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Generalized q-Legendre polynomials

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Abstract

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This article gives a q-version of the generalized Legendre polynomials recently introduced by the author. The generalization makes use of the little q-Jacobi polynomials. In conclusion some open problems are posed.

Keywords: Orthogonal polynomials; Legendre polynomials

1. Introduction

In an earlier paper [4] we considered the problem of finding all polynomials f_n in the variable n such that the recursion formula

$$(n+1)u_{n+1} - f_n u_n + n u_{n-1} = 0, \quad \text{for } n \geqslant 0,$$

has an integral solution (u) with $u_{-1} = 0$ and $u_0 = 1$. With a slight change of notation the main result obtained was the following.

Theorem 1.1. *Let* $(x) = (x_0, x_1, \dots, x_j, \dots)$ *be a sequence of complex numbers. Let*

$$f_n(x) = (2n+1)\left(1 - 2\sum_{j=0}^{n} \binom{n}{j}\binom{n+j}{j} x_j\right), \tag{1}$$

and consider sequences $(u) = (u_0, u_1, \dots, u_n, \dots)$ *satisfying the recursion formula*

$$(n+1)u_{n+1} - f_n u_n + n u_{n-1} = 0, \quad \textit{for } n \geqslant 1. \tag{2}$$

Then (2) *has two independent solutions* (p) *and* (q) *as follows. The element* p_n *is represented as*

$$p_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} c_k, \tag{3}$$

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where

$$c_0 = 1,$$

$$c_{k+1} = \sum_{i=0}^{k} (-1)^{k-i} \sum_{j=k-i}^{k} \binom{i+j}{k}\binom{k}{i}\binom{k}{j} x_j c_i, \quad \textit{for } k \geqslant 0. \tag{4}$$

The element q_n *is represented as*

$$q_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{n+k}{k} d_k, \tag{5}$$

where

$$d_0 = 0,$$

$$d_{k+1} = \sum_{i=0}^{k} (-1)^{k-i} \sum_{j=k-i}^{k} \binom{i+j}{k}\binom{k}{i}\binom{k}{j} x_j d_i + \frac{1}{k+1}, \quad \textit{for } k \geqslant 0. \tag{6}$$

As a corollary of this theorem, formula (1) will provide a solution to the problem when $(x) = (x_0, x_1, \ldots, x_N, 0, 0, \ldots)$ and $x_i \in \mathbb{Z}$ for $0 \leqslant i \leqslant N$.

The purpose of the present paper is to give a q-version of this theorem, and to pose some open problems.

2. Notation and preliminary results

In the sequel, q is a fixed real number in $]0, 1[$, and f, g, etc. are complex polynomials in one variable.

To fix the notation we recall a number of definitions and results (see in particular [1,3]).

For $a \in \mathbb{C}$ the *q-shifted factorial* is defined by

$$(a; q)_n = \begin{cases} 1, & \text{if } n = 0, \\ (1-a)(1-aq)\cdots(1-aq^{n-1}), & \text{if } n \in \mathbb{N}, \\ ((1-aq^{-1})(1-aq^{-2})\cdots(1-aq^{n}))^{-1}, & \text{if } -n \in \mathbb{N}. \end{cases}$$

We need the elementary formula

$$(a^{-1}; q)_n = \left(-\frac{1}{a}\right)^n q^{\binom{n}{2}} (aq^{1-n}; q)_n, \quad \text{for } a \neq 0, \ n \in \mathbb{N}_0. \tag{7}$$

The *Gaussian polynomial* or *q-binomial coefficient* is defined by

$$\begin{bmatrix} \alpha \\ m \end{bmatrix}_q = \frac{(q^{\alpha-m+1}; q)_m}{(q; q)_m}, \quad \text{for } \alpha \in \mathbb{R}, \ m \in \mathbb{N}_0.$$

The *Heine* (or *basic*) *series* ${}_{r+1}\phi_r$ is defined by

$${}_{r+1}\phi_r(a_0, \ldots, a_r; b_1, \ldots, b_r; q, x) = \sum_{n=0}^{\infty} \frac{(a_0; q)_n \cdots (a_r; q)_n}{(q; q)_n (b_1; q)_n \cdots (b_r; q)_n} x^n,$$

for $a_i, b_i \in \mathbb{C}, |x| < 1$.

We shall make essential use of the *q-Pfaff–Saalschütz identity* (see [5])

$$ {}_3\phi_2(q^{-n}, a, b; c, d; q, q) = \frac{(c/a; q)_n (c/b; q)_n}{(c; q)_n (c/ab; q)_n}, \quad \text{if } cd = abq^{1-n}. $$

The *Jackson integral* J_q and the *q-derivative* ϑ_q are defined by

$$ (J_q f)(x) = \int_0^x f(t)\, \mathrm{d}_q t = x(1-q) \sum_{n=0}^{\infty} f(xq^n) q^n, $$

$$ (\vartheta_q f)(x) = \frac{f(x) - f(qx)}{x(1-q)}. $$

Both J_q and ϑ_q are linear operators on $\mathbb{Z}[x]$ satisfying the following rules:

$$ \vartheta_q(J_q(f)) = f, \tag{8} $$

$$ J_q(\vartheta_q(f)) = f - f(0), \tag{9} $$

$$ \vartheta_q(fg)(x) = f(qx)\vartheta_q(g)(x) + \vartheta_q(f)(x)g(x), \tag{10} $$

$$ \vartheta_q^n(fg)(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \vartheta_q^k(f)(xq^{n-k})\vartheta_q^{n-k}(g)(x), \quad \text{for } n \in \mathbb{N}, \tag{11} $$

$$ \vartheta_q(x^n) = \begin{bmatrix} n \\ 1 \end{bmatrix}_q x^{n-1}, \quad \text{for } n \in \mathbb{N}, \tag{12} $$

$$ \vartheta_q((x; q^{-1})_n) = -\begin{bmatrix} n \\ 1 \end{bmatrix}_q q^{1-n}(x; q^{-1})_{n-1}, \quad \text{for } n \in \mathbb{N}, \tag{13} $$

$$ \mu_n = \int_0^1 x^n\, \mathrm{d}_q x = \frac{1-q}{1-q^{n+1}}, \quad \text{for } n \in \mathbb{N}_0. \tag{14} $$

The *little q-Jacobi polynomials* introduced by Hahn [3] (see [2]) are given by

$$ P_n(x; \alpha, \beta \mid q) = {}_2\phi_1(q^{-n}, \alpha\beta q^{n+1}; \alpha q; q, qx), \quad \text{for } \alpha, \beta \in \mathbb{C}. $$

We shall need only the special case $P_n(x \mid q) = P_n(x; 1, 1 \mid q)$ which consequently will be called the *little q-Legendre polynomials.* By means of the q-binomial coefficients this can be written as

$$ P_n(x \mid q) = \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2} - kn} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q x^k. $$

Some fundamental properties of the little q-Legendre polynomials are the following.

(i) The sequence $u_n = P_n(x \mid q)$ satisfies a three-term recurrence relation

$$ q^n(1-q^{n+1})(1+q^n)u_{n+1} - (1-q^{2n+1})(2q^n - (1+q^n)(1+q^{n+1})x)u_n + q^n(1-q^n)(1+q^{n+1})u_{n-1} = 0, \quad \text{for } n \geqslant 0, $$

with initial conditions $u_{-1} = 0$, $u_0 = 1$.

(ii) They are given by a *q-Rodrigues formula*

$$ P_n(x \mid q) = \frac{(1-q)^n}{(q; q)_n} \vartheta_q^n(x^n(x; q^{-1})_n), \quad \text{for } n \geqslant 0. $$

(iii) They satisfy the following orthogonality relations:

$$\int_0^1 P_m(x\mid q)P_n(x\mid q)\,\mathrm{d}_q x = \delta_{mn} q^n \frac{1-q}{1-q^{2n+1}}, \quad \text{for } m, n \geqslant 0.$$

Lemma 2.1. *The following identity is valid for n, i, j* $\in \mathbb{N}_0$:

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} i+j \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q q^{(n-k)(i+j-k)} = \begin{bmatrix} n+i \\ i \end{bmatrix}_q \begin{bmatrix} n+j \\ j \end{bmatrix}_q \begin{bmatrix} n \\ i \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q.$$

Proof. When we apply the q-Pfaff–Saalschütz identity for

$$a = q^{n+1}, \qquad b = q^{-i-j+2\epsilon}, \qquad c = q^{1-i+\epsilon}, \qquad d = q^{1-j+\epsilon},$$

where $0 < \epsilon < \frac{1}{2}$, we get

$$\sum_{k=0}^{n} \frac{(q^{-n};q)_k (q^{n+1};q)_k (q^{-i-j+2\epsilon};q)_k}{(q;q)_k (q^{1-i+\epsilon};q)_k (q^{1-j+\epsilon};q)_k} q^k = \frac{(q^{-i-n+\epsilon};q)_n (q^{j+1-\epsilon};q)_n}{(q^{1-i+\epsilon};q)_n (q^{j-n-\epsilon};q)_n}.$$

By applying (7) twice on each side and rearranging, we get

$$\sum_{k=0}^{n} \frac{\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} i+j-2\epsilon \\ k \end{bmatrix}_q (q;q)_k^2}{(q^{1+\epsilon};q)_{k-i}(q;q)_i(q^{1+\epsilon};q)_{k-j}(q;q)_j} q^{(n-k)(i+j-k-2\epsilon)}$$

$$= \frac{\begin{bmatrix} n+i-\epsilon \\ n \end{bmatrix}_q \begin{bmatrix} n+j-\epsilon \\ n \end{bmatrix}_q (q;q)_n^2}{(q^{1+\epsilon};q)_{n-i}(q;q)_i(q^{1+\epsilon};q)_{n-j}(q;q)_j}.$$

From this the result is obtained by letting $\epsilon \to 0$. □

Lemma 2.2. *For* $n \geqslant 1$ *the following identity is valid*:

$$\sum_{k=0}^{n} (-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q \frac{1-q}{1-q^{k+1}} = 0.$$

Proof. For the q-Legendre polynomial $P_n(x\mid q)$ we get, by (14),

$$\int_0^1 P_n(x\mid q)\,\mathrm{d}_q x = \sum_{k=0}^{n} (-1)^k q^{\binom{k+1}{2}-kn} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q \frac{1-q}{1-q^{k+1}}.$$

On the other hand, it follows by the q-Rodrigues formula and (9) that

$$\int_0^1 P_n(x\mid q)\,\mathrm{d}_q x = \frac{(1-q)^n}{(q;q)_n}(f(1)-f(0)),$$

where

$$f(x) = \vartheta_q^{n-1}(x^n(x;q^{-1})_n).$$

However, it follows by (11)–(13) that $f(0) = f(1) = 0$, and multiplication by $q^{\binom{n}{2}}$ then ends the proof. □

3. Solutions of the recursion relation

Theorem 3.1. *Let* $(x)=(x_0, x_1, \ldots, x_j, \ldots)$ *be a sequence of complex numbers. Let*

$$f_n(x\,|\,q)=(1-q^{2n+1})\left(2q^n-(1+q^n)(1+q^{n+1})\sum_{j=0}^{n} q^{-jn}\begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n+j \\ j \end{bmatrix}_q x_j\right), \tag{15}$$

and consider sequences $(u)=(u_0, u_1, \ldots, u_n, \ldots)$ *satisfying the recursion formula*

$$(1-q^{n+1})(1+q^n)u_{n+1}-f_n(x\,|\,q)u_n+q^{2n-1}(1-q^n)(1+q^{n+1})u_{n-1}=0, \quad \textit{for } n \geqslant 0. \tag{16}$$

Then (16) *has two independent solutions* (p) *and* (q) *as follows. The element* p_n *is represented as*

$$p_n=\sum_{k=0}^{n}(-1)^k q^{\binom{n-k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q c_k, \tag{17}$$

where

$$\begin{aligned} &c_0=1, \\ &c_{k+1}=\sum_{i=0}^{k}(-1)^{k-i}q^{\binom{k-i}{2}}\sum_{j=k-i}^{k} q^{-jk}\begin{bmatrix} i+j \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}\begin{bmatrix} k \\ j \end{bmatrix}_q x_j c_i, \quad \textit{for } k \geqslant 0. \end{aligned} \tag{18}$$

The element q_n *is represented as*

$$q_n=\sum_{k=0}^{n}(-1)^k q^{\binom{n-k}{2}}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q d_k, \tag{19}$$

where

$$\begin{aligned} &d_0=0, \\ &d_{k+1}=\sum_{i=0}^{k}(-1)^{k-i}q^{\binom{k-i}{2}}\sum_{j=k-i}^{k} q^{-jk}\begin{bmatrix} i+j \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q x_j d_i+\frac{1-q}{1-q^{k+1}}, \quad \textit{for } k \geqslant 0. \end{aligned} \tag{20}$$

Corollary 3.2. *If* $x_j \in \mathbb{Z}$ *for* $j \in \mathbb{N}_0$, *then* $q^{(n-1)n(2n-1)/6}p_n$, $q^{(n-1)n(2n-1)/6}[1-q, 1-q^2, \ldots, 1-q^n]q_n \in \mathbb{Z}[q]$ *for all* n, *the symbol* $[\cdot]$ *denoting the least common multiple in* $\mathbb{Z}[q]$.

Remark 3.3. It follows immediately that by letting $q \to 1$ (after dividing by $1-q$ in (15) and (16)) the previous theorem is obtained.

Proof of Theorem 3.1. Let us denote the left-hand side of (16) by r_n. We shall first show that $r_n=0$ for $n \in \mathbb{N}$ when $(u)=(p)$. In r_n we substitute for p_n and f_n the expressions in (17) and

(15). This gives

$$
\begin{aligned}
r_n = {} & (1-q^{n+1})(1+q^n) \sum_{k=0}^{n+1} (-1)^k q^{\binom{n+1-k}{2}} \begin{bmatrix} n+1 \\ k \end{bmatrix}_q \begin{bmatrix} n+k+1 \\ k \end{bmatrix}_q c_k \\
& -(1-q^{2n+1}) \left(2q^n - (1+q^n)(1+q^{n+1}) \sum_{j=0}^{n} q^{-jn} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n+j \\ j \end{bmatrix}_q x_j \right) \\
& \times \sum_{k=0}^{n} (-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q c_k \\
& + q^{2n-1}(1-q^n)(1+q^{n+1}) \sum_{k=0}^{n-1} (-1)^k q^{\binom{n-k-1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q c_k .
\end{aligned}
$$

If we collect all terms with c_k and without any explicit x_j and perform some reductions and a shift of index, we obtain

$$
r_n = (1-q^{2n+1})(1+q^n)(1+q^{n+1}) s_n ,
$$

where

$$
\begin{aligned}
s_n = {} & - \sum_{k=0}^{n} (-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q c_{k+1} \\
& + \sum_{j=0}^{n} q^{-jn} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n+j \\ j \end{bmatrix}_q x_j \sum_{k=0}^{n} (-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q c_k .
\end{aligned}
$$

In the first sum we now substitute for c_{k+1}, $0 \leqslant k \leqslant n$, the expression (18). Rearranging the triple sum, we obtain

$$
\begin{aligned}
s_n = {} & - \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^i c_i x_j \sum_{k=0}^{n} q^{\binom{n-k}{2} + \binom{k-i}{2} - jk} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} i+j \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \\
& + \sum_{j=0}^{n} q^{-jn} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n+j \\ j \end{bmatrix}_q \sum_{k=0}^{n} (-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q x_j c_k .
\end{aligned}
$$

By Lemma 2.1 we finally get

$$
\begin{aligned}
& \sum_{k=0}^{n} q^{\binom{n-k}{2} + \binom{k-i}{2} - jk} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q \begin{bmatrix} i+j \\ k \end{bmatrix}_q \begin{bmatrix} k \\ i \end{bmatrix}_q \begin{bmatrix} k \\ j \end{bmatrix}_q \\
& \quad = q^{\binom{n-i}{2} - jn} \begin{bmatrix} n+j \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n+i \\ i \end{bmatrix}_q \begin{bmatrix} n \\ i \end{bmatrix}_q ,
\end{aligned}
$$

so that $s_n = 0$. This proves the first part of the theorem.

Proceeding in the same way with $(u) = (q)$, we obtain an additional term in s_n, namely

$$
- \sum_{k=0}^{n} (-1)^k q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n+k \\ k \end{bmatrix}_q \frac{1-q}{1-q^{k+1}} = 0,
$$

by Lemma 2.2. This proves the second part of the theorem. □

4. Open problems

The first two problems concern the generalized Legendre polynomials, cf. [4].

(1) Prove that the polynomials f_n in (1) provide all polynomials such that the corresponding sequence (u_n) is integral.

(2) For $x=(x_0, x_1, \ldots, x_N, 0, 0, \ldots)$ find — as in the case $N=0$ — explicit expressions for α in terms of $x_0, x_1, \ldots, x_N$. This would be very interesting in view of [4, Theorem 3].

(3) For $r=2, 3, \ldots$ the sequence $a_n=a_n(r)$ given by

$$a_n=\sum_{k=0}^{n}\binom{n}{k}^r\binom{n+k}{k}^r$$

can be written uniquely as

$$a_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}c_k,$$

where $c_k=c_k(r)\in\mathbb{Q}$ is independent on n. Note that the formula for a_0 determines c_0, the formula for a_1 then determines c_1, etc. Show that the sequence (c_k) is integral for all r. For $r=2$, (a_n) is the famous Apéry-sequence for $\zeta(3)$.

Added June 1992. Problem (3) $(r=2)$ was solved independently by Strehl (University of Erlangen-Nürnberg) and myself with

$$c_k(2)=\sum_{j=0}^{k}\binom{k}{j}^3,$$

shortly after that this formula had been observed numerically by Deuber, Thumser and Voigt (University of Bielefeld). Later Strehl solved the problem for $r=3$ with

$$c_k(3)=\sum_{j=0}^{k}\binom{2j}{j}^2\binom{k}{j}^2\binom{2j}{k-j}.$$

The problem remains unsolved for $r\geqslant 4$.

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