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Determinacy of a rational moment problem

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Abstract

We discuss a moment problem of Stieltjes type that is related to the theory of orthogonal rational functions. We obtain results which lead to a sufficient condition of Carleman type for determinacy of the moment problem. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

The moments c_n of a positive measure μ on the nonnegative real line $[0,\infty)$ are defined as

$$c_n = \int_0^\infty t^n \,\mathrm{d}\mu(t), \quad n = 0, 1, \ldots, \tag{1.1}$$

if these integrals are finite (see [1,26–28]). The *Stieltjes moment problem* associated with the sequence $\{c_n\}$ is called *determinate* if there is only one measure μ (with support in $[0,\infty)$) giving

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rise to the moments c_n through the formula (1.1). A well-known criterion by Carleman gives a sufficient condition for determinacy: *If*

$$\sum_{n=1}^{\infty} c_n^{-1/2n} = \infty, \tag{1.2}$$

then the Stieltjes moment problem is determinate. See e.g. [10–12,20,26,27]. Our treatment in this paper of a rational Stieltjes problem is especially indebted to a method developed by Karlsson and von Sydow in [20]. By an argument in [1, p. 25–26, Problem 4] (building on results by Wouk [29]) for a similar situation, and also from results in [20], it can even be shown that *if the Stieltjes moment problem is solvable and* (1.2) *is satisfied*, *then the Hamburger moment problem is determinate* (i.e., there is only one measure with support in $(-\infty,\infty)$ giving rise to the moments c_n through the formula $c_n = \int_{-\infty}^{\infty} t^n \,\mathrm{d}\mu(t)$, $n=0,1,\ldots$). We shall discuss in detail the analogous situation for the rational moment problem treated in this paper in Section 5.

The *strong Stieltjes moment problem* arises when also moments with negative index are given:

$$c_n = \int_0^{\infty} t^n \,\mathrm{d}\mu(t), \quad n = 0, \pm 1, \pm 2, \ldots \tag{1.3}$$

(See e.g. [17,19,25].)

A result analogous to Carleman's criterion was proved by Aldén [2,3]: If

$$\sum_{n=0}^{\infty} [(c_n)^{-1/2n} + (c_{-n})^{-1/2n}] = \infty, \tag{1.4}$$

then the strong Stieltjes moment problem is determinate (see also [15,24]). By an approach following that of Karlsson and von Sydow, two of the present authors [14] obtained essentially the same result. A condition equivalent to (1.4) is

$$\sum_{m=0}^{\infty} [(c_{2m})^{-1/4m} + (c_{-2m})^{-1/(4m-2)}] = \infty. \tag{1.5}$$

The classical moment problems are related to the theory of orthogonal polynomials. Similarly, the strong moment problems are related to the theory of orthogonal Laurent polynomials (see [13,17,18]). In this paper, we discuss a moment problem of Stieltjes type that is related to the theory of orthogonal rational functions. The basic theory of this problem is developed in [9]. We obtain results which also in this situation lead to a sufficient condition of Carleman type for determinacy of the moment problem.

Related problems have been treated by different methods, and related Carleman type conditions obtained. See especially the work by Lopez [21–23].

2. Orthogonal rational functions

The basic theory of orthogonal rational functions can be found in [8] and also in [4–6] for the situation discussed here where interpolation points may belong to the support of the measure.

Let $\{\alpha_k\}_{k=1}^{\infty}$ be a sequence of points on the real line $(-\infty,\infty)$. For some nonessential technical reasons, we assume that $\alpha_k \neq 0$ for all k. We might include the possibility that some or all of the points α_k could be the point at infinity (on the Riemann sphere), but that situation will not be considered in this paper.

We define

$$\omega_0 = 1, \quad \omega_n(z) = (z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n), \;\; n = 1, 2, \ldots \tag{2.1}$$

The space $\mathscr{L}$ is defined to be the linear span of the sequence $\{1/\omega_n\}_{n=0}^{\infty}$. We note that $\mathscr{L}$ consists of all functions of the form s_n/ω_n for some n, where s_n is a polynomial of degree at most n.

Other bases for $\mathscr{L}$ than $\{1/\omega_n\}_{n=0}^{\infty}$ may be useful. We shall in this paper mostly consider the situation when $\alpha_{2p} \neq \alpha_{2q+1}$ for all p and q. It is readily verified that the sequence $\{b_n\}_{n=0}^{\infty}$ is a basis for $\mathscr{L}$, where

$$b_0 = 1, \tag{2.2}$$

$$b_{2m}(z) = \frac{z^{2m}}{(z - \alpha_1)(1 - z/\alpha_2)\cdots(z - \alpha_{2m-1})(1 - z/\alpha_{2m})}, \tag{2.3}$$

$$b_{2m+1}(z) = \frac{1}{(z - \alpha_1)(1 - z/\alpha_2)\cdots(1 - z/\alpha_{2m})(z - \alpha_{2m+1})}. \tag{2.4}$$

In the limiting situation $\alpha_{2m} \to \infty, \alpha_{2m+1} \to 0$, these functions become $b_0 = 1$, $b_{2m} = z^m$, $b_{2m+1} = 1/z^{m+1}$, which is a standard basis for the Laurent polynomials.

Let M be a positive linear functional on the product space $\mathscr{L} \cdot \mathscr{L}$ (consisting of products $f \cdot g$, with $f, g \in \mathscr{L}$). The *moments* $\mu_{m,n}$ are defined by

$$\mu_{m,n} = M[b_m \cdot b_n], \quad m, n = 0, 1, 2, \ldots \,. \tag{2.5}$$

For simplicity we normalize such that $\mu_{0,0} = 1$.

A positive measure with infinite support in $(-\infty, \infty)$ is said to solve the *rational* (*Hamburger*) *moment problem* (*RMP*) *on* $\mathscr{L} \cdot \mathscr{L}$ if

$$\mu_{m,n} = \int_{-\infty}^{\infty} b_m(t) b_n(t) \, \mathrm{d}\mu(t), \quad m, n = 0, 1, 2, \ldots, \tag{2.6}$$

and to solve the rational moment problem (*Hamburger*) (RMP) on $\mathscr{L}$ if

$$\mu_{0,n} = \int_{-\infty}^{\infty} b_n(t) \, \mathrm{d}\mu(t), \quad n = 0, 1, 2, \ldots \,. \tag{2.7}$$

A measure which solves the moment problem on $\mathscr{L} \cdot \mathscr{L}$ obviously also solves the moment problem on $\mathscr{L}$.

We shall in the following assume that the moment problem on $\mathscr{L} \cdot \mathscr{L}$ has at least one solution μ. An inner product $\langle \cdot, \cdot \rangle$ is then defined on $\mathscr{L}$ by

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} \, \mathrm{d}v(t), \quad f, g \in \mathscr{L}, \tag{2.8}$$

where v is *any solution of the moment problem on* $\mathscr{L} \cdot \mathscr{L}$.

Let $\{\phi_n\}_{n=0}^{\infty}$ denote an orthonormal sequence in $\mathscr{L}$ obtained by orthonormalization with respect to $\langle \cdot, \cdot \rangle$ of the basis $\{1/\omega_n\}$, (or equivalently of the basis $\{b_n\}$). The associated orthogonal rational functions σ_n are defined by

$$\sigma_n(z) = M\left[\frac{\phi_n(t) - \phi_n(z)}{t - z}\right], \quad n = 0, 1, 2, \ldots \tag{2.9}$$

(the functional operating on its argument as a function of t). Equivalently,

$$\sigma_n(z) = \int_{-\infty}^{\infty} \frac{\phi_n(t) - \phi_n(z)}{t - z} \, \mathrm{d}v(t), \tag{2.10}$$

where v is *any solution of the moment problem on* $\mathscr{L}$.

By the *Stieltjes transform* of a finite measure μ, we shall here mean the function

$$S(z, \mu) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\mu(t)}{z - t}. \tag{2.11}$$

The measure μ is determined by its Stieltjes transform $S(z, \mu)$, which means that if two measures have the same Stieltjes transform, then they are equal.

The rational function $\sigma_n(z)/\phi_n(z)$ is an $[n-1/n]$ multi-point Padé approximant of $S(z, \mu)$ at the table $\{\infty, \alpha_1, \alpha_1, \ldots, \alpha_{n-1}, \alpha_{n-1}, \alpha_n\}$. See [7, Theorem 5.2].

We will need an estimate of the *error term* $\sigma_n(z)/\phi_n(z) - S(z, \mu)$. We have the following result.

Proposition 2.1. *Let μ be an arbitrary solution of the RMP on $\mathscr{L}$. Then the following formula is valid*:

$$\frac{\sigma_n(z)}{\phi_n(z)} - S(z, \mu) = \frac{1}{\phi_n(z)} \int_{-\infty}^{\infty} \frac{\phi_n(t)}{t - z} \, \mathrm{d}\mu(t). \tag{2.12}$$

Proof. This follows immediately from (2.10) and (2.11). □

3. A Stieltjes problem

We deal in this paper with a special situation, which may be considered as an extension of the Stieltjes situation for the classical (one-point) and strong (two-point) problems. Contributions to a theory of this rational Stieltjes problem are given in [9].

We assume the existence of real numbers α and β such that

$$\alpha_{2m} \leqslant \alpha < \beta \leqslant \alpha_{2m-1} \leqslant 0, \quad m = 1, 2, \ldots \tag{3.1}$$

and such that for all measures μ considered, the support is contained in the nonnegative real line:

$$\mathrm{supp}(\mu) \subset [0, \infty). \tag{3.2}$$

Thus we assume the existence of at least one positive measure μ with support in $[0, \infty)$ which solves the moment problem on $\mathscr{L} \cdot \mathscr{L}$ for the sequence $\{\mu_{m,n}\}_{m,n}$, and we only consider solutions of the moment problem on $\mathscr{L}$ and on $\mathscr{L} \cdot \mathscr{L}$ with support in $[0, \infty)$. Such solutions are called solutions of the *rational Stieltjes moment problem* (*RSMP*) on $\mathscr{L}$ and $\mathscr{L} \cdot \mathscr{L}$, respectively. (In [9] we considered a slightly more general situation, where 0 in (3.1) and (3.2) is replaced by an arbitrary real point γ.)

The orthogonal rational function ϕ_n may be written in the form

$$\phi_n(z) = \frac{p_n(z)}{\omega_n(z)}, \tag{3.3}$$

where $p_n(z)$ is a polynomial of exact degree n. The polynomial p_n has n simple zeros in $(0,\infty)$. We assume that ϕ_n is normalized such that

$$\phi_{2m}(x) > 0, \quad \phi_{2m+1}(x) < 0, \quad x \in (\alpha,\beta). \tag{3.4}$$

(This determination is unambiguous, $\phi_n(x)$ having constant sign in (α,β).)

The function ϕ_n can be written as

$$\phi_n = v_n b_n + \cdots + u_n b_0. \tag{3.5}$$

(We may call v_n the *leading coefficient* and u_n the *trailing coefficient* with respect to the basis $\{b_n\}$.) It easily follows from the definitions (2.1), (2.3)–(2.4), and (3.4) and (3.5) that we have

$$p_{2m}(\alpha_{2m}) = (-1)^m \alpha_2\alpha_4\cdots\alpha_{2m}(\alpha_{2m})^{2m} v_{2m}, \tag{3.6}$$

$$p_{2m+1}(\alpha_{2m+1}) = (-1)^m \alpha_2\alpha_4\cdots\alpha_{2m} v_{2m+1}. \tag{3.7}$$

The functions $\{\phi_m, \sigma_n\}$ satisfy a three-term recurrence relation of the following form (see [4,6–8]):

$$\begin{bmatrix}\sigma_n(z)\\ \phi_n(z)\end{bmatrix} = \frac{U_n(z-\alpha_{n-2}) + V_n(z-\alpha_{n-1})}{z-\alpha_n}\begin{bmatrix}\sigma_{n-1}(z)\\ \phi_{n-1}(z)\end{bmatrix} + \frac{W_n(z-\alpha_{n-2})}{z-\alpha_n}\begin{bmatrix}\sigma_{n-2}(z)\\ \phi_{n-2}(z)\end{bmatrix}, \quad n = 3,4,\ldots, \tag{3.8}$$

$$\begin{bmatrix}\sigma_2(z)\\ \phi_2(z)\end{bmatrix} = \frac{U_2(z-\alpha_2) + V_2(z-\alpha_1)}{z-\alpha_2}\begin{bmatrix}\sigma_1(z)\\ \phi_1(z)\end{bmatrix} + \frac{W_2}{z-\alpha_2}\begin{bmatrix}\sigma_0(z)\\ \phi_0(z)\end{bmatrix}, \tag{3.9}$$

$$\begin{bmatrix}\sigma_1(z)\\ \phi_1(z)\end{bmatrix} = \frac{U_1(z-\alpha_1) + V_1(z-\alpha_2)}{z-\alpha_1}\begin{bmatrix}\sigma_0(z)\\ \phi_0(z)\end{bmatrix} + \frac{W_1}{z-\alpha_1}\begin{bmatrix}\sigma_{-1}(z)\\ \phi_{-1}(z)\end{bmatrix}, \tag{3.10}$$

$$\begin{bmatrix}\sigma_0(z)\\ \phi_0(z)\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}, \qquad \begin{bmatrix}\sigma_{-1}(z)\\ \phi_{-1}(z)\end{bmatrix} = \begin{bmatrix}-\mu_{0,1}\\ 0\end{bmatrix}. \tag{3.11}$$

Here

$$U_n < 0 \quad \text{for } n = 1,2,3,\ldots \tag{3.12}$$

$$V_n > 0 \quad \text{for } n = 1,3,4,\ldots \tag{3.13}$$

$$W_n > 0 \quad \text{for } n = 1,2,3,\ldots \tag{3.14}$$

We also find that (see [7, Proposition 3.4])

$$W_1W_2\cdots W_n = (-1)^n(\mu_{0,1})^{-1}(\alpha_{n-2}-\alpha_{n-1})U_n. \tag{3.15}$$

The coefficients U_n, V_n can be expressed as follows:

$$U_1 = \frac{p_1(\alpha_2)}{\alpha_2-\alpha_1}, \quad U_2 = \frac{p_2(\alpha_1)}{p_1(\alpha_1)(\alpha_1-\alpha_2)},$$

$$U_n = \frac{p_n(\alpha_{n-1})}{p_{n-1}(\alpha_{n-1})(\alpha_{n-1}-\alpha_{n-2})}, \quad n = 3,4,\ldots \tag{3.16}$$

$$V_1 = \frac{p_1(\alpha_1)}{\alpha_1-\alpha_2}, \quad V_n = \frac{p_n(\alpha_{n-1})}{p_{n-1}(\alpha_{n-2})(\alpha_{n-2}-\alpha_{n-1})}, \quad n = 3,4,\ldots \tag{3.17}$$

For the approximants $\{\sigma_n(z)/\phi_n(z)\}$ of the continued fraction associated with the three-term recurrence relation given by (3.8)–(3.11) the following fundamental result holds (see [7]):

Theorem 3.1. *For $x \in (\alpha, \beta)$ the sequence $\{\sigma_{2m}(x)/\phi_{2m}(x)\}$ is decreasing and the sequence $\{\sigma_{2m+1}(x)/\phi_{2m+1}(x)\}$ is increasing. For $z \in \mathbb{C} \setminus [0, \infty)$, $\{\sigma_{2m}(z)/\phi_{2m}(z)\}$ converges to $S(z, \mu^{(0)})$ and $\{\sigma_{2m+1}(z)/\phi_{2m+1}(z)\}$ converges to $S(z, \mu^{(\infty)})$, where $\mu^{(0)}$ and $\mu^{(\infty)}$ are solutions of the RSMP on $\mathscr{L}$.*

4. Estimates on coefficients

We shall in the following restrict ourselves to consider the following situation:

$$\text{The sequence } \{\alpha_{2m}\}_{m=1}^{\infty} \text{ decreases to } -\infty, \tag{4.1}$$

$$\text{The sequence } \{\alpha_{2m+1}\}_{m=0}^{\infty} \text{ increases to } 0. \tag{4.2}$$

To discuss determinacy of the moment problem, we need estimates on the differences $\sigma_n(x)/\phi_n(x) - \sigma_{n-2}(x)/\phi_{n-2}(x)$. It follows from (3.15) and [7, formula (4.10)] that for $n = 3, 4, \dots$ we may write

$$\frac{\sigma_n(z)}{\phi_n(z)} - \frac{\sigma_{n-2}(z)}{\phi_{n-2}(z)} = \Delta_n(z) \frac{1}{\phi_n(z)\phi_{n-2}(z)}, \tag{4.3}$$

where

$$\Delta_n(z) = \frac{U_{n-1}(\alpha_{n-3} - \alpha_{n-2})[U_n(z - \alpha_{n-2}) + V_n(z - \alpha_{n-1})]}{(z - \alpha_n)(z - \alpha_{n-1})(z - \alpha_{n-2})}. \tag{4.4}$$

Lemma 4.1. *For $x \in (\alpha, \beta)$ and $n = 3, 4, \dots$ we have*

$$|\Delta_n(z)| > \frac{V_n |U_{n-1}(\alpha_{n-3} - \alpha_{n-2})|}{|(x - \alpha_n)(x - \alpha_{n-2})|}. \tag{4.5}$$

Proof. It follows from (3.1) and (3.12)–(3.13) that $U_n(x - \alpha_{n-2})$ and $V_n(x - \alpha_{n-1})$ have the same sign for $x \in (\alpha, \beta)$. Thus $|U_n(x - \alpha_{n-2}) + V_n(x - \alpha_{n-1})| > |V_n(x - \alpha_{n-1})|$. From this and the positivity of $\mu_{0,1}$ and V_n the inequality follows. □

Lemma 4.2. *The following results hold*:

(A) *The function $|p_n(x)|$ is decreasing with increasing x for $x < 0$.*
(B) *The function $|p_n(x)/x^n|$ is increasing with increasing x for $x < 0$.*

Proof. The polynomial p_n has n simple zeros in $(0, \infty)$, hence p_n' has $n - 1$ zeros in $(0, \infty)$, and so p_n is monotonic for $x < 0$. Since $|p_n(x)| \to \infty$ as $x \to -\infty$, (A) follows.

We have $[p_n(x)/x^n]' = (x p_n'(x) - n p_n(x))/x^{n+1}$. The numerator polynomial has degree at most $n - 1$. Since $p_n(x)/x^n$ has at least $n - 1$ extrema for $x > 0$, it follows that $p_n(x)/x^n$ is monotonic for $x < 0$. Since $|p_n(x)/x^n| \to \infty$ for $x \to 0^-$, we conclude that (B) holds. □

It follows from (3.16) and (3.17) that we may write

$$V_n U_{n-1}(\alpha_{n-3} - \alpha_{n-2}) = \frac{p_n(\alpha_{n-2})}{p_{n-2}(\alpha_{n-2})(\alpha_{n-1} - \alpha_{n-2})}, \quad n = 3, 4, \dots \tag{4.6}$$

Lemma 4.3. *For $x \in (\alpha, \beta)$, the following inequalities hold*:
(A)

$$|V_{2m}U_{2m-1}(\alpha_{2m-3}-\alpha_{2m-2})| \geqslant \frac{|\alpha_{2m-2}|^2|\alpha_{2m}|}{|\alpha_{2m-2}-\alpha_{2m-1}|}\frac{|v_{2m}|}{|v_{2m-2}|}, \quad m \geqslant 2, \tag{4.7}$$

(B)

$$|V_{2m+1}U_{2m}(\alpha_{2m-2}-\alpha_{2m-1})| \geqslant \frac{|\alpha_{2m}|}{|\alpha_{2m-1}-\alpha_{2m}|}\frac{|v_{2m+1}|}{|v_{2m-1}|}, \quad m \geqslant 1. \tag{4.8}$$

Proof. (A) It follows from (3.6) and (4.6) that we may write

$$V_{2m}U_{2m-1}(\alpha_{2m-3}-\alpha_{2m-2}) = \frac{\alpha_2\alpha_4\cdots\alpha_{2m}(\alpha_{2m})^{2m}v_{2m}}{\alpha_2\alpha_4\cdots\alpha_{2m-2}(\alpha_{2m-2})^{2m-2}v_{2m-2}(\alpha_{2m-2}-\alpha_{2m-1})}\frac{p_{2m}(\alpha_{2m-2})}{p_{2m}(\alpha_{2m})}. \tag{4.9}$$

Lemma 4.2(B) together with (4.1) implies that

$$\left|\frac{p_{2m}(\alpha_{2m-2})}{p_{2m}(\alpha_{2m})}\right| \geqslant \frac{|\alpha_{2m-2}|^{2m}}{|\alpha_{2m}|^{2m}}.$$

This together with (4.9) gives (4.7).
(B) Similarly from (3.7) and (4.6) we find that

$$V_{2m+1}U_{2m}(\alpha_{2m-2}-\alpha_{2m-1}) = -\frac{\alpha_2\alpha_4\cdots\alpha_{2m}v_{2m+1}}{\alpha_2\alpha_4\cdots\alpha_{2m-2}v_{2m-1}(\alpha_{2m-1}-\alpha_{2m})}\frac{p_{2m+1}(\alpha_{2m-1})}{p_{2m+1}(\alpha_{2m+1})}. \tag{4.10}$$

Lemma 4.2(A) together with (4.2) implies that $p_{2m+1}(\alpha_{2m-1}) > p_{2m+1}(\alpha_{2m+1})$. This together with (4.10) gives (4.8). □

Proposition 4.4. *For $x \in (\alpha, \beta)$, the following inequalities hold*:
(A)

$$|\Delta_{2m}(x)| > |v_{2m}/v_{2m-2}|, \quad m = 2, 3, \ldots \tag{4.11}$$

(B)

$$|\Delta_{2m+1}(x)| > \frac{1}{\alpha^2}|v_{2m+1}/v_{2m-1}|, \quad m = 1, 2, \ldots \tag{4.12}$$

with $\Delta_n(x)$ as given by (4.4).

Proof. Because of (3.1) we have $|\alpha_{2p}-\alpha_{2q+1}| < |\alpha_{2p}|$, $|\alpha_{2p}-x| < |\alpha_{2p}|$, and $|\alpha_{2q+1}-x| < |\alpha|$ for all p and q and for all $x \in (\alpha, \beta)$. We find from (4.5) and (4.7) that

$$|\Delta_{2m}(x)| > \frac{|\alpha_{2m-2}|^2|\alpha_{2m}|}{|\alpha_{2m-2}||\alpha_{2m}||\alpha_{2m-2}|}\left|\frac{v_{2m}}{v_{2m-2}}\right|,$$

which is (4.11). Similarly, we find from (4.5) and (4.8) that

$$|\Delta_{2m+1}(x)| > \frac{1}{\alpha^2}\frac{|\alpha_{2m}|}{|\alpha_{2m}|}\left|\frac{v_{2m+1}}{v_{2m-1}}\right|,$$

which is (4.12). □

5. Uniqueness of solutions

We shall in this section give sufficient conditions in terms of the leading coefficients v_n for determinacy of the RSMP on $\mathscr{L}\cdot\mathscr{L}$.

Proposition 5.1. *The following implications hold*:
(A) *If*

$$\sum_{m=1}^{\infty}\left|\frac{v_{2m}}{v_{2m-2}}\right|^{1/2}=\infty, \quad \textit{then}\sum_{m=0}^{\infty}|\phi_{2m}(x)|^2=\infty \ \ \textit{for } x\in(\alpha,\beta), \tag{5.1}$$

(B) *If*

$$\sum_{m=1}^{\infty}\left|\frac{v_{2m+1}}{v_{2m-1}}\right|^{1/2}=\infty, \quad \textit{then}\sum_{m=0}^{\infty}|\phi_{2m+1}(x)|^2=\infty \ \ \textit{for } x\in(\alpha,\beta). \tag{5.2}$$

Proof. It follows from Proposition 4.4 that there exists a constant K independent of n and x such that

$$\left|\frac{v_n}{v_{n-2}}\right|\leqslant K|\varDelta_n(x)|. \tag{5.3}$$

Recall from (4.3) that

$$\varDelta_n(x)=\left[\frac{\sigma_n(x)}{\phi_n(x)}-\frac{\sigma_{n-2}(x)}{\phi_{n-2}(x)}\right]\phi_n(x)\phi_{n-2}(x). \tag{5.4}$$

If

$$\sum_{m=1}^{\infty}\left|\frac{v_{2m}}{v_{2m-2}}\right|^{1/2}=\infty \quad \text{then} \sum_{m=1}^{\infty}|\varDelta_{2m}(x)|^{1/2}=\infty$$

and hence by (5.4), Schwartz' inequality and Theorem 3.1

$$\left\{\sum_{m=1}^{\infty}\left[\frac{\sigma_{2m-2}(x)}{\phi_{2m-2}(x)}-\frac{\sigma_{2m}(x)}{\phi_{2m}(x)}\right]\right\}\left\{\sum_{m=1}^{\infty}|\phi_{2m}(x)\phi_{2m-2}(x)|\right\}=\infty. \tag{5.5}$$

Similarly, we find that $\sum_{m=1}^{\infty}|v_{2m+1}/v_{2m-1}|^{1/2}=\infty$ implies

$$\left\{\sum_{m=1}^{\infty}\left[\frac{\sigma_{2m+1}(x)}{\phi_{2m+1}(x)}-\frac{\sigma_{2m-1}(x)}{\phi_{2m-1}(x)}\right]\right\}\left\{\sum_{m=1}^{\infty}|\phi_{2m+1}(x)\phi_{2m-1}(x)|\right\}=\infty. \tag{5.6}$$

Again by Theorem 3.1 we conclude that the series to the left in (5.5) and (5.6) converge to finite values, hence $\sum_{m=1}^{\infty}|\phi_{2m}(x)\phi_{2m-2}(x)|=\infty$ holds if (5.5) holds and $\sum_{m=1}^{\infty}|\phi_{2m+1}(x)\phi_{2m-1}(x)|=\infty$ if (5.6) holds. Divergence of $\sum_{m=0}^{\infty}|\phi_{2m}(x)|^2$ and $\sum_{m=0}^{\infty}|\phi_{2m+1}(x)|^2$, respectively, now easily follows by Schwartz' inequality. □

Let v be an arbitrary solution of the RSMP on $\mathscr{L}\cdot\mathscr{L}$. Recall that the *error term* $E_n(z,v)$ is defined to be

$$E_n(z,v)=\frac{\sigma_n(x)}{\phi_n(x)}-S(z,v), \tag{5.7}$$

(cf. Section 2).

Proposition 5.2. *Let v be an arbitrary solution of the RSMP on $\mathscr{L}\cdot\mathscr{L}$, and let $x\in(\alpha,\beta)$.*

(A) *If $\sum_{m=0}^{\infty}|\phi_{2m}(x)|^2=\infty$, then $E_{2m}(x,v)$ tends to zero as $m\to\infty$,*
(B) *If $\sum_{m=0}^{\infty}|\phi_{2m+1}(x)|^2=\infty$, then $E_{2m+1}(x,v)$ tends to zero as $m\to\infty$.*

Proof. According to Proposition 2.1 we have

$$E_n(z,v)=\frac{1}{\phi_n(x)}\int_0^{\infty}\frac{\phi_n(t)}{t-z}\,\mathrm{d}v(t). \tag{5.8}$$

Let $x\in(\alpha,\beta)$. Then the function $t\to 1/(t-x)$ is square integrable with respect to v. Its Fourier coefficients with respect to the system $\{\phi_n\}$ are $E_n(x,v)\phi_n(x)$ according to (5.8), hence by Bessel's inequality

$$\sum_{n=0}^{\infty}|E_n(z,v)|^2|\phi_n(x)|^2<\infty. \tag{5.9}$$

(Note that the use of Bessel's inequality requires that the system $\{\phi_n\}$ is orthogonal with respect to v, and hence that v is a solution of the RSMP on $\mathscr{L}\cdot\mathscr{L}$, not only on $\mathscr{L}$.)

Assume e.g., that $\sum_{m=0}^{\infty}|\phi_{2m}(x)|^2=\infty$. It then follows from (5.9) that at least a subsequence of $\{E_{2m}(x,v)\}$ must tend to zero, and hence the whole sequence tends to zero since it is monotonic by Theorem 3.1. Similarly we find that $\{E_{2m+1}(x,v)\}$ tends to zero if $\sum_{m=0}^{\infty}|\phi_{2m+1}(x)|^2=\infty$. □

Theorem 5.3. *Assume that $\sum_{n=2}^{\infty}|v_n/v_{n-2}|^{1/2}=\infty$. Then the RSMP on $\mathscr{L}\cdot\mathscr{L}$ is determinate.*

Proof. If $\sum_{n=2}^{\infty}|v_n/v_{n-2}|^{1/2}=\infty$, then at least one of the series

$$\sum_{m=1}^{\infty}|v_{2m}/v_{2m-2}|^{1/2}\quad\text{and}\quad\sum_{m=1}^{\infty}|v_{2m+1}/v_{2m-1}|^{1/2}$$

diverges. It follows from Proposition 5.1 that at least one of the series

$$\sum_{m=0}^{\infty}|\phi_{2m}(x)|^2\quad\text{and}\quad\sum_{m=0}^{\infty}|\phi_{2m+1}(x)|^2,$$

diverges for all $x\in(\alpha,\beta)$. Proposition 5.2 then implies that at least one of the sequences $\{E_{2m}(x,v)\}$ and $\{E_{2m+1}(x,v)\}$ tends to zero for all solutions v of the RSMP on $\mathscr{L}\cdot\mathscr{L}$. Consequently at least one of the sequences $\{\sigma_{2m}(x)/\phi_{2m}(x)\}$ and $\{\sigma_{2m+1}(x)/\phi_{2m+1}(x)\}$ tends to $S(x,v)$ for all solutions of the RSMP on $\mathscr{L}\cdot\mathscr{L}$ and all $x\in(\alpha,\beta)$. In any case, $S(x,v)$ is then the same function on (α,β) for all solutions v of the RSMP on $\mathscr{L}\cdot\mathscr{L}$, which implies that the solution is unique. □

Remark 5.4. It follows from Lemma 11.7.3 and Theorem 11.7.5 of [8] that if $\sum_{n=0}^{\infty}|\phi_n(z)|^2$ converges for some $z\in\mathbb{C}\setminus\{\mathbb{R}\cup\{\boldsymbol{i}\}\cup\{-\boldsymbol{i}\}\}$, then $\sum_{n=0}^{\infty}|\phi_n(z)|^2$ converges for every $z\in\mathbb{C}\setminus\{\mathbb{R}\cup\{\boldsymbol{i}\}\cup\{-\boldsymbol{i}\}\}$. Actually the proof shows that if $\sum_{n=0}^{\infty}|\phi_n(z)|^2$ converges for some $z\in\mathbb{C}\setminus\{\mathbb{R}\cup\{\boldsymbol{i}\}\cup\{-\boldsymbol{i}\}\}$, then $\sum_{n=0}^{\infty}|\phi_n(z)|^2$ converges for every $z\in\mathbb{C}_\alpha:=\mathbb{C}\setminus\{\hat{\mathbb{A}}\cup\{\boldsymbol{i}\}\cup\{-\boldsymbol{i}\}\}$ where $\hat{\mathbb{A}}$ is the closure of the set $\mathbb{A}$ of interpolation points α_n. Furthermore, it follows from Lemma 11.7.3 and Corollary 11.8.3 of [8] that if $\sum_{n=0}^{\infty}|\phi_n(z)|^2$ diverges for some (or all) $z\in\mathbb{C}\setminus\{\mathbb{R}\cup\{\boldsymbol{i}\}\cup\{-\boldsymbol{i}\}\}$, then the rational

(Hamburger) moment problem on $\mathscr{L} \cdot \mathscr{L}$ (see Section 1) is determinate. Thus we may conclude from Proposition 5.1 that the following stronger version of Theorem 5.3 holds.

Assume that (4.1) *and* (4.2) *hold, that the RSMP on* $\mathscr{L} \cdot \mathscr{L}$ *is solvable and that*

$$\sum_{n=2}^{\infty} |v_n/v_{n-2}|^{1/2} = \infty.$$

Then the rational (*Hamburger*) *moment problem on* $\mathscr{L} \cdot \mathscr{L}$ *is determinate.*

6. A Carleman condition

We shall briefly point out how we can obtain from Theorem 5.3 a sufficient condition of Carleman type for the determinacy of the RSMP on $\mathscr{L} \cdot \mathscr{L}$. Let μ be any solution of the RSMP on $\mathscr{L} \cdot \mathscr{L}$. We recall the definition (2.5), which means that we may write the moments $\mu_{m,n}$ as

$$\mu_{m,n} = \int_0^{\infty} b_m(t) b_n(t)\, \mathrm{d}\mu(t), \quad m,n = 0,1,2,\ldots \tag{6.1}$$

Lemma 6.1. *For* $n = 0,1,2,\ldots$ *we have*

$$|v_n| \geqslant \frac{1}{\sqrt{\mu_{n,n}}}. \tag{6.2}$$

Proof. We find from the orthogonality of $\{\phi_n\}$ and the representation (3.5) that we may write

$$1 = v_n \int_0^{\infty} b_n(t) \phi_n(t)\, \mathrm{d}\mu(t). \tag{6.3}$$

By using Schwartz' inequality in the integral we obtain

$$1 \leqslant |v_n| \sqrt{\mu_{n,n}},$$

which is (6.2). $\square$

Theorem 6.2. *Assume that*

$$\sum_{n=0}^{\infty} \frac{1}{(\mu_{n,n})^{1/2n}} = \infty. \tag{6.4}$$

Then the RSMP on $\mathscr{L} \cdot \mathscr{L}$ *is determinate.*

Proof. Carleman's inequality for nonnegative numbers $a_1, a_2, \ldots, a_k, \ldots$ may be written

$$\sum_{k=1}^{\infty} a_k \geqslant \frac{1}{e} \sum_{m=1}^{\infty} (a_1 a_2 \cdots a_m)^{1/m} \tag{6.5}$$

(see e.g. [16]). Applying this inequality to the situation $a_k = |v_{2k}/v_{2k-2}|^{1/2}$ we obtain

$$\sum_{k=1}^{\infty} \left| \frac{v_{2k}}{v_{2k-2}} \right|^{1/2} \geqslant \frac{1}{e} \sum_{m=1}^{\infty} |v_{2m}|^{1/2m}. \tag{6.6}$$

Similarly by applying the inequality to the situation $a_k = |v_{2k-1}/v_{2k-3}|^{1/2}$ (with $v_{-1} = 1$) we obtain (with m replaced by $m+1$)

$$\sum_{k=1}^{\infty} \left|\frac{v_{2k-1}}{v_{2k-3}}\right|^{1/2} \geqslant \frac{1}{e} \sum_{m=1}^{\infty} |v_{2m+1}|^{1/(2m+2)}. \tag{6.7}$$

If (6.4) holds, then at least one of the series

$$\sum_{m=1}^{\infty} \frac{1}{(\mu_{2m,2m})^{1/4m}} \quad \text{and} \quad \sum_{m=1}^{\infty} \frac{1}{(\mu_{2m+1,2m+1})^{1/4m+4}}$$

diverges. (Note that $(\mu_{2m+1,2m+1})^{1/4m+2} > (\mu_{2m+1,2m+1})^{1/4m+4}$ if $\mu_{2m+1,2m+1} > 1$. If $\mu_{2m+1,2m+1} \leqslant 1$ for infinitely many m, then the series

$$\sum_{m=1}^{\infty} \frac{1}{(\mu_{2m+1,2m+1})^{(1/4m+4)}}$$

trivially diverges.) It follows from (6.2) and (6.6), (6.7) that also at least one of the series

$$\sum_{k=1}^{\infty} \left|\frac{v_{2k}}{v_{2k-2}}\right|^{1/2} \quad \text{and} \quad \sum_{k=1}^{\infty} \left|\frac{v_{2k+1}}{v_{2k-1}}\right|^{1/2}$$

diverges, which means that the series $\sum_{n=1}^{\infty} |v_n/v_{n-2}|^{1/2}$ diverges. The result now follows from Theorem 5.3. $\square$

Remark 6.3. In the limiting situation $\alpha_{2m} = -\infty$, $\alpha_{2m+1} = 0$ for all m, the moments $\mu_{n,n}$ become $\mu_{2m-1,2m-1} = c_{-2m}$ and $\mu_{2m,2m} = c_{2m}$. The condition $\sum_{n=0}^{\infty} 1/(\mu_{n,n})^{1/2n}$ becomes

$$\sum_{m=0}^{\infty} \frac{1}{(c_{-2m})^{1/4m-2}} + \sum_{m=0}^{\infty} \frac{1}{(c_{2m})^{1/4m}} = \infty.$$

(see Section 1.)

Remark 6.4. Taking into account the Remark 5.4, we find that Theorem 6.2 can be strengthened as follows.

Assume that (4.1) *and* (4.2) *hold, that the RSMP on* $\mathscr{L} \cdot \mathscr{L}$ *is solvable and that*

$$\sum_{n=0}^{\infty} \frac{1}{(\mu_{n,n})^{1/2n}} = \infty.$$

Then the rational (*Hamburger*) *moment problem on* $\mathscr{L} \cdot \mathscr{L}$ *is determinate.*

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