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Short communication

# Comments on the spectrum of a tridiagonal $k$-Toeplitz matrix

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In this note we will discuss some errors presented in a recent paper on $r$-periodic tridiagonal matrices. In addition, we present several results that had been previously reported in mathematical literature.



In the recent paper [1], M. Şahin and S. Yılmaz claim the introduction of the notion of $r$-periodic tridiagonal matrix, for $r \geqslant 2$. The authors mention that in these matrices the entries on the main diagonal can be any $r$-periodic sequence. Indeed, the periodicity is not reduced to the main diagonal.

This short communication aims to stress that this concept was established more than 3 decades ago. In fact, in 1985, M.J.C. Gover and S. Barnett in [2] explicitly refer to the concept of *tridiagonal k-Toeplitz matrix*: an $n \times n$ complex tridiagonal matrix $A = (a_{ij})$ such that

$$a_{i+k,j+k} = a_{ij}\,, \quad \text{for } i, j = 1, 2, \ldots, n-k\,,$$

for a given positive integer number $k \leqslant n$. Analogously,

$$A = \begin{pmatrix} a_1 & b_1 & & & & & & & \\ c_1 & \ddots & \ddots & & & & & & \\ & \ddots & a_k & b_k & & & & & \\ & & c_k & a_1 & b_1 & & & & \\ & & & c_1 & \ddots & \ddots & & & \\ & & & & \ddots & a_k & b_k & & \\ & & & & & c_k & a_1 & b_1 & \\ & & & & & & c_1 & \ddots & \ddots \\ & & & & & & & \ddots & \end{pmatrix}_{n\times n} .$$

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For $k = 1$, we get a tridiagonal Toeplitz matrix and its determinant was known in [3] as a *continuant*. The spectrum of a tridiagonal 2-Toeplitz matrix was studied by M.J.C. Gover in 1994 [4]. Notwithstanding, the characteristic polynomial of such a matrix was found by V. Lovass-Nagy and P. Rózsa [5] in 1963. Before that, the particular case when $k = 2$ and the two subdiagonals are constant equal to 1, was first considered in 1947 in D.E. Rutherford's seminal paper [6], followed soon after by J.F. Elliott with his Master's thesis [7, Section IV.4]. In 1966, L. Elsner and R.M. Redheffer [8] studied $A$ for special cases of $k$ and, two years later, P. Rózsa in [3] originally proved a general formula for the determinant of $A$, which was called *periodic continuant*. Independently, the author and J. Petronilho [9] considered the case when $k = 3$ and, later on, the characteristic polynomial of $A$ was stated, for any $k$, when analysing the invertibility conditions for $A$ based on orthogonal polynomials theory (cf. [10]). Namely, setting $\mu^2 = \prod_{i=1}^{k} b_i c_i$, for $j > i$, define

$$\Delta_{i,j}(x) = \begin{vmatrix} x-a_i & 1 & & & \\ b_i c_i & x-a_{i+1} & 1 & & \\ & b_{i+1}c_{i+1} & \ddots & \ddots & \\ & & \ddots & \ddots & 1 \\ & & & b_{j-1}c_{j-1} & x-a_j \end{vmatrix},$$

so $\Delta_{i,j}$ is a polynomial of degree $j - i + 1$ in $x$, and the polynomial $\varphi_k$ of degree $k$

$$\varphi_k(x) = \frac{1}{2\mu}\left(D_k(x) + (-1)^k (b_1 c_1 \cdots b_{k-1}c_{k-1} + b_k c_k)\right),$$

where $D_k$ is the monic polynomial of degree $k$

$$D_k(x) = \begin{vmatrix} x-a_1 & 1 & 0 & \cdots & 0 & 1 \\ b_1 c_1 & x-a_2 & 1 & \cdots & 0 & 0 \\ 0 & b_2 c_2 & x-a_3 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & x-a_{k-1} & 1 \\ b_k c_k & 0 & 0 & \cdots & b_{k-1}c_{k-1} & x-a_k \end{vmatrix}.$$

Then, from [10, Theorem 5.1], for $0 \leqslant r \leqslant k-1$ and $n \equiv r \pmod{k}$, the characteristic polynomial of $A$ is

$$p(x) = \mu^{\lfloor n/k \rfloor}\left\{\Delta_{1,r}(x)\, U_{\lfloor n/k \rfloor}(\varphi_k(x)) + \frac{b_k c_k}{\mu}\left(\prod_{i=1}^{r} b_i c_i\right)\Delta_{r+2,k-1}(x)\, U_{\lfloor (n-k)/k \rfloor}(\varphi_k(x))\right\},$$

where $U_n$ is the Chebyshev polynomial of second kind of degree $n$. For the particular case when $n \equiv k-1 \pmod{k}$, we have

$$p(x) = \mu^{\lfloor n/k \rfloor}\,\Delta_{1,k-1}(x)\, U_{\lfloor n/k \rfloor}(\varphi_k(x)),$$

and the eigenvalues of $A$ are the $k-1$ zeros of $\Delta_{1,k-1}(x)$ together with all the solutions of the $\lfloor n/k \rfloor$ algebraic equations of degree $k$

$$\varphi_k(x) = \cos\frac{jk\pi}{n+1}, \quad j = 1, 2, \ldots, \lfloor n/k \rfloor. \tag{1}$$

When we want to analyse the spectrum of a tridiagonal $k$-Toeplitz matrix, we consider $b_i c_i \neq 0$, for all $i = 1, \ldots, k$, i.e., $A$ is irreducible. Otherwise, the spectral problem is reduced to a block decomposition.

It is clear that the instance $n \equiv k-1 \pmod{k}$ is the only case when one can say something more substantial regarding the spectrum of $A$. This should be the appropriate formulation to the [1, Theorem 12]. For some perturbations in the (1, 1) and (2, 1)-entries, the reader is referred to [11–13]. Note that *all* cases and examples treated in [1] fell in this particular situation which are either trivial reducible cases or imprecise.

For example, Theorem 22 is inaccurate. The eigenvalues of a tridiagonal 2-Toeplitz of even order can be found in [4, Theorem 2.4]. Explicit formulas are known for particular cases as we can find for instance in [7, p. 32]. Moreover, if the condition $b_1 = 0$ is satisfied, the matrix and its spectrum become, respectively, reducible and trivial.

Another mistaken formula can be found in page 103 regarding the determinant of $T(n)$: the expansion is neither correct nor the final formula. Indeed, the matrix considered is not of order $n-1$ and it is not the right one. The correct formula is $\det T(n) = a_n \det T(n-1) + b_n \det T(n-2)$, according to the notation used in the paper.

The formula found for $q_k(x)$ and used throughout the manuscript can be easily recognized as $\Delta_{1,k}(x)$, as we can see from [1, Example 11], where

$$\begin{aligned} q_3(x) &= ((a_2 - x)(a_3 - x) + b_3)(v_1 - x) + b_2(a_3 - x)v_0 \\ &= -\begin{vmatrix} x - v_1 & 1 & 0 \\ -v_0 b_2 & x - a_2 & 1 \\ 0 & -b_3 & x - a_3 \end{vmatrix} \\ &= -\Delta_{1,3}(x) \end{aligned}$$

So, Algorithm 21, brings no novelty. For example, the eigenvalues a tridiagonal 3-Toeplitz matrix of order 30 002 can be easily derived from the solutions of the cubic algebraic equations (1) and the zeros of the quadratic polynomial $\Delta_{1,2}$. Observe that $30002 \equiv 2 \pmod 3$. The same comment can be made on Example 18 where we have a tridiagonal 4-Toeplitz matrix of order $11 \equiv 3 \pmod 4$. For the general case, no computational efficiency is provided.

For Theorem 23 and the case $r = 4$ in page 112, if one considers $b_1 = 0$, then as mentioned earlier we have a reducible matrix and the eigenvalues follow in a straightforward way. Note that in Theorem 23, the eigenvalues $\xi$'s do not depend on $k$, since $d_k$ is constant. In the other cases, we have again the order of the matrix congruent to $k - 1 \pmod k$.

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