A$ is symmetric positive semidefinite and $B+\Delta B$ is symmetric positive definite. Let X and $\tilde{X}$ be, respectively, the unique positive definite solution of Eq. (1.1) and the unique positive definite solution of the perturbed matrix equation*

$$\tilde{X}^p = \hat{A} + M^T(\tilde{X}\#\hat{B})M. \tag{4.5}$$

Then,

$$\frac{\|\tilde{X}-X\|}{\|X\|} \le e^{\sigma} - 1, \tag{4.6}$$

where $\sigma = \frac{1}{p}d(\hat{A}, A) + \frac{1}{2p-1}d(\hat{B}, B)$.

Proof. Let Z be the unique positive definite solution of the nonlinear matrix equation

$$Z^p = A + M^T(Z\#\hat{B})M. \tag{4.7}$$

Then Eq. (4.5) can be regarded as a perturbed equation of Eq. (4.7) where only the matrix A is perturbed. It follows from (4.3) that $d(\tilde{X}, Z) \le \frac{1}{p}d(\hat{A}, A)$. From the triangle inequality and (4.4), we have

$$d(\tilde{X}, X) \le d(\tilde{X}, Z) + d(Z, X) \le \frac{1}{p}d(\hat{A}, A) + \frac{1}{2p-1}d(\hat{B}, B),$$

which, together with Lemma 4.1, implies (4.6). $\square$

When matrices A, B and M are all perturbed, the technique used above cannot apply. Here, a perturbation bound which is measured by spectral norm is obtained.

Let $\hat{X}$ be the unique positive definite solution of Eq. (4.1). As in Theorem 3.6, it can be proved that $\hat{\beta}_1 I \le \hat{X} \le \hat{\beta}_2 I$, where $\hat{\beta}_1$ and $\hat{\beta}_2$ are, respectively, the unique positive zeros of the functions

$$\hat{k}_1(x) = x^p - \lambda_{\min}(\hat{M}^T\hat{M})\lambda_{\min}^{\frac{1}{2}}(\hat{B})x^{1/2} - \lambda_{\min}(\hat{A}),$$

and

$$\hat{k}_2(x) = x^p - \lambda_{\max}(\hat{M}^T\hat{M})\lambda_{\max}^{\frac{1}{2}}(\hat{B})x^{1/2} - \lambda_{\max}(\hat{A}).$$

Let $\tilde{\beta}_1 = \min\{\beta_1, \hat{\beta}_1\}$ and $\tilde{\beta}_2 = \max\{\beta_2, \hat{\beta}_2\}$, then $\tilde{\beta}_1 I \le X, \hat{X} \le \tilde{\beta}_2 I$.

Theorem 4.4. *Let X and $\hat{X}$ be the unique positive definite solutions of Eqs.* (1.1) and (4.1), *respectively. Set $\Delta X = \hat{X} - X$. Suppose $\xi = 1 - \frac{\tilde{\beta}_1^{\frac{1-2p}{2}}\|M\|^2\kappa(B)\|B\|^{1/2}}{2p} > 0$, where $\kappa(B) = \|B\|\|B^{-1}\|$. Let $b = \min\{\lambda_{\min}(\hat{B}), \lambda_{\min}(B)\}$, then*

$$\begin{aligned}\|\Delta X\| \le \frac{\tilde{\beta}_1^{1-p}\tilde{\beta}_2^{1/2}}{2p\xi}\big(&2\tilde{\beta}_2^{-1/2}\|\Delta A\| + \tilde{\beta}_1^{-1}\tilde{\beta}_2 b^{-1/2}\|M\|^2\|\Delta B\| \\ &+ 2\|\Delta M\|\|B+\Delta B\|^{1/2}(2\|M\| + \|\Delta M\|)\big).\end{aligned} \tag{4.8}$$

Proof. It holds that

$$\hat{A} + \hat{M}^T(\hat{X}\#\hat{B})\hat{M} \ge \tilde{\beta}_1^p I,$$

and

$$A + M^T(X\#B)M \ge \tilde{\beta}_1^p I.$$

Applying Lemma 4.2 yields

$$\begin{aligned}\|\hat{X}-X\| &\le \frac{1}{p}\tilde{\beta}_1^{1-p}\|\hat{A} + \hat{M}^T(\hat{X}\#\hat{B})\hat{M} - A - M^T(X\#B)M\| \\ &\le \frac{1}{p}\tilde{\beta}_1^{1-p}\big(\|\Delta A\| + \|\hat{M}^T(\hat{X}\#\hat{B})\hat{M} - M^T(X\#B)M\|\big) \\ &\le \frac{1}{p}\tilde{\beta}_1^{1-p}\big(\|\Delta A\| + \|M^T(\hat{X}\#\hat{B} - X\#B)M\| + \|\Delta M^T(\hat{X}\#\hat{B})M\| \\ &\quad + \|M^T(\hat{X}\#\hat{B})\Delta M\| + \|\Delta M^T(\hat{X}\#\hat{B})\Delta M\|\big).\end{aligned} \tag{4.9}$$

Note that $\hat{X}^{-1/2}\hat{B}\hat{X}^{-1/2} \geq \tilde{\beta}_2^{-1}bI$ and $\hat{X}^{-1/2}B\hat{X}^{-1/2} \geq \tilde{\beta}_2^{-1}bI$, according to Lemma 4.2 again, we have

$$\begin{aligned}
\|\hat{X}\#\hat{B} - \hat{X}\#B\| &= \|\hat{X}^{1/2}\left((\hat{X}^{-1/2}\hat{B}\hat{X}^{-1/2})^{1/2} - (\hat{X}^{-1/2}B\hat{X}^{-1/2})^{1/2}\right)\hat{X}^{1/2}\| \\
&\leq \|\hat{X}\| \ \ \|(\hat{X}^{-1/2}\hat{B}\hat{X}^{-1/2})^{1/2} - (\hat{X}^{-1/2}B\hat{X}^{-1/2})^{1/2}\| \\
&\leq \frac{1}{2}\ \|\hat{X}\| \ \ (\tilde{\beta}_2^{-1}b)^{-1/2} \ \ \|\hat{X}^{-1/2}\hat{B}\hat{X}^{-1/2} - \hat{X}^{-1/2}B\hat{X}^{-1/2}\| \\
&\leq \frac{1}{2}\ (\tilde{\beta}_2^{-1}b)^{-1/2}\ \|\hat{X}\| \ \|\hat{X}^{-1}\| \ \|\Delta B\| \\
&\leq \frac{1}{2}\tilde{\beta}_2^{3/2}\tilde{\beta}_1^{-1}b^{-1/2}\|\Delta B\|.
\end{aligned} \tag{4.10}$$

Since $B^{-1/2}\hat{X}B^{-1/2} \geq \tilde{\beta}_1\lambda_{\max}^{-1}(B)I$ and $B^{-1/2}XB^{-1/2} \geq \tilde{\beta}_1\lambda_{\max}^{-1}(B)I$, we have

$$\begin{aligned}
\|\hat{X}\#B - X\#B\| &= \|B\#\hat{X} - B\#X\| \\
&= \|B^{1/2}\left((B^{-1/2}\hat{X}B^{-1/2})^{1/2} - (B^{-1/2}XB^{-1/2})^{1/2}\right)B^{1/2}\| \\
&\leq \|B\| \ \|(B^{-1/2}\hat{X}B^{-1/2})^{1/2} - (B^{-1/2}XB^{-1/2})^{1/2}\| \\
&\leq \frac{1}{2}\ \|B\| \ (\tilde{\beta}_1\|B\|^{-1})^{-1/2} \ \|B^{-1/2}\hat{X}B^{-1/2} - B^{-1/2}XB^{-1/2}\| \\
&\leq \frac{1}{2}\ \|B\| \ (\tilde{\beta}_1\|B\|^{-1})^{-1/2} \ \|B^{-1}\| \ \|\hat{X} - X\| \\
&= \frac{1}{2}\ \tilde{\beta}_1^{-1/2} \ \|B^{-1}\| \ \|B\|^{3/2} \ \|\Delta X\|.
\end{aligned} \tag{4.11}$$

It follows from (4.10) and (4.11) that

$$\begin{aligned}
&\|M^T((\hat{X}\#\hat{B}) - (X\#B))M\| \\
&\leq \|M\|^2\|\hat{X}\#\hat{B} - X\#B\| \\
&\leq \|M\|^2\left(\|\hat{X}\#\hat{B} - \hat{X}\#B\| + \|\hat{X}\#B - X\#B\|\right) \\
&\leq \frac{\|M\|^2}{2\tilde{\beta}_1^{1/2}}\left(\tilde{\beta}_2^{3/2}(\tilde{\beta}_1 b)^{-1/2}\|\Delta B\| + \kappa(B)\|B\|^{1/2}\|\Delta X\|\right).
\end{aligned} \tag{4.12}$$

Moreover, since $\hat{X} \leq \tilde{\beta}_2 I$, it follows from the monotone property [3] of the geometric mean that $\hat{X}\#\hat{B} \leq (\tilde{\beta}_2 I)\#\hat{B} = (\tilde{\beta}_2)^{1/2}\hat{B}^{1/2}$, from which we have $\|\hat{X}\#\hat{B}\| \leq \tilde{\beta}_2^{1/2}\|\hat{B}^{1/2}\|$. Subsequently, it yields

$$\begin{aligned}
&\|\Delta M^T(\hat{X}\#\hat{B})M\| + \|M^T(\hat{X}\#\hat{B})\Delta M\| + \|\Delta M^T(\hat{X}\#\hat{B})\Delta M\| \\
&\leq \tilde{\beta}_2^{1/2}\|\Delta M\|\|\hat{B}^{1/2}\|(2\|M\| + \|\Delta M\|).
\end{aligned} \tag{4.13}$$

Set $\xi = 1 - \frac{\tilde{\beta}_1^{\frac{1-2p}{2}}\|M\|^2\kappa(B)\|B\|^{1/2}}{2p}$, according to (4.9), (4.12) and (4.13), it yields

$$\begin{aligned}
\|\Delta X\| \leq \frac{\tilde{\beta}_1^{1-p}\tilde{\beta}_2^{1/2}}{2p\xi}&\left(2\tilde{\beta}_2^{-1/2}\|\Delta A\| + \tilde{\beta}_1^{-1}\tilde{\beta}_2 b^{-1/2}\|M\|^2\|\Delta B\|\right. \\
&\left.+\ 2\|\Delta M\|\|B + \Delta B\|^{1/2}(2\|M\| + \|\Delta M\|)\right).
\end{aligned}$$

5. Numerical examples

In this section, we present some test results which illustrate the performance of the fixed-point iteration (2.2), the tightness of the lower and upper bounds (3.6) and (3.7), and the perturbation bounds (4.6) and (4.8). The computations were performed in MATLAB R2017b with unit roundoff $\mu \approx 10^{-16}$ and the iterations terminate if the relative residual $\rho(X_k)$ satisfies

$$\rho(X_k) = \frac{\|fl(X_k^p - A - M^T(X_k\#B)M)\|_F}{\|X_k\|_F^p + \|A\|_F + \|M\|_F^2\|X_k\#B\|_F} \leq 10^{-15}.$$

Example 5.1 (*[10]*). Let

$$A = \begin{pmatrix} 0.0120 & -0.0030 & 0.0010 \\ -0.0030 & 0.0210 & 0.0020 \\ 0.0010 & 0.0020 & 0.0070 \end{pmatrix}, \quad B = \begin{pmatrix} 1.1231 & 0.4497 & 0.9024 \\ 0.4497 & 0.8283 & 0.7254 \\ 0.9024 & 0.7254 & 1.0292 \end{pmatrix},$$

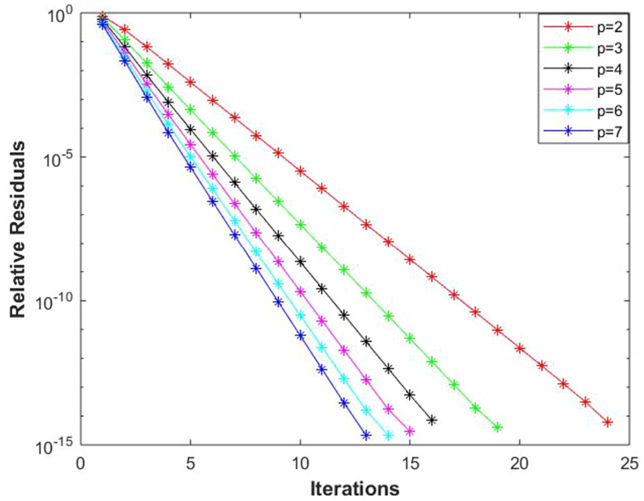

Fig. 1. Convergence behaviour of the fixed-point iteration (2.2).

Table 2
Lower and upper bounds of the unique positive definite solution of Eq. (1.1).

n	$\alpha_1 I$	$\beta_1 I$	$\lambda_{\min}(X_+)I$	$\lambda_{\max}(X_+)I$	$\beta_2 I$	$\alpha_2 I$
	$\alpha = 1$					
10	1.0030I	1.0056I	1.0666I	1.4115I	1.5014I	1.4115I
30	1.0066I	1.0066I	1.0578I	1.4313I	1.5447I	1.6631I
60	1.0001I	1.0001I	1.0660I	1.4588I	1.5467I	1.6875I
80	1.0007I	1.0007I	1.0439I	1.4596I	1.5509I	1.6952I
	$\alpha = 10$					
10	1.0582I	1.0701I	1.1090I	6.1949I	6.5998I	9.5424I
30	1.0975I	1.1728I	1.3049I	6.9752I	7.2032I	10.7039I
60	1.0081I	1.0082I	1.1195I	6.8818I	7.1923I	10.6199I
80	1.0001I	1.0001I	1.1368I	6.8693I	7.2287I	10.6840I
	$\alpha = 30$					
10	1.4330I	1.7467I	1.8833I	15.5767I	16.2707I	28.42643I
30	1.0176I	1.0177I	1.1161I	16.2378I	16.9957I	29.8444I
60	1.0062I	1.0067I	1.1745I	16.3280I	17.0233I	29.8836I
80	1.0492I	1.0931I	1.2177I	16.4255I	17.4167I	30.7172 I

$M = \begin{pmatrix} 0.7922 & 0.0357 & 0.6787 \\ 0.9594 & 0.8491 & 0.7577 \\ 0.6557 & 0.9339 & 0.7431 \end{pmatrix}$, and $p = 2, 3, 4, 5, 6, 7$. We apply the fixed-point iteration (2.2) with starting point $X_0 = I$ to Eq. (1.1). The results are shown in Fig. 1.

It shows that the fixed-point iteration (2.2) works well for obtaining the symmetric positive definite solution of Eq. (1.1) and the iteration number decreases as p increases.

Example 5.2. Let A, B and M be the same as those in Example 3.5. We compute the lower bound $\beta_1 I$ and upper bound $\beta_2 I$ in (3.7) and compare them with $\alpha_1 I$ and $\alpha_2 I$ which are obtained in Theorem 3.2. The results are shown in Table 2.

We can see from Table 2 that the lower bound $\alpha_1 I$ is not as sensitive to the eigenvalues of M as $\alpha_2 I$ is. Both $\alpha_1 I$ and $\beta_1 I$ can be tight lower bounds. However, if $\lambda_{\max}(M^T M)$ is very large, then $\beta_2 I$ can be a tighter upper bound than $\alpha_2 I$.

Example 5.3. Set $p = 2$ and $M = \mathtt{rand}(n)$. Let $A = Q_1^* \Lambda_1 Q_1$, $B = Q_2^* \Lambda_2 Q_2$, $\Delta A = Q_3^* \Lambda_3 Q_3 \times 10^{-j}$, $\Delta B = Q_4^* \Lambda_4 Q_4 \times 10^{-j}$, where, in MATLAB commands, $\Lambda_i = \mathtt{diag}(1 + \mathtt{rand(n, 1)}) \in \mathbb{R}^{n\times n}$, $Q_i = \mathtt{orth}(\mathtt{rand(n)}) \in \mathbb{R}^{n\times n}$, $i = 1, 2, 3, 4$.

Set $n = 10, 20, 30, 40, 50$ and $j = 3, 7, 10$. The perturbation bound (4.6) is displayed in Table 3.

Table 3 shows that for the case where only the matrices A and B are perturbed, (4.6) provides a sharp perturbation bound.

Table 3
Perturbation bounds for the different n and j.

n	$j=3$		$j=7$		$j=10$	
	$\frac{\Vert\varDelta X\Vert}{\Vert X\Vert}$	$e^{\sigma}-1$	$\frac{\Vert\varDelta X\Vert}{\Vert X\Vert}$	$e^{\sigma}-1$	$\frac{\Vert\varDelta X\Vert}{\Vert X\Vert}$	$e^{\sigma}-1$
10	3.60e−04	1.22e−03	3.32e−08	1.30e−07	4.84e−10	1.34e−09
20	3.10e−04	1.41e−03	3.50e−08	1.44e−07	3.20e−10	1.35e−09
30	3.86e−04	1.41e−03	3.46e−08	1.30e−07	3.22e−10	1.31e−09
40	3.45e−04	1.35e−03	3.41e−08	1.38e−07	3.36e−10	1.34e−09
50	3.62e−04	1.34e−03	3.63e−08	1.34e−07	3.72e−10	1.42e−09

Table 4
Comparison of the perturbation error with the perturbation bound (4.8).

p	$j=3$		$j=5$		$j=7$	
	$\Vert\varDelta X\Vert$	Bound (4.8)	$\Vert\varDelta X\Vert$	Bound (4.8)	$\Vert\varDelta X\Vert$	Bound (4.8)
1	1.00e−02	2.61e−02	1.00e−04	3.48e−04	1.25e−06	3.24e−06
3	1.04e−03	1.84e−03	1.28e−05	2.84e−05	8.32e−08	1.98e−07
5	3.98e−04	9.31e−04	6.56e−06	1.29e−05	3.21e−08	1.02e−07
7	4.26e−04	9.42e−04	3.73e−06	9.05e−06	3.80e−08	6.77e−08

Example 5.4. Set $n=3$, let A, B, $\varDelta A$ and $\varDelta B$ be defined by using the same technique as in Example 5.3. Let

$$M=\begin{pmatrix} 0.3169 & 0.0143 & 0.2715 \\ 0.3838 & 0.3396 & 0.3031 \\ 0.2623 & 0.3736 & 0.2972 \end{pmatrix}.$$

Suppose M is perturbed by $\varDelta M=\mathtt{rand}(3)\times 10^{-j}$. Set $p=1,3,5,7$ and $j=3,5,7$, respectively, it can be proved that all the conditions in Theorem 4.4 are satisfied. We compare the perturbation error $\Vert\varDelta X\Vert$ with the perturbation bound (4.8) in Theorem 4.4. The results are listed in Table 4.

Under the condition $\xi>0$, where ξ is defined in Theorem 4.4, we can see from Table 4 that Theorem 4.4 gives a very sharp and revealing perturbation bound.

6. Conclusion

In this paper, we consider the nonlinear matrix equation (1.1). Based on the elegant properties of the Thompson metric, we prove that the nonlinear matrix equation always has a unique positive definite solution and we compare it with the unique positive definite solution of equation $X^p=A+M^T(X+B)M$. A fixed-point iteration method is employed to compute the positive definite solution and elegant estimates of the solution are given. Perturbation analysis of the unique positive definite solution is presented.

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References

[1] T. Ando, Concavity of certain maps on positive definite matrices and applications to Hadamard products, Linear Algebra Appl. 26 (1979) 203–241.
[2] R. Bhatia, On the exponential metric increasing property, Linear Algebra Appl. 375 (2003) 211–220.
[3] J.D. Lawson, Y. Lim, The geometric mean, matrices, metrics, and more, Am. Math. Mon. 108 (2001) 797–812.
[4] R. Bhatia, Positive Definite Matrices, Princeton Univrsity Press, Princeton, 2007.
[5] T. Ando, Topics on Operator Inequalities, in: Lecture Notes Hokkaido University, Sapporo, 1978.
[6] T. Ando, On the arithmetic-geometric-harmonic mean inequalities for positive definite matrices, Linear Algebra Appl. 52–53 (1983) 31–37.
[7] Yu.E. Nesterov, M.J. Todd, Self-scaled barriers and interior-point methods for convex programming, Math. Oper. Res. 22 (1997) 1–42.
[8] G. Corach, H. Porta, L. Recht, Geodesics and operator means in the space of positive operators, Int. J. Math. 4 (1993) 193–202.

[9] Y. Lim, The inverse mean problem of geometric and contraharmonic means, Linear Algebra Appl. 408 (2005) 221–229.
[10] C. Jung, H.-M. Kim, Y. Lim, On the solution of the nonlinear matrix equation $X^n = f(X)$, Linear Algebra Appl. 430 (2009) 2042–2052.
[11] C.N. Yang, Some exact results for the many-body problem in one dimension with repulsive delta-function interaction, Phys. Rev. Lett. 19 (1967) 1312–1315.
[12] R.J. Baxter, Partition function of the eight-vertex lattice model, Ann. Phys. 70 (1972) 193–228.
[13] F. Felix, Nonlinear Equations, Quantum Groups and Duality Theorems: A Primer on the Yang–Baxter Equation, VDM Verlag, 2009.
[14] C. Yang, M. Ge, Braid Group, Knot Theory, and Statistical Mechanics, World Scientific, 1989.
[15] H. Lee, Y. Lim, Invariant metrics, contractions and nonlinear matrix equations, Nonlinearity 21 (2008) 857–878.
[16] Z. Jia, M. Zhao, M. Wang, S. Ling, Solvability theory and iteration method for one self-adjoint polynomial matrix equation, J. Appl. Math. (2014) 7, Art. ID 681605.
[17] H. Abou-Kandil, G. Freiling, V. Ionesu, G. Jank, Matrix Riccati Equations in Control and Systems Theory, Birkhauser Verlag, 2003.
[18] X. Zhan, Matrix Inequalities, Springer-Verlag, Berlin, 2002.
[19] R.D. Nussbaum, Hilbert's projective metric and iterated nonlinear maps, Mem. Amer. Math. Soc. 75 (391) (1988).
[20] A.C. Thompson, On Certain Contraction Mappings in a Partially Ordered Vector Space, Vol. 14, Pro. Amer. Math. Soc., 1963, pp. 438–443.
[21] Y. Lim, Solving the nonlinear matrix equation $X = Q + \sum_{i=1}^{m} A_i^* X^{\delta_i} A_i$ via a contraction principle, Linear Algebra Appl. 430 (2009) 1380–1383.
[22] G. Corach, H. Porta, L. Recht, Convexity of the geodesic distance on spaces of positive operators, Illinois J. Math. 38 (1994) 87–94.
[23] J.D. Lawson, Y. Lim, Symmetric spaces with convex metrics, Forum Math. 19 (2007) 571–602.
[24] J. Meng, H.-Y. Kim, The positive definite solution of the nonlinear matrix equation $X^p = A + M(B + X^{-1})^{-1}M^*$, J. Comput. Appl. Math. 322 (2017) 139–147.
[25] Y. Lim, Stopping criteria for the Ando-Li-Mathias and Bini-Meini-Poloni geometric means, Linear Algebra Appl. 434 (2011) 1884–1892.
[26] R. Bhatia, Matrix Analysis, in: Graduate Texts in Mathematics, Spring, Berlin, 1997.

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